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Boundary parametrization of self-affine tiles

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Abstract. A standard way to parametrize the boundary of a connected fractal tile T is proposed. The parametrization is Hölder continuous from $\boldsymbol{R}/\boldsymbol{Z}$ to ∂T and fixed points of ∂T have algebraic preimages. A class of planar tiles is studied in detail as sample cases and a relation with the recurrent set method by Dekking is discussed. When the tile T is a topological disk, this parametrization is a bi-Hölder homeomorphism.

1. Introduction.

A tiling consists of a compact set $T \subset \boldsymbol{R}^d$ and translation vectors $\mathscr{J}$ such that $T + \mathscr{J}$ covers $\boldsymbol{R}^d$ without overlaps of positive measure. Here T is equal to the closure of its interior. We also assume that T is *self-affine*, that is, there exists an expanding $d \times d$ matrix $\boldsymbol{A}$ (each eigenvalue is greater than one in modulus) such that $\boldsymbol{A}T$ is divided into some translates of T:

$$\boldsymbol{A}T = T + \mathscr{D} = \bigcup_{a \in \mathscr{D}} (T + a), \tag{1.1}$$

where $\mathscr{D} \subset \mathscr{J}$. This is called 'inflation-subdivision' principle. Self-affine tiles give a higher dimensional analogue of substitution dynamical systems, a model of self-similar structures that appears in many branches of mathematics. A self-affine tile T and its boundary ∂T often show fractal shapes, and their topological study is much more difficult than for a substitution dynamical system where the associated tiles are intervals.

In this paper, we are interested in a detailed description of the boundary of T, especially in giving a parametrization of ∂T. It was shown by Tang [**48**] that the boundary of a connected self-affine tile is a locally connected continuum. Therefore, by the theorem of Hahn-Mazurkewicz [**52**], it is the continuous image of

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the unit interval. The goal of this paper is to construct an explicit parametrization of ∂T which we think to be *standard* in topological, geometrical and measure theoretical aspects. Moreover, it will be applicable to further study of T and ∂T. Our parametrization is almost bijective and the set of non-injective points are recognized by a Büchi automaton. As a result, this gives a direct way to show that T is a topological disk when we treat planar tiles, and the parametrization is a homeomorphism in this case. We shall give such an application in the last section. Indeed, we will give a new short proof for the characterization of disk-like tiles among the class of tiles associated to quadratic number systems (see also [**4**]). Our construction also allows to obtain fractal tiles by the recurrent set method introduced by Dekking in [**8**], [**9**]. This method consists in constructing tiles from a suitable substitution on a free group whose generators represent the boundary of a polygon. For a given tile it is a hard task to find out the appropriate substitution. From the construction of our parametrization, we will see that as a basic polygon, the hexagon works better than the square that is used in the literature. For the class of tiles mentioned above, we will introduce a tiling by hexagons and obtain the right substitution without effort. We are expecting this parametrization to give a precise description of the topology of non disk-like tiles. We already found a description of the Heighway dragon but we leave this discussion for a forthcoming paper.

We shall now briefly review several historical approaches to describe the fractal boundary of T. They are algebraic, geometric and analytic.

An algebraic method is proposed by Indlekofer-Katai-Racsko [**22**] and Katai [**24**]. The basic idea is as follows. By definition, ∂T is the union of $T \cap (T+s)$ with $s \in \mathscr{J}$. As T is compact, there are only finitely many candidates s such that $T \cap (T+s) \neq \emptyset$. Using (1.1), we draw a finite directed graph whose vertices are these s, and which describes how the inflation-subdivision process acts on $T \cap (T+s)$. The graph given by this procedure is called 'neighbor graph'. This graph usually becomes larger than necessary to describe the boundary but it is used in Section 4 to describe the set of non-trivial identifications in our parametrization.

A brilliant topological idea is introduced by Dekking [**8**], [**9**]. He implicitly introduced hierarchical substitution structures on homological complexes associated to planar tiles. One substitution structure naturally arises from (1.1) by approximation $AB_{n+1} = B_n + D$ together with a rectangular fundamental domain B_1. The other is a consistent substitution σ acting on $aba^{-1}b^{-1}$ representing ∂B_1, which is a homomorphism on the free group on two letters a, b. Under a certain condition on cancellation, this framework gives an efficient way to approximate the boundary of T. In an unpublished paper, Song identified the class of tiles associated to Dekking homomorphisms on 2-letters with short range cancellation ([**46**]). It follows that the boundary of many planar fractal tiles can not be described. An elaborate effort to associate a Dekking homomorphism on three letters a, b, c to planar tiles associated to canonical number systems (CNS tiles for short) is found in [**23**]. However it turned out that cancellation occurrences in the free group are still mysterious and difficult to handle. For a further development, Sano-Arnoux-Ito [**5**] constructed a nice complex structure for Pisot substitutions. The problem of cancellation remains as long as the recurrent set method is concerned.

An analytic tool called 'contact matrix' is developed by Gröchenig and Haas [**16**] to compute the fractal dimension of ∂T. Their essential idea is to recursively construct box union approximations of tiles and count the number of surrounding boxes. Vince [**50**] and Duvall-Keesling-Vince [**12**] gave a more geometric understanding of the method. We will interpret this idea in our paper (see the proof of Proposition 2.1) to fit into our framework. It gives a so called 'contact automaton', whose underlying graph is a subgraph of the neighbor graph, but still gives the whole set of points in ∂T. This contact automaton usually has much smaller number of states (vertices) than the neighbor graph.

In this paper, we shall combine the later two ideas to parametrize the boundary ∂T for a connected digit tile T in the sense of Lagarias-Wang [**31**]. We shall prove that if the contact automaton is strongly connected and has an oriented extension with cyclic matching condition which is essentially due to Hata [**19**], then there is a standard parametrization by using a Dumont-Thomas number system (Theorem 1). The parametrization is a concrete surjective continuous mapping from $\boldsymbol{R}/\boldsymbol{Z}$ to ∂T. It is Hölder continuous by the natural metrics and the fixed points of the GIFS associated to the contact automaton have natural algebraic addresses. Hata's matching condition is easily confirmed for a given ordered extension of the contact automaton. Therefore by way of brute force, we get an algorithm to check whether there exists such a good extension of the contact automaton. For connected planar tiles, we conjecture that this matching condition is always fulfilled by some suitable choice of orientation. In higher dimensions, it is not clear whether such an orientation can be found to obtain space filling curves.

By a recent development by Lau-He [**20**] to tweak Hausdorff measure by pseudo norm, one can treat self-affine tile almost as easily as self-similar ones. The result of Luo-Yang [**34**] under the same line allows us to show that the GIFS of ∂T satisfies an open set condition. As a result by our parametrization, the segment $[0, t]$ is mapped to an arc (not necessarily simple) in ∂T of Hausdorff measure t with respect to the pseudo norm (Theorem 2).

Finally we apply the above theory to the CNS tiles (Theorem 4). Under some minor restriction, the strong connectedness of the contact automaton is verified and we can show that the matching condition holds. In this case, the result is naturally understood under Dekking's framework but by taking a hexagonal fundamental domain and using substitution on three letters a, b, c acting on $abca^{-1}b^{-1}c^{-1}$ which represents the boundary of this hexagon (see Proposition 6.2). We could confirm that the cancellation occurs only in a short range, even if the tile is not a topological disk. Showing that the parametrization is bijective, one can reprove directly a result of Akiyama-Thuswaldner [**4**], characterizing the disk-like tiles among the class of CNS tiles.

For simplicity of presentation, among Lagarias-Wang tiles, we have chosen CNS tiles because the expanding matrix has positive determinants. This makes easy the description of the oriented automata, in the sense that switching the direction of the pieces is not necessary. Basically this direction change is not necessary, since one can take the square the matrix in negative determinants case at the cost of lengthy description. Since we do not use the specific properties of the canonical number systems, the method should work in wider classes of tiles, even in the aperiodic tiles with several protiles.

Several results around the boundary parametrization of fractal tiles are based on a similar method that is, finding an appropriate ordering of an automaton, under the implicit assumption that the tiles are disks. For a class of Rauzy fractals, Messaoudi [**36**], [**37**] used an associated periodic tiling to parametrize the boundary via an automaton. He could prove that the corresponding tiles are disk-like. More recently, Bandt and Mesing [**6**] studied large classes of self-affine sets. They constructed a neighbor graph in a slightly different way as the above definition and analyzed the language associated to each boundary piece. They showed that a self-affine tile is disk-like if and only if these languages have some properties, algorithmically checkable. Comparing with these results, one advantage of our method is that we will be dealing with non disk-like tiles as well. In this case many identifications can occur between different subpieces. Consequently, the structure of the language of the neighbor graph may be more intricate. However, we will show that the identifications in the parametrization are still recognized by an automaton (see Theorem 3).

We expect our parametrization to be a helpful tool in the study of the topology of self-affine tiles. Not only for deciding whether a tile is disk-like or not, as done in Section 5 for a class of examples. But also for the investigation of non disk-like tiles, whose topology is by now less understood. Some examples were treated in details [**40**], [**32**]. Our parametrization should help to get further. Indeed, as mentioned above, the identifications in the parametrization are recognized by an automaton. This automaton is trivial whenever the tile is a topological disk. We think that the careful study of this automaton for non-disk like tiles will help in, for example, quickly finding out the cut points of the tiles, or describing their interior components.

2. Main results.

Our purpose is to construct an explicit parametrization for the boundary of so-called integral self-affine $\boldsymbol{Z}^d$-tiles. Let us specify this class of attractors. Let $\boldsymbol{A}$ be a $d \times d$ real *expanding* matrix (that is, the eigenvalues of $\boldsymbol{A}$ are greater than 1 in modulus) and $\mathscr{D} \subset \boldsymbol{R}^d$ a finite set. Then there is a unique nonempty compact set $T = T(\boldsymbol{A}, \mathscr{D})$ satisfying

$$\boldsymbol{A}T = \bigcup_{a \in \mathscr{D}} (T + a) \tag{2.1}$$

(see [**21**]). T is a *self-affine tile* if it has positive Lebesgue measure, *i.e.*, $\lambda_d(T) > 0$, where λ_d is the d-dimensional Lebesgue measure, and if the *digit tiles* $T + a (a \in \mathscr{D})$ are essentially disjoint:

$$\lambda_d\big((T + a) \cap (T + a')\big) = 0 \ \text{ if } \ a \neq a' \in \mathscr{D}.$$

Fundamental properties of self-affine tiles were established in [**16**], [**28**], [**29**], [**31**]. Among them is the tiling property. We say that $T + \mathscr{J}$ ($\mathscr{J} \subset \boldsymbol{R}^d$) is a *tiling of* $\boldsymbol{R}^d$ if it covers the space without overlapping:

$$\bigcup_{s \in \mathscr{J}} (T + s) = \boldsymbol{R}^d \ \text{ and } \ \lambda_d\big((T + s) \cap (T + s')\big) = 0 \ \text{ if } \ s \neq s' \in \mathscr{J}.$$

Suppose now that the self-affine tile T is obtained from $(\boldsymbol{A}, \mathscr{D})$, where $\boldsymbol{A}$ is an integer matrix and $\mathscr{D} \subset \boldsymbol{Z}^d$ is a complete residue system of $\boldsymbol{Z}^d/\boldsymbol{A}\boldsymbol{Z}^d$. Then T is called *integral self-affine tile with digit set* $\mathscr{D}$. In this case, the tiling set $\mathscr{J}$ is a sublattice of $\boldsymbol{Z}^d$ (see [**31**]). The property that $\mathscr{J} = \boldsymbol{Z}^d$ is closely related to the behavior of natural approximations of the boundary ∂T of T. The connecting tool is the contact set. Let $e_1, \ldots, e_d$ be the canonical basis of $\boldsymbol{Z}^d$ and $R_0 := \{0, \pm e_1, \ldots, \pm e_d\}$. Define recursively the sets

$$R_n := \big\{k \in \boldsymbol{Z}^d ; (\boldsymbol{A}k + \mathscr{D}) \cap (l + \mathscr{D}) \neq \emptyset \ \text{ for some } \ l \in R_{n-1}\big\}$$

and $R := \bigcup_{n \geq 0} R_n \setminus \{0\}$. Then the so-called *contact set* R is a finite set ([**16**]). Moreover, we call *contact matrix* the $|R| \times |R|$ integer matrix $\boldsymbol{C}$ with coefficients

$$c_{kl} := |(\boldsymbol{A}k + \mathscr{D}) \cap (l + \mathscr{D})| \quad (k, l \in R).$$

Let T_0 be the unit d-dimensional cube spanned by the canonical basis and T_n

defined recursively by

$$\boldsymbol{A}T_n = \bigcup_{a\in\mathscr{D}} (T_{n-1} + a).$$

Then the following assertions are equivalent.

(1) $T + \boldsymbol{Z}^d$ is a tiling of $\boldsymbol{R}^d$.
(2) $\lim_{n\to\infty} \partial T_n = \partial T$ (Hausdorff metric).
(3) Let β be the spectral radius of the contact matrix $\boldsymbol{C}$. Then $\beta < |\det(\boldsymbol{A})|$.

This *Tiling Theorem* can be found in [**12**], [**16**], [**50**]. If the above assertions hold, we call T an *integral self-affine $\boldsymbol{Z}^d$-tile.* We make the following remarks.

1. By [**51**], the assertions of the Tiling Theorem are also equivalent to:
   (4) $\lambda_d(T) = 1$.
   (5) $\lim_{n\to\infty} \partial T_n$ *(Hausdorff metric) is not space filling.*
2. The approximations T_n always converge to the attractor T in the Hausdorff metric. Moreover,

$$T_n = \boldsymbol{A}^{-n}Q + \sum_{i=1}^{n} \boldsymbol{A}^{-i}\mathscr{D}$$

   and $T_n + \boldsymbol{Z}^d$ is a tiling of $\boldsymbol{R}^d$ for all n.
3. For the boundary of the natural approximations, the relation

$$\partial T_n = \bigcup_{s\in R} T_n \cap (T_n + s) \tag{2.2}$$

   holds and connects the items of the Tiling Theorem ([**16**]). Under the assumptions of the theorem, it also enables the computation of the Hausdorff dimension of ∂T in the case that $\boldsymbol{A}$ is a similarity (see [**12**], [**47**]).

Let $T(\boldsymbol{A},\mathscr{D})$ be an integral self-affine $\boldsymbol{Z}^d$-tile. The contact set R can be used to describe the boundary of T. Let $M \subset \boldsymbol{Z}^d$ and $G(M)$ be the graph defined as follows. The vertices of $G(M)$ are the elements of M, and for $k, l \in M$, there is a transition $k \xrightarrow{a|a'} l$ ($a, a' \in \mathscr{D}$) if and only if $\boldsymbol{A}k + a' = l + a$. We may also simply write $k \xrightarrow{a} l$ for such a transition, since a' is then uniquely determined. Also, if $M := \{m_1, \ldots, m_p\} \subset \boldsymbol{Z}^d$ and d_{kl} the number of transitions in $G(M)$ from m_l to m_k, we define

$$\boldsymbol{D}(M) := (d_{kl})_{1\le k,l\le p}.$$

It is the *incidence matrix* of $G(M)$. It is a non negative matrix, and therefore it has a dominant positive eigenvalue β_M, called *Perron-Frobenius eigenvalue*. Taking $M = R$, the contact set, $G(R)$ is called the *contact automaton* of T. Note that $\boldsymbol{D}(R) = {}^t\boldsymbol{C}$, the transpose of the contact matrix. We will prove the following proposition. It is the starting point of our construction. Its non trivial proof includes several considerations found in previous papers [**16**], [**43**], [**50**].

PROPOSITION 2.1. *Let $T(\boldsymbol{A}, \mathscr{D})$ be an integral self-affine $\boldsymbol{Z}^d$-tile. Then there is a set $\mathscr{R} \subset R$ and non-empty compact sets $(K_s)_{s\in\mathscr{R}}$ such that*

$$\partial T = \bigcup_{s\in\mathscr{R}} K_s \tag{2.3}$$

and

$$K_s = \bigcup_{s \xrightarrow{a} s' \in G(\mathscr{R})} \boldsymbol{A}^{-1}(K_{s'} + a). \tag{2.4}$$

Moreover, $\beta_{\mathscr{R}} = \beta_R =: \beta$.

A property of the above $G(\mathscr{R})$ is that each state has at least one outgoing and one incoming transition (it is a trim automaton). This may not hold for $G(R)$.

Since T is a self-affine tile, iterating (2.1) gives T explicitly:

$$T = \left\{ \sum_{j\geq 1} \boldsymbol{A}^{-j} a_j; (a_j)_{j\geq 1} \in \mathscr{D}^{\boldsymbol{N}} \right\}.$$

Moreover, by Proposition 2.1, we have a natural onto mapping

$$\begin{aligned} \psi:\ G(\mathscr{R}) &\to \partial T \\ w &\mapsto \sum_{j\geq 1} \boldsymbol{A}^{-j} a_j, \end{aligned}$$

where $w : s_1 \xrightarrow{a_1} s_2 \xrightarrow{a_2} \cdots$ is an infinite walk in the automaton $G(\mathscr{R})$.

