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Higher homotopy commutativity and the resultohedra

Dedicated to Professor James P. Lin on his sixtieth birthday

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Abstract. We define a higher homotopy commutativity for the multiplication of a topological monoid. To give the definition, we use the resultohedra constructed by Gelfand, Kapranov and Zelevinsky. Using the higher homotopy commutativity, we have necessary and sufficient conditions for the classifying space of a topological monoid to have a special structure considered by Félix, Tanré and Aguadé. It is also shown that our higher homotopy commutativity is rationally equivalent to the one of Williams.

1. Introduction.

Félix-Tanré [**7**] studied a condition for a pointed mapping space to be an H-space. To give the condition, they introduced the concept of $H(n)$-space for $n \geq 1$. Then by their result [**7**, Proposition 1], if Y is a space with $\mathrm{cat}(Y) \leq n$ and Z is an $H(n)$-space, then $\mathrm{Map}_*(Y, Z)$ is an H-space, where $\mathrm{cat}(Y)$ denotes the Lusternik-Schnirelmann category of Y. From the definition, any space is an $H(1)$-space, and a space Z is an $H(\infty)$-space if and only if Z is an H-space.

Aguadé [**1**] also considered another criterion for a space to be an H-space. He first defined a T-space as a space Z such that the fibration

$$\Omega Z \longrightarrow \mathrm{Map}(S^1, Z) \xrightarrow{e} Z$$

is fiber homotopy equivalent to the trivial fibration, where ΩZ is the based loop space of Z and $e\colon \mathrm{Map}(S^1, Z) \to Z$ denotes the evaluation map at the base point. While an H-space is always a T-space, the converse is not true. To study when a T-space is an H-space, he also introduced the concept of T_k-space for $k \geq 1$. Then his result [**1**, Proposition 4.1] implies that a T_1-space and a T_∞-space are the same as a T-space and an H-space, respectively.

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Generalizing both of the definitions by Félix-Tanré and Aguadé, we introduce the concept of $H_k(n)$-space for $n \geq 1$ and $1 \leq k \leq n$ (see Definition 5.1). Then it is easy to see that an $H_n(n)$-space is just an $H(n)$-space, and an $H_k(\infty)$-space is the same as a T_k-space. In particular, a space Z is an $H_\infty(\infty)$-space if and only if Z is an H-space.

Sugawara [**19**] gave a criterion for the classifying space of a topological monoid to be an H-space. His criterion is a higher homotopy commutativity for the multiplication (see Theorem 4.1). In this paper, we define a higher homotopy commutativity of a topological monoid, and generalize the result by Sugawara to the case of $H_k(n)$-spaces. The polytopes used in the definition are called the resultohedra, which are constructed by Gelfand-Kapranov-Zelevinsky [**8**].

A topological monoid with a multiplication admitting our higher homotopy commutativity is called a $C_k(n)$-space for $n \geq 1$ and $1 \leq k \leq n$ (see Definition 4.3). From the definition, any topological monoid is a $C_1(1)$-space, and a topological monoid X is a $C_k(2)$-space if and only if the multiplication of X is homotopy commutative for $k = 1, 2$. Moreover, any abelian topological monoid is a $C_\infty(\infty)$-space.

Our main result is stated as follows:

Theorem A. *Let $n \geq 1$ and $1 \leq k \leq n$. Assume that X is a connected topological monoid. Then X is a $C_k(n)$-space if and only if the classifying space BX is an $H_k(n)$-space.*

From Theorem A, we have the following corollary:

Corollary 1.1. *Let X be a connected topological monoid.*

(1) *X is a $C_k(\infty)$-space if and only if BX is a T_k-space for $k \geq 1$. In particular, X is a $C_1(\infty)$-space if and only if BX is a T-space.*
(2) *X is a $C_n(n)$-space if and only if BX is an $H(n)$-space for $n \geq 1$.*

Stasheff [**17**] expanded the theory of Sugawara into the concept of A_n-map for $n \geq 1$ (see Section 4). Then by Corollary 1.1(2) and Proposition 4.2, we see that a topological monoid X is a $C_n(n)$-space if and only if the multiplication of X is an A_n-map for $n \geq 1$.

Williams [**22**] also considered another type of higher homotopy commutativity of a topological monoid. The polytopes used in his definition are called the permutohedra, which are introduced by Milgram [**16**] to construct approximations to iterated loop spaces. A topological monoid with a multiplication of this sort is called a C_n-space for $n \geq 1$. While a $C_k(n)$-space is always a C_n-space by Proposition 4.5, the converse is not true (see Propositions 5.3 and 5.5). However, when the spaces are assumed to be rationalized, we have the following result:

Theorem B. *Let $n \geq 1$ and $1 \leq k \leq n$. Assume that X is a connected topological monoid. Then $X_{(0)}$ is a $C_k(n)$-space if and only if $X_{(0)}$ is a C_n-space, where $X_{(0)}$ denotes the rationalization of X.*

Throughout the paper, all spaces are assumed to be pointed, connected and of the homotopy type of CW-complexes.

This paper is organized as follows: In Section 2, we recall the definition and properties of the resultohedra which are used in the latter sections. In Section 3, we regard the resultohedron as a subspace of the permutohedron (see Proposition 3.1). From this interpretation, the permutohedron is decomposed by the resultohedra combinatorially (see Proposition 3.3). In Section 4, we define a $C_k(n)$-space using the resultohedra, and show that a $C_k(n)$-space is always a C_n-space by Proposition 3.3 (see Proposition 4.5). Section 5 is devoted to the proofs of Theorems A and B. We recall the projective spaces of a topological monoid, and define an $H_k(n)$-space. To prove Theorem A, we generalize the definition of the projective space to be compatible with a $C_k(n)$-structure. Using Theorem A, Proposition 4.5 and the result by Félix-Tanré [**7**], we prove Theorem B. In Section 6, we show that a $C_k(n)$-structure is preserved by the homotopy localizations introduced by Bousfield [**2**] and Dror Farjoun [**6**] (see Theorem 6.2). Then we have that a $C_k(n)$-structure is compatible with the Postnikov systems and the higher connected coverings (see Corollary 6.5).

2. Resultohedra.

Let $\mu_n\colon X^n \to X$ be the n-fold multiplication of a topological monoid X given by $\mu_n(x_1, \ldots, x_n) = x_1 \cdots x_n$. Then Williams [**22**] considered a higher homotopy between the maps $\{\mu_n\sigma \mid \sigma \in \Sigma_n\}$, where Σ_n denotes the n-th symmetric group which acts on X^n by the permutation of the factors. The polytopes to describe this higher homotopy are called the permutohedra, which are introduced by Milgram [**16**]. The n-th permutohedron P_n has vertices corresponding to Σ_n.

Now, if BX is an H-space, then the multiplication of X satisfies the higher homotopy commutativity of Williams in the infinite level. Unfortunately, the converse is not true. To make BX an H-space, we need to consider higher homotopy commutativity given by shuffles, where $\sigma \in \Sigma_{m+n}$ is called an (m, n)-shuffle if

$$\sigma(1) < \cdots < \sigma(m) \quad \text{and} \quad \sigma(m+1) < \cdots < \sigma(m+n) \quad \text{for } m, n \geq 1.$$

For example, for the second level, we consider higher homotopy commutativity corresponding to the $(1, 2)$ and $(2, 1)$ shuffles. For these cases, the polytopes representing the higher homotopy are the 2-simplex Δ^2. For the third level, we consider three types corresponding to the $(1, 3)$, $(2, 2)$ and $(3, 1)$ shuffles. The polytopes for the higher homotopy commutativity corresponding to the $(1,3)$ and $(3,1)$ shuffles are the 3-simplex Δ^3, while for the $(2,2)$ shuffle, we need to consider a more complicated polytope illustrated in [**8**, p. 240, Figure 1] (see also [**9**, p. 414, Figure 61]).

In this section, we introduce the polytopes to describe our higher homotopy commutativity. The polytopes are called the resultohedra, which are constructed by Gelfand-Kapranov-Zelevinsky [**8**]. Since these polytopes are very complicated, we first describe the vertices of them by lattice paths. Our description is an analogy of the one of the vertices of the permutohedron P_n by the lattice paths in I^n described by Milgram.

Let $m, n \geq 1$. A lattice path in the rectangle $[0,m] \times [0,n]$ is a map $\ell\colon [0, m+n] \to [0,m] \times [0,n]$ such that $\ell(0) = (0,0)$, $\ell(m+n) = (m,n)$ and if we write $\ell(s) = (\ell_1(s), \ell_2(s))$ for $s \in [0, m+n]$, then $\ell(i+t)$ is either $(\ell_1(i) + t, \ell_2(i))$ or $(\ell_1(i), \ell_2(i) + t)$ for $0 \leq i < m+n$ and $t \in I$. We denote the set of all lattice paths in $[0,m] \times [0,n]$ by $\mathscr{L}_{m,n}$.

For any two words $x_1 \cdots x_m$ and $y_1 \cdots y_n$, we have a new word w of length $m+n$ containing $x_1 \cdots x_m$ and $y_1 \cdots y_n$ as subsequences. In other words, if we put $z_i = x_i$ for $1 \leq i \leq m$ and $z_{m+j} = y_j$ for $1 \leq j \leq n$, then w is given by

$$w = z_{\sigma^{-1}(1)} \cdots z_{\sigma^{-1}(m+n)} \quad \text{for some } (m,n)\text{-shuffle } \sigma.$$

We call such a word w a shuffle of $x_1 \cdots x_m$ and $y_1 \cdots y_n$. In $[0,m] \times [0,n]$, we label the interval $[i-1, i] \times \{j\}$ by x_i for $1 \leq i \leq m$, $0 \leq j \leq n$ and the interval $\{i\} \times [j-1, j]$ by y_j for $0 \leq i \leq m$, $1 \leq j \leq n$ as in Figure 1. Then each lattice path $\ell \in \mathscr{L}_{m,n}$ is labeled by a shuffle of $x_1 \cdots x_m$ and $y_1 \cdots y_n$. In this label of ℓ, the symbol x_i means the horizontal unit move from the line $x = i-1$ to the line $x = i$ for $1 \leq i \leq m$, and y_j is the vertical move between two lines $y = j-1$ and $y = j$ for $1 \leq j \leq n$. For example, the lattice path $\ell \in \mathscr{L}_{4,3}$ in Figure 1 is labeled by $x_1 y_1 x_2 x_3 y_2 x_4 y_3$.

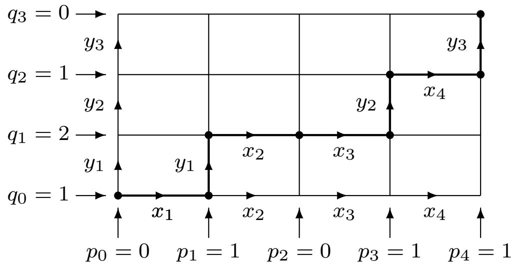

Figure 1. The lattice path $\ell = x_1 y_1 x_2 x_3 y_2 x_4 y_3$.

Given a lattice path $\ell \in \mathscr{L}_{m,n}$, let p_i^ℓ and q_j^ℓ be the lengths of the intersections of ℓ with the lines $x = i$ for $0 \le i \le m$ and $y = j$ for $0 \le j \le n$, respectively. Then in the corresponding shuffle of $x_1 \cdots x_m$ and $y_1 \cdots y_n$, p_i^ℓ is the number of y_js between x_i and x_{i+1} for $0 \le i \le m$, and q_j^ℓ is the number of x_is between y_j and y_{j+1} for $0 \le j \le n$. For example, $(p_0^\ell, \ldots, p_4^\ell, q_0^\ell, \ldots, q_3^\ell) = (0,1,0,1,1,1,2,1,0)$ for $\ell = x_1y_1x_2x_3y_2x_4y_3$ in Figure 1.

