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On removable singularities of an analytic function of several complex variables.

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1. Let $u(P)=u(x_1,\dots\dots\dots, x_n)$ be defined in a domain D in an n-dimensional space and all its partial derivatives of the second order be continuous and satisfy the equation :

$$\Delta u=\frac{\partial^2 u}{\partial x_1^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}=0, \qquad (1)$$

then $u(P)$ is called a harmonic function in D. It is easily seen that $u(P)=\overline{OP}^{n-2}$ ($n\geqq 3$) is a harmonic function, where P is a variable point and O is a fixed point.

Let $\sum$ be a sphere in an n-dimensional space with O as its center and R be its radius and S be its boundary. Let Q be a point of S and $\varphi(Q)$ be an integrable function on S. We define a Poisson integral with $\varphi(Q)$ as its boundary value :[1]

$$u(P)=\frac{1}{RS_n}\int_S \varphi(Q)\,\frac{R^2-\overline{OP}^2}{\overline{PQ}^n}\,d\sigma_Q, \qquad (2)$$

where S_n is the surface area of a unit sphere and $d\sigma_Q$ is the surface element of S at Q. Then $u(P)$ is harmonic in $\sum$.

We can prove that $u(P)$ tends to $\varphi(Q)$ almost everywhere on S, when P tends to Q non-tangentially to S. If $\varphi(Q)$ is continuous at Q_0, then $u(P)$ tends to $\varphi(Q_0)$, when P tends to Q_0 from the inside of $\sum$. Let $u(P)$ be a bounded harmonic function in $\sum$, then $\lim u(P)=\varphi(Q)$ exists almost everywhere on S, when P tends to Q non-tangentially to S and $u(P)$ can be expressed by (2)[2].

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1) For an n-dimensional Poisson integral, c. f. C. Carathéodory: On Dirichlet problem. Amer. Jour. Math. **59**(1937).

2) The case n=2 is the well known theorems of Fatou and Schwarz. For the case n=3 I have proved analogous theorems in a paper: On Fatou's theorem on Poisson integrals. Jap. Jour-Math. **15**(1938). The method can be applied for the general case.

We will prove:

Lemma 1. *Let $\sum$ be a sphere