To speak roughly, we will construct a mapping $[0,1] \to G(\mathscr{R})$ and connect it to the boundary ∂T via ψ in such a way that the resulting mapping $C : [0,1] \to \partial T$ is continuous. For the mapping $[0,1] \to G(\mathscr{R})$, we will require the irreducibility of the incidence matrix $\boldsymbol{D}(\mathscr{R})$. This is equivalent to $G(\mathscr{R})$ being strongly connected. For the connection to the boundary, compatibility conditions are needed. We can formulate them as follows. Consider the subdivisions of ∂T as in Equations (2.3) and (2.4). We order these subdivisions, by ordering the states and the transitions of $G(\mathscr{R})$. We order the states of $\mathscr{R}$ from 1 to $p := |\mathscr{R}|$. Also the transitions starting from each state $s \in \mathscr{R}$ are given an order, from 1 to $l_{\max}$ (the number of these transitions, depending on s). Thus we have a bijection:

$$G(\mathscr{R}) \to G(\mathscr{R})^o$$
$$s^i \xrightarrow{a} s^j \mapsto i \xrightarrow{a|\boldsymbol{o}} j =: (i;\boldsymbol{o}).$$

We denoted by $G(\mathscr{R})^o$ this arbitrary ordered extension of $G(\mathscr{R})$. There are finitely many possible such extensions. We extend the bijection for consecutive transitions.

$$P : G(\mathscr{R})^o \to G(\mathscr{R})$$
$$(i;\boldsymbol{o_1}\boldsymbol{o_2}\dots) \mapsto w : s^i \xrightarrow{a_1} s^{j_1} \xrightarrow{a_2} \cdots$$

whenever $i \xrightarrow{a_1|\boldsymbol{o_1}} j_1 \xrightarrow{a_2|\boldsymbol{o_2}} \cdots \in G(\mathscr{R})^o$. We say that $G(\mathscr{R})^o$ is a *compatible* ordered extension of $G(\mathscr{R})$ if

$$\psi\big(P(i;\overline{l_{\max}})\big) = \psi\big(P(i+1;\overline{1})\big) \qquad (1 \le i \le p-1) \tag{2.5}$$

$$\psi\big(P(p;\overline{l_{\max}})\big) = \psi\big(P(1;\overline{1})\big) \tag{2.6}$$

$$\psi\big(P(i;\boldsymbol{o},\overline{l_{\max}})\big) = \psi\big(P(i;o+1,\overline{1})\big) \quad (1 \le i \le p,\ 1 \le o < l_{\max}). \tag{2.7}$$

Here, $\overline{\boldsymbol{o}}$ is the infinite repetition of $\boldsymbol{oo}\dots$. Thus the above conditions result in checking finitely many equalities between points having eventually periodic expansion in the basis $\boldsymbol{A}$. Like in [**19**] they can also be expressed as equalities between images of fixed points of contractions

$$(f_{a_1} \circ \cdots \circ f_{a_l})\big(\mathrm{Fix}(f_{a_{l+1}} \circ \cdots \circ f_{a_{l+n}})\big),$$

where $f_a(x) := \boldsymbol{A}^{-1}(x+a)$. Consequently, whether an ordered extension is compatible or not can be checked algorithmically.

THEOREM 1. *Let $T(\boldsymbol{A},\mathscr{D})$ be an integral self-affine $\boldsymbol{Z}^d$-tile and the set $\mathscr{R}$ as in Proposition* 2.1. *Suppose that $G(\mathscr{R})$ is strongly connected, i.e., its incidence matrix is irreducible. Moreover, suppose that there exists a compatible ordered extension of $G(\mathscr{R})$. Then there exist a Hölder continuous onto mapping $C : [0,1] \to \partial T$ with $C(0) = C(1)$ and a sequence $(\Delta_n)_{n \ge 0}$ of polygonal curves with the following properties.*

(1) $\lim_{n\to\infty} \Delta_n = \partial T$ *(Hausdorff metric).*
(2) *Denote by* V_n *the set of vertices of* Δ_n*. For all* $n \in \boldsymbol{N}$, $V_n \subset V_{n+1} \subset C(\boldsymbol{Q}(\beta) \cap [0,1])$ *(i.e., the vertices have* $\boldsymbol{Q}(\beta)$*-addresses in the parametrization).*

An immediate consequence is that if $T(\boldsymbol{A}, \mathscr{D})$ satisfies the assumptions of the theorem, then ∂T (thus T itself) is connected. We will even see that the boundary parts K_s ($s \in \mathscr{R}$) are then connected (see Proposition 3.11). They correspond to intersections of T with some of its neighboring tiles $T + s$. This non trivial topological result will easily follow from the construction of the parametrization. Conversely, we conjecture that there exists always a compatible ordered extension of the automaton $G(\mathscr{R})$ whenever T is a planar connected tile (see also [**33**], [**48**]).

We mention that $G(\mathscr{R})$ may not be strongly connected in general. However, this assumption is valid for the class of plane canonical number system tiles presented in Section 5. For this class also a compatible ordered extension of $G(\mathscr{R})$ is found. Thus we are able to perform the boundary parametrization. In this case, the approximating curves (Δ_n) are even simple closed curves, a property that we think will remain in many cases.

Recent developments on generalized Hausdorff measure and the open set condition for self-affine sets allow us to compare our parametrization of the boundary ∂T with an appropriate Hausdorff measure on ∂T. For any expanding matrix $\boldsymbol{A}$, a pseudo-norm w exists for which $\boldsymbol{A}$ becomes a similarity:

$$w(\boldsymbol{A}x) = |\det(\boldsymbol{A})|^{1/d} w(x) \quad (x \in \boldsymbol{R}^d). \tag{2.8}$$

For this pseudo norm, Hausdorff measures $\mathscr{H}_w^\alpha$ ($\alpha > 0$) and dimensions can be defined in a similar way as for the Euclidean norm (see [**20**]). Showing that ∂T satisfies an open set condition will lead us to the following theorem.

THEOREM 2. *Let* $T(\boldsymbol{A}, \mathscr{D})$ *satisfy the assumptions of Theorem* 1, C *be the corresponding parametrization. Furthermore, let* w *be a pseudo-norm such that* (2.8) *holds,*

$$\alpha := d \frac{\log(\beta)}{\log(|\det(\boldsymbol{A})|)}.$$

and $\mathscr{H}_w^\alpha$ *the associated Hausdorff measure. Then, for each boundary part* K_s ($s \in \mathscr{R}$) *as in Proposition* 2.1,

$$\infty > \mathscr{H}_w^\alpha(K_s) > 0.$$

Moreover, there is a subdivision of the interval $[0,1]$, $t_0 := 0 < t_1 < \cdots < t_{|\mathscr{R}|} := 1$

such that

$$\frac{1}{c}\mathscr{H}_w^\alpha\big(C([t_i,t))\big) = t - t_i \quad (t_i \le t \le t_{i+1}),$$

where $c := \sum_{s\in\mathscr{R}} \mathscr{H}_w^\alpha(K_s)$. *If the additional separation condition*

$$\mathscr{H}_w^\alpha(K_s \cap K_{s'}) = 0 \tag{2.9}$$

holds for all $s \neq s'$, *then*

$$\frac{1}{c}\mathscr{H}_w^\alpha\big(C([0,t))\big) = t \quad (t \in [0,1])$$

with $c = \mathscr{H}_w^\alpha(\partial T)$.

The separation condition (2.9) is related to the Hausdorff dimension of the triple points in the tiling induced by T, compared with the dimension of the boundary. In the case of plane canonical number system tiles, we will show that it is always satisfied.

The paper is now organized as follows. In Section 3, we construct the boundary parametrization for self-affine tiles. We introduce a numeration system induced by strongly connected automata and extend it under compatibility conditions to a continuous mapping from the unit interval to the boundary of tiles. In Section 4, we describe a way to check the compatibility conditions by automata. In Section 5, we apply our considerations to the case of canonical number system tiles. We obtain a parametrization of their boundary with standard properties. In Section 6, we consider the relation to the recurrent set method. We end up in Section 7 with some comments and questions on the generalization of this procedure to larger classes of tiles.

3. Parametrization of graph directed sets.

The boundary of a self-affine tile is the attractor of a graph directed iterated function system. This is described by an automaton $\mathscr{B}$, which is usually non deterministic. To define a numeration system on $\mathscr{B}$, we need a weak deterministic version $\mathscr{B}^o$. This will be obtained by ordering the states and transitions of $\mathscr{B}$. If the automaton is strongly connected, a numeration system of Dumont-Thomas type [**11**] can be introduced and results in a injective mapping $[0,1] \to \mathscr{B}^o$. However, any ordering may not allow us to extend this mapping to the attractor set. We will give conditions for which this extension exists and obtain a continuous mapping from the unit interval to the boundary of the tile. This follows a work of Hata [**19**] concerning a criterion for iterated function systems to be a simple arc.

Let us start with several definitions and fundamental facts on automata. Let Λ be a finite set, or *alphabet*. Its elements are *letters* and sequences of letters are *words*. Λ^* denotes the set of finite words, Λ^ω the set of infinite words. If $l = (l_1, \ldots, l_n) \in \Lambda^*$, we write $|l| = n$ for the *length* of l and $l_{|m} = (l_1, \ldots, l_m)$ for the *prefix* of l of length $m \le n$. If $l \in \Lambda^\omega$, then $|l| = \infty$ and prefixes $l_{|m}$ are defined for all $m \ge 1$. The concatenation of two words a and b is denoted by $a\&b$. If a word a is repeated infinitely many times, we write $\overline{a}$, meaning $a\&a\&a\ldots$.

An *automaton* is a triple $\mathscr{A} = (S, \Lambda, E)$. S is a finite set of *states* and Λ an alphabet. $E \subset S \times \Lambda \times S$ is the set of *transitions*. If $(s, l, s') \in E$, we write $s \xrightarrow{l} s'$. If for each $(s, l) \in S \times \Lambda$ a transition $s \xrightarrow{l} s'$ exists for at most one $s' \in S$, we will say that the automaton is *weak deterministic*; if such a transition exists for exactly one $s' \in S$, the automaton is *deterministic*. In the other cases, we will call the automaton *non-deterministic*. A *walk* w in the automaton $\mathscr{A}$ is a finite or infinite sequence of transitions $(s_n, l_n, s'_n)_{n\ge1}$ such that $s'_n = s_{n+1}$. We write

$$w : s_1 \xrightarrow{l_1} s_2 \xrightarrow{l_2} s_3 \xrightarrow{l_3} \cdots$$

We say that w *starts* from s_1, and if w is finite of the form $(s_n, l_n, s'_n)_{1\le n\le m}$, we say it *ends* at s'_m. Having two walks w and w' such that w ends where w' starts, we may concatenate them and write $w\&w'$. The associated sequence $l = (l_n)$ of letters of a walk w is the *label* of w. If the automaton is deterministic or weakly deterministic, then the walk w is completely defined by its starting state s_1 and its label l, hence we may simplify the notation and write $w = (s_1; l)$. As for words, we can define length and prefixes of the walk: the length of w is simply $|w| = |l|$ and a prefix $w_{|m}$ $(m \le |w|)$ consists of the first m transitions of w: $w_{|m} : s_1 \xrightarrow{l_1} s_2 \xrightarrow{l_2} \cdots \xrightarrow{l_m} s_m$.

Let now $\mathscr{B} = (M, \Sigma, E_B)$ be an automaton such that:

- Σ is a finite set of contractions on $\boldsymbol{R}^d$;
- each $s \in M$ has an outgoing transition: $s \xrightarrow{f} s' \in E_B$, for at least one $f \in \Sigma$ and one $s' \in M$.

Then we call $\mathscr{B}$ a *graph iterated function system* or *GIFS*. By [**13**], [**35**], there exists a unique vector of non-empty compact sets $(K_s)_{s\in M}$ with

$$K_s = \bigcup_{s \xrightarrow{f} s' \in E_B} f(K_{s'}) \tag{3.1}$$

$(s \in M)$. $(K_s)_{s\in M}$ is called *system of graph directed sets.* If f is an affine contraction for all $f \in \Sigma$, it is called *system of graph directed self-affine sets.*

For $w : s \xrightarrow{f_1} s_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} s_n \in \mathscr{B}$, we write $f_w = f_1 \circ f_2 \circ \cdots \circ f_n$. Then for any infinite walk

$$w : s \xrightarrow{f_1} s_1 \xrightarrow{f_2} \cdots ,$$

the intersection $\bigcap_{n\geq 1} f_{w|_n}(K_{s_n})$ consists of exactly one point, $x_w = \lim_{n\to\infty} f_{w|_n}(0) \in K_s$. Note that x_w depends only on the label $(f_n)_{n\geq 1}$ of w. This gives a well-defined onto mapping

$$\begin{aligned} \psi : \mathscr{B} &\to K := \bigcup_{s\in M} K_s \\ w &\mapsto x_w. \end{aligned}$$

Definition 3.1. Let $\mathscr{B} = (M, \Sigma, E_B)$ a GIFS. We call $\mathscr{B}^o = (S, \Lambda, E)$ *an ordered extension of* $\mathscr{B}$ any weak deterministic automaton constructed from $\mathscr{B}$ in the following way.

- The states of $\mathscr{B}$ are ordered, from 1 to $p = |M|$, to provide the states of $\mathscr{B}^o$.
- Also the transitions starting from a given state s are given an order, from 1 to l_s (the number of these transitions).

Hence $S = \{1, \ldots, p\}$, $\Lambda = \{1, \ldots, m\}$ for $m = \max\{l_s; s \in M\}$, and the transitions have the form $(i; o) = i \xrightarrow{o} j$ for some $i, j \in S$ and $o \in \Lambda$.

From now on, we consider that $\mathscr{B} = (M, \Sigma, E_B)$ is a GIFS and $\mathscr{B}^o = (S, \Lambda, E)$ is an associated ordered automaton. For convenience, we suppose that $M = \{1, \ldots, p\}$ $(= S)$. Thus, to each transition $i \xrightarrow{f} j \in \mathscr{B}$ corresponds a unique transition $i \xrightarrow{o} j \in \mathscr{B}^o$. This sometimes will be condensed to: $i \xrightarrow{f|\boldsymbol{o}} j$. We call P the natural bijection:

$$\begin{aligned} P : \mathscr{B}^o &\to \mathscr{B} \\ (i; o_1, o_2, \ldots) &\mapsto w : i \xrightarrow{f_1} s_1 \xrightarrow{f_2} \cdots \end{aligned}$$

whenever $(i; o_1, o_2, \ldots) = i \xrightarrow{f_1|\boldsymbol{o_1}} s_1 \xrightarrow{f_2|\boldsymbol{o_2}} \cdots \in \mathscr{B}^o$.

In $\mathscr{B}^o$, the set of transitions is an ordered set, from the transition $(1; 1)$ to the transition $(p; l_p)$. A direct consequence is that for all n the set of walks of length

n is lexicographically ordered, as well as the set of infinite walks. For convenience, we will write $l_{\max}$ if we refer to the value l_i, independently of the value of i. The minimal and maximal infinite walks are then respectively

$$w^{\min} = (1;1,1,1,\ldots) = (1;\overline{1}) \quad \text{and } w^{\max} : p \xrightarrow{l_p} s_1 \xrightarrow{l_{s_1}} s_2 \xrightarrow{l_{s_2}} \cdots = \big(p;\overline{l_{\max}}\big)$$

(that is, all the transitions are labeled by the maximal order).

Our aim is to map the automaton $\mathscr{B}^o$ into $[0,1]$. We will need the following assumption. Let $l_{i,j}$ be the number of transitions from j to i (by convention, $l_{ij} := 0$ if there is no such transition). The *incidence matrix* is $\boldsymbol{L} = (l_{ij})_{1\le i,j\le p}$. We suppose that $\boldsymbol{L}$ is *irreducible*. We call β its Perron-Frobenius eigenvalue. We choose the corresponding positive left eigenvector $u = (u_1,\ldots,u_p)$ satisfying $u_1 + \cdots + u_p = 1$.

In the mapping, the walks $w = (i; o_1, o_2, \ldots)$ starting from the state i will be sent to a subinterval of $[0,1]$ of length u_i. We define a function on the transitions:

$$\phi^0(i;o) = \begin{cases} 0, & \text{if } o = 1 \\ \displaystyle\sum_{\substack{1\le k<o,\\ i\xrightarrow{k}j}} u_j, & \text{if } o \neq 1. \end{cases}$$

Thus $\phi^0(i;o) < \sum_{1\le k\le l_i, i\xrightarrow{k}j} u_j = \beta u_i$ for all transitions $(i;o)$.

We set $u_0 := 0$ and map the infinite walks to $[0,1]$.

Proposition 3.2. *The mapping*

$$\begin{aligned} \phi : \mathscr{B}^o &\to [0,1] \\ w &\mapsto \lim_{n\to\infty}\Big(u_0 + u_1 + \cdots + u_{i-1} \\ &\qquad + \frac{1}{\beta}\phi^0(i;o_1) + \frac{1}{\beta^2}\phi^0(s_1;o_2) + \cdots + \frac{1}{\beta^n}\phi^0(s_{n-1};o_n)\Big) \end{aligned}$$

whenever w is the infinite walk:

$$w : i \xrightarrow{o_1} s_1 \xrightarrow{o_2} \cdots \xrightarrow{o_n} s_n \xrightarrow{o_{n+1}} \cdots .$$

is well-defined, increasing and onto.

Proof. ϕ is well-defined and the straight forward proof of monotony is omitted. To show the surjectivity, one constructs directly an inverse as follows. We use the fact that the subdivisions of $[0,1]$ of length u_i/β^n are ordered by the automaton. We define a piecewise linear expanding mapping H. Consider the intervals of length u_i:

$$I_i := [u_0 + \cdots + u_{i-1}, u_0 + \cdots + u_i) \quad (1 \leq i \leq p).$$

For each i, I_i is subdivided in l_i subintervals according to the transitions of the automaton starting from the state i:

$$I_i = I_{(i;1)} \cup \cdots \cup I_{(i;l_i)},$$

where we have, whenever $i \xrightarrow{o} j$:

$$I_{(i;o)} = \left[u_0 + \cdots + u_{i-1} + \frac{1}{\beta}\phi^0(i;o),\ u_0 + \cdots + u_{i-1} + \frac{1}{\beta}\phi^0(i;o) + \frac{u_j}{\beta}\right).$$

H is then the piecewise increasing affinity, expanding by a factor β on each $I_{(i;o)}$ and onto I_j. We add the convention $H(1) = 1$. Note that, if $t \in [0,1)$, then there are unique integers $d_0(t), d_1(t)$ such that $t \in I_{(d_0(t);d_1(t))}$. Thus let

$$\begin{aligned}\phi^{(1)} : [0,1] &\to \mathscr{B}^o \\ t &\mapsto \big(d_0(t); d_1(t), d_1(H(t)), \dots, d_1(H^n(t)), \dots\big).\end{aligned}$$

Then one can check that, for all $t \in [0,1]$, $\phi \circ \phi^{(1)}(t) = t$. Hence ϕ is onto. $\square$

By this proposition, any $t \in [0,1]$ has a β-representation whose coefficients are lead by an infinite walk in the automaton $\mathscr{B}^o$. It is the *number system induced by the automaton* $\mathscr{B}^o$. A similar result is well-known as Dumont-Thomas substitutive numeration system [**11**]. Note that also in our case a substitution σ on the set of states $\{1, \dots, p\}$ can be associated to the automaton $\mathscr{B}^o$. The image $\sigma(i)$ of a state i is the sequence of states j such that there is transition from i to j; the sequence of states in $\sigma(i)$ is ordered by the labels of the transitions. In [**11**], the definitions are more restrictive: a unique fixed point is considered, the corresponding mapping ϕ is always injective. However, in the construction of our parametrization, the non-injectivity of ϕ will play an essential rôle (see Proposition 3.5). The following lemma shows that the identifications occur exactly on lexicographically consecutive walks.