For $m, n \ge 1$, Gelfand-Kapranov-Zelevinsky [**8**, Theorem 4] defined $N_{m,n}$ as the subspace of $\boldsymbol{R}^{m+n+2}$ consisting of all points $(p_0, \ldots, p_m, q_0, \ldots, q_n) \in (\boldsymbol{R}^+)^{m+n+2}$ with the relations:

$$\sum_{0 \le i \le m} p_i = n, \quad \sum_{0 \le j \le n} q_j = m, \quad h_{i,j} \ge 0 \quad \text{and} \quad h_{m,n} = 0, \tag{2.1}$$

where $\boldsymbol{R}^+ = \{t \in \boldsymbol{R} \mid t \ge 0\}$ and

$$h_{i,j} = \sum_{0 \le k \le i} (i-k)p_k + \sum_{0 \le l \le j} (j-l)q_l - ij \quad \text{for } 0 \le i \le m \text{ and } 0 \le j \le n.$$

Then by their result [**8**, Theorems 2' and 6], $N_{m,n}$ is an $(m+n-1)$-dimensional polytope such that the set of all vertices is given by

$$v(N_{m,n}) = \left\{ \left(p_0^\ell, \ldots, p_m^\ell, q_0^\ell, \ldots, q_n^\ell\right) \in \boldsymbol{R}^{m+n+2} \mid \ell \in \mathscr{L}_{m,n} \right\}.$$

According to Kapranov-Voevodsky [**14**, p. 242, 6.2], the polytope $N_{m,n}$ is called the resultohedron. By [**8**, Proposition 13], $N_{m,1}$ and $N_{1,n}$ are the simplices Δ^m and Δ^n, respectively (see (2.4)). For convenience, we put $N_{m,0} = N_{0,n} = \{*\}$ for $m, n \ge 1$.

Consider the subspaces $N(p_i)$, $N(q_j)$ and $N(h_{i,j})$ of $N_{m,n}$ defined by

$$N(p_i) = \left\{ (p_0, \ldots, p_m, q_0, \ldots, q_n) \in N_{m,n} \mid p_i = 0 \right\} \quad \text{for } 0 \le i \le m,$$
$$N(q_j) = \left\{ (p_0, \ldots, p_m, q_0, \ldots, q_n) \in N_{m,n} \mid q_j = 0 \right\} \quad \text{for } 0 \le j \le n$$

and

$$N(h_{i,j}) = \left\{ (p_0, \ldots, p_m, q_0, \ldots, q_n) \in N_{m,n} \mid h_{i,j} = 0 \right\}$$

for $0 < i < m$ and $0 < j < n$.

Proposition 2.1 ([**9**, Chapter 12, Corollary 2.17, Theorem 2.18]).

(1) *The boundary of* $N_{m,n}$ *is given by*

$$\partial N_{m,1} = \bigcup_{0 \le i \le m} N(p_i), \quad \partial N_{1,n} = \bigcup_{0 \le j \le n} N(q_j)$$

and

$$\partial N_{m,n} = \bigcup_{0 \le i \le m} N(p_i) \cup \bigcup_{0 \le j \le n} N(q_j) \cup \bigcup_{0<i<m, 0<j<n} N(h_{i,j}) \quad \textit{for } m, n > 1.$$

(2) *The facets* $N(p_i)$, $N(q_j)$ *and* $N(h_{i,j})$ *are affinely homeomorphic to* $N_{m-1,n}$, $N_{m,n-1}$ *and* $N_{i,j} \times N_{m-i,n-j}$ *by the face operators*

$$\varepsilon^{(p_i)} \colon N_{m-1,n} \to N_{m,n} \quad \textit{for } 0 \le i \le m,$$
$$\varepsilon^{(q_j)} \colon N_{m,n-1} \to N_{m,n} \quad \textit{for } 0 \le j \le n$$

and

$$\varepsilon^{(h_{i,j})} \colon N_{i,j} \times N_{m-i,n-j} \to N_{m,n} \quad \textit{for } 0 < i < m \textit{ and } 0 < j < n,$$

respectively.

Using the same way as the proof of [**16**, Lemma 4.5], we have the following lemma:

LEMMA 2.2. *There are degeneracy operators* $\{\delta_k \colon N_{m,n} \to N_{m-1,n}\}_{1 \le k \le m}$ *and* $\{\delta'_l \colon N_{m,n} \to N_{m,n-1}\}_{1 \le l \le n}$ *with the following relations*:

$$\begin{aligned}
\delta_k \varepsilon^{(p_i)}(a) &= \begin{cases} \varepsilon^{(p_i)} \delta_{k-1}(a) & \textit{if } 0 \le i < k-1 \\ a & \textit{if } i = k-1, k \\ \varepsilon^{(p_{i-1})} \delta_k(a) & \textit{if } k < i \le m, \end{cases} \\
\delta_k \varepsilon^{(q_j)}(a) &= \varepsilon^{(q_j)} \delta_k(a) \quad \textit{for } 0 \le j \le n, \\
\delta_k \varepsilon^{(h_{i,j})}(a, b) &= \begin{cases} \varepsilon^{(h_{i,j})}(a, \delta_{k-i}(b)) & \textit{if } 0 < i < k \\ \varepsilon^{(h_{i-1,j})}(\delta_k(a), b) & \textit{if } k \le i < m. \end{cases}
\end{aligned} \tag{2.2}$$

$$\delta'_l \varepsilon^{(p_i)}(a) = \varepsilon^{(p_i)} \delta'_l(a) \quad \textit{for } 0 \le i \le m,$$

$$\delta'_l \varepsilon^{(q_j)}(a) = \begin{cases} \varepsilon^{(q_j)} \delta'_{l-1}(a) & \textit{if } 0 \le j < l-1 \\ a & \textit{if } j = l-1, l \\ \varepsilon^{(q_{j-1})} \delta'_l(a) & \textit{if } l < j \le n, \end{cases} \tag{2.3}$$

$$\delta'_l \varepsilon^{(h_{i,j})}(a, b) = \begin{cases} \varepsilon^{(h_{i,j})}(a, \delta'_{l-j}(b)) & \textit{if } 0 < j < l \\ \varepsilon^{(h_{i,j-1})}(\delta'_l(a), b) & \textit{if } l \le j < n. \end{cases}$$

PROOF. We prove the case of $\{\delta_k\}_{1 \le k \le m}$ by induction on m and n. When $m = 1$ or $n = 0$, we put $\delta_k(a) = *$ for $1 \le k \le m$. Let $m > 1$ and $n > 0$. Assume inductively that $\{\delta_k \colon N_{m',n'} \to N_{m'-1,n'}\}_{1 \le k \le m'}$ are constructed for $m' \le m$ and $n' \le n$ with $(m', n') \ne (m, n)$.

Now we define $\widetilde{\delta}_k \colon \partial N_{m,n} \to N_{m-1,n}$ by (2.2) for $1 \le k \le m$. Since $N_{m,n}$ is the reduced cone of $\partial N_{m,n}$, if $a \in N_{m,n}$, then we can write $a = (b, t)$ with $b \in \partial N_{m,n}$ and $t \in I$. Set $\widetilde{\delta}_k(b) = (c, u)$ with $c \in \partial N_{m-1,n}$ and $u \in I$. Then we can define $\delta_k \colon N_{m,n} \to N_{m-1,n}$ by $\delta_k(a) = (c, tu)$, and $\{\delta_k\}_{1 \le k \le m}$ satisfies the required conditions. In the case of $\{\delta'_l\}_{1 \le l \le n}$, the proof is similar. This completes the proof. $\square$

Let Δ^m denote the m-simplex:

$$\Delta^m = \left\{ (t_0, \dots, t_m) \in (\boldsymbol{R}^+)^{m+1} \;\middle|\; \sum_{0 \le i \le m} t_i = 1 \right\} \quad \text{for } m \ge 0 \tag{2.4}$$

with the vertices $v_i = (\overbrace{0, \dots, 0}^{i}, 1, \overbrace{0, \dots, 0}^{m-i})$ for $0 \le i \le m$. Then we have the face operators $\{\partial_i \colon \Delta^{m-1} \to \Delta^m\}_{0 \le i \le m}$ and the degeneracy operators $\{s_k \colon \Delta^m \to \Delta^{m-1}\}_{1 \le k \le m}$ (cf. [**11**, p. 109]). We define $\rho_m \colon \Delta^m \to [0, m]$ by

$$\rho_m(t_0, \dots, t_m) = \sum_{0 \le i \le m} i t_i,$$

and identify the image $\rho_m(\Delta^m) = [0, m]$ with the edge $v_0 v_m \subset \Delta^m$ (see Figure 2).

Consider the quotient space

$$\Delta^{m,n} = \Delta^m \times \Delta^n / \sim \quad \text{for } m, n \ge 0 \text{ with } m + n \ge 1$$

and the projection $\pi_{m,n} \colon \Delta^m \times \Delta^n \to \Delta^{m,n}$, where the relation " $\sim$ " is given by $(a_1, v_j) \sim (a_2, v_j)$ if $\rho_m(a_1) = \rho_m(a_2)$ for $a_1, a_2 \in \Delta^m$ and $0 \le j \le n$, and

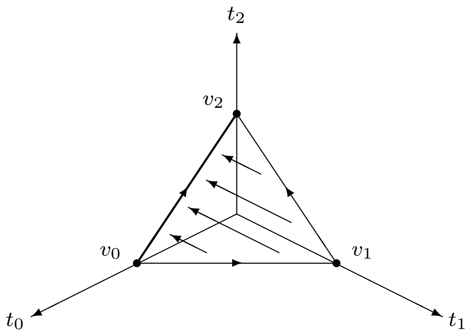

Figure 2. The projection ρ_2.

$(v_i, b_1) \sim (v_i, b_2)$ if $\rho_n(b_1) = \rho_n(b_2)$ for $b_1, b_2 \in \Delta^n$ and $0 \leq i \leq m$ (see Figure 3).

Denote $\pi_{m,n}(a, b) \in \Delta^{m,n}$ by $\langle a, b\rangle$ for $(a, b) \in \Delta^m \times \Delta^n$. Then we have the face operators $\{\beta_i \colon \Delta^{m-1,n} \to \Delta^{m,n}\}_{0\leq i\leq m}$ and $\{\beta'_j \colon \Delta^{m,n-1} \to \Delta^{m,n}\}_{0\leq j\leq n}$ given by $\beta_i(\langle a, b\rangle) = \langle\partial_i(a), b\rangle$ and $\beta'_j(\langle a, b\rangle) = \langle a, \partial_j(b)\rangle$. Moreover, the degeneracy operators $\{\gamma_k \colon \Delta^{m,n} \to \Delta^{m-1,n}\}_{1\leq k\leq m}$ and $\{\gamma'_l \colon \Delta^{m,n} \to \Delta^{m,n-1}\}_{1\leq l\leq n}$ are defined by $\gamma_k(\langle a, b\rangle) = \langle s_k(a), b\rangle$ and $\gamma'_l(\langle a, b\rangle) = \langle a, s_l(b)\rangle$.