LEMMA 3.3. *Let $w \neq w' \in \mathscr{B}^o$, say for example $w >_{lex} w'$. Then $\phi(w) = \phi(w')$ if and only if*

$$1.\ \begin{cases} w = \big(i+1;\overline{1}\big) \\ w' = \big(i;\overline{l_{\max}}\big) \end{cases} \quad or \quad 2.\ \begin{cases} w = \big(j;o_1,\ldots,o_m,o+1,\overline{1}\big) \\ w' = \big(j;o_1,\ldots,o_m,o,\overline{l_{\max}}\big) \end{cases}$$

holds for some state $i = 1,\ldots,p$ or some prefix $(j;o_1,\ldots,o_m)$ and an order o.

PROOF. First note that the equality $\phi(w) = \phi(w')$ holds for the above pairs of walks w, w'. Consider for example

$$\begin{cases} w \ = i+1 \ \xrightarrow{1} \ s_1 \ \xrightarrow{1} \ s_2 \ \xrightarrow{1} \ \cdots \\ w' = \quad i \quad \xrightarrow{l_{\max}} s'_1 \xrightarrow{l_{\max}} s_2 \xrightarrow{l_{\max}} \cdots \end{cases}$$

for some state $i \leq p-1$. Then for all n,

$$\begin{aligned} \phi_n(w'|_n) &= u_0 + \cdots + u_{i-1} + \frac{1}{\beta}\big(\beta u_i - u_{s_1}\big) + \cdots + \frac{1}{\beta^n}\big(\beta u_{s_{n-1}} - u_{s_n}\big) \\ &= u_0 + \cdots + u_{i-1} + u_i - \frac{u_{s_n}}{\beta^n} \\ &= \phi(i+1;\overline{1}) - \frac{u_{s_n}}{\beta^n}, \end{aligned}$$

which converges to $\phi(i+1;\overline{1})$ as $n \to \infty$. The other pairs of walks identify through the mapping ϕ in the same way.

Let us show that no other identifications occur. Suppose that

$$w = (i;w_1,w_2,\ldots) >_{lex} w' = (i';w'_1,w'_2,\ldots)$$

are two walks in the automaton. Assume first $i > j$. Note that

$$\phi(w) \geq \phi(i;\overline{1}) = u_0 + \cdots + u_{i-1}$$

and this inequality is strict as soon as $i' > i+1$ or $w_k \neq 1$ for some k. On the contrary,

$$\phi(w') \leq \phi\big(i';\overline{l_{\max}}\big) = u_0 + \cdots + u_{i'-1} + u_{i'},$$

with strict inequality if $w_k \neq l_{\max}$ for some k. Indeed, let k be the index of the first such occurrence, then

$$\begin{aligned}\phi(w') &\leq \phi\big(i'; l_{\max}, \dots, l_{\max}, w'_k, \overline{l_{\max}}\big)\\ &= \phi\big(i'; l_{\max}, \dots, l_{\max}, w'_k + 1, \overline{1}\big)\\ &< u_0 + \dots + u_{i'-1} + u_{i'}.\end{aligned}$$

Therefore, the difference $\phi(w) - \phi(w')$ vanishes only if $i = j + 1$ and $w_n = 1$, $w'_n = l_{\max}$ for all n. This gives the first kind of pair of walks identified by ϕ.

Assume now $i = i'$, $w|_n = w|_{n'}$ for some $n \geq 0$ and $w_{n+1} > w'_{n+1}$. We denote by j the ending state of $w|_n$. Then

$$\begin{aligned}\phi(w) - \phi(w') &= \frac{1}{\beta^{n+1}}\big(\phi(j; w_{n+1}, \dots) - \phi(j; w'_{n+1}, \dots)\big)\\ &\geq \frac{1}{\beta^{n+1}}\big(\phi(j; w_{n+1}, \overline{1}) - \phi(j; w'_{n+1}, \overline{l_{\max}})\big)\\ &= \frac{1}{\beta^{n+1}}\big(\phi(j; w_{n+1}, \overline{1}) - \phi(j; w'_{n+1} + 1, \overline{1})\big)\\ &> 0\end{aligned}$$

as soon as $w_{n+1} \neq w'_{n+1} + 1$. Thus $w_{n+1} = w'_{n+1} + 1$ must hold. The necessary conditions $w_k = 1$ and $w'_k = l_{\max}$ for $k \geq n + 2$ follow. This gives the second kind of pairs of identified walks. $\square$

By this lemma, if $t \in [0, 1]$, then $\phi^{-1}(t)$ consists of at most two elements. Therefore an inverse of ϕ can be defined as

$$\begin{aligned}\phi^{(1)} : [0, 1] &\to \mathscr{B}^o\\ t &\mapsto \max{}^{lex} \phi^{-1}(t),\end{aligned} \tag{3.2}$$

where $\max^{lex}$ maps a finite set of walks to its lexicographically maximal walk.

We are now able to link the unit interval to the GIFS attractor $K = \bigcup_{i=1}^{p} K_i$.

Proposition 3.4. *The mapping* $C : [0, 1] \xrightarrow{\phi^{(1)}} \mathscr{B}^o \xrightarrow{P} \mathscr{B} \xrightarrow{\psi} K$ *is well-defined and surjective. Furthermore, let*

$$A := \Big\{ t \in [0,1]; t = u_0 + u_1 + \cdots + u_{i-1} \\ + \frac{1}{\beta}\phi^0(i;o_1) + \frac{1}{\beta^2}\phi^0(s_1;o_2) + \cdots + \frac{1}{\beta^n}\phi^0(s_{n-1};o_n) \\ \text{for some finite walk } i \xrightarrow{f_1|o_1} s_1 \xrightarrow{f_2|o_2} \cdots \xrightarrow{f_n|o_n} s_n \in \mathscr{B}^o \Big\}.$$

Then C is continuous on $[0,1]\setminus A$, and right continuous on A. Also, if $t \in A$, $\lim_{t^-} C$ exists.

Proof. Let us first prove the continuity of C on $[0,1] \setminus A$. Suppose that $t \in [0,1] \setminus A$. Then $w = \phi^{(1)}(t) = (i; o_1, o_2, \ldots)$ is a walk of $\mathscr{B}^o$ which does not end up in $\overline{1}$ nor in $\overline{l_{\max}}$. Let us write

$$P(w) : i \xrightarrow{f_1} s_1 \xrightarrow{f_2} s_2 \xrightarrow{f_3} \cdots$$

for the corresponding walk in $\mathscr{B}$. Now, for any given $\epsilon > 0$, there exists n such that

$$\operatorname{diam}(f_1 \circ \cdots \circ f_n(K)) < \frac{\epsilon}{2}.$$

Let $n_0, n_1 \geq n$ such that

$$\begin{cases} t - \phi(i; o_1, \ldots, o_{n_0}, \overline{1}) \geq \dfrac{u}{\beta^{n_0}} \\ \phi(i; o_1, \ldots, o_{n_1}, \overline{l_{\max}}) - t \geq \dfrac{u}{\beta^{n_1}} \end{cases}$$

where $u = \min\{u_1, \ldots, u_p\}$. We set $N := \max\{n_0, n_1\}$ and $N' := \min\{n_0, n_1\}$. Let $\delta := u/\beta^N$. Then for any t' with $|t' - t| < \delta$,

$$\phi(i; o_1, \ldots, o_{N'}, \overline{1}) \leq t' \leq \phi\big(i; o_1, \ldots, o_{N'}, \overline{l_{\max}}\big).$$

We used here the monotony of ϕ. Hence, since $\phi(\phi^{(1)}(t')) = t'$, and again by monotony of ϕ, we obtain that

$$(i; o_1, \ldots, o_{N'}, \overline{1}) \leq \phi^{(1)}(t') \leq \big(i; o_1, \ldots, o_{N'}, \overline{l_{\max}}\big).$$

This implies that $\psi(P(\phi^{(1)}(t'))) \in f_1 \circ \cdots \circ f_{N'}(K) \subset f_1 \circ \cdots \circ f_n(K)$, thus

$$\|C(t') - C(t)\| \le \epsilon.$$

We now prove the right continuity of C on A. Let $t \in A$. Then $w = \phi^{(1)}(t) = (i; o_1, \dots, o_{n_0}, \overline{1})$ is a corresponding walk in the automaton $\mathscr{B}^o$, to which corresponds

$$P(w) = i \xrightarrow{f_1} s_1 \xrightarrow{f_2} s_2 \xrightarrow{f_3} \cdots$$

in $\mathscr{B}$. Again, for a given $\epsilon > 0$, there exists $n \ge n_0$ such that

$$\mathrm{diam}(f_1 \circ \cdots \circ f_n(K)) < \frac{\epsilon}{2}.$$

We choose $N \ge n$ such that

$$t = \phi(i; o_1, \dots, o_{n_0}, \overline{1}) < \phi\big(i; \underbrace{o_1, \dots, o_{n_0}, 1, \dots, 1}_{N \text{ digits}}, \overline{l_{\max}}\big) =: t_1.$$

In this case, if $\delta := u/\beta^N$ (u defined as above), then for all t' with $0 \le t' - t < \delta$ we have $t \le t' < t_1$, hence

$$(i; o_1, \dots, o_{n_0}, \overline{1}) \le \phi^{(1)}(t') \le \big(i; \underbrace{o_1, \dots, o_{n_0}, 1, \dots, 1}_{N \text{ digits}}, \overline{l_{\max}}\big).$$

This insures that

$$C(t') \in \psi\big(P(i; \underbrace{o_1, \dots, o_{n_0}, 1, \dots, 1}_{N \text{ digits}})\big) \subset f_1 \circ \cdots \circ f_n(K),$$

thus

$$\|C(t') - C(t)\| \le \epsilon.$$

Finally, if $t \in A$, then C admits a limit in t^-. This is obtained by a similar argument as above but taking the representatives $w = (i; o_1, \dots, o_{n_0}, \overline{l_{\max}})$ such that $\phi(w) = t$, instead of $w = \phi^{(1)}(t)$. The limit is then

$$C(t-) = \psi(P(w)) = \lim_{n \to \infty} f'_1 \circ f'_2 \circ \cdots \circ f'_n(0),$$

where the contractions f_i' are defined via the label of $P(w) = i \xrightarrow{f_1'} s_1 \xrightarrow{f_2'} s_2 \xrightarrow{f_3'} \cdots$. □

The above proposition means that discontinuities of C may occur if ψ does not identify walks that are "trivially" identified by the number system ϕ. The following proposition indicates that only finitely many conditions can be checked to insure continuity.

PROPOSITION 3.5. *C is continuous on $[0,1]$ with $C(0) = C(1)$ if and only if the following equalities hold*:

$$\psi\big(P(i;\overline{l_{\max}})\big) = \psi\big(P(i+1;\overline{1})\big) \qquad (1 \le i \le p-1) \tag{3.3}$$

$$\psi\big(P(p;\overline{l_{\max}})\big) = \psi\big(P(1;\overline{1})\big) \tag{3.4}$$

$$\psi\big(P(i;o,\overline{l_{\max}})\big) = \psi\big(P(i;o+1,\overline{1})\big) \quad (1 \le i \le p,\ 1 \le o < l_i). \tag{3.5}$$

In this case, it is even $-(\log(\delta)/\log(\beta))$-Hölder continuous ($\delta$ is the maximal contraction factor among all the contractions $f \in \Sigma$).

PROOF. We first prove the continuity statement. By Proposition 3.4, C is continuous on $[0,1]$ if and only if it is left continuous on the countable set A. Note that (3.5) means that $C(0) = C(1)$. Also, (3.3) and (3.4) mean that C is left continuous at the points associated to walks of length $n = 0$ and $n = 1$ in the definition of A. Hence we just need to prove that this is sufficient for C to be continuous on the whole set A. But this follows from the definition of ψ. Indeed, let $t \in \mathscr{C}$ associated to a walk of length $n \ge 2$ but not to a walk of smaller length. Thus

$$t = \phi\big(\underbrace{i;o_1,\ldots,o_n,\overline{1}}_{w}\big)$$

with $o_n \ne 1$. We write $(f_1, f_2, \ldots)$ for the labeling sequence of $P(w)$. Then,

$$\begin{aligned} C(t) &= \psi(P(w)) = f_1 \circ \cdots \circ f_n \circ \psi\big(P(j;o_n,\overline{1})\big) \\ &= f_1 \circ \cdots \circ f_n \circ \psi\big(P(j;o_n-1,\overline{l_{\max}})\big) \qquad \text{(by Condition (3.4))} \\ &= C(t^-) \end{aligned}$$

(here j is the ending state of the walk $w|_{n-1}$ in the automaton $\mathscr{B}^o$). Thus C is left continuous in t.

We now show the Hölder continuity property. We set $\alpha := -(\log(\delta)/\log(\beta))$. The parametrization C has the following property. For $w \in \mathscr{B}^o$ of length $|w| = n$ and ending at the state j, let I_w be the following subinterval of $[0,1]$:

$$I_w = \left[\phi(w\&\overline{1}),\ \phi(w\&\overline{1}) + \frac{u_j}{\beta^n}\right).$$

Then

$$\bigcup_{w \in \mathscr{B}^o, |w|=n} I_w = [0,1)$$

(disjoint union), and

$$C(I_w) = f_w(K_j).$$

Note that by the left continuity condition, if $w \in \mathscr{B}^o$ with $|w| = n$ starts in i and ends in j, and $w^{+1} \in \mathscr{B}^o$ is the "next walk" of length n in the lexicographical order (starting in i' and ending in j'), then

$$f_w(K_j) \cap f_{w^{+1}}(K_{j'}) \neq \emptyset.$$

In this way, if $t \neq t' \in [0,1]$ with $|t - t'| \leq \min\{u_1, \ldots, u_p\}$, and n is such that

$$\frac{\min\{u_1, \ldots, u_p\}}{\beta^{n+1}} \leq |t - t'| \leq \frac{\min\{u_1, \ldots, u_p\}}{\beta^n},$$

then $C(t)$ and $C(t')$ belong to two touching subpieces of K:

$$C(t) \in f_w(K_j),\ C(t') \in f_{w'}(K_{j'}), \quad f_w(K_j) \cap f_{w'}(K_{j'}) \neq \emptyset,$$

w, w' being two walks of length n ending at the state j and j', respectively ($w = w'$ is allowed). This leads to

$$\|C(t) - C(t')\| \leq c_0 |t - t'|^\alpha$$

for some constant c_0. □

REMARK 3.6.

1. The walks $P(i; \overline{l_{\max}})$ etc. involved in Conditions (3.3)–(3.5) end up in cycles in

$\mathscr{B}$. Thus these conditions can be written as finitely many equalities between fixed points of contractions (see example in Section 5).

2. This result is similar to Hata's result ([**19**]) on iterated function systems ($T = \cup f_i(T)$). This result included the case where the subpieces $f_i(T)$ are flipped with respect to T. This is also possible in our setting by allowing increasing as well as decreasing piecewise affinities in the definition of the mapping H.

Finally, we give a characterization for K to be a circle.

CHARACTERIZATION 3.7. *Suppose that Conditions* (3.3)–(3.5) *are fulfilled. Then K is a simple closed curve if and only if for all $w, w' \in \mathscr{B}$,*

$$\psi(w) = \psi(w') \Rightarrow \phi(P^{-1}(w)) = \phi(P^{-1}(w')). \tag{3.6}$$

Indeed, (3.6) means that if ψ does not identify "too many" walks, that is, not more than these already identified by the number system ϕ, then the mapping C becomes injective on $[0, 1)$.

For the remaining part of this section we suppose the mapping $C : [0, 1] \to K$ to be continuous with $C(0) = C(1)$. We now construct a sequence of polygonal closed curves Δ_n converging to K with respect to the Hausdorff metric. For N points $M_1, \ldots, M_N$ of $\boldsymbol{R}^d$, we denote by $[M_1, \ldots, M_N]$ the curve joining $M_1, \ldots, M_N$ in this order by straight lines.

DEFINITION 3.8. Let $w_1^{(n)}, \ldots, w_{N_n}^{(n)}$ be the walks of length n in the automaton $\mathscr{B}^o$, written in the lexicographical order:

$$(1; 1, \ldots, 1) = w_1^{(n)} \leq_{lex} w_2^{(n)} \leq_{lex} \cdots \leq_{lex} w_{N_n}^{(n)} = (p; l_{\max}, \ldots, l_{\max}),$$

where $N_n := |\mathscr{B}_n^o|$ is the number of these walks. For $n = 0$, these are just the states $1, \ldots, p$. Let

$$C_j^{(n)} := C\big(\phi(w_j^{(n)} \& \bar{1})\big) \in \partial T \quad (1 \leq j \leq N_n).$$

Then we call

$$\Delta_n := \big[C_1^{(n)}, C_2^{(n)}, \ldots, C_{N_n}^{(n)}, C_1^{(n)}\big],$$

the *n-th approximation of K*.