Now as in the case of $[0, m] \times [0, n]$, we label the edge $v_{i-1}v_i \times \{v_j\}$ of $\Delta^{m,n}$ by x_i for $1 \leq i \leq m$, $0 \leq j \leq n$ and the edge $\{v_i\} \times v_{j-1}v_j$ of $\Delta^{m,n}$ by y_j for $0 \leq i \leq m$, $1 \leq j \leq n$ (see Figure 3). Put

$$\mathscr{K}_{m,n} = \bigl\{\ell \colon [0, m+n] \to \Delta^{m,n} \mid \ell(0) = \langle v_0, v_0\rangle \text{ and } \ell(m+n) = \langle v_m, v_n\rangle\bigr\}.$$

Then any lattice path $\ell \in \mathscr{L}_{m,n}$ can be regarded as $\ell \in \mathscr{K}_{m,n}$ (see Figure 4). Let $\widetilde{\kappa}_{m,n} \colon v(N_{m,n}) \to \mathscr{K}_{m,n}$ be defined by $\widetilde{\kappa}_{m,n}((p^\ell_0, \ldots, p^\ell_m, q^\ell_0, \ldots, q^\ell_n)) = \ell$. Since $N_{m,n}$ is the convex hull of $v(N_{m,n})$:

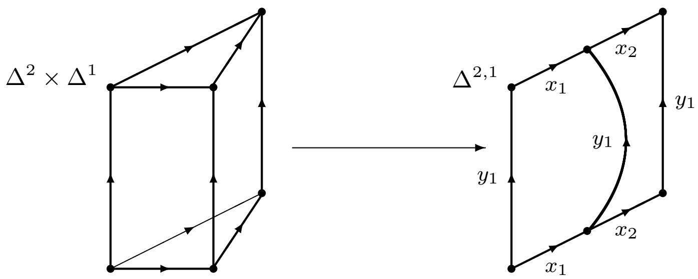

Figure 3. The projection $\pi_{2,1}$.

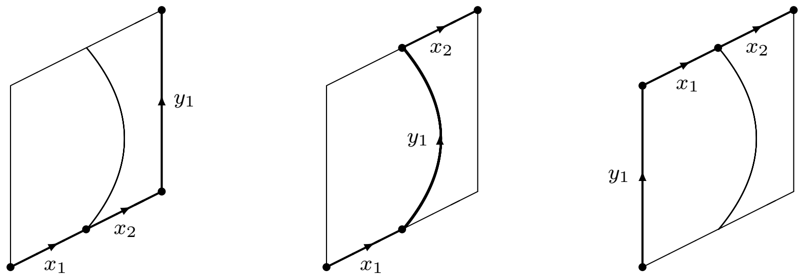

Figure 4. The lattice paths $\ell_1 = x_1x_2y_1$, $\ell_2 = x_1y_1x_2$ and $\ell_3 = y_1x_1x_2$ in $\mathscr{K}_{2,1}$.

$$N_{m,n} = \left\{ \sum_{1\le i\le k} t_i a_i \,\middle|\, a_i \in v(N_{m,n}) \text{ and } t_i \in \boldsymbol{R}^+ \text{ with } \sum_{1\le i\le k} t_i = 1 \right\},$$

we extend $\widetilde{\kappa}_{m,n}$ to $\kappa_{m,n}\colon N_{m,n} \to \mathscr{K}_{m,n}$ by

$$\kappa_{m,n}\left(\sum_{1\le i\le k} t_i a_i \right)(s) = \sum_{1\le i\le k} t_i \widetilde{\kappa}_{m,n}(a_i)(s) \quad \text{for } s \in [0, m+n]. \tag{2.5}$$

3. Permutohedra.

The n-th symmetric group Σ_n acts on $\boldsymbol{R}^n$ by the permutation of the factors. Put $\boldsymbol{n} = (1, \dots, n) \in \boldsymbol{R}^n$. According to Milgram [**16**, Definition 4.1], the permutohedron P_n is an $(n-1)$-dimensional polytope defined by the convex hull of $\{\sigma(\boldsymbol{n}) \in \boldsymbol{R}^n \mid \sigma \in \Sigma_n\}$ for $n \ge 1$. From the construction, there is a natural way to describe all the faces of P_n.

Let $u_1, \dots, u_m \ge 1$ with $u_1 + \cdots + u_m = n$. A partition of $\boldsymbol{n}$ of type $(u_1, \dots, u_m)$ is an ordered sequence $(\alpha_1, \dots, \alpha_m)$ consisting of disjoint subsequences α_i of length u_i for $1 \le i \le m$ with $\alpha_1 \cup \cdots \cup \alpha_m = \boldsymbol{n}$ as sets (see [**11**, p. 107], [**12**, p. 3826]). Then there is a correspondence between the faces of P_n and the partitions of $\boldsymbol{n}$ into at least two disjoint parts (see [**11**, p. 107]). In particular, a facet of P_n is represented by a partition of $\boldsymbol{n}$ into just two disjoint parts.

Consider the subspace T_n of $\boldsymbol{R}^n$ defined by

$$T_n = \left\{ (t_1, \dots, t_n) \in \boldsymbol{R}^n \,\middle|\, \sum_{1\le i\le n} t_i = \frac{n(n+1)}{2} \right\} \quad \text{for } n \ge 1.$$

Put

$$T(\alpha_1, \alpha_2) = \left\{ (t_1, \ldots, t_n) \in \boldsymbol{R}^n \,\middle|\, \sum_{1 \le i \le u_1} t_{\alpha_1(i)} \ge \frac{u_1(u_1+1)}{2} \right\}$$

and

$$\partial T(\alpha_1, \alpha_2) = \left\{ (t_1, \ldots, t_n) \in \boldsymbol{R}^n \,\middle|\, \sum_{1 \le i \le u_1} t_{\alpha_1(i)} = \frac{u_1(u_1+1)}{2} \right\},$$

where (α_1, α_2) is a partition of $\boldsymbol{n}$ of type (u_1, u_2). From the definition,

$$P_n = T_n \cap \bigcap_{(\alpha_1, \alpha_2)} T(\alpha_1, \alpha_2)$$

whose boundary ∂P_n is given by

$$\partial P_n = \bigcup_{(\alpha_1, \alpha_2)} P(\alpha_1, \alpha_2) \quad \text{with} \quad P(\alpha_1, \alpha_2) = P_n \cap \partial T(\alpha_1, \alpha_2),$$

where (α_1, α_2) covers all partitions of $\boldsymbol{n}$ into two disjoint parts (see Figure 5). By [**16**, Lemma 4.2], the facet $P(\alpha_1, \alpha_2)$ is affinely homeomorphic to $P_{u_1} \times P_{u_2}$ by the face operator $\varepsilon^{(\alpha_1, \alpha_2)} \colon P_{u_1} \times P_{u_2} \to P(\alpha_1, \alpha_2)$. Moreover, we have the degeneracy operators $\{d_k \colon P_n \to P_{n-1}\}_{1 \le k \le n}$ with the relations in [**16**, Lemma 4.5].

Now we recall that a permutation $\sigma \in \Sigma_{m+n}$ is called an (m,n)-shuffle if

$$\sigma(1) < \cdots < \sigma(m) \quad \text{and} \quad \sigma(m+1) < \cdots < \sigma(m+n) \quad \text{for } m, n \ge 1.$$

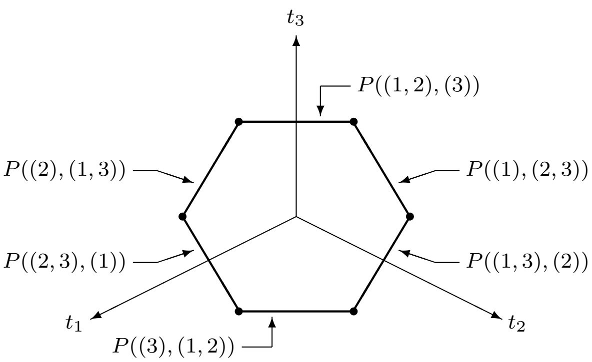

Figure 5. The permutohedron P_3.

We denote the set of all (m,n)-shuffles by $\mathscr{S}_{m,n}$. Then there is a bijection between $\mathscr{S}_{m,n}$ and $\mathscr{L}_{m,n}$. In fact, if $\sigma \in \mathscr{S}_{m,n}$, then putting x_i on the $\sigma(i)$-th place for $1 \leq i \leq m$ and y_j on the $\sigma(m+j)$-th place for $1 \leq j \leq n$, we have a shuffle of $x_1 \cdots x_m$ and $y_1 \cdots y_n$ which is the label of some lattice path $\ell \in \mathscr{L}_{m,n}$. For example, the $(4,3)$-shuffle

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 6 & 2 & 5 & 7 \end{pmatrix} \in \mathscr{S}_{4,3}$$

is corresponding to the lattice path $\ell \in \mathscr{L}_{4,3}$ labeled by $x_1y_1x_2x_3y_2x_4y_3$ (see Figure 1).

Proposition 3.1 ([**9**, Chapter 12, Proposition 2.6]). *The resultohedron $N_{m,n}$ is embedded in P_{m+n} as*

$$N_{m,n} = P_{m+n} \cap \bigcap_{1 \leq i \leq m-1} H_i \cap \bigcap_{1 \leq j \leq n-1} H'_j \quad \textit{for } m,n \geq 1,$$

which is the convex hull of $\{\sigma(1,\ldots,m+n) \in \boldsymbol{R}^{m+n} \mid \sigma \in \mathscr{S}_{m,n}\}$, *where*

$$H_i = \bigl\{(t_1,\ldots,t_{m+n}) \in \boldsymbol{R}^{m+n} \mid t_{i+1} \geq t_i + 1\bigr\} \quad \textit{for } 1 \leq i \leq m-1$$

and

$$H'_j = \bigl\{(t_1,\ldots,t_{m+n}) \in \boldsymbol{R}^{m+n} \mid t_{m+j+1} \geq t_{m+j} + 1\bigr\} \quad \textit{for } 1 \leq j \leq n-1.$$

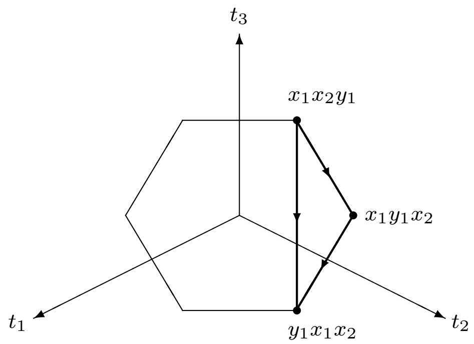

Figure 6. The resultohedron $N_{2,1}$.

Remark 3.2. In (2.1), the resultohedron $N_{m,n}$ is defined in $\boldsymbol{R}^{m+n+2}$. Proposition 3.1 implies that $N_{m,n}$ is considered as a subspace of $\boldsymbol{R}^{m+n}$.

In the proof of Proposition 4.5, we need the following result proved by Hemmi [**11**] and Kapranov-Voevodsky [**14**]:

Proposition 3.3 ([**11**, p. 108, (5.1)], [**14**, Theorem 6.5]).

(1) *The permutohedron* P_{n+1} *is decomposed by the subspaces* $\Gamma(\alpha_1, \dots, \alpha_m)$ *as*

$$P_{n+1} = \bigcup_{(\alpha_1, \dots, \alpha_m)} \Gamma(\alpha_1, \dots, \alpha_m) \quad \text{for } n \geq 1,$$

where $(\alpha_1, \dots, \alpha_m)$ *covers all partitions of* $\boldsymbol{n}$ *with* $m \geq 1$.

(2) *If* $(\alpha_1, \dots, \alpha_m)$ *is a partition of* $\boldsymbol{n}$ *of type* $(u_1, \dots, u_m)$, *then* $\Gamma(\alpha_1, \dots, \alpha_m)$ *is affinely homeomorphic to* $N_{m,1} \times P_{u_1} \times \cdots \times P_{u_m}$ *by an operator* $\iota^{(\alpha_1, \dots, \alpha_m)} \colon N_{m,1} \times P_{u_1} \times \cdots \times P_{u_m} \to \Gamma(\alpha_1, \dots, \alpha_m)$.

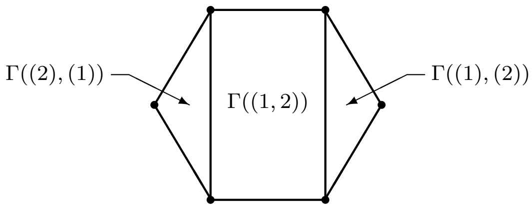

Figure 7. The decomposition of P_3.

For the decomposition of the 4-th permutohedron P_4, see [**14**, p. 245, Figure 15]. By Proposition 3.1, $N_{m,1}$ is embedded in P_{m+1}. Then the inclusion $N_{m,1} \subset P_{m+1}$ is corresponding to the operator $\iota^{((1), \dots, (m))} \colon N_{m,1} \times P_1 \times \cdots \times P_1 \to \Gamma((1), \dots, (m)) \subset P_{m+1}$ in Proposition 3.3 (see Figures 6 and 7).

4. Higher homotopy commutativity.

Sugawara [**19**] introduced the concept of strongly homotopy multiplicativity for maps between topological monoids. Later Stasheff [**17**] expanded his definition, and introduced the concept of A_n-map for $n \geq 1$. Let X and Y be topological monoids and $n \geq 1$. A map $\phi \colon X \to Y$ is called an A_n-map if there is a family of maps $\{F_i \colon I^{i-1} \times X^i \to Y\}_{1 \leq i \leq n}$ such that $F_1(x) = \phi(x)$ and

$$F_i(t_1, \dots, t_{i-1}, x_1, \dots, x_i) = \begin{cases} F_{i-1}(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_{i-1}, x_1, \dots, x_j \cdot x_{j+1}, \dots, x_i) & \text{if } t_j = 0 \\ F_j(t_1, \dots, t_{j-1}, x_1, \dots, x_j) \cdot F_{i-j}(t_{j+1}, \dots, t_{i-1}, x_{j+1}, \dots, x_i) & \text{if } t_j = 1 \end{cases}$$

for $1 \le j \le i-1$.

From the definition, an A_2-map is just an H-map, and an A_3-map is an H-map preserving the homotopy associativity. Moreover, an A_∞-map is the same as a strongly homotopy multiplicative map.

Using the strongly homotopy multiplicativity, Sugawara gave a criterion for the classifying space of a topological monoid to be an H-space (see also Stasheff [**18**, p. 71, Theorem 14.1]):

THEOREM 4.1 ([**19**]). *Let X be a topological monoid. The multiplication $\mu\colon X^2 \to X$ is strongly homotopy multiplicative if and only if the classifying space BX is an H-space.*

In Theorem 4.1, the condition of strongly homotopy multiplicativity for $\mu\colon X^2 \to X$ can be regarded as a higher homotopy commutativity for μ. In fact, we see that $\mu\colon X^2 \to X$ is an H-map if and only if μ is a homotopy commutative multiplication of X.

Generalizing Theorem 4.1, we have the following result:

PROPOSITION 4.2. *Let X be a topological monoid. The multiplication $\mu\colon X^2 \to X$ is an A_n-map if and only if BX is an $H(n)$-space for $n \ge 1$.*

The proof of Proposition 4.2 is given in Section 5.

Now we define a $C_k(n)$-space. Let $n \ge 1$ and $1 \le k \le n$. Put

$$\Lambda_k(n) = \bigl\{(r,s) \in \boldsymbol{Z}^2 \mid r, s \ge 0,\ 1 \le r+s \le n \text{ and } s \le k\bigr\}.$$

DEFINITION 4.3. Let $n \ge 1$ and $1 \le k \le n$. A topological monoid X is called a $C_k(n)$-space if there is a family of maps $\{Q_{r,s}\colon N_{r,s} \times X^{r+s} \to X\}_{(r,s)\in\Lambda_k(n)}$ with the following relations:

$$Q_{r,0}(*, x_1, \dots, x_r) = x_1 \cdots x_r \quad \text{and} \quad Q_{0,s}(*, y_1, \dots, y_s) = y_1 \cdots y_s. \tag{4.1}$$

$$Q_{r,s}(\varepsilon^{(p_i)}(a), x_1, \dots, x_r, y_1, \dots, y_s) = \begin{cases} x_1 \cdot Q_{r-1,s}(a, x_2, \dots, x_r, y_1, \dots, y_s) & \text{if } i = 0 \\ Q_{r-1,s}(a, x_1, \dots, x_i \cdot x_{i+1}, \dots, x_r, y_1, \dots, y_s) & \text{if } 0 < i < r \\ Q_{r-1,s}(a, x_1, \dots, x_{r-1}, y_1, \dots, y_s) \cdot x_r & \text{if } i = r. \end{cases} \tag{4.2}$$

$$
\begin{aligned}
&Q_{r,s}(\varepsilon^{(q_j)}(a), x_1, \dots, x_r, y_1, \dots, y_s) \\
&\quad = \begin{cases} y_1 \cdot Q_{r,s-1}(a, x_1, \dots, x_r, y_2, \dots, y_s) & \text{if } j = 0 \\ Q_{r,s-1}(a, x_1, \dots, x_r, y_1, \dots, y_j \cdot y_{j+1}, \dots, y_s) & \text{if } 0 < j < s \\ Q_{r,s-1}(a, x_1, \dots, x_r, y_1, \dots, y_{s-1}) \cdot y_s & \text{if } j = s. \end{cases}
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
&Q_{r,s}(\varepsilon^{(h_{i,j})}(a, b), x_1, \dots, x_r, y_1, \dots, y_s) \\
&\quad = Q_{i,j}(a, x_1, \dots, x_i, y_1, \dots, y_j) \cdot Q_{r-i,s-j}(b, x_{i+1}, \dots, x_r, y_{j+1}, \dots, y_s)
\end{aligned}
\tag{4.4}
$$

for $0 < i < r$ and $0 < j < s$.

$$
\begin{aligned}
&Q_{r,s}(a, x_1, \dots, x_{i-1}, *, x_{i+1}, \dots, x_r, y_1, \dots, y_s) \\
&\quad = Q_{r-1,s}(\delta_i(a), x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r, y_1, \dots, y_s) \quad \text{for } 1 \le i \le r, \\
&Q_{r,s}(a, x_1, \dots, x_r, y_1, \dots, y_{j-1}, *, y_{j+1}, \dots, y_s) \\
&\quad = Q_{r,s-1}(\delta'_j(a), x_1, \dots, x_r, y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_s) \quad \text{for } 1 \le j \le s.
\end{aligned}
\tag{4.5}
$$

Remark 4.4.

(1) Any topological monoid is a $C_1(1)$-space, and a $C_k(2)$-space is a topological monoid whose multiplication is homotopy commutative for $k = 1, 2$.
(2) An abelian topological monoid has a $C_\infty(\infty)$-structure:

$$
Q_{r,s}(a, x_1, \dots, x_r, y_1, \dots, y_s) = x_1 \cdots x_r \cdot y_1 \cdots y_s \quad \text{for } r, s \ge 1.
$$

In particular, Eilenberg-Mac Lane spaces have the homotopy type of $C_\infty(\infty)$-spaces.

Williams [**22**] considered another type of higher homotopy commutativity using the permutohedra. Let $n \ge 1$. A topological monoid X is called a C_n-space if there is a family of maps $\{Q_i : P_i \times X^i \to X\}_{1 \le i \le n}$ with the following relations:

$$
Q_1(*, x) = x. \tag{4.6}
$$

$$
\begin{aligned}
&Q_i(\varepsilon^{(\alpha_1,\alpha_2)}(c_1, c_2), x_1, \dots, x_i) \\
&\quad = Q_{u_1}(c_1, x_{\alpha_1(1)}, \dots, x_{\alpha_1(u_1)}) \cdot Q_{u_2}(c_2, x_{\alpha_2(1)}, \dots, x_{\alpha_2(u_2)}),
\end{aligned}
\tag{4.7}
$$

where (α_1, α_2) is a partition of $\boldsymbol{i}$ of type (u_1, u_2).

$$
Q_i(c, x_1, \dots, x_{j-1}, *, x_{j+1}, \dots, x_i) = Q_{i-1}(d_j(c), x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_i) \tag{4.8}
$$

for $1 \leq j \leq i$.

Proposition 4.5. *Let $n \geq 1$ and $1 \leq k \leq n$. If X is a $C_k(n)$-space, then X is a C_n-space.*

Proof. Since a $C_k(n)$-space is a $C_{k-1}(n)$-space for $1 < k \leq n$, it is enough to prove the case of $k = 1$.

We work by induction on n. The result is clear for $n = 1$. Assume that the result is proved for n, and consider the case of $n+1$. Let X be a $C_1(n+1)$-space. Since a $C_1(n+1)$-space is a $C_1(n)$-space, by inductive hypothesis, there is a C_n-structure $\{Q_i\}_{1 \leq i \leq n}$ on X. By Proposition 3.3, we can define $Q_{n+1} \colon P_{n+1} \times X^{n+1} \to X$ by

$$\begin{aligned} &Q_{n+1}\big(\iota^{(\alpha_1,\dots,\alpha_m)}(a, c_1, \dots, c_m), x_1, \dots, x_{n+1}\big) \\ &\quad = Q_{m,1}\big(a, Q_{u_1}\big(c_1, x_{\alpha_1(1)}, \dots, x_{\alpha_1(u_1)}\big), \dots, \\ &\qquad\qquad\qquad Q_{u_m}\big(c_m, x_{\alpha_m(1)}, \dots, x_{\alpha_m(u_m)}\big), x_{n+1}\big), \end{aligned}$$

where $(\alpha_1, \dots, \alpha_m)$ is a partition of $\boldsymbol{n}$ of type $(u_1, \dots, u_m)$ with $m \geq 1$ (see Figure 8). Then $\{Q_i\}_{1 \leq i \leq n+1}$ is a C_{n+1}-structure on X. This completes the proof. $\square$

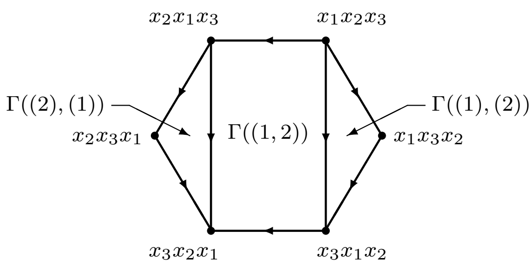

Figure 8. The C_3-structure on X.

Let S^{2t-1} denote the $(2t-1)$-sphere for $t \geq 1$. Then the p-completion $(S^{2t-1})^\wedge_p$ is a topological monoid if and only if $t = 1, 2$ for $p = 2$ and $t \mid (p-1)$ for $p > 2$, where p is a prime (cf. [**13**, pp. 172–173, Section 24–2]).

Proposition 4.6.

(1) *$(S^1)^\wedge_p$ is a $C_\infty(\infty)$-space.*
(2) *$(S^3)^\wedge_2$ is a $C_1(1)$-space, but not a $C_1(2)$-space.*
(3) *Let $p > 2$ and $t > 1$ with $t \mid (p-1)$. Put $n = (p-1)/t$. Then $(S^{2t-1})^\wedge_p$ is a $C_n(n)$-space, but not a $C_1(n+1)$-space.*

Proof. We have (1) and (2) by Remark 4.4.