PROPOSITION 3.9. *Δ_n is a polygonal closed curve and its vertices have $\boldsymbol{Q}(\beta)$-addresses. Moreover, (Δ_n) converges to K in Hausdorff metric.*

Proof. By construction, Δ_n is a polygonal closed curve, and for all $1 \leq j \leq N_n$,

$$\phi\big(w_j^{(n)}\&\overline{1}\big) =: t_j^{(n)} \in \boldsymbol{Q}(\beta).$$

Hence the vertices $C_j^{(n)} = C(t_j^{(n)})$ have $\boldsymbol{Q}(\beta)$-addresses.

The convergence in Hausdorff distance now results from the Hölder continuity of the parametrization. Indeed, let α be a Hölder exponent and c_0 a constant such that for all $t, t' \in [0,1]$

$$\|C(t') - C(t)\| \leq c_0 |t' - t|^{\alpha}.$$

Moreover, note that $t_1^{(n)} = 0$, and setting $t_{n+1}^{(n)} := 1$, we have

$$\big|t_{j+1}^{(n)} - t_j^{(n)}\big| \leq \frac{u}{\beta^n}$$

for all $0 \leq j \leq n$ (with $u := \max\{u_1, \ldots, u_p\}$). This implies that

$$\sup_{x \in K} \inf_{y \in \Delta_n} \{\|x - y\|\} \xrightarrow{n \to \infty} 0.$$

Indeed, let $\epsilon > 0$ and n_0 such that $u/\beta^{n_0} < (\epsilon/c_0)^{1/\alpha}$. Then for all $n \geq n_0$ and $t \in [0,1]$, there is a $t_j^{(n)}$ with

$$\big\|C(t) - C\big(t_j^{(n)}\big)\big\| < \epsilon.$$

Thus, by surjectivity of C, $d(x, \Delta_n) \leq \epsilon$ for all $x \in K$.

Similarly, $\sup_{y \in \Delta_n} \inf_{x \in K} \{\|x - y\|\} \xrightarrow{n \to \infty} 0$. Consequently,

$$d_H(\Delta_n, K) \xrightarrow{n \to \infty} 0,$$

where d_H denotes the Hausdorff distance associated to $\|\cdot\|$. □

If the contractions of the GIFS are affine mappings, then $(\Delta_n)_{n \geq 0}$ becomes the natural approximation sequence for the graph directed set K. For $w \in \mathscr{B}_n^o$, we write w^{+1} the next walk in the lexicographical order. We make the convention that the maximal walk has the minimal one as follower. For $i = 1, \dots, p$, let

$$\Delta_i^{(n)} := \bigcup_{\substack{w \in \mathscr{B}^o, |w|=n \\ w \text{ starts in } i}} \big[C(\phi(w\&\overline{1})), C(\phi(w^{+1}\&\overline{1}))\big].$$

PROPOSITION 3.10. *For all $n \in \boldsymbol{N}$,*

$$\begin{cases} \Delta_n = \displaystyle\bigcup_{i=1}^{p} \Delta_i^{(n)} \\ \Delta_i^{(n+1)} = \displaystyle\bigcup_{i \xrightarrow{f} j \in \mathscr{B}} f\big(\Delta_j^{(n)}\big). \end{cases} \tag{3.7}$$

PROOF. This follows from the left continuity of C. Indeed, for $n = 0$, that is for $w = (i)$ and $w^{(+1)} = (i+1)$, we have $C(\phi(i; \overline{l_{\max}})) = C(\phi(i+1; \overline{1}))$ by (3.3). Also,

$$\begin{aligned} \bigcup_{i \xrightarrow{f} j \in \mathscr{B}} f\big(\Delta_j^{(0)}\big) &= \bigcup_{i \xrightarrow{f|o} j \in \mathscr{B}^o} f\big(\big[\psi(P(j;\overline{1})), \psi(P(j+1;\overline{1}))\big]\big) \\ &= \bigcup_{i \xrightarrow{f|o} j \in \mathscr{B}^o} f\big(\big[\psi(P(j;\overline{1})), \psi(P(j;\overline{l_{\max}}))\big]\big) && \text{by (3.3)} \\ &= \bigcup_{i \xrightarrow{f|o} j \in \mathscr{B}^o} \big[\psi(P(i;o,\overline{1})), \psi(P(i;o,\overline{l_{\max}}))\big] \\ &= \bigcup_{i \xrightarrow{f|o} j \in \mathscr{B}^o} \Big[\psi\Big(P(\underbrace{i;o}_{w},\overline{1})\Big), \psi(P(w^{+1}\&\overline{1}))\Big] && \text{by (3.5)} \\ &= \Delta_i^{(1)}. \end{aligned}$$

If (3.7) is now true for all $n \le N$, then by iteration we have

$$\bigcup_{i \xrightarrow{f} j_1 \in \mathscr{B}} f(\Delta_{j_1}^{(N)}) = \bigcup_{i \xrightarrow{f_1|o_1} j_1 \rightarrow \cdots \xrightarrow{f_N|o_N} j_N \in \mathscr{B}^o} f_1 \circ \cdots \circ f_N\big(\Delta_{j_N}^{(0)}\big).$$

Using again (3.3) and (3.5) successively, we obtain (3.7) for $n = N+1$. $\square$

We have now the tools to prove Theorem 1. Let $T(\boldsymbol{A}, \mathscr{D})$ be an integral self-affine $\boldsymbol{Z}^d$-tile. We first show that $\mathscr{B} = G(\mathscr{R})$ is a good candidate to perform the parametrization, by giving a proof of Proposition 2.1.

Proof of Proposition 2.1. The proof runs as follows. A reduction procedure for $G(R)$ was introduced in [**43**]. The resulting reduced automaton is then a GIFS, for which a unique attractor K exists. On the other hand, it is proved in [**50**] that for a $\boldsymbol{Z}^d$-tile T, if (T_n) converges to T and the sequence of boundaries (∂T_n) converges to some K, then $K = \partial T$. Thus we will show that the GIFS obtained after reduction still describes boundary approximations of ∂T and apply the result of [**50**].

Let $\mathrm{Red}(G(M))$ be the graph emerging from $G(M)$ when all states that are not the starting state of an infinite walk in $G(M)$ are removed. Define $\mathscr{R}$ to be the subset of R such that $\mathrm{Red}(G(R)) = G(\mathscr{R})$. This reduction procedure does not affect the spectral radius of the incidence matrix: $\beta_R = \beta_{\mathscr{R}}$. Also, $G(\mathscr{R})$ is a GIFS. Now, let T_0 be the cube spanned by the canonical basis and $(T_n)_{n\geq 0}$ be the corresponding sequence of approximations of T. Then, for n large enough, setting $K_s^{(n)} := T_n \cap (T_n + s)$, we have

$$\begin{cases} \partial T_n = \bigcup\limits_{s\in\mathscr{R}} K_s^{(n)} \\ K_s^{(n)} = \bigcup\limits_{s\xrightarrow{a}s'\in G\left(\mathscr{R}\right)} \boldsymbol{A}^{-1}\big(K_{s'}^{(n-1)} + a\big). \end{cases} \tag{3.8}$$

This is proved as follows. By definition, for all n,

$$\begin{cases} \partial T_n = \bigcup\limits_{s\in R_n} K_s^{(n)} \\ K_s^{(n)} = \bigcup\limits_{s\xrightarrow{a}s'\in G(R)} \boldsymbol{A}^{-1}\big(K_{s'}^{(n-1)} + a\big). \end{cases}$$

We first show that the set $\mathscr{R}$ is sufficient to describe the boundary of a fine enough approximation: there is n_0 such that $R_{n_0} \subset \mathscr{R}$. Suppose that this does not hold, that is, there is a sequence of $s_n \in R_n \setminus \mathscr{R} \subset R$. By the finiteness of R, one can find a subsequence s_{n_i} with $s_{n_i} = s_{n_1} =: s$ for all i. By construction of R_n, for all i we obtain a walk in $G(R)$, from $s_{n_i} = s$ to some $s_i \in R_{n_1}$, of length $n_i - n_1$. Thus there are walks of arbitrary length in $G(R)$ starting from s, and again by finiteness of R, this implies that s is the starting state of an infinite walk in $G(R)$. Therefore $s \in \mathscr{R}$, a contradiction.

This approximation satisfies

$$\partial T_{n_0} = \bigcup_{s\in\mathscr{R}} K_s^{(n_0)}.$$

Moreover, $T_{n_0} + \boldsymbol{Z}^d$ is a tiling by polyhedra. Thus the intersection $T_{n_0} \cap (T_{n_0} + s)$ has at most dimension $d-2$ for $s \notin \mathscr{R}$. (3.8) is now true for all $n \geq n_0 + 1$. This can be shown inductively, using the following stability property of $\mathscr{R}$.

$$\mathscr{R} = \big\{s \in \boldsymbol{Z}^d \setminus \{0\}; (\boldsymbol{A}s + \mathscr{D}) \cap (s' + \mathscr{D}) \neq \emptyset \text{ for some } s' \in \mathscr{R}\big\} =: \mathscr{R}'.$$

Indeed, $\mathscr{R}$ was obtained from the contact set R by removing the elements that are not starting states of an infinite walk in $G(R)$. Thus $\mathscr{R} \subset \mathscr{R}'$: any $s \in \mathscr{R}$ must be the starting state of a transition $s \xrightarrow{d|d'} s' \in G(\mathscr{R})$, meaning that $\boldsymbol{A}s + d' = s' + d$, that is, $(\boldsymbol{A}s + \mathscr{D}) \cap (s' + \mathscr{D}) \neq \emptyset$. On the contrary, if $s \in \mathscr{R}'$, then there is some $s' \in \mathscr{R}$ and $d, d' \in \mathscr{D}$ such that $s \xrightarrow{d|d'} s'$ is a transition in $G(R)$. But s' being the starting state of an infinite walk in $G(\mathscr{R})$, this is also true for s, hence $s \in \mathscr{R}$.

We show that (3.8) is true for $n = n_0 + 1$. The proof can be extended to the induction step. Since T_{n_0+1} tiles by $\boldsymbol{Z}^d$,

$$\partial T_{n_0+1} = \bigcup_{s\in\boldsymbol{Z}^d} T_{n_0+1} \cap (T_{n_0+1} + s).$$

For all $s \in \boldsymbol{Z}^2$,

$$T_{n_0+1} \cap (T_{n_0+1} + s) = \bigcup_{a,a'\in\mathscr{D}} \boldsymbol{A}^{-1}\big(T_{n_0} \cap (T_{n_0} + \boldsymbol{A}s + a' - a) + a\big),$$

where an intersection on the right part is at most $d-2$-dimensional if $s' = \boldsymbol{A}s + a' - a \notin \mathscr{R}$. Moreover, by the stability property of $\mathscr{R}$, if $s \xrightarrow{a} s' \in G(R)$ with $s' \in \mathscr{R}$, then $s \in \mathscr{R}$. It follows that

$$T_{n_0+1} \cap (T_{n_0+1} + s) = \bigcup_{s \xrightarrow{a} s' \in G(\mathscr{R})} \boldsymbol{A}^{-1}\big(T_{n_0} \cap (T_{n_0} + s') + a\big),$$

and

$$\partial T_{n_0+1} = \bigcup_{s\in\mathscr{R}} T_{n_0+1} \cap (T_{n_0+1} + s).$$

Thus (3.8) holds for $n = n_0 + 1$, and in a similar way for all $n \geq n_0 + 1$.

Now, since $G(\mathscr{R})$ is a GIFS, the vector of sets $(K_s^{(n)})_{s\in\mathscr{R}}$ converges in Hausdorff metrics to some vector of non-empty compact sets $(K_s)_{s\in\mathscr{R}}$. Let $K := \bigcup_{s\in\mathscr{R}} K_s$. The property that $K = \partial T$ is now implied by the fact that T induces a $\boldsymbol{Z}^d$-tiling. The proof can be found in [**50**, Theorem 3]. $\square$

Proof of Theorem 1. Take $\mathscr{B} = G(\mathscr{R})$ as in Proposition 2.1. Then the theorem is a consequence of Propositions 3.4, 3.5 and 3.9. $\square$

The following topological result is a consequence of the construction of the parametrization.

Proposition 3.11. *Let $T(\boldsymbol{A}, \mathscr{D})$ satisfy the assumptions of Theorem* 1 *and $\mathscr{R}$ as in Proposition* 2.1. *Then the set $K_s \subset T\cap(T+s)$ is connected for all $s \in \mathscr{R}$.*

Proof. It follows from the construction of the parametrization C that the set $K_s \subset T\cap(T+s)$ $(s \in \mathscr{R})$ is the image of a subinterval of $[0, 1]$ by C. Therefore, it is connected. $\square$

Note that the set $\mathscr{R}$ must contain all the adjacent neighbors of T, that is, all $s \in \boldsymbol{Z}^d$ such that

$$T \cap (T + s) \cap \operatorname{int}(T \cup (T + s)) \neq \emptyset.$$

Thus we obtained that each intersection of T with an adjacent neighbor $T + s$ contains an "essential" part K_s, which is connected, and

$$\partial T = \bigcup_{\substack{s;T+s \\ \text{adjacent neighbor}}} K_s.$$

We end up this section by the measure comparison between subsets of $[0, 1]$ and their images by the parametrization C (see Theorem 2). Let $T(\boldsymbol{A}, \mathscr{D}) \subset \boldsymbol{R}^d$ be any self-affine set satisfying the open set condition. If $\boldsymbol{A}$ is a similarity matrix, the Hausdorff dimension of $T = T(\boldsymbol{A}, \mathscr{D})$ is equal to its similarity dimension and the corresponding Hausdorff measure of T is positive (and finite). If $\boldsymbol{A}$ is not a similarity, a pseudo-norm w such that

$$w(\boldsymbol{A}x) = |\det(\boldsymbol{A})|^{1/d} w(x) \quad (x \in \boldsymbol{R}^d)$$

can be used. The Hausdorff measures $\mathscr{H}_w^\alpha$ $(\alpha > 0)$ can be defined in terms of the pseudo-norm, and $0 < \mathscr{H}_w^\alpha(T) < \infty$ for some $\alpha > 0$. We refer to [**20**] for the details. In [**34**], this work is extended to graph directed self-affine sets with open set condition when the contractions involve a single matrix $\boldsymbol{A}$ in the following way. We recall that the GIFS $\mathscr{B}$ is said to satisfy the open set condition (OSC) if there exist open sets $V_1, \dots, V_p$ such that for all $i = 1, \dots, p$,

$$\bigcup_{j=1}^{p} \bigcup_{i \xrightarrow{f} j \in \mathscr{B}} f(V_j) \subset V_i,$$

the above union being disjoint. Let

$$\alpha := d \frac{\log(\beta)}{\log(|\det(\boldsymbol{A})|)}.$$

Suppose that the incidence matrix of $\mathscr{B}$ is irreducible. Then $\mathscr{H}_w^{\alpha}(K_i) > 0$ for $i = 1, \dots, p$. Moreover, for this class of GIFS, the OSC is related to the structure of the translation sets carrying the approximations of K.

Proposition 3.12. *Let $\mathscr{B}$ be a strongly connected GIFS. Suppose that the contraction labels are affinities of the form $f(x) = \boldsymbol{A}^{-1}(x + a)$, where $\boldsymbol{A}$ does not depend on the transition. Let*

$$\mathscr{D}^{(n)}(i,j) := \left\{ \sum_{k=0}^{n-1} \boldsymbol{A}^k a_{n-k} ; i \xrightarrow{a_1} \cdots \xrightarrow{a_n} j \in \mathscr{B}_n(i,j) \right\},$$

where $\mathscr{B}_n(i,j)$ is the set of walks of length n in $\mathscr{B}$ starting from i and ending at j. Then the following properties are equivalent.

(i) *$\mathscr{B}$ satisfies the open set condition.*
(ii) *If $i, j \in \{1, \dots, p\}$, then $|\mathscr{B}_n(i,j)| = |\mathscr{D}^{(n)}(i,j)|$ for all $n \geq 1$ and $\bigcup_{n \geq 1} \mathscr{D}^{(n)}(i,j)$ is uniformly discrete.*

Also this result had been first obtained in the case of self-similar tiles (see [**44**]), then extended to self-affine tiles in [**20**] and remains valid in the above case of GIFS (see [**34**]).

Let now $T(\boldsymbol{A}, \mathscr{D}) \subset \boldsymbol{R}^d$ be an integral self-affine $\boldsymbol{Z}^d$-tile. We show that the boundary GIFS $G(\mathscr{R})$ satisfies the OSC. The proof of Theorem 2 then follows.

Proposition 3.13. *Let $G(\mathscr{R})$ be the reduced contact automaton of an integral self-affine $\boldsymbol{Z}^d$-tile. Suppose that it is strongly connected. Then it satisfies the open set condition.*

Proof. Since $\boldsymbol{A}$ and $\mathscr{D}$ have integral coefficients, the uniform discreteness property is trivially true. The equality of cardinalities follows from the fact that $\mathscr{D}$ is a complete residue system of $\boldsymbol{Z}^d$ modulo $\boldsymbol{A}\boldsymbol{Z}^d$. Indeed, suppose that there are two states s, s' of the contact automaton and two walks $w \neq w'$ of the same length starting at s and ending at s' leading to the same integral point:

$$\sum_{k=0}^{n-1} \boldsymbol{A}^k a_{n-k} = \sum_{k=0}^{n-1} \boldsymbol{A}^k a'_{n-k}.$$

Since $\mathscr{D}$ is a complete residue system of $\boldsymbol{Z}^d$ modulo $\boldsymbol{A}\boldsymbol{Z}^d$, we have $a_k = a'_k$ for all k. However, consider the automaton obtained by reversing the transitions of the contact automaton. In this automaton, if $s' \xrightarrow{a} s'_1$, then $s' + a = \boldsymbol{A}s'_1 + b$ for some $b \in \mathscr{D}$. Again, the property of $\mathscr{D}$ implies that s_1 is completely determined by the pair (s', a). In other words, this automaton is weakly deterministic. It follows that the digit sequence $(a_n, \dots, a_1)$ defines a unique walk starting from s' in this automaton. This contradicts $w \neq w'$. □

Proof of Theorem 2. The Hausdorff measure $\mathscr{H}_w^\alpha$ is translation invariant and has the following scaling property:

$$\mathscr{H}_w^\alpha(\boldsymbol{A}^{-1}(E)) = \frac{1}{\beta}\mathscr{H}_w^\alpha(E) \tag{3.9}$$

for all $E \subset \boldsymbol{R}^d$ (see [**20**]). The following can be found in [**34**]. Since the OSC is satisfied and $\mathscr{B} = G(\mathscr{R})$ has irreducible incidence matrix,

$$\infty > \mathscr{H}_w^\alpha(K_i) > 0$$

for all $i \in \mathscr{R}$. The second inequality is an application of the mass distribution principle. Moreover, for each i, the union is disjoint in the sense of the measure $\mathscr{H}_w^\alpha$. Indeed, for $i = 1, \dots, p$,

$$\begin{aligned}\mathscr{H}_w^\alpha(K_i) &= \mathscr{H}_w^\alpha\Bigg(\bigcup_{i \xrightarrow{a} j \in G(\mathscr{R})} \boldsymbol{A}^{-1}(K_j + a)\Bigg)\\ &\le \frac{1}{\beta}\sum_{i \xrightarrow{a} j \in G(\mathscr{R})} \mathscr{H}_w^\alpha(K_j)\\ &= \sum_{j=1}^{p} d_{ji}\mathscr{H}_w^\alpha(K_j),\end{aligned}$$

where d_{ij} are the coefficients of the incidence matrix of $G(\mathscr{R})$. Since β is the Perron Frobenius eigenvalue of this primitive matrix, the above inequality is in fact an equality:

$$\mathscr{H}_w^\alpha(K_i) = \frac{1}{\beta} \sum_{i \xrightarrow{a} j \in G(\mathscr{R})} \mathscr{H}_w^\alpha(K_j), \tag{3.10}$$

and $(\mathscr{H}_w^\alpha(K_i))/(\sum_{i=1}^p \mathscr{H}_w^\alpha(K_i)) = u_i$, where $u_1, \ldots, u_p$ are the values used to construct the parametrization C.

Let now $t = \phi(i; o_1, o_2, \ldots) \in [0,1]$. We set $t_i := u_0 + \cdots + u_{i-1}$ ($u_0 = 0$). Then

$$C([t_i, t)) = \bigcup_{n \in \boldsymbol{N}} \bigcup_{\substack{i \xrightarrow{a_1|o'_1} s_1 \xrightarrow{a_2|o'_2} \cdots \xrightarrow{a_n|o'_n} s_n \in \mathscr{B}^o \\ (i;o'_1,\ldots,o'_n) <_{lex} (i;o_1,\ldots,o_n)}} f_{a_1} \circ \cdots \circ f_{a_n}(K_{s_n}).$$

Here, $f_a(x) = \boldsymbol{A}^{-1}(x+a)$. For each n, let G_n denote the second union of sets in the above equality. Then $(G_n)_{n\geq 1}$ is increasing. Hence it follows from the separation conditions (3.10) that

$$\mathscr{H}_w^\alpha\big(C([t_i, t))\big) = \lim_{n\to\infty} \sum \frac{\mathscr{H}_w^s(K_{s_n})}{\beta^n}.$$

Here, for each n the sum is taken over the walks $i \xrightarrow{a_1|o'_1} \cdots \xrightarrow{a_n|o'_n} s_n \in G(\mathscr{R})^o$ satisfying

$$(i; o'_1, \ldots, o'_n) <_{lex} (i; o_1, \ldots, o_n).$$

By definition, this sum is equal to $c(\phi_n(i; o_1 \ldots, o_n) - t_i)$, where $c := \sum_{j=1}^p \mathscr{H}_w^s(K_j)$. Thus

$$\frac{1}{c}\mathscr{H}_w^\alpha\big(C([t_i, t))\big) = t - t_i$$

and the first part of the theorem is proved.

If now (2.9) is assumed, that is, if the sets K_i, K_j are also measure disjoint for $i \neq j$, then $c = \mathscr{H}_w^\alpha(\partial T)$ and

$$\frac{1}{c}\mathscr{H}_w^\alpha\big(C([0, t))\big) = t. \qquad \square$$

4. Automata.

In this section, we construct the automata that enable to check the compatibility conditions of our parametrization. These conditions assure that walks trivially identified by the number system ϕ give rise to the same boundary point. The section will end up with a theoretical result: the existence of a Büchi automaton that gives the non trivial identifications.

A Büchi automaton is a quintuple $\mathscr{A} = (S, \Lambda, E, I, F)$ where (S, Λ, E) is an automaton, $I \subset S$ a set of *initial* states and $F \subset S$ a set of *final* states. An infinite walk

$$w : s_1 \xrightarrow{l_1} s_2 \xrightarrow{l_2} s_3 \xrightarrow{l_3} \cdots \in \mathscr{A}$$

is *admissible* if $s_1 \in I$ and $\{s_n; n \geq 1\} \cap F$ is infinite, *i.e.*, if w visits F infinitely often. The *language* $\mathscr{L}(\mathscr{A}) \subset \Lambda^\omega$ of the Büchi automaton $\mathscr{A}$ is the set of labels of the admissible walks. Conversely, a subset L of Λ^ω is *recognized by a Büchi automaton* if there is a Büchi automaton $\mathscr{A}$ such that $L = \mathscr{L}(\mathscr{A})$.

Let $T(\boldsymbol{A}, \mathscr{D})$ be an integral self-affine $\boldsymbol{Z}^d$-tile and $G(\mathscr{R})$ as in Proposition 2.1. We denote by $G(\mathscr{R})^o$ any ordered extension $G(\mathscr{R})$, giving rise to a Dumont Thomas number system ϕ (see Definition 3.1 and *seq.*). For simplicity, we will consider that $\mathscr{R} = \{1, \ldots, p\} = S$, the set of states of $G(\mathscr{R})^o$, unless the elements of $\mathscr{R}$ obviously refer to integer vectors. Remember that we have a natural bijection $P : G(\mathscr{R})^o \to G(\mathscr{R})$. Finally, the mapping $\psi : G(\mathscr{R}) \to \partial T$ reads $\psi(w) = \sum_{n \geq 1} \boldsymbol{A}^{-n} a_n$, whenever $(a_n)_{n \geq 1} \in \mathscr{D}^{\boldsymbol{N}}$ is the labeling sequence of w.

We will construct the following three automata. They contain all the informations on the occurring identifications.

- $\mathscr{A}^\phi$: pairs of walks $(w, w') \in G(\mathscr{R})^o \times G(\mathscr{R})^o$ leading to trivial identifications in the Dumont Thomas numeration system $\phi : G(\mathscr{R})^o \to [0,1]$, *i.e.*, to $\phi(w) = \phi(w')$. See Proposition 4.1.
- $\mathscr{A}^{\mathrm{sl}}$: pairs $(w, w') \in G(\mathscr{R})^o \times G(\mathscr{R})^o$ of distinct walks having the same contraction labels, *i.e.*, such that $P(w)$ and $P(w')$ have the same labeling sequences. See Proposition 4.2.
- $\mathscr{A}^\psi$: pairs $(w, w') \in G(\mathscr{R})^o \times G(\mathscr{R})^o$ arising from the identifications on ∂T, that is, such that $\psi(P(w)) = \psi(P(w'))$ where $P(w)$ and $P(w')$ have different labeling sequences. See Proposition 4.5.

Proposition 4.1. *The set*

$$\big\{(w, w') \in G(\mathscr{R})^o \times G(\mathscr{R})^o; w \neq w', \phi(w) = \phi(w')\big\}$$

is given by a weak deterministic Büchi automaton $\mathscr{A}^\phi$.

PROOF. Using Lemma 3.3, this Büchi automaton can be constructed from the automaton $G(\mathscr{R})^o = (S, \Lambda, E)$. We easily get an automaton $\mathscr{A}^{\phi}_{>_{lex}}$ that gives the walks

$$\big\{(w, w') \in G(\mathscr{R})^o \times G(\mathscr{R})^o; w >_{lex} w', \phi(w) = \phi(w')\big\}.$$

The alphabet is $\Lambda \times \Lambda$. We first create the states

$$i \mid i \ \text{ and } \ i \parallel j$$

for all $i, j \in S$. The union of these states is the set S'. The transition set E' is then constructed in the following three steps.

1. For each state $i \in S$, we set a transition

$$i \mid i \xrightarrow{o|o} j \mid j \ \text{ if and only if } \ i \xrightarrow{o} j \in G(\mathscr{R})^o.$$

2. Also, for each $i \in S$, the transition

$$i \mid i \xrightarrow{o+1|o} s \parallel s' \ \text{ if and only if } \begin{cases} i \xrightarrow{o+1} s \\ i \ \xrightarrow{o} \ s' \end{cases} \in G(\mathscr{R})^o.$$

3. Moreover, for each $s \parallel s'$,

$$s \parallel s' \xrightarrow{1|l_{s'}} t \parallel t' \ \text{ if and only if } \begin{cases} s \ \xrightarrow{1} \ t \\ s' \xrightarrow{l_{s'}} t' \end{cases} \in G(\mathscr{R})^o.$$

($l_{s'}$ is the number of transitions starting from s' in $G(\mathscr{R})^o$). We now define

- the states $i \mid i$ $(1 \leq i \leq p)$ and $i + 1 \parallel i$ $(1 \leq i \leq p - 1)$ as initial states (set I'),
- all the states $i \parallel j$ as final states (set F').

Then it follows from Lemma 3.3 that $((S', \Lambda \times \Lambda, E', I'), F') =: \mathscr{A}^{\phi}_{>_{lex}}$ is the required Büchi automaton. Note that some states $s \parallel s'$ may be on none of the admissible walks. Therefore they can be erased from the automaton.

In a similar way, one gets the automaton $\mathscr{A}^{\phi}_{<_{lex}}$ that gives the walks

$$\big\{(w, w') \in G(\mathscr{R})^o \times G(\mathscr{R})^o; w <_{lex} w', \ \phi(w) = \phi(w')\big\}.$$

Some pairs of states are found in both automata: they should be marked differently. $\mathscr{A}^{\phi}$ is then the (disjoint) union $\mathscr{A}^{\phi}_{>_{lex}} \cup \mathscr{A}^{\phi}_{<_{lex}}$. $\square$

Proposition 4.2. *The set*

$$\bigl\{(w,w') \in G(\mathscr{R})^o \times G(\mathscr{R})^o; w \neq w', P(w) \text{ and } P(w') \text{ have the same label}\bigr\}$$

is given by a Büchi automaton $\mathscr{A}^{sl}$.

Proof. We create the states

$$i \mid i \quad \text{and} \quad i \parallel j$$

for all $i, j \in S$, and define the transitions as follows.

1. $i \mid i \xrightarrow{o|o} j \mid j$ if and only if $i \xrightarrow{o} j \in G(\mathscr{R})^o$.
2. $i \mid i \xrightarrow{o|o'} j \parallel j'$ if and only if $o \neq o'$ and there is $a \in \mathscr{D}$ such that $\begin{cases} i \xrightarrow{a|\boldsymbol{o}} j \\ i \xrightarrow{a|\boldsymbol{o'}} j' \end{cases} \in G(\mathscr{R})^o$.
3. $i \parallel i' \xrightarrow{o|o'} j \parallel j'$ if and only if there is $a \in \mathscr{D}$ such that $\begin{cases} i \xrightarrow{a|\boldsymbol{o}} j \\ i' \xrightarrow{a|\boldsymbol{o'}} j' \end{cases} \in G(\mathscr{R})^o$.

We now set

- $i \mid i$ $(i \in S)$, $i \parallel i'$ $(i, i' \in N$ with $i > i')$ and $1||p$ as initial states;
- $i \parallel i'$ $(i, i' \in S)$ as final states.

The corresponding Büchi automaton has the required property. $\square$

In the $\boldsymbol{Z}^d$-tiling induced by $T(\boldsymbol{A}, \mathscr{D})$, we denote by $\mathscr{S}$ the set of neighbors of T:

$$\mathscr{S} = \{s \in \boldsymbol{Z}^d; (T+s) \cap T \neq \emptyset\}.$$

Then $G(\mathscr{S})$ is called *neighbor automaton*. It can be obtained algorithmically from the data $\boldsymbol{A}, \mathscr{D}$. Indeed, the relation $T \cap (T+s) \neq \emptyset$ bounds the norm of s and implies that there are finitely many candidates for $s \in \mathscr{S}$. Call this set of candidates M_0. Then $G(M_0)$ is easily computed. The automaton obtained from $G(M_0)$ after erasing the states that are not the starting state of an infinite walk is exactly $G(\mathscr{S})$.

$G(\mathscr{S})$ can be seen as the maximal GIFS describing the boundary of T. In particular, we have $G(\mathscr{R}) \subset G(\mathscr{S})$. The mapping ψ naturally extends to $G(\mathscr{S})$. The following lemma can be found in the literature [**22**], [**24**].

LEMMA 4.3. *$x = \sum_{j=1}^{\infty} \boldsymbol{A}^{-j} a_j = s + \sum_{j=1}^{\infty} \boldsymbol{A}^{-j} a'_j \in T \cap (T+s)$ if and only if there is an infinite walk*

$$s \xrightarrow{a_1|a'_1} s_1 \xrightarrow{a_2|a'_2} s_2 \xrightarrow{a_3|a'_3} \cdots$$

in $G(\mathscr{S})$.

LEMMA 4.4. *There is a Büchi automaton $\mathscr{I}^{\psi}$ on the alphabet $\mathscr{D} \times \mathscr{D}$ such that for w, w' two walks in $G(\mathscr{S})$ labeled by $(a_n)_{n\geq 1}, (a'_n)_{n\geq 1}$, then $\psi(w) = \psi(w')$ if and only if $(a_n)_{n\geq 1} = (a'_n)_{n\geq 1}$ or there is a walk labeled by $a_n \mid a'_n$ in this automaton.*

PROOF. Suppose $\psi(w) = \psi(w')$, that is, $\sum_{n\geq 1} \boldsymbol{A}^{-n} a_n = \sum_{n\geq 1} \boldsymbol{A}^{-n} a'_n$, with different labeling sequences. Then, this point being on ∂T, there is an $s \in \mathscr{S}$ such that it belongs to $T \cap (T+s)$. Thus there is a sequence $(a''_n)_{n\geq 1}$ such that

$$\sum_{n\geq 1} \boldsymbol{A}^{-n} a_n = \sum_{n\geq 1} \boldsymbol{A}^{-n} a'_n = s + \sum_{n\geq 1} \boldsymbol{A}^{-n} a''_n.$$

By Lemma 4.3, this means that two walks

$$\begin{cases} s \xrightarrow{a_1|a''_1} s_1 \xrightarrow{a_2|a''_2} \cdots \\ s \xrightarrow{a'_1|a''_1} s'_1 \xrightarrow{a'_2|a''_2} \cdots \end{cases}$$

exist in $G(\mathscr{S})$.

Consequently, we construct the automaton as follows.

1. $s \mid s \xrightarrow{a|a} t \mid t$ if and only if $s \xrightarrow{a} t \in G(\mathscr{S})$.
2. $s \mid s \xrightarrow{a|a'} t \parallel t'$ if and only if $a \neq a'$ and there is a'' such that $\begin{cases} s \xrightarrow{a|a''} t \\ s \xrightarrow{a'|a''} t' \end{cases} \in G(\mathscr{S})$.
3. $s \parallel s' \xrightarrow{a|a'} t \parallel t'$ if and only if there is a'' such that $\begin{cases} s \xrightarrow{a|a''} t \\ s' \xrightarrow{a'|a''} t' \end{cases} \in G(\mathscr{S})$.

We set

- $s \mid s$ as initial states;
- $t \parallel t'$ as final states.

This defines a Büchi automaton $\mathscr{I}^\psi$. By construction, if two walks $w, w' \in G(\mathscr{S})$ with different labelings $(a_n), (a'_n)$ lead to the same boundary point ($\psi(w) = \psi(w')$), then the sequence $(a_n|a'_n)$ is the label of a walk in $\mathscr{I}^\psi$.

Reciprocally, if $(a_n|a'_n)$ is the labeling sequence of a walk W in $\mathscr{I}^\psi$, then W provides two walks in $G(\mathscr{S})$. These two walks lead to the same boundary point by Lemma 4.3. Since the value $\psi(w)$ only depends on the label of w, this remains true for any two walks with the same labels. $\square$

PROPOSITION 4.5. *The set*

$$\begin{aligned}\big\{(w,w') \in G(\mathscr{R})^o \times G(\mathscr{R})^o;&\\ &P(w) \textit{ and } P(w') \textit{ have different labels and } \psi(P(w)) = \psi(P(w'))\big\}\end{aligned}$$

is given by a Büchi automaton $\mathscr{A}^\psi$.

PROOF. First we need to erase from the automaton $\mathscr{I}^\psi$ the pairs (w, w') whose labels (a_n), (a'_n) can not be found as labeling sequences of walks in $G(\mathscr{R})$. This is done by the following intersection procedure. We prepare an automaton G_2 which is a product of $G(\mathscr{R})$ with itself. The states of G_2 are the pairs $s \mid s'$ of states of $G(\mathscr{R})$, and all states of G_2 are both initial and final. Moreover, there is a transition

$$s \mid s' \xrightarrow{a|a'} s_1 \mid s'_1 \in G_2 \text{ if and only if } \begin{cases} s \xrightarrow{a} s_1 \in G(\mathscr{R}) \\ s' \xrightarrow{a'} s'_1 \in G(\mathscr{R}) \end{cases}.$$

Now we intersect the languages of the automata $\mathscr{I}^\psi$ and G_2. Note that a walk in $\mathscr{I}^\psi$, as soon as it has visited its first final state, visits afterward only final states. Thus the intersection is constructed via the following automaton $G_2 \cap \mathscr{I}^\psi$. The states are the pairs (x, y) where x is a state of G_2 and y a state of $\mathscr{I}^\psi$. There is a transition

$$(x,y) \xrightarrow{a|a'} (x_1,y_1) \text{ if and only if } \begin{cases} x \xrightarrow{a|a'} x_1 \in G_2 \\ y \xrightarrow{a|a'} y_1 \in \mathscr{I}^\psi \end{cases}.$$

We set

- (x, y), where $x \in G_2$ and y is initial state of $\mathscr{I}^\psi$, as initial states;
- (x, y), where $x \in G_2$ and y is final state of $\mathscr{I}^\psi$, as final states.