We consider the case of (3). Put $W = (S^{2t-1})^\wedge_p$. We first construct a $C_n(n)$-structure $\{Q_{r,s}\}_{1\le r+s\le n}$ on W. Assume inductively that $\{Q_{r,s}\}_{1\le r+s<m}$ are constructed for some $m \le n$. Then the obstructions to the existence of $Q_{r,s}$ with $r+s=m$ belong to the cohomology groups:

$$\begin{aligned} H^{j+1}\big(N_{r,s}\times W^m, \partial N_{r,s}\times W^m \cup N_{r,s}\times W^{[m]}; \pi_j(W)\big) \\ \cong \widetilde{H}^{j+2}\big((S^{2tm})^\wedge_p; \pi_j(W)\big) \quad \text{for } j \ge 1 \end{aligned} \tag{4.9}$$

since $N_{r,s}\times W^m/(\partial N_{r,s}\times W^m \cup N_{r,s}\times W^{[m]}) \simeq (S^{2tm-1})^\wedge_p$, where $Y^{[m]}$ denotes the m-fold fat wedge of a space Y given by

$$Y^{[m]} = \big\{(y_1,\dots,y_m)\in Y^m \mid y_i = * \text{ for some } 1\le i\le m\big\} \quad \text{for } m\ge 1.$$

This implies that (4.9) is non-trivial only if j is an even integer with $j < 2p-2$ since $tm \le tn = p-1$. On the other hand, $\pi_j(W) = 0$ for any even integer j with $j < 2p-2$ by Toda [**20**, Theorem 13.4]. Thus (4.9) is trivial for all j, and we have a map $Q_{r,s}$ with $r+s=m$. This completes the induction, and we have a $C_n(n)$-structure $\{Q_{r,s}\}_{1\le r+s\le n}$ on W.

We next show that W is not a $C_1(n+1)$-space. Assume contrarily that W is a $C_1(n+1)$-space. Then by Proposition 4.5, W is a C_{n+1}-space, which is a contradiction by [**11**, Theorems 2.2 and 2.4(4)]. This completes the proof. □

An H-space X is called $\boldsymbol{F}_p$-finite if the cohomology $H^*(X;\boldsymbol{F}_p)$ is finite dimensional, and is called Postnikov if the homotopy groups $\pi_j(X)$ vanish above some dimension. For example, any Lie group is an $\boldsymbol{F}_p$-finite H-space. On the other hand, Eilenberg-Mac Lane spaces $K(\boldsymbol{Z},n)$ are always Postnikov, but not $\boldsymbol{F}_p$-finite for $n>1$.

By Hemmi-Kawamoto [**12**, Corollaries 1.1 and 3.6] and Kawamoto [**15**, Theorem B], Proposition 4.5 implies the following corollary:

Corollary 4.7. *Let X be a connected $C_k(p)$-space, where p is a prime and $1\le k\le p$.*

(1) *If X is $\boldsymbol{F}_p$-finite, then the p-completion $X^\wedge_p$ is a p-completed torus.*
(2) *If the cohomology $H^*(X;\boldsymbol{F}_p)$ of X is finitely generated as an algebra over the Steenrod algebra $\mathscr{A}^*_p$, then the p-completion $X^\wedge_p$ is Postnikov.*

Bousfield [**3**, Theorem 7.2] determined the $K(n)_*$-localizations for Postnikov H-spaces, where $K(n)_*$ denotes the Morava K-homology theory for $n\ge 1$. By his result and Corollary 4.7(2), if X is a connected $C_k(p)$-space with finitely generated cohomology over $\mathscr{A}_p^*$, then the $K(n)_*$-localization $L_{K(n)_*}(X_p^\wedge)$ of $X_p^\wedge$ is the $(n+1)$-st stage for the modified Postnikov system of $X_p^\wedge$ (see [**3**, p. 2408]).

5. Proofs of Theorems A and B.

Consider the loop space ΩZ of a space Z in the sense of Moore (cf. [**13**, p. 45, Section 5–3 (iii)], [**18**, p. 14, Definition 4.1]). Then we may assume that the multiplication of ΩZ is strictly associative. Recall the definition of the projective spaces $\{P_n(\Omega Z)\}_{n\geq 0}$ of ΩZ. Put $P_0(\Omega Z) = \{*\}$, and define $P_n(\Omega Z)$ for $n \geq 1$ by

$$P_n(\Omega Z) = P_{n-1}(\Omega Z) \cup_{\Psi_n} \Delta^n \times (\Omega Z)^n,$$

where $\Psi_n\colon \partial\Delta^n \times (\Omega Z)^n \cup \Delta^n \times (\Omega Z)^{[n]} \to P_{n-1}(\Omega Z)$ is given by the following relations:

$$\Psi_n(\partial_i(a), \omega_1, \dots, \omega_n) = \begin{cases} \Psi_{n-1}(a, \omega_2, \dots, \omega_n) & \text{if } i = 0 \\ \Psi_{n-1}(a, \omega_1, \dots, \omega_i \cdot \omega_{i+1}, \dots, \omega_n) & \text{if } 0 < i < n \\ \Psi_{n-1}(a, \omega_1, \dots, \omega_{n-1}) & \text{if } i = n. \end{cases} \tag{5.1}$$

$$\begin{aligned} &\Psi_n(a, \omega_1, \dots, \omega_{j-1}, *, \omega_{j+1}, \dots, \omega_n) \\ &\quad = \Psi_{n-1}(s_j(a), \omega_1, \dots, \omega_{j-1}, \omega_{j+1}, \dots, \omega_n) \quad \text{for } 1 \leq j \leq n. \end{aligned} \tag{5.2}$$

Then we have the inclusions $P_1(\Omega Z) = \Sigma\Omega Z \subset P_2(\Omega Z) \subset P_3(\Omega Z) \subset \cdots$. Put

$$P_\infty(\Omega Z) = \bigcup_{n\geq 1} P_n(\Omega Z).$$

Let $\eta_n = \widetilde{\varepsilon}_n(\rho_n \times 1_{(\Omega Z)^n})\colon \Delta^n \times (\Omega Z)^n \to Z$, where $\widetilde{\varepsilon}_n\colon [0, n] \times (\Omega Z)^n \to Z$ is defined by $\widetilde{\varepsilon}_n(t, \omega_1, \dots, \omega_n) = \omega_i(t - i + 1)$ if $t \in [i-1, i]$ for $1 \leq i \leq n$. Then $\{\eta_n\}_{n\geq 1}$ induces a family of maps $\{\varepsilon_n\colon P_n(\Omega Z) \to Z\}_{n\geq 1}$ such that $\varepsilon_1\colon \Sigma\Omega Z \to Z$ is the evaluation map and $\varepsilon_n|_{P_{n-1}(\Omega Z)} = \varepsilon_{n-1}\colon P_{n-1}(\Omega Z) \to Z$ for $n > 1$. Moreover, $\varepsilon_\infty\colon P_\infty(\Omega Z) \to Z$ is a homotopy equivalence (cf. [**13**, p. 55, Section 6–5], [**18**, p. 18, Theorem 4.8]).

If Z is an H-space, then identifying Z with $P_\infty(\Omega Z)$, we can restrict the multiplication $Z^2 \to Z$ to an axial map $P_m(\Omega Z) \times P_n(\Omega Z) \to Z$ for any $m, n \geq 1$. From this fact, we introduce the concept of $H_k(n)$-space.

Definition 5.1. Let $n \geq 1$ and $1 \leq k \leq n$. A space Z is called an $H_k(n)$-space if there is a map

$$\psi_k(n)\colon \bigcup_{0\le s\le k} P_{n-s}(\Omega Z)\times P_s(\Omega Z)\to Z$$

with $\psi_k(n)(z,*)=\varepsilon_n(z)$ for $z\in P_n(\Omega Z)$ and $\psi_k(n)(*,w)=\varepsilon_k(w)$ for $w\in P_k(\Omega Z)$.

Let BX denote the classifying space of a topological monoid X with $X\simeq\Omega(BX)$. From the above construction, we have the projective spaces $\{P_n(X)\}_{n\ge 0}$ with the maps $\{\varepsilon_n\colon P_n(X)\to BX\}_{n\ge 1}$ such that $\varepsilon_1\colon \Sigma X\to BX$ is the adjoint of the homotopy equivalence $X\simeq\Omega(BX)$ and $\varepsilon_\infty\colon P_\infty(X)\to BX$ is a homotopy equivalence.

Now we prove Proposition 4.2 as follows:

Proof of Proposition 4.2. If $\mu\colon X^2\to X$ is an A_n-map, then by [**17**, p. 300, Theorem 4.5], we have the induced map $P_n(\mu)\colon P_n(X^2)\to P_n(X)$ (see also [**18**, p. 34, Theorem 8.4]). Put $\psi(n)=\varepsilon_n P_n(\mu)\colon P_n(X^2)\to BX$. Then $\psi(n)$ is an $H(n)$-structure on BX by [**7**, Definition 3].

Conversely, we assume that there is an $H(n)$-structure $\psi(n)\colon P_n(X^2)\to BX$ on BX. Then we can write $\mu=\Omega(\psi(n))\iota_n\colon X^2\to\Omega(BX)\simeq X$, where $\iota_n\colon X^2\to\Omega P_n(X^2)$ denotes the adjoint of the inclusion $\Sigma(X^2)\subset P_n(X^2)$. Since ι_n is an A_n-map by [**18**, p. 34, Theorem 8.6], so is μ. This completes the proof. $\square$

To prove Theorem A, we generalize the definition of the projective spaces, and construct a family of spaces $\{P_{m,n}(X)\}_{m,n\ge 0}$. Put $P_{0,0}(X)=\{*\}$, and define $P_{m,n}(X)$ for $m,n\ge 0$ with $m+n\ge 1$ by

$$P_{m,n}(X)=P_{m-1,n}(X)\cup P_{m,n-1}(X)\cup_{\Psi_{m,n}}\Delta^{m,n}\times X^{m+n},$$

where $\Psi_{m,n}\colon \partial\Delta^{m,n}\times X^{m+n}\cup\Delta^{m,n}\times X^{[m+n]}\to P_{m-1,n}(X)\cup P_{m,n-1}(X)$ is given by the following relations:

$$\begin{aligned}&\Psi_{m,n}(\beta_i(a),x_1,\dots,x_m,y_1,\dots,y_n)\\&\quad=\begin{cases}\Psi_{m-1,n}(a,x_2,\dots,x_m,y_1,\dots,y_n) & \text{if } i=0\\ \Psi_{m-1,n}(a,x_1,\dots,x_i\cdot x_{i+1},\dots,x_m,y_1,\dots,y_n) & \text{if } 0<i<m\\ \Psi_{m-1,n}(a,x_1,\dots,x_{m-1},y_1,\dots,y_n) & \text{if } i=m.\end{cases}\end{aligned}\tag{5.3}$$

$$\begin{aligned}&\Psi_{m,n}\big(\beta'_j(a),x_1,\dots,x_m,y_1,\dots,y_n\big)\\&\quad=\begin{cases}\Psi_{m,n-1}(a,x_1,\dots,x_m,y_2,\dots,y_n) & \text{if } j=0\\ \Psi_{m,n-1}(a,x_1,\dots,x_m,y_1,\dots,y_j\cdot y_{j+1},\dots,y_n) & \text{if } 0<j<n\\ \Psi_{m,n-1}(a,x_1,\dots,x_m,y_1,\dots,y_{n-1}) & \text{if } j=n.\end{cases}\end{aligned}\tag{5.4}$$

$$\begin{aligned}
&\Psi_{m,n}(a, x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_m, y_1, \ldots, y_n) \\
&\quad = \Psi_{m-1,n}(\gamma_i(a), x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m, y_1, \ldots, y_n) \quad \text{for } 1 \le i \le m, \\
&\Psi_{m,n}(a, x_1, \ldots, x_m, y_1, \ldots, y_{j-1}, *, y_{j+1}, \ldots, y_n) \\
&\quad = \Psi_{m,n-1}(\gamma'_j(a), x_1, \ldots, x_m, y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n) \quad \text{for } 1 \le j \le n.
\end{aligned} \tag{5.5}$$

From the definition, we have $P_{1,0}(X) = P_{0,1}(X) = \Sigma X$. Since the projection $\pi_{m,n}\colon \Delta^m \times \Delta^n \to \Delta^{m,n}$ is compatible with the face operators and the degeneracy operators, $\pi_{m,n}$ induces a map $\widetilde{\pi}_{m,n}\colon P_m(X) \times P_n(X) \to P_{m,n}(X)$ for $m, n \ge 0$. In particular, we see that $\widetilde{\pi}_{1,0}\colon \Sigma X \times \{*\} \to \Sigma X$ and $\widetilde{\pi}_{0,1}\colon \{*\} \times \Sigma X \to \Sigma X$ are the projections.