Restricting the states of this automaton $G_2 \cap \mathscr{I}^\psi$ to their first components, we have an automaton $G(\mathscr{R})^\psi$ (after marking differently the pairs $s \mid s'$ that appear several times). A walk in $G(\mathscr{R})^\psi$ consists in a pair of walks of $G(\mathscr{R})$. Thus the automaton $\mathscr{A}^\psi$ is easily obtained via the correspondence P^{-1} between $G(\mathscr{R})$ and $G(\mathscr{R})^o$. $\square$

We now describe the use of the automata $\mathscr{A}^\phi, \mathscr{A}^{sl}$ and $\mathscr{A}^\psi$. Let Atm denote one of these automata. By definition, $\mathscr{L}(\text{Atm}(\text{state}))$ is the language of the set of walks in "Atm" that start at "state". It consists of sequences of pairs of orders $(o_n \mid o'_n)_{n \geq 1}$. Also, each state x in Atm is associated to a pair (s, s') of states of $G(\mathscr{R})$. Given two automata Atm_1, Atm_2, we will write $\text{Atm}_1 \subset \text{Atm}_2$ if for each initial state x of Atm_1, there is an initial state x' of Atm_2 associated to the same pair of states as x such that $\mathscr{L}(\text{Atm}_1(x)) \subset \mathscr{L}(\text{Atm}_2(x'))$.

It follows that the compatibility conditions for parametrization (3.3) to (3.5) are equivalent to $\mathscr{A}^\phi \subset \mathscr{A}^\psi$. If moreover $\mathscr{A}^\psi \cup \mathscr{A}^{sl} \subset \mathscr{A}^\phi$, then ∂T is a simple closed curve (Condition (3.6)). In this case, all the identifications on the boundary are trivial with respect to the numeration system ϕ.

For further topological information in the case that ∂T is not a simple closed curve, it will be of interest to compute an automaton giving the non trivial identifications. The following theorem assures the existence of such an automaton. It relies on the complementation procedure of Büchi automata, which happens to be very difficult to perform in practice. We refer to [**41**], [**49**] for details on the construction. However, we have some examples where this complementation is tractable (see forthcoming papers).

THEOREM 3. *The set*

$$\left\{(w, w') \in G(\mathscr{R})^o \times G(\mathscr{R})^o ; \psi(P(w)) = \psi(P(w')) \text{ and } \phi(w) \neq \phi(w')\right\}$$

is given by a Büchi automaton $\mathscr{A}^{\psi \setminus \phi}$.

PROOF. We wish to express that the difference $\mathscr{A}^\psi \cup \mathscr{A}^{sl} \setminus \mathscr{A}^\phi$ is again an automaton. For convenience, we write $\mathscr{A}_1 = \mathscr{A}^\psi \cup \mathscr{A}^{sl}$, $\mathscr{A}_2 = \mathscr{A}^\phi$. For each initial state x of the automaton $\mathscr{A}_i$ $(i = 1, 2)$, $\mathscr{A}_i(x)$ is a Büchi automaton with unique initial state x. $\mathscr{L}(\mathscr{A}_i(x))$ is a subset of $\Lambda \times \Lambda$, where $\Lambda = \{1, \ldots, m\}$, m being the maximal order on the transitions of $G(\mathscr{R})^o$. Let now x be an initial state of $\mathscr{A}_1$ and (s, s') the pair of states associated to x. For y initial state of $\mathscr{A}_2$ associated to (s, s'), the language

$$\mathscr{L}(\mathscr{A}_1(x)) \setminus \mathscr{L}(\mathscr{A}_2(y)) = \mathscr{L}(\mathscr{A}_1(x)) \cap (\Lambda \times \Lambda)^\infty \setminus \mathscr{L}(\mathscr{A}_2(y))$$

is recognized by a Büchi automaton $\mathscr{A}_{xy}$ (see [**41**, Chapter 1–Theorem 9.4] or [**49**, Theorem 2.1]). This language consists in a set of sequences $(o_n \mid o'_n)$ that are such that

- $w := (s; o_1, o_2, \dots, o_n, \dots)$ and $w' := (s'; o'_1, o'_2, \dots, o'_n, \dots)$ are infinite walks in $G(\mathscr{R})^o$.
- $\phi(w) \neq \phi(w')$.
- $\psi(P(w)) = \psi(P(w'))$.

Consequently, the automaton $\mathscr{A}^{\psi\setminus\phi}$ is the (disjoint) union $\bigcup_{x,y} \mathscr{A}_{xy}$, where x describes the initial states of $\mathscr{A}_1$ and y is an initial state of $\mathscr{A}_2$ associated to the same pair of states as x. $\square$

5. Application to the CNS-tiles.

Our main result is now applied to a special class of integral self-affine plane tiles. For each tile in this class, we obtain a continuous parametrization together with a sequence of approximations of the boundary having standard properties. Let us first recall the definitions and well-known properties of the class. Let

$$\boldsymbol{A} = \begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix} \quad \text{and} \quad \mathscr{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} B-1 \\ 0 \end{pmatrix} \right\},$$

where $-1 \leq A \leq B$ and $B \geq 2$. Exactly for this choice of coefficients, the pair $(\boldsymbol{A}, \mathscr{D})$ has an algebraic property: it is called a *canonical number system*. We will not use this property and simply refer to [**7**], [**15**], [**26**], [**25**] for more informations on the subject. We choose this class because the main tool we need –the automaton of Proposition 2.1– was already computed in [**3**].

The set $T = T(\boldsymbol{A}, \mathscr{D})$ satisfying $\boldsymbol{A}T = T + \mathscr{D}$ is the associated *quadratic CNS tile* $T = T(\boldsymbol{A}, \mathscr{D})$. The Knuth dragon is an example of quadratic CNS tile $(A = B = 2)$. It is known from [**25**] that T is a $\boldsymbol{Z}^2$-tile. Topological properties of these tiles can be found in [**2**], [**4**], [**38**], [**42**] or in the survey [**3**]. In fact, ∂T is a simple closed curve if $2A < B + 3$. This was proved in [**4**] "from inside": the authors showed that the interior of T is connected and used results of plane topology to conclude. Otherwise $(2A \geq B + 3)$, T has disconnected interior and not much is known. It is proved in [**39**] that for $A = 4$, $B = 5$, T has no cut point and the infinitely many components of T^o are closed disks. A description of these components can be found in [**32**]. A more complicated st! ructure (like the existence of cut points) is suspected for other cases.

We will prove the following theorem.

THEOREM 4. *Let T be a quadratic CNS tile. Let β be the spectral radius of the corresponding contact matrix. Then there exists $C : [0,1] \to \partial T$ Hölder continuous onto mapping and a hexagon $Q \subset \boldsymbol{R}^2$ with the following properties. Let $T_0 := Q$ and*

$$\boldsymbol{A}T_n = \bigcup_{a \in \mathscr{D}} (T_{n-1} + a).$$

be the sequence of approximations of T associated to Q. Then:

(1) $\lim_{n\to\infty} \partial T_n = \partial T$ *(Hausdorff metric).*
(2) *For all $n \in \boldsymbol{N}$, ∂T_n is a polygonal simple closed curve.*
(3) *Denote by V_n the set of vertices of ∂T_n. For all $n \in \boldsymbol{N}$, $V_n \subset V_{n+1} \subset C(\boldsymbol{Q}(\beta) \cap [0,1])$ (i.e., the vertices have $\boldsymbol{Q}(\beta)$-addresses in the parametrization).*

Moreover, let $\alpha := 2(\log(\beta)/\log(B))$. Then there is a Hausdorff measure $\mathscr{H}_w^\alpha$ with respect to a pseudo-norm w such that

$$\frac{1}{c}\mathscr{H}_w^\alpha\big(C([0,t))\big) = t \quad (t \in [0,1])$$

with $c = \mathscr{H}_w^\alpha(\partial T) > 0$.

REMARK 5.1.

1. Apart from the measure theoretical considerations, this description can be compared to [**23**]. The boundary of the tiles was approximated by a sequence of simple closed curves, but through iteration on a rectangle instead of a hexagon. The parametrization relied on a modification of the recurrent set method in order to avoid some cancellation problems. Our theorem gives a geometrical interpretation of this modification (see also Section 6).
2. By construction, the vertices V_n of the approximations have eventually periodic expansion in the base $\boldsymbol{A}$. We will see that the vertices V_0 of the first approximation are even purely periodic. The vertices of the further approximations have the same ending periods. Hence these points can be interpreted as control points in the sense of Solomyak [**45**].
3. If $\boldsymbol{A}$ is a similarity, the pseudo-norm w can be taken simply as the Euclidean norm, and the Hausdorff measure is the usual one. This includes the case of the well-known Twin Dragon.

We derive from the previous sections a parametrization of the boundary of canonical number system tiles. Let T be a CNS tile. The automaton $G(\mathscr{R})$ was computed in [**4**]. It is depicted on Figure 1. We denoted the digits simply by the

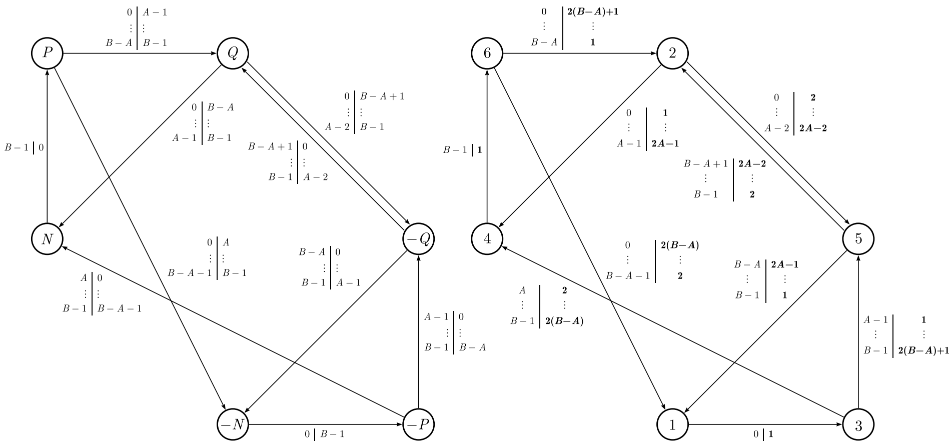

Figure 1. A strongly connected automaton for the boundary (left) and a compatible ordered extension $G(\mathscr{R})^o$ (right).

letter a instead of the vector $\left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right)$. We have

$$\mathscr{R} = \{\pm P, \pm Q, \pm N\},$$

where

$$P = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Q = \begin{pmatrix} A-1 \\ 1 \end{pmatrix}, \quad N = \begin{pmatrix} -A \\ -1 \end{pmatrix}.$$

Note that in the depicted automaton the most general case $B \geq A - 1 \geq 1$ is assumed. For simplicity, we will detail the proofs only for this case. If $B = A \geq 2$, no transition exists from $-P$ to N and from P to $-N$, hence the automaton is even lighter. However, the irreducibility of its incidence matrix is not affected (see Proposition 5.2). The case $|A| = 1$ is treated in a similar way. For $A = 0$, the tile is just a rectangle.

The following proposition is an easy consequence of Proposition 2.1.

Proposition 5.2. *Let T be a CNS-tile and $\boldsymbol{D} := \boldsymbol{D}(\mathscr{R})$ the incidence matrix of $G(\mathscr{R})$. Then $\boldsymbol{D}$ is irreducible. Moreover, there exists a unique family of non-empty compact sets $(K_s)_{s\in\mathscr{R}}$ such that*

$$\begin{cases} \partial T = \bigcup_{s \in \mathscr{R}} K_s, \\ K_s = \bigcup_{s \xrightarrow{a} s' \in G(\mathscr{R})} \boldsymbol{A}^{-1}(K_{s'} + a). \end{cases}$$

That is, ∂T is the GIFS directed by the graph $G(\mathscr{R})$.

Consider the ordered extension $G(\mathscr{R})^o$ on the right side of Figure 1. We use the notations of Section 3. Let C be the onto mapping

$$C : [0,1] \xrightarrow{\phi^{(1)}} G(\mathscr{R})^o \xrightarrow{P} G(\mathscr{R}) \xrightarrow{\psi} \partial T$$

obtained in Proposition 3.4. We show that it is a parametrization of the boundary.

Lemma 5.3. *The mapping $C : [0,1] \to \partial T$ is Hölder continuous. Moreover, $C(0) = C(1)$.*

Proof. We check the conditions of Proposition 3.5:

$$\psi\big(P(i;\overline{l_{\max}})\big) = \psi\big(P(i+1;\overline{1})\big) \qquad (1 \le i \le 5) \tag{5.1}$$

$$\psi\big(P(i;o,\overline{l_{\max}})\big) = \psi\big(P(i;o+1,\overline{1})\big) \quad (1 \le i \le 6,\ 1 \le o < l_i). \tag{5.2}$$

Condition (5.1) is easily seen. Indeed, for $i = 1$, we have the equalities :

$$P(1;\overline{l_{\max}}) = 1 \xrightarrow{0} 3 \xrightarrow{B-1} 5 \xrightarrow{B-A} 1 \xrightarrow{0} \cdots$$

$$P(2;\overline{1}) = 2 \xrightarrow{0} 4 \xrightarrow{B-1} 6 \xrightarrow{B-A} 2 \xrightarrow{0} \cdots$$

The walks on the right side are cycles of length 3 with the same labels. Hence the equality

$$\psi\big(P(1;\overline{l_{\max}})\big) = \psi(P(2;\overline{1}))$$

holds trivially. This is also the case for the other values of $i = 2,3,4,5$.

In the same way as above, we have $\psi(P(6;\overline{l_{\max}})) = \psi(P(1;\overline{1}))$, giving $C(0) = C(1)$.

For Condition (5.2), only the values $i = 2,3,5,6$ have to be considered, since $l_1 = l_4 = 1$. Note that if $B = A$, then only the values $i = 2,5$ make sense.

We treat the case $i = 2$. The parity of o ($1 \le o < l_2 = 2A-1$) is of importance.

If o is odd, then

$$P(2; o, \overline{l_{\max}}) = 2 \xrightarrow{a} P(4; \overline{l_{\max}})$$
$$P(2; o + 1\overline{1}) = 2 \xrightarrow{a} P(5; \overline{1}).$$

for some digit a. Hence, by (5.1), the equality $\psi(P(2; o, \overline{l_{\max}})) = \psi(P(2; o + 1, \overline{1}))$ again holds trivially. If o is even, we have

$$P(2; o, \overline{l_{\max}}) = 2 \xrightarrow{a} P(5; \overline{l_{\max}})$$
$$P(2; o + 1, \overline{1}) = 2 \xrightarrow{a} P(2; \overline{1}).$$

for some $a \in \{0, \dots, A-2\}$. The label of $P(5; \overline{l_{\max}})$ is $(a_n)_{n\geq 1} = \overline{(B-A)0(B-1)}$, and the label of $P(2; \overline{1})$ is $(a'_n)_{n\geq 1} = \overline{(B-1)(B-A)0}$. Hence it remains to check that these digit sequences lead to the same boundary point, that is $\sum_{n\geq 1} \boldsymbol{A}^{-n} a_n = \sum_{n\geq 1} \boldsymbol{A}^{-n} a'_n$. This is a consequence of Lemma 4.3, because the walks

$$\begin{cases} Q \xrightarrow{a|a''} -Q \xrightarrow{B-A|0} -N \xrightarrow{0|B-1} -P \xrightarrow{B-1|B-A} -Q \xrightarrow{B-A|0} \cdots \\ Q \xrightarrow{a+1|a''} N \xrightarrow{B-1|0} P \xrightarrow{B-A|B-1} Q \xrightarrow{0|B-A} N \xrightarrow{B-1|0} \cdots \end{cases}$$

both exist in $G(\mathscr{R})$ for some digit a'' (remember that $G(\mathscr{R}) \subset G(\mathscr{S})$).

The proof is similar for $i = 3, 5, 6$. □

REMARK 5.4.

- The compatibility conditions (5.1) and (5.2) may also be checked by constructing and comparing the automaton of trivial identifications emerging from the number system (see Proposition 4.1) and the automaton of boundary identifications (see Proposition 4.4). Note that the construction of the latter requires the whole boundary automaton $G(\mathscr{S})$.
- As we mentioned, (5.1) and (5.2) can be written as fixed points equalities. Indeed, let $\mathrm{Fix}(f)$ denote the fixed point of a contraction f and f_a the contraction associated to the digit a. Then the equality $\sum_{n\geq 1} \boldsymbol{A}^{-n} a_n = \sum_{n\geq 1} \boldsymbol{A}^{-n} a'_n$, where $(a_n)_{n\geq 1}$ and $(a'_n)_{n\geq 1}$ are the ultimately periodic sequences

$$(a_n)_{n\geq 1} = a\overline{(B-A)0(B-1)},$$
$$(a'_n)_{n\geq 1} = (a+1)\overline{(B-1)(B-A)0},$$

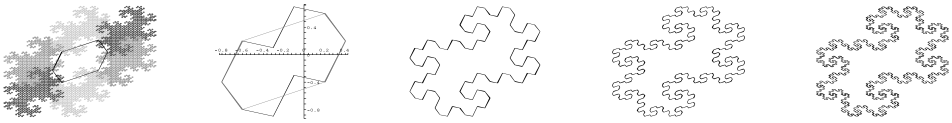

Figure 2. Knuth dragon ($A = B = 2$): tiling with Δ_0; $\Delta_0 \cup \Delta_1$, Δ_5, Δ_8, Δ_{11}.