Lemma 5.2. *Let $n \ge 1$ and $1 \le k \le n$. If X is a topological monoid such that BX has an $H_k(n)$-structure $\psi_k(n)$, then there is a map*

$$\widetilde{\psi}_k(n)\colon \bigcup_{0 \le s \le k} P_{n-s,s}(X) \to BX \quad \textit{with} \quad \widetilde{\psi}_k(n)\Bigl(\bigcup_{0 \le s \le k} \widetilde{\pi}_{n-s,s}\Bigr) = \psi_k(n).$$

Proof. Let $\theta_{r,s}\colon \Delta^r \times \Delta^s \times X^{r+s} \to BX$ be the composite of $\psi_k(n)$ with the inclusion

$$\begin{aligned}
&\Delta^r \times \Delta^s \times X^{r+s} \to \Delta^r \times X^r \times \Delta^s \times X^s \\
&\qquad \subset P_r(X) \times P_s(X) \subset \bigcup_{0 \le s \le k} P_{n-s}(X) \times P_s(X) \quad \text{for } (r,s) \in \Lambda_k(n),
\end{aligned}$$

where the first arrow denotes the appropriate switching map. From the definition of $\psi_k(n)$, we have that

$$\theta_{r,s}(a, v_j, x_1, \ldots, x_r, y_1, \ldots, y_s) = \eta_r(a, x_1, \ldots, x_r) = \widetilde{\varepsilon}_r(\rho_r(a), x_1, \ldots, x_r)$$

for $0 \le j \le s$ and

$$\theta_{r,s}(v_i, b, x_1, \ldots, x_r, y_1, \ldots, y_s) = \eta_s(b, y_1, \ldots, y_s) = \widetilde{\varepsilon}_s(\rho_s(b), y_1, \ldots, y_s)$$

for $0 \le i \le r$, which implies that there is a map $\widetilde{\theta}_{r,s}\colon \Delta^{r,s} \times X^{r+s} \to BX$ with $\widetilde{\theta}_{r,s}(\pi_{r,s} \times 1_{X^{r+s}}) = \theta_{r,s}$. Then $\{\widetilde{\theta}_{r,s}\}_{(r,s) \in \Lambda_k(n)}$ induces a map

$$\widetilde{\psi}_k(n)\colon \bigcup_{0 \le s \le k} P_{n-s,s}(X) \to BX$$

with the required conditions. This completes the proof. □

Proof of Theorem A. Assume that X is a $C_k(n)$-space and $\{Q_{r,s}\}_{(r,s)\in\Lambda_k(n)}$ is the $C_k(n)$-structure. From the same reason as in [**23**, p. 250], we may assume without loss of generality that the image of $Q_{r,s}$ lies in the set of loops of length $r+s$ in $X \simeq \Omega(BX)$. Consider the adjoint $\psi_{r,s}\colon [0,r+s]\times N_{r,s} \times X^{r+s} \to BX$ of $Q_{r,s}$.

Let $\Phi_{r,s}\colon [0,r+s]\times N_{r,s} \to \Delta^{r,s}$ be the adjoint of $\kappa_{r,s}\colon N_{r,s} \to \mathscr{K}_{r,s}$ given in (2.5). Put $\widetilde{\Phi}_{r,s} = \Phi_{r,s}|_{\partial([0,r+s]\times N_{r,s})}\colon \partial([0,r+s]\times N_{r,s}) \to \partial\Delta^{r,s}$. From the definition, we have $\partial\Delta^{r,s}\cup_{\widetilde{\Phi}_{r,s}}[0,r+s]\times N_{r,s} = \Delta^{r,s}$, and so $\Phi_{r,s}\colon ([0,r+s] \times N_{r,s}, \partial([0,r+s]\times N_{r,s})) \to (\Delta^{r,s},\partial\Delta^{r,s})$ is a relative homeomorphism. Then we have inductively a family of maps $\{\widetilde{\theta}_{r,s}\colon \Delta^{r,s}\times X^{r+s} \to BX\}_{(r,s)\in\Lambda_k(n)}$ with $\widetilde{\theta}_{r,0} = \widetilde{\varepsilon}_r$ and $\widetilde{\theta}_{0,s} = \widetilde{\varepsilon}_s$, which implies that $\{\widetilde{\theta}_{r,s}\}_{(r,s)\in\Lambda_k(n)}$ induces a map

$$\widetilde{\psi}_k(n)\colon \bigcup_{0\le s\le k} P_{n-s,s}(X) \to BX$$

such that

$$\psi_k(n) = \widetilde{\psi}_k(n)\Big(\bigcup_{0\le s\le k}\widetilde{\pi}_{n-s,s}\Big)\colon \bigcup_{0\le s\le k} P_{n-s}(X)\times P_s(X) \to BX$$

is an $H_k(n)$-structure on BX.

Conversely, we assume that BX is an $H_k(n)$-space. Let $\widetilde{\theta}_{r,s}\colon \Delta^{r,s}\times X^{r+s} \to BX$ denote the composite of $\widetilde{\psi}_k(n)$ with the inclusion

$$\Delta^{r,s}\times X^{r+s} \subset P_{r,s}(X) \subset \bigcup_{0\le s\le k} P_{n-s,s}(X) \quad \text{for } (r,s)\in\Lambda_k(n),$$

where

$$\widetilde{\psi}_k(n)\colon \bigcup_{0\le s\le k} P_{n-s,s}(X) \to BX$$

is given by Lemma 5.2. Consider the adjoint $Q_{r,s}\colon N_{r,s}\times X^{r+s} \to X$ of $\widetilde{\theta}_{r,s}(\Phi_{r,s}\times 1_{X^{r+s}})\colon [0,r+s]\times N_{r,s}\times X^{r+s} \to BX$. Then $\{Q_{r,s}\}_{(r,s)\in\Lambda_k(n)}$ is a $C_k(n)$-structure on X. This completes the proof of Theorem A. □

Let $\boldsymbol{CP}^\infty$ be the infinite dimensional complex projective space. Then the cohomology is given by $H^*(\boldsymbol{CP}^\infty;\boldsymbol{F}_p) \cong \boldsymbol{F}_p[u]$ with $\deg u = 2$, where p is a prime.

Consider the homotopy fiber Z_t of the map $\phi_t\colon \boldsymbol{C}P^\infty \to K(\boldsymbol{Z}/p, 2t)$ corresponding to the class $u^t \in H^{2t}(\boldsymbol{C}P^\infty; \boldsymbol{F}_p)$ for $t \geq 1$. Put $X_t = \Omega Z_t$ for $t \geq 1$.

PROPOSITION 5.3.

(1) *If $t = p^a$ for some $a \geq 0$, then X_t is a $C_\infty(\infty)$-space.*
(2) *Assume $t = p^a b$ for $a \geq 0$ and $b > 1$ with $b \not\equiv 0 \bmod p$. Then X_t is a $C_k(n)$-space if $k < p^a$ or $n < t$, but not a $C_{p^a}(t)$-space.*

We remark that Proposition 5.3 is a generalization of the result by Aguadé [**1**, Proposition 4.2].

To prove Proposition 5.3, we need the following lemma:

LEMMA 5.4. *Consider the homotopy commutative diagram*:

$$\begin{tikzcd} \Omega B \arrow[r] & F \arrow[r, "\iota"] & X \arrow[r] & B \\ & L \arrow[u, "g"] \arrow[r] & K, \arrow[u, "f"'] & \end{tikzcd} \tag{5.6}$$

where the top horizontal arrow is a fibration sequence and (K, L) is a relative CW-complex. Assume that (K, L) has the extension property with respect to ΩB, that is, for any map $d\colon L \to \Omega B$, there is a map $\widetilde{d}\colon K \to \Omega B$ with $\widetilde{d}|_L = d$. If there is a lift $\widetilde{f}\colon K \to F$ with $\iota\widetilde{f} \simeq f$, then we have a map $h\colon K \to F$ with $\iota h \simeq f$ and $h|_L = g$.

PROOF. Let $\nu\colon \Omega B \times F \to F$ be the natural action of the principal fibration (5.6). Since $\iota\widetilde{f}|_L \simeq f|_L \simeq \iota g$, there is a map $d\colon L \to \Omega B$ with $\nu(d \times \widetilde{f}|_L)\Delta_L \simeq g$. From the assumption, we have a map $\widetilde{d}\colon K \to \Omega B$ with $\widetilde{d}|_L = d$. Put $\widetilde{g} = \nu(\widetilde{d} \times \widetilde{f})\Delta_K\colon K \to F$. Then $\iota\widetilde{g} = \iota\nu(\widetilde{d} \times \widetilde{f})\Delta_K \simeq \iota\widetilde{f} \simeq f$ and $\widetilde{g}|_L = \nu(d \times \widetilde{f}|_L)\Delta_L \simeq g$. From the homotopy extension property with respect to (K, L), we have a map $h\colon K \to F$ with $h \simeq \widetilde{g}$ and $h|_L = g$. This completes the proof. $\square$

PROOF OF PROPOSITION 5.3.

(1) If $t = p^a$ for some $a \geq 0$, then Z_t is an H-space, and so the result follows from Corollary 1.1.

(2) We first prove that if $k < p^a$ or $n < t$, then X is a $C_k(n)$-space. Put

$$K = \bigcup_{0 \leq s \leq k} P_{n-s}(X_t) \times P_s(X_t) \quad \text{and} \quad L = P_n(X_t) \vee P_k(X_t).$$

Let $f\colon K \to \boldsymbol{C}P^\infty$ be the composite of $\mu(\iota_t)^2\colon (Z_t)^2 \to \boldsymbol{C}P^\infty$ with the inclusion

$K \subset (Z_t)^2$, where μ is the multiplication of $\boldsymbol{C}P^\infty$ and $\iota_t\colon Z_t \to \boldsymbol{C}P^\infty$ denotes the fiber inclusion. We define $g\colon L \to Z_t$ by $g(z,*) = \varepsilon_n(z)$ for $z \in P_n(X_t)$ and $g(*,w) = \varepsilon_k(w)$ for $w \in P_k(X_t)$. Then $f|_L \simeq \iota_t g$. Put $\xi_i = (\iota_t\varepsilon_i)^\#(u) \in H^2(P_i(X_t);\boldsymbol{F}_p)$ for $i \geq 1$.

If $k < p^a$, then

$$\begin{aligned}(\phi_t f)^\#(\iota_{2t}) = f^\#(u)^t &= (\xi_n \otimes 1 + 1 \otimes \xi_k)^{p^a b} \\ &= \big((\xi_n)^{p^a} \otimes 1 + 1 \otimes (\xi_k)^{p^a}\big)^b \\ &= (\xi_n)^t \otimes 1 = (\varepsilon_n)^\#((\iota_t)^\#(u)^t) \otimes 1 = 0,\end{aligned}$$

and so there is a map $\psi_k(n)\colon K \to Z_t$ with $\psi_k(n)|_L = g$ and $\iota_t\psi_k(n) \simeq f$ by Lemma 5.4. This implies that Z_t is an $H_k(n)$-space, and so X_t is a $C_k(n)$-space by Theorem A.