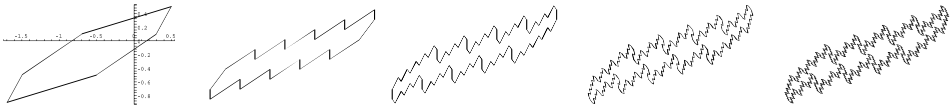

Figure 3. $A = 4, B = 5$: $\Delta_0, \ldots, \Delta_4$.

Figure 4. $A = 5, B = 5$: $\Delta_0, \ldots, \Delta_3$.

reads

$$f_a\big(\mathrm{Fix}(f_{B-A} \circ f_0 \circ f_{B-1})\big) = f_{a+1}\big(\mathrm{Fix}(f_{B-1} \circ f_{B-A} \circ f_0)\big).$$

As seen in Section 3, we obtain together with the parametrization a sequence of approximations (Δ_n) converging to the boundary of T in the Hausdorff distance (see Proposition 3.9). The first terms of the approximation sequence are depicted in Figures 2 to 4 for some examples.

We now show that the polygonal approximations Δ_n are simple closed curves equal to the boundary ∂T_n of the natural approximations of T, when starting from a special hexagon. We use the notations of the previous section, but will simply write $C_1, \ldots, C_6$ instead of $C_1^{(0)}, \ldots, C_6^{(0)}$. Hence we have:

$$\begin{aligned} C_1 &= \psi\big(\overline{0(A-1)(B-1)}\big), & C_2 &= \psi\big(\overline{0(B-1)(B-A)}\big),\\ C_3 &= \psi\big(\overline{(A-1)(B-1)0}\big), & C_4 &= \psi\big(\overline{(B-1)(B-A)0}\big),\\ C_5 &= \psi\big(\overline{(B-1)0(A-1)}\big), & C_6 &= \psi\big(\overline{(B-A)0(B-1)}\big). \end{aligned}$$

Proposition 5.5. *$[C_1, \ldots, C_6, C_1]$ is a simple closed curve. Let Q be the closure of its bounded complementary component. Then $Q + \boldsymbol{Z}^2$ is a tiling of* *the plane. Two neighboring tiles have* 1*-dimensional intersection. Moreover, the neighbors of Q are the tiles $Q+s$ with $s \in \mathscr{R}$, that is,*

$$\partial Q = \bigcup_{s \in \mathscr{R}} Q \cap (Q+s).$$

Proof. We study the relative positions of the points, whose coordinates are easily computable and have rational expressions in A, B. We have the following relations.

$$\begin{aligned} C_1 &= C_3 + (1,0)^T, \quad C_6 = C_4 + (1,0)^T, \\ C_2 &= C_6 + (A-1,1)^T, \quad C_3 = C_5 + (A-1,1)^T. \end{aligned} \tag{5.3}$$

Moreover, $C_6 - C_5 = (A(A+1)/(A+B+1), A/(A+B+1))^T$. The hexagon is depicted on the left part of Figure 5. Note that $0 < A/(A+B+1) =: x < 1/2$.

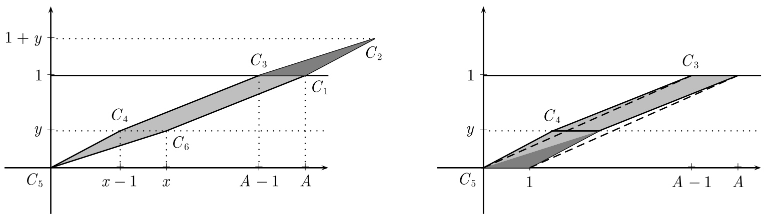

Figure 5. Tiling property of Q: proof of Proposition 5.5.

After a translation of the triangle $[C_1, C_2, C_3]$ (gray part) by $-(A-1,1)^T$, we obtain a union of two parallelograms (see the right part of Figure 5). Now a translation of the triangle $[C_3, C_4, C_5]$ by $+(1,0)^T$ leads to a single parallelogram which obviously tiles the plane by $\boldsymbol{Z}^2$. Hence this is also the case for the hexagon Q. The neighboring tiles are then obtained from the relations (5.3). □

We consider the approximations $(T_n)_{n \geq 0}$ of the tile T obtained by starting with $T_0 := Q$, the hexagon of the above proposition. Thus for all $n \geq 0$,

$$\boldsymbol{A} T_{n+1} = \bigcup_{a \in \mathscr{D}} (T_n + a).$$

Proposition 5.6. *For all n, ∂T_n is a simple closed curve.*

Proof. One can show inductively that $\boldsymbol{A}^n T_n$ is a connected union of translates of Q and it tiles the plane by $\boldsymbol{A}^n \boldsymbol{Z}^2$. Since the tiling by Q has only 1-dimensional intersections, $\boldsymbol{A}^n T_n$ has no cut point. Moreover, by the tiling property, also $\boldsymbol{R}^2 \setminus \boldsymbol{A}^n T_n$ is connected. Indeed, suppose it has a bounded component R_1. Then R_1 intersects some translate $z_1 + \boldsymbol{A}^n T_n$ ($z_1 \neq 0$). It follows that $z_1 + \boldsymbol{A}^n T_n \subset R_1$. Iterating this argument gives a sequence of translates $z_p + \boldsymbol{A}^n T_n$ ($p \geq 1$) inside R_1, where $z_p \neq z_q$ for $p \neq q$. This contradicts the local finiteness of the tiling. Thus $\boldsymbol{A}^n T_n$ is a locally connected continuum without cut point and with connected complement. By a result of Torhorst (see [**27**, X, II, Theorem 4]), it is homeomorphic to a disk. □

By definition, ∂T_0 and Δ_0 are both equal to ∂Q. We will now prove the equality of the whole sequences. Recall that Δ_n satisfies the GIFS equation of ∂T (see (3.7)). We will show that this is also true for ∂T_n.

Proposition 5.7. *For all $n \geq 0$, $\Delta_n = \partial T_n$.*

Proof. We write $B_s^{(n)} := T_n \cap (T_n + s)$ for $s \in \mathscr{R}$. Then

$$\begin{cases} \partial T_n = \bigcup_{s \in \mathscr{R}} B_s^{(n)} \\ B_s^{(n+1)} = \bigcup_{s \xrightarrow{a} s' \in G(\mathscr{R})} \boldsymbol{A}^{-1}\big(B_{s'}^{(n)} + a\big). \end{cases} \tag{5.4}$$

First note the following stability property of the set $\mathscr{R}$ due to its definition.

$$\mathscr{R} = \big\{k \in \boldsymbol{Z}^2 \setminus \{0\}; (\boldsymbol{A}k + \mathscr{D}) \cap (l + \mathscr{D}) \neq \emptyset \text{ for some } l \in \mathscr{R}\big\} =: \mathscr{R}'.$$

This general property can be found in the proof of Proposition 2.1. It has the following consequence on the $\boldsymbol{Z}^2$-tiling induced by the approximation T_n. Indeed, for $k \in \boldsymbol{Z}^2$,

$$(T_n + k) \cap T_n \neq \emptyset \Longleftrightarrow k \in \mathscr{R}.$$

This can be shown by induction (the case $n = 0$ is given by Proposition 5.5).

Thus, by the tiling property,

$$\partial T_n = \bigcup_{s \in \mathscr{R}} B_s^{(n)},$$

and

$$
\begin{aligned}
B_s^{(n+1)} &= \boldsymbol{A}^{-1} \bigcup_{d,d' \in \mathscr{D}} \underbrace{(T_n + d) \cap (T_n + d' + \boldsymbol{A}s)}_{\neq \emptyset \Leftrightarrow \boldsymbol{A}s + d' - d \in \mathscr{R}} \\
&= \bigcup_{s \xrightarrow{a} s' \in G(\mathscr{R})} \boldsymbol{A}^{-1}\big(B_{s'}^{(n)} + a\big).
\end{aligned}
$$

We eventually notice that with the correspondence $i \leftrightarrow s$ between the orders and the states of $G(\mathscr{R})$, we have

$$
[C_i, C_{i+1}] = B_s^{(0)}.
$$

Thus, Δ_n and ∂T_n satisfying the same recursion, they are equal for all n. $\square$

We are now able to prove our theorem.

Proof of Theorem 4. It follows from Lemma 5.3 that $C : [0,1] \to \partial T$ is a Hölder continuous onto mapping. The hexagon Q defined in Proposition 5.5 defines a sequence $(T_n)_{n \geq 0}$ of approximation of T with the required properties. Indeed, by Proposition 5.6, ∂T_n is a simple closed curve. Moreover, by Proposition 5.7, the vertices of ∂T_n have $\boldsymbol{Q}(\beta)$-addresses. Finally, the convergence of ∂T_n toward ∂T in Hausdorff metrics is a consequence of Proposition 3.9.

The measure theoretical part is an application of Theorem 2. We just show that the additional separation condition is fulfilled, that is,

$$
\mathscr{H}_w^\alpha(K_s \cap K_{s'}) = 0
$$

for $s \neq s'$. Note that by the proof of Theorem 2, this measure separation condition is satisfied for subsets of a single boundary part. More precisely, if w and w' are two distinct walks in $G(\mathscr{R})^o$ given by

$$
i \xrightarrow{a_1|o_1} s_1 \xrightarrow{a_2|o_2} \cdots \xrightarrow{a_n|o_n} s_n
$$

and

$$
i \xrightarrow{a_1'|o_1'} s_1' \xrightarrow{a_2'|o_2'} \cdots \xrightarrow{a_n'|o_n'} s_n'
$$

respectively, then

$$
\mathscr{H}_w^\alpha\big(f_{a_1} \circ \cdots \circ f_{a_n}(K_{s_n}) \cap f_{a_1'} \circ \cdots \circ f_{a_n'}(K_{s_n'})\big) = 0,
$$

where $f_a(x) = \boldsymbol{A}^{-1}(x+a)$. Thus it is sufficient to show that for $s \neq s'$, a state i and a common sequence $a_1 = a'_1, \ldots, a_n = a'_n$ can be found such that $s_n = s$ and $s'_n = s'$. Indeed, in this case,

$$\mathscr{H}_w^\alpha\big(f_{a_1} \circ \cdots \circ f_{a_n}(K_s) \cap f_{a_1} \circ \cdots \circ f_{a_n}(K_{s'})\big) = 0,$$

thus K_s and $K_{s'}$ must also be separated with respect to $\mathscr{H}_w^\alpha$, because of the scaling property $\mathscr{H}_w^\alpha(f_a(E)) = 1/\beta \mathscr{H}_w^\alpha(E)$.

We check in Figure 1 this property for every pair of states $\{s, s'\}$. We illustrate the case of the pair $\{1, 6\}$. If $B \geq A-1$, we have walks $6 \xrightarrow{B-1} 4 \xrightarrow{0} 3 \xrightarrow{B-1} 1$ and $6 \xrightarrow{B-1} 2 \xrightarrow{0} 1 \xrightarrow{B-1} 6$, thus $\mathscr{H}_w^s(K_1 \cap K_6) = 0$. If $A = B$, we have walks $2 \xrightarrow{0} 1 \xrightarrow{B-1} 3 \xrightarrow{0} 2 \xrightarrow{B-1} 1$ and $2 \xrightarrow{0} 5 \xrightarrow{B-1} 2 \xrightarrow{0} 1 \xrightarrow{B-1} 6$, thus again $\mathscr{H}_w^s(K_1 \cap K_6) = 0$. We can check that this property holds for every pair $\{s, s'\}$, except for $\{3, 6\}$. However, this case corresponds to the intersection $T \cap (T-P) \cap (T+P)$, which is empty (see [**4**], where the whole list of the neighbors of T was computed). $\square$

We now give a proof of disk-likeness for canonical number system tiles with polynomial $x^2 + Ax + B$ with $2A - B < 3$. In this case, the contact set and the neighbor set are equal: $\mathscr{R} = \mathscr{S}$, hence the automaton of Figure 1 is the boundary automaton. We show that for this choice of coefficients, no identifications occur in the parametrization other than the trivial identifications (see Characterization 3.7). We recall that this can be done by comparing three automata: $\mathscr{A}^\phi$ of trivial identifications, $\mathscr{A}^{sl}$ of walks in $G(\mathscr{R})^o$ carrying the same contraction labels, and $\mathscr{I}^\psi$ of boundary identifications. Following the constructions of Section 3, we obtain the automata of Figures 6 to 8. The initial states are colored, the final states are circled by a double line. Remember that a single walk in these automata corresponds to a pair of walks in the automaton $G(\mathscr{R})^o$ (or in $G(\mathscr{S}) = G(\mathscr{R})$). In the depicted automata $\mathscr{A}^\phi$ and $\mathscr{A}^{sl}$, the transitions carry both order (letters $\boldsymbol{o}$) and contraction labels (letters a). To get a lighter picture, the walks in these automata correspond only to the pairs (w, w') with $w >_{lex} w'$. For simplicity, we also avoided repetition of the labels in the core of the automata, where the two walks do not yet distinguish.

"Projecting" the identification automaton $\mathscr{A}^\phi$ on the boundary, that is, looking only at the contraction labels, we obtain the automaton $\mathscr{I}^\psi$ together with walks whose labeling sequences have the form $(a_n|a_n)_{n\geq 1}$ (hence the corresponding boundary points are trivially the same). This confirms the continuity of the parametrization. Also, the walks in $\mathscr{I}^\psi$ can all be found in $\mathscr{A}^\phi$, meaning that two walks having distinct contraction digit sequences but leading to the same boundary points are trivially identified by the number system ϕ. Considering $\mathscr{A}^{sl}$, we

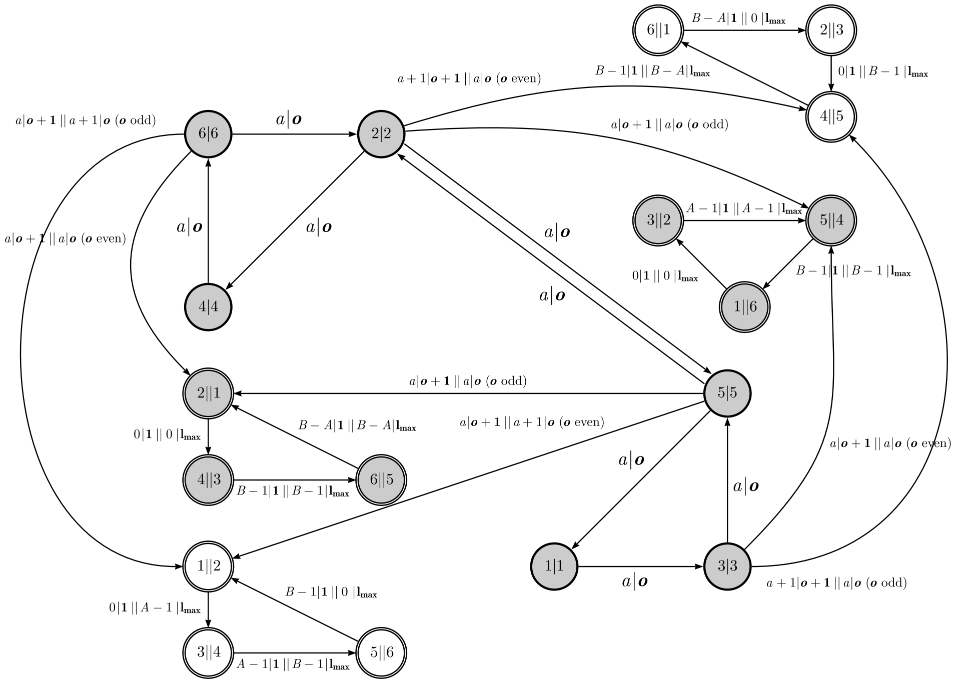

Figure 6. Automaton $\mathscr{A}^{\phi}_{>_{lex}}$.

see that this is also the case for two walks carrying the same contraction digit sequences. Thus $C([0,1])$ is a simple closed curve and the tile T is homeomorphic to a disk.

In the case $2A-B\geq 3$, non trivial identifications occur, that is, we can find $t\neq t'\in[0,1]$ such that $C(t)=C(t')$. Indeed, consider the contact graph of Figure 1. There exists a digit a' such that the transitions

$$Q\xrightarrow{0|B-A+1}-Q\quad\text{and}\quad -Q\xrightarrow{a'|B-A+1}Q.$$

This is because $B-A+1\leq A-2$. This implies the existence of the following infinite walk in $\mathscr{I}^{\psi}$.

$$Q\mid Q\xrightarrow{1\|0}N\parallel -Q\xrightarrow{B-1\|B-A+1}P\parallel Q\xrightarrow{B-A\|A-2}Q\parallel -Q\underset{\xleftarrow{a'\|0}}{\xrightarrow{0\|a'}}-Q\parallel Q.$$

This leads to a non trivial identification

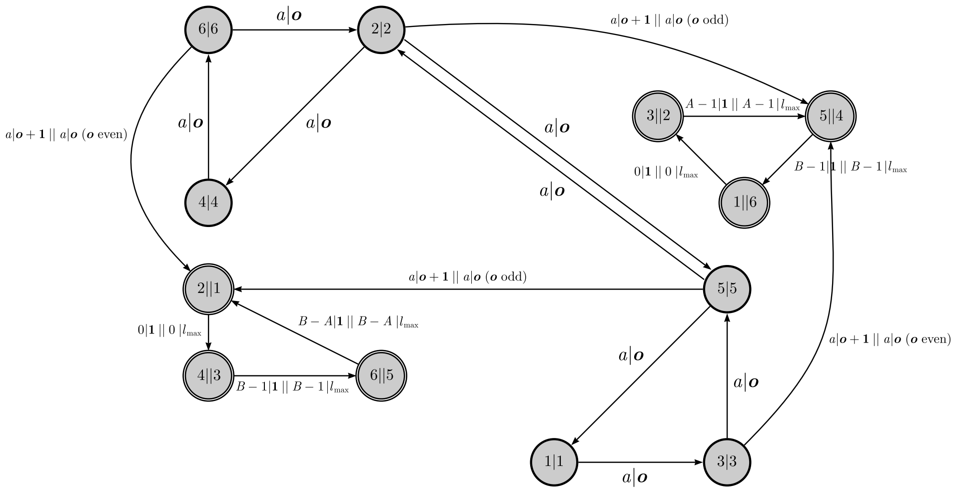

Figure 7. Automaton $\mathscr{A}^{sl}_{>_{lex}}$.

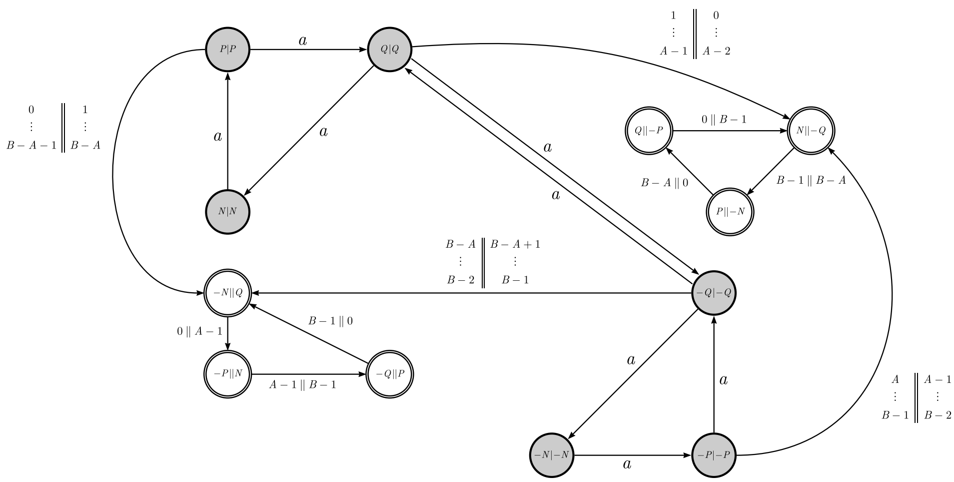

Figure 8. Automaton $\mathscr{I}^{\psi}$ for $2A < B + 3$.

$$C(\phi(2; 3, 1, 1, \overline{2\boldsymbol{o}})) = C(\phi(2; 2, l_{\max} - 1, l_{\max} - 1), \overline{\boldsymbol{o}2}).$$

Thus the corresponding tiles are not disk-like.

6. A relation to the recurrent set method.

The recurrent set method was introduced in [**8**], [**9**] by Dekking. Let $\langle a, b\rangle$ be the free group generated by two letters. The recurrent set method associates to a given endomorphism

$$\sigma : \langle a, b\rangle \to \langle a, b\rangle$$

and a homomorphism

$$g : \langle a, b\rangle \to \boldsymbol{R}^2$$

the boundary of a replication fractal tile. This boundary curve is approximated by polygonal curves enclosing square-like tiles (contracted copies of the parallelogram generated by $g(a), g(b)$). The approximations are geometrical realizations of the iterates of the substitution σ on the initial word $aba^{-1}b^{-1}$. Under some conditions, they converge in the Hausdorff distance to a boundary curve. The corresponding class of Dekking endomorphisms on 2-letters was characterized in [**46**] in terms of digit systems. It followed that for example no 2-letter endomorphism can be found in order to obtain the boundary of the canonical number system tiles associated to the base $-n+i$, $n \geq 3$.

In this section, we consider the substitution given by the contact automaton of a canonical number system tile T and adapt the recurrent set method to recover the boundary of T. A suitable mapping g will be defined in terms of a hexagonal tiling corresponding to the tile T, obtained in the last section.

Let T be a canonical number system tile defined by the matrix $\boldsymbol{A}$ and the digit set $\mathscr{D}$. We consider the following substitution, associated to its ordered automaton $G(\mathscr{R})^o$. It is the endomorphism of the free group over three letters $\langle 1, 2, 3\rangle$, first defined for $1, 2, 3$ according to Figure 1:

$$1 \to 3 \quad 2 \to \dot{1}(\dot{2}\dot{1})^{A-1} \quad 3 \to (\dot{2}\dot{1})^{B-A}\dot{2},$$

where $\dot{1}, \dot{2}, \dot{3}$ stand for the inverses of the letters $1, 2, 3$ and replace $4, 5, 6$ of the automaton. This definition is then extended to $\langle 1, 2, 3\rangle$ by concatenation. We call this substitution σ.

We now map the words into the plane. To each letter $1, 2, 3, \dot{1}, \dot{2}, \dot{3}$, we associate a vector of $\boldsymbol{R}^2$, corresponding to a side of the hexagon Q of Proposition 5.5:

$$\boldsymbol{v}_1 := C_2 - C_1, \qquad \boldsymbol{v}_2 := C_3 - C_2, \qquad \boldsymbol{v}_3 := C_4 - C_3,$$
$$\boldsymbol{v}_{\dot{1}} := C_5 - C_4 = -\boldsymbol{v}_1, \quad \boldsymbol{v}_{\dot{2}} := C_6 - C_1 = -\boldsymbol{v}_2, \quad \boldsymbol{v}_{\dot{3}} := C_1 - C_6 = -\boldsymbol{v}_3,$$

and construct directed curves in the plane via two mappings. The first one is the homomorphism

$$\begin{aligned} g &: \langle 1,2,3\rangle \to \boldsymbol{R}^2 \\ o_1 o_2 \dots o_n &\to \boldsymbol{v}_{o_1} + \dots + \boldsymbol{v}_{o_n}. \end{aligned}$$

The mapping g connects the action of σ on the words and the action of $\boldsymbol{A}$ on the plane.

LEMMA 6.1. *For all words $w \in \langle 1,2,3\rangle$,*

$$g(\sigma(w)) = \boldsymbol{A} g(w).$$

PROOF. It is sufficient to show this property on the letters $1,2,3$. For this, observe that

$$\begin{gathered} \boldsymbol{A}C_1 = C_3, \quad \boldsymbol{A}C_2 = C_4, \quad \boldsymbol{A}C_3 = C_5 + (A-1,0)^T, \\ \boldsymbol{A}C_4 = C_6 + (B-1,0)^T, \quad \boldsymbol{A}C_5 = C_1 + (B-1,0)^T, \\ \boldsymbol{A}C_6 = C_2 + (B-A,0)^T. \end{gathered} \tag{6.1}$$

Thus

$$g(\sigma(1)) = g(3) = \boldsymbol{v}_3 = C_4 - C_3 = \boldsymbol{A}(C_2 - C_1) = \boldsymbol{A}\boldsymbol{v}_1 = \boldsymbol{A} g(1).$$

Also,

$$g(\sigma(2)) = -\boldsymbol{v}_1 - (A-1)(\boldsymbol{v}_2 + \boldsymbol{v}_1) = C_5 - C_4 - (A-1)(C_3 - C_1).$$

By the relations (5.3), $C_3 - C_1 = -(1,0)^T$, thus

$$g(\sigma(2)) = C_5 - C_4 + (A-1,0)^T = \boldsymbol{A}(C_3 - C_2) = \boldsymbol{A} g(2).$$

A similar computation gives $g(\sigma(3)) = \boldsymbol{A} g(3)$. Thus the equality $g(\sigma(o)) = \boldsymbol{A} g(o)$ holds for each letter $o \in \{1,2,3\}$, and by extension for each word of $\langle 1,2,3\rangle$. $\square$

The second mapping is $p : \langle 1,2,3\rangle \to \boldsymbol{R}^2$. If $W = o_1 \dots o_n$ is a reduced word of $\langle 1,2,3\rangle$, $p(W)$ is the directed polygonal curve joining the end points of the vectors

$$0,\ g(o_1),\ g(o_1o_2),\dots,g(o_1o_2\dots o_n).$$

For example, if $W_0 := 123\dot{1}\dot{2}\dot{3}$, then the curve $p(W_0)$ is the boundary of the hexagon Q up to a translation by $-C_1$. We say that a directed curve *encloses* a bounded set Q_0 if it is a simple closed curve oriented counterclockwise and equal to the boundary of Q_0.

Proposition 6.2. *For all $n \geq 1$, $p(\sigma^n(W_0))$ encloses $Q - C_1 + \mathscr{D} + \cdots + \boldsymbol{A}^{n-1}\mathscr{D} + \sum_{k=0}^{n-1}\boldsymbol{A}^k(1,0)^T$.*

Proof. First note that

$$p(W_0) = p(123\dot{1}\dot{2}\dot{3}) = p(3\dot{1}\dot{2}\dot{3}12) + g(12).$$

Since $g(12) = C_3 - C_1 = -(1,0)^T$ by the relations (5.3), this means that $p(3\dot{1}\dot{2}\dot{3}12)$ encloses $Q - C_1 + (1,0)^T$.

Now we prove the above statement by induction on n. For $n = 1$,

$$\begin{aligned}
p\big(\sigma(123\dot{1}\dot{2}\dot{3})\big) &= p\big(3(\dot{1}\dot{2})^B\dot{3}(12)^B\big)\\
&= p\big(3\dot{1}(\dot{2}\dot{1})^{B-1}\dot{2}\dot{3}(12)^B\big)\\
&= p\big(3\dot{1}(\dot{2}\dot{3}3\dot{1})^{B-1}\dot{2}\dot{3}(12)^B\big)\\
&= p\big((3\dot{1}\dot{2}\dot{3})^B(12)^B\big)\\
&= \bigcup_{a=0}^{B-1}\big[p(3\dot{1}\dot{2}\dot{3}12) + ag(3\dot{1}\dot{2}\dot{3})\big] \setminus \bigcup_{a=1}^{B-1}\big[p(\dot{3}) + ag(3\dot{1}\dot{2}\dot{3})\big]\\
&= \bigcup_{a=0}^{B-1}\big[p(3\dot{1}\dot{2}\dot{3}12) + (a,0)^T\big] \setminus \bigcup_{a=1}^{B-1}\big[p(\dot{3}) + (a,0)^T\big].
\end{aligned}$$

We made a slight abuse of notation, since in the last lines, the endpoints of the translates of $p(\dot{3})$ are in fact included in the curve $p(\sigma(123\dot{1}\dot{2}\dot{3}))$. Each $p(3\dot{1}\dot{2}\dot{3}12) + (a,0)^T$ encloses the boundary of the hexagon

$$Q - C_1 + (1,0)^T + (a,0)^T,$$

and these hexagons are essentially disjoint by the tiling property of Q. Thus $p((\sigma(123\dot{1}\dot{2}\dot{3}))$ is the boundary of the union

$$Q - C_1 + (1,0)^T + \mathscr{D}$$

of hexagons glued together through the edges $p(\dot{3}) + (a,0)^T$. The intersections are one-dimensional. In other words, $p((\sigma(123\dot{1}\dot{2}\dot{3}))$ encloses $Q - C_1 + (1,0)^T + \mathscr{D}$.

Suppose now the statement true for some $n \geq 1$. Then

$$\begin{aligned}
&p\big(\sigma^{n+1}(123\dot{1}\dot{2}\dot{3})\big)\\
&\quad = p\big(\sigma^n((3\dot{1}\dot{2}\dot{3})^B)\sigma^n((12)^B)\big)\\
&\quad = p\big((\sigma^n(3\dot{1}\dot{2}\dot{3}))^B(\sigma^n(12))^B\big)\\
&\quad = \bigcup_{a=0}^{B-1}\big[p(\sigma^n(3\dot{1}\dot{2}\dot{3}12)) + ag(\sigma^n(3\dot{1}\dot{2}\dot{3}))\big] \setminus \bigcup_{a=1}^{B-1}\big[p(\sigma^n(\dot{3})) + ag(\sigma^n(3\dot{1}\dot{2}\dot{3}))\big]\\
&\quad = \bigcup_{a=0}^{B-1}\big[p(\sigma^n(3\dot{1}\dot{2}\dot{3}12)) + \boldsymbol{A}^n(a,0)^T\big] \setminus \bigcup_{a=1}^{B-1}\big[p(\sigma^n(\dot{3})) + \boldsymbol{A}^n(a,0)^T\big].
\end{aligned}$$

We observe again that $p(\sigma^n(123\dot{1}\dot{2}\dot{3})) = p(\sigma^n(3\dot{1}\dot{2}\dot{3}12)) + g(\sigma^n(12))$. Thus by induction hypothesis, $p(\sigma^n(3\dot{1}\dot{2}\dot{3}12))$ encloses

$$Q - C_1 + \mathscr{D} + \cdots + \boldsymbol{A}^{n-1}\mathscr{D} + \sum_{k=0}^{n-1}\boldsymbol{A}^k(1,0)^T + \boldsymbol{A}^n(1,0)^T,$$

which tiles the plane by $\boldsymbol{A}^n\boldsymbol{Z}^2$. Consequently, $p(\sigma^{n+1}(123\dot{1}\dot{2}\dot{3}))$ encloses the union of tiles

$$Q - C_1 + \mathscr{D} + \cdots + \boldsymbol{A}^{n-1}\mathscr{D} + \sum_{k=0}^{n}\boldsymbol{A}^k(1,0)^T + \boldsymbol{A}^n\mathscr{D},$$

and we are done. $\square$

Let $(T'_n)_{n\geq 0}$ be the sequence defined by $T'_0 = Q - C_1$ and $\boldsymbol{A}T'_{n+1} = T'_n + \mathscr{D}$. That is to say, $T'_n = T_n - \boldsymbol{A}^{-n}C_1$, where $(T_n)_{n\geq 0}$ satisfies the same recurrence relation but starts with $T_0 = Q$. Then it follows from our parametrization in the last section that the curves $K_n := \boldsymbol{A}^{-n}p(\sigma^n(W_0))$ converge in Hausdorff distance to a curve K. Moreover, K is the boundary of the self-affine set $T + \sum_{k=1}^{\infty}\boldsymbol{A}^{-k}(1,0)^T$.

7. Concluding remarks.

In this paper, we assumed that the "reduced" contact automaton $G(\mathscr{R})$ of $T(\boldsymbol{A},\mathscr{D})$ is strongly connected. Unfortunately this is not always valid. However, it follows from Section 3 that this automaton can be replaced by any strongly connected GIFS containing enough information to describe the boundary. For a given tile, a way to obtain this minimal automaton is to replace $\mathscr{R}$ by a smaller set with the minimal property $\mathscr{D}+\mathscr{R}' \subset \boldsymbol{A}\mathscr{R}'+\mathscr{D}$. We should also require that there is a polygonal fundamental domain Q of the lattice with the set $\mathscr{R}'$ as "adjacent" neighbors (translates of Q having $d-1$-dimensional intersection with Q). Then all approximations T_n of T admit $\mathscr{R}'$ as set of adjacent neighbors. Thus we may perform the recurrent set method on this polygonal tiling. Note that a good candidate, if it can be guessed, is the set of adjacent neighbors of the tile itself (in this case s is adjacent to T if $(T+s)\cap T\cap((T+s)\cup T)^o \neq \emptyset$).

The following example shows that sometimes the lack of strong connectedness can not be avoided. Consider the tile $T(\boldsymbol{A},\mathscr{D})$, where

$$\boldsymbol{A}=\begin{pmatrix}2 & -1\\ 0 & 3\end{pmatrix} \quad\text{and}\quad \mathscr{D}=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right\}.$$

It is depicted in Figure 9. Its contact automaton consists of two disjoint components. Let

$$\mathscr{R}':=\left\{\begin{pmatrix}-1\\0\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\}.$$

Then $G(\mathscr{R}')$ is enough to describe the boundary, since the other two states lead just to two boundary points. Note that the attractor corresponding to the state $(-1,-1)$ is the interval $[0,1]$ of the x-axis. $G(\mathscr{R}')$ is still not strongly connected: it is possible to parametrize ∂T, but in a non-uniform way, which does not fit exactly to the framework of this paper. We refer to [**1**] for a large class of examples whose boundary parametrization could be performed despite the disconnectedness of the contact automaton.

By our parametrization, we are expecting to obtain deep topological information of ∂T and T when it is not a disk. The advantage of this method is that the set of non-trivial identifications occurs in the parametrization is recognized by a Büchi automaton (see p. 559, Theorem 3). Though in practical computation languages of Büchi automaton are difficult to handle, we found several applications when the size of this Büchi automaton is small. This topic will be discussed in a forthcoming paper.

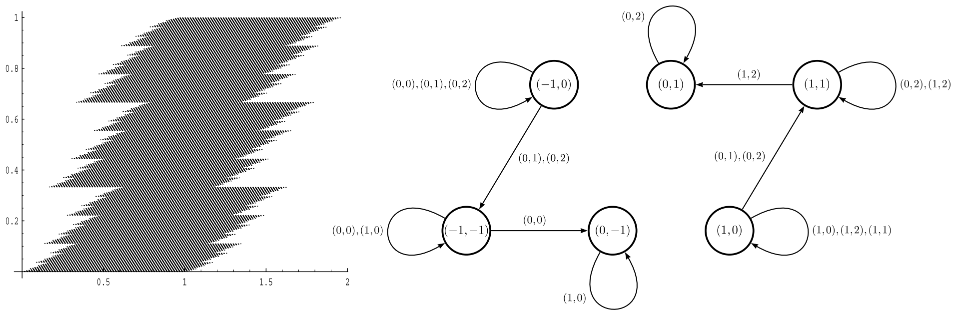

Figure 9. A tile with non primitive reduced contact automaton.

It is important to extend our result to a suitable class of (not necessarily periodic) planar self-affine tilings by several different tile shapes. We believe that this method gives an idea to define the fundamental polygons to start with and to extend the recurrent set method to such tilings. Also the boundary of tiles that do not fulfill any tiling condition may be considered. Indeed, even if overlaps happen, the boundary of the tile is still described by an automaton (see [**10**], [**20**]). Another challenge would be to generalize the result in higher dimensional case. It is not clear whether we will obtain one dimensional parametrizations $[0,1] \to \partial T$, that is, space-filling curves. We rather expect the standard parametrization of the boundary of a d-dimensional tile by the boundary of the d-dimensional unit cube such that the natural projections on lower dimensional faces preserve the complex structure. An example for our motivation is the neighbor graph of a 3-dimensional twin dragon obtained in [**6**].

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