In the case of $n < t$, $(\phi_t f)^\#(\iota_{2t}) = f^\#(u)^t = 0$ since $\mathrm{cat}(K) \leq n$, and so by the same reason as above, X_t is a $C_k(n)$-space.

We next show that X_t is not a $C_{p^a}(t)$-space. Assume contrarily that X_t is a $C_{p^a}(t)$-space. Then Z_t is an $H_{p^a}(t)$-space by Theorem A. Let $f\colon P_{p^a(b-1)}(X_t) \times P_{p^a}(X_t) \to Z_t$ denote the composite of $\psi_{p^a}(t)$ with the inclusion

$$P_{p^a(b-1)}(X_t) \times P_{p^a}(X_t) \subset \bigcup_{0 \leq s \leq p^a} P_{t-s}(X_t) \times P_s(X_t),$$

where $\psi_{p^a}(t)$ is the $H_{p^a}(t)$-structure on Z_t. Then we have

$$\begin{aligned}(\phi_t\iota_t f)^\#(\iota_{2t}) &= \big(\xi_{p^a(b-1)} \otimes 1 + 1 \otimes \xi_{p^a}\big)^t \\ &= \binom{t}{p^a}(\xi_{p^a(b-1)})^{p^a(b-1)} \otimes (\xi_{p^a})^{p^a} \quad \text{with } \binom{t}{p^a} \equiv b \not\equiv 0 \bmod p.\end{aligned}$$

Since $\phi_t\iota_t f \simeq *$, we have a contradiction, which implies that X_t is not a $C_{p^a}(t)$-space. This completes the proof. □

Proposition 5.5.
(1) *If $1 < t < p$, then X_t is a C_{t-1}-space, but not a C_t-space.*
(2) *If $t = 1$ or $t \geq p$, then X_t is a C_∞-space.*

Recall the following result proved by Williams [**21**]:

Theorem 5.6 ([**21**, Theorem 2]). *Let $n \geq 1$. A topological monoid X is a C_n-space if and only if there is a map $\psi_n\colon J_n(\Sigma X) \to BX$ with $\psi_n|_{\Sigma X} =$*

$\varepsilon_1\colon \Sigma X \to BX$, *where* $J_n(Y)$ *denotes the* n*-th James reduced product space of a space* Y *for* $n \geq 1$.

PROOF OF PROPOSITION 5.5. (1) By Propositions 4.5 and 5.3(2), X_t is a C_{t-1}-space.

If we assume that X_t is a C_t-space, then there is a map $\psi_t\colon J_n(\Sigma X_t) \to Z_t$ with $\psi_t|_{\Sigma X_t} = \varepsilon_1$ by Theorem 5.6. Let $f\colon (\Sigma X_t)^t \to Z_t$ denote the composite of ψ_t with the projection $(\Sigma X_t)^t \to J_t(\Sigma X_t)$. Then we have

$$\begin{aligned}(\phi_t \iota_t f)^{\#}(\iota_{2t}) &= (\xi_1 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \xi_1)^t \\ &= t!\xi_1 \otimes \cdots \otimes \xi_1 \quad \text{with } t! \not\equiv 0 \bmod p.\end{aligned}$$

Since $\phi_t \iota_t f \simeq *$, we have a contradiction, and so X_t is not a C_t-space.

(2) By Propositions 4.5 and 5.3(1), X_1 is a C_∞-space.

Since S^1 is a C_∞-space, there is a map $\psi'_n\colon J_n(S^2) \to \boldsymbol{CP}^\infty$ with $\psi'_n|_{S^2} = \varepsilon'_1\colon S^2 \to \boldsymbol{CP}^\infty$ for any $n \geq 1$ by Theorem 5.6.

Now we prove that there is a family of maps $\{\psi_n\colon J_n(\Sigma X_t) \to Z_t\}_{n\geq 1}$ with the following relations:

$$\begin{aligned}&\psi_1 = \varepsilon_1\colon \Sigma X_t \to Z_t,\\ &\psi_n|_{J_{n-1}(\Sigma X_t)} = \psi_{n-1} \quad \text{for } n > 1, \\ &\iota_t \psi_n \simeq \psi'_n J_n(\Sigma\Omega\iota_t) \quad\ \text{for } n \geq 1.\end{aligned} \tag{5.7}$$

We work by induction on n. The result is clear for $n = 1$. Assume that the result is proved for $n - 1$. Put $K = (\Sigma X_t)^n$ and $L = (\Sigma X_t)^{[n]}$. Let $f\colon K \to \boldsymbol{CP}^\infty$ be the composite of $\psi'_n J_n(\Sigma\Omega\iota_t)$ with the projection $K \to J_n(\Sigma X_t)$. Then by inductive hypothesis, there is a map $\psi_{n-1}\colon J_{n-1}(\Sigma X_t) \to Z_t$ with (5.7).

Consider the composite $g\colon L \to Z_t$ of ψ_{n-1} with the projection $L \to J_{n-1}(\Sigma X_t)$. Then $f|_L \simeq \iota_t g$. If $t \geq p$, then

$$\begin{aligned}(\phi_t f)^{\#}(\iota_{2t}) = f^{\#}(u)^t = (\xi_1 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \xi_1)^{t-p} \\ \cdot ((\xi_1)^p \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes (\xi_1)^p) = 0,\end{aligned}$$

and so there is a map $\widetilde{\psi}_n\colon K \to Z_t$ with $\widetilde{\psi}_n|_L = g$ and $\iota_t\widetilde{\psi}_n \simeq f$ by Lemma 5.4. Since $\widetilde{\psi}_n|_L = g$, we have a map $\psi_n\colon J_n(\Sigma X_t) \to Z_t$ with (5.7), which implies that X_t is a C_∞-space by Theorem 5.6. This completes the proof. $\square$

REMARK 5.7. Let W_t be the homotopy fiber of the map $\phi'_t\colon \boldsymbol{CP}^\infty \to$

$K(\boldsymbol{Q}, 2t)$ corresponding to the class $v^t \in H^{2t}(\boldsymbol{CP}^\infty; \boldsymbol{Q})$ for $t > 1$, where $v \in H^2(\boldsymbol{CP}^\infty; \boldsymbol{Q})$ denotes the generator. Put $Y_t = \Omega W_t$ for $t > 1$. Using the same way as the proofs of Propositions 5.3 and 5.5, we can prove that Y_t is a $C_k(t-1)$-space for any $1 \leq k \leq t-1$, but not a C_t-space.

Now we proceed to the proof of Theorem B.

Proof of Theorem B. If $X_{(0)}$ is a $C_k(n)$-space, then $X_{(0)}$ is a C_n-space by Proposition 4.5.

Now we consider the converse. Let S be the set of all generators for $H^*(BX_{(0)}; \boldsymbol{Q})$ as a $\boldsymbol{Q}$-algebra. Consider the free $\boldsymbol{Q}$-algebra A^* generated by S with the projection $\omega\colon A^* \to H^*(BX_{(0)}; \boldsymbol{Q})$. Since $X_{(0)}$ is a C_n-space, there is a map $\psi_n\colon J_n(\Sigma X_{(0)}) \to BX_{(0)}$ with $\psi_n|_{\Sigma X_{(0)}} = \varepsilon_1$ by Theorem 5.6.

From the same reason as the proof of [**11**, Lemma 4.7], we have

$$\ker \psi_n^{\#} \omega \subset D^{n+1} A^*, \tag{5.8}$$

where $D^{n+1}A^*$ denotes the $(n+1)$-fold decomposable module of A^*. Since $\ker \omega \subset \ker \psi_n^{\#}\omega$, we have $\ker \omega \subset D^{n+1}A^*$ by (5.8). This implies that $BX_{(0)}$ is an $H(n)$-space by [**7**, Proposition 8]. Then by Theorem A, $X_{(0)}$ is a $C_k(n)$-space for any $1 \leq k \leq n$. This completes the proof of Theorem B. $\square$

6. Homotopy localizations.

Let A and B be spaces and $f \in \mathrm{Map}_*(A, B)$. According to Dror Farjoun [**6**, p. 2, A.1], a space Z is called f-local if the induced map $f^{\#}\colon \mathrm{Map}_*(B, Z) \to \mathrm{Map}_*(A, Z)$ is a homotopy equivalence. In particular, when $B = \{*\}$ and $f\colon A \to \{*\}$ is the constant map, Z is called A-local, that is, $\mathrm{Map}_*(A, Z)$ is contractible.

Bousfield [**2**, Section 2] and Dror Farjoun [**6**, Section 1] constructed the A-localization $L_A(X)$ with the universal map $\phi_X\colon X \to L_A(X)$ for a space X. By their results [**6**, p. 4, A.4] and [**2**, Theorem 2.10(ii)], $L_A(X)$ is A-local and ϕ_X induces a homotopy equivalence

$$(\phi_X)^{\#}\colon \mathrm{Map}_*(L_A(X), Z) \longrightarrow \mathrm{Map}_*(X, Z) \tag{6.1}$$

for any A-local space Z (see also [**5**, Theorem 14.1]).

Definition 6.1. Let $n \geq 1$ and $1 \leq k \leq n$. Assume that X and Y are $C_k(n)$-spaces with the $C_k(n)$-structures $\{Q^X_{r,s}\}_{(r,s)\in\Lambda_k(n)}$ and $\{Q^Y_{r,s}\}_{(r,s)\in\Lambda_k(n)}$. A homomorphism $\phi\colon X \to Y$ is called a $C_k(n)$-map if there is a family of maps $\{D_{r,s}\colon I \times N_{r,s} \times X^{r+s} \to Y\}_{(r,s)\in\Lambda_k(n)}$ with the following relations:

$$D_{r,0}(*, x_1, \ldots, x_r) = \phi(x_1 \cdots x_r) \quad \text{and} \quad D_{0,s}(*, y_1, \ldots, y_s) = \phi(y_1 \cdots y_s). \tag{6.2}$$

$$D_{r,s}(t, \varepsilon^{(p_i)}(a), x_1, \ldots, x_r, y_1, \ldots, y_s)$$
$$= \begin{cases} \phi(x_1) \cdot D_{r-1,s}(t, a, x_2, \ldots, x_r, y_1, \ldots, y_s) & \text{if } i = 0 \\ D_{r-1,s}(t, a, x_1, \ldots, x_i \cdot x_{i+1}, \ldots, x_r, y_1, \ldots, y_s) & \text{if } 0 < i < r \\ D_{r-1,s}(t, a, x_1, \ldots, x_{r-1}, y_1, \ldots, y_s) \cdot \phi(x_r) & \text{if } i = r. \end{cases} \tag{6.3}$$

$$D_{r,s}(t, \varepsilon^{(q_j)}(a), x_1, \ldots, x_r, y_1, \ldots, y_s)$$
$$= \begin{cases} \phi(y_1) \cdot D_{r,s-1}(t, a, x_1, \ldots, x_r, y_2, \ldots, y_s) & \text{if } j = 0 \\ D_{r,s-1}(t, a, x_1, \ldots, x_r, y_1, \ldots, y_j \cdot y_{j+1}, \ldots, y_s) & \text{if } 0 < j < s \\ D_{r,s-1}(t, a, x_1, \ldots, x_r, y_1, \ldots, y_{s-1}) \cdot \phi(y_s) & \text{if } j = s. \end{cases} \tag{6.4}$$

$$D_{r,s}(t, \varepsilon^{(h_{i,j})}(a, b), x_1, \ldots, x_r, y_1, \ldots, y_s)$$
$$= D_{i,j}(t, a, x_1, \ldots, x_i, y_1, \ldots, y_j) \cdot D_{r-i,s-j}(t, b, x_{i+1}, \ldots, x_r, y_{j+1}, \ldots, y_s) \tag{6.5}$$

for $0 < i < r$ and $0 < j < s$.

$$\begin{aligned} &D_{r,s}(t, a, x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_r, y_1, \ldots, y_s) \\ &\quad = D_{r-1,s}(t, \delta_i(a), x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r, y_1, \ldots, y_s) \quad \text{for } 1 \leq i \leq r, \\ &D_{r,s}(t, a, x_1, \ldots, x_r, y_1, \ldots, y_{j-1}, *, y_{j+1}, \ldots, y_s) \\ &\quad = D_{r,s-1}(t, \delta'_j(a), x_1, \ldots, x_r, y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_s) \quad \text{for } 1 \leq j \leq s. \end{aligned} \tag{6.6}$$

$$D_{r,s}(0, a, x_1, \ldots, x_r, y_1, \ldots, y_s) = \phi(Q^X_{r,s}(a, x_1, \ldots, x_r, y_1, \ldots, y_s)). \tag{6.7}$$

$$D_{r,s}(1, a, x_1, \ldots, x_r, y_1, \ldots, y_s) = Q^Y_{r,s}(a, \phi(x_1), \ldots, \phi(x_r), \phi(y_1), \ldots, \phi(y_s)). \tag{6.8}$$

THEOREM 6.2. *Let $n \geq 1$ and $1 \leq k \leq n$. If X is a $C_k(n)$-space, then the A-localization $L_A(X)$ is a $C_k(n)$-space such that the universal map $\phi_X \colon X \to L_A(X)$ is a $C_k(n)$-map.*

Using the same way as the proof of [**15**, Proposition 4.1], we have the following proposition:

PROPOSITION 6.3. *Let $n \geq 1$ and $1 \leq k \leq n$. Assume that X and Y are* *topological monoids and $\phi\colon X \to Y$ is a homomorphism. If X is a $C_k(n)$-space and Y is ϕ-local, then Y is a $C_k(n)$-space such that ϕ is a $C_k(n)$-map.*

We give an outline of the proof of Proposition 6.3.

Proof of Proposition 6.3. We work by induction on n. The result is clear for $n = 1$. Assume that the result is proved for $n - 1$.

Let $\{Q^X_{r,s}\}_{(r,s)\in\Lambda_k(n)}$ be a $C_k(n)$-structure on X, and put $k' = \min\{k, n-1\}$. By inductive hypothesis, we have that Y is a $C_{k'}(n-1)$-space and $\phi\colon X \to Y$ is a $C_{k'}(n-1)$-map whose $C_{k'}(n-1)$-structures are given by $\{Q^Y_{r,s}\}_{(r,s)\in\Lambda_{k'}(n-1)}$ and $\{D_{r,s}\}_{(r,s)\in\Lambda_{k'}(n-1)}$, respectively. Put

$$U_{r,s} = (I \times \partial N_{r,s} \cup \{0\} \times N_{r,s}) \times X^n \cup I \times N_{r,s} \times X^{[n]}$$

for $r, s \in \Lambda_k(n)$ with $r + s = n$, and let $E_{r,s}\colon U_{r,s} \to Y$ be defined by (6.2)–(6.7). From the homotopy extension property, there is a map $\widetilde{E}_{r,s}\colon I \times N_{r,s} \times X^n \to Y$ with $\widetilde{E}_{r,s}|_{U_{r,s}} = E_{r,s}$.

Consider the maps $F_{r,s}\colon N_{r,s} \times X^n \to Y$ and $G_{r,s}\colon \partial N_{r,s} \times Y^n \to Y$ given by

$$F_{r,s}(a, x_1, \ldots, x_r, y_1, \ldots, y_s) = \widetilde{E}_{r,s}(1, a, x_1, \ldots, x_r, y_1, \ldots, y_s)$$

and (4.1)–(4.4), respectively. Let $\mu_n\colon Y^n \to Y$ be the n-fold multiplication of Y given by $\mu_n(y_1, \ldots, y_n) = y_1 \cdots y_n$. We denote the adjoint of $F_{r,s}$ and $G_{r,s}$ by $\eta_{r,s}\colon N_{r,s} \to \mathrm{Map}_*(X^n, Y)_{(\phi^n)^\#(\mu_n)}$ and $\lambda_{r,s}\colon \partial N_{r,s} \to \mathrm{Map}_*(Y^n, Y)_{\mu_n}$, respectively. Then $(\phi^n)^\#(\lambda_{r,s}) = \eta_{r,s}|_{\partial N_{r,s}}$, which implies that there is a map $\widetilde{\lambda}_{r,s}\colon N_{r,s} \to \mathrm{Map}_*(Y^n, Y)_{\mu_n}$ with $\widetilde{\lambda}_{r,s}|_{\partial N_{r,s}} = \lambda_{r,s}$ and $(\phi^n)^\#(\widetilde{\lambda}_{r,s}) \simeq \eta_{r,s}$ rel $\partial N_{r,s}$ by [**15**, Lemmas 4.2 and 4.3]. Consider the adjoint $\widetilde{G}_{r,s}\colon N_{r,s} \times Y^n \to Y$ of $\widetilde{\lambda}_{r,s}$. Using the same way as the proof of [**15**, Proposition 4.1], we modify $\widetilde{G}_{r,s}$ and $\widetilde{E}_{r,s}$ to have maps $Q^Y_{r,s}\colon N_{r,s} \times Y^n \to Y$ and $D_{r,s}\colon I \times N_{r,s} \times X^n \to Y$ with (4.1)–(4.5) and (6.2)–(6.8). Then $\{Q^Y_{r,s}\}_{(r,s)\in\Lambda_k(n)}$ and $\{D_{r,s}\}_{(r,s)\in\Lambda_k(n)}$ are $C_k(n)$-structures on Y and ϕ, respectively. This completes the proof. □

Proof of Theorem 6.2. According to Dror Farjoun [**6**, p. 59, A.1], there is a homotopy equivalence $L_A(X) \simeq \Omega L_{\Sigma A}(BX)$ such that the universal map $\phi_X\colon X \to L_A(X)$ is identified with $\Omega(\phi_{BX})\colon X \to \Omega L_{\Sigma A}(BX)$. Then we may assume that $L_A(X)$ is a topological monoid and ϕ_X is a homomorphism. Since $L_A(X)$ is ϕ_X-local by (6.1), we have the required conclusion by Proposition 6.3. This completes the proof of Theorem 6.2. □

Proposition 6.4. *Let $n \geq 1$ and $1 \leq k \leq n$. Assume that X and B are*

$C_k(n)$-spaces and $\phi\colon X \to B$ is a $C_k(n)$-map. Then the homotopy fiber $F(\phi)$ of ϕ is a $C_k(n)$-space such that the fiber inclusion $\iota\colon F(\phi) \to X$ is a $C_k(n)$-map.

PROOF. Recall that

$$F(\phi) = \{(x, \omega) \in X \times \mathrm{Map}(I, B) \mid \omega(0) = \phi(x) \text{ and } \omega(1) = *\}$$

and $\iota\colon F(\phi) \to X$ is given by $\iota(x, \omega) = x$ (cf. [**10**, p. 407]). Let $\mu\colon F^2 \to F$ be the multiplication defined by $\mu((x_1, \omega_1), (x_2, \omega_2)) = (x_1 \cdot x_2, \omega_1 * \omega_2)$, where $\omega_1 * \omega_2 \in \mathrm{Map}(I, B)$ is given by $(\omega_1 * \omega_2)(t) = \omega_1(t) \cdot \omega_2(t)$ for $t \in I$. Then $F(\phi)$ is a topological monoid and $\iota\colon F(\phi) \to X$ is a homomorphism.

Let $\{Q^X_{r,s}\}_{(r,s)\in\Lambda_k(n)}$ and $\{Q^B_{r,s}\}_{(r,s)\in\Lambda_k(n)}$ denote the $C_k(n)$-structures on X and B, respectively. Since $\phi\colon X \to B$ is a $C_k(n)$-map, we have the $C_k(n)$-structure $\{D_{r,s}\}_{(r,s)\in\Lambda_k(n)}$. Define $Q^{F(\phi)}_{r,s}\colon N_{r,s} \times F(\phi)^{r+s} \to F(\phi)$ by

$$\begin{aligned} &Q^{F(\phi)}_{r,s}\big(a, (x_1, \omega_1), \ldots, (x_r, \omega_r), (y_1, \omega'_1), \ldots, (y_s, \omega'_s)\big) \\ &\quad = \big(Q^X_{r,s}(a, x_1, \ldots, x_r, y_1, \ldots, y_s), \zeta_{r,s}(a, \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s)\big), \end{aligned} \tag{6.9}$$

where

$$\begin{aligned} &\zeta_{r,s}(a, \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s)(t) \\ &= \begin{cases} D_{r,s}(2t, a, x_1, \ldots, x_r, y_1, \ldots, y_s) & \text{if } t \in [0, 1/2], \\ Q^B_{r,s}(a, \omega_1(2t-1), \ldots, \omega_r(2t-1), \omega'_1(2t-1), \ldots, \omega'_s(2t-1)) & \text{if } t \in [1/2, 1] \end{cases} \end{aligned}$$

for $(r, s) \in \Lambda_k(n)$. Then $\{Q^{F(\phi)}_{r,s}\}_{(r,s)\in\Lambda_k(n)}$ satisfies (4.1)–(4.5), and so $F(\phi)$ is a $C_k(n)$-space. Moreover, we see that $\iota\colon F(\phi) \to X$ is a $C_k(n)$-map by (6.9). This completes the proof. □

According to Dror Farjoun [**6**, p. 26, E.1], the localization $L_{S^{t+1}}(X)$ with respect to the $(t+1)$-sphere is the t-th stage $X[t]$ for the Postnikov system of X. Then by Theorem 6.2 and Proposition 6.4, we have the following corollary:

COROLLARY 6.5. *Let X be a connected $C_k(n)$-space, where $n \geq 1$ and $1 \leq k \leq n$.*

(1) *The t-th stage $X[t]$ for the Postnikov system of X is a $C_k(n)$-space and the projection $X \to X[t]$ is a $C_k(n)$-map.*
(2) *The t-connected covering $X\langle t\rangle$ of X is a $C_k(n)$-space and the fiber inclusion $X\langle t\rangle \to X$ is a $C_k(n)$-map.*

Castellana-Crespo-Scherer [**4**, Theorem 7.3] proved that if X is a connected H-space whose cohomology $H^*(X; \boldsymbol{F}_p)$ is finitely generated as an algebra over the Steenrod algebra $\mathscr{A}_p^*$, then the $B\boldsymbol{Z}/p$-localization $L_{B\boldsymbol{Z}/p}(X)$ is $\boldsymbol{F}_p$-finite and the homotopy fiber $F(\phi_X)$ of the universal map $\phi_X \colon X \to L_{B\boldsymbol{Z}/p}(X)$ is Postnikov. By their result, Theorem 6.2 and Proposition 6.4, if X is a connected $C_k(n)$-space with finitely generated cohomology over $\mathscr{A}_p^*$, then $L_{B\boldsymbol{Z}/p}(X)$ is an $\boldsymbol{F}_p$-finite $C_k(n)$-space and $F(\phi_X)$ is a Postnikov $C_k(n)$-space.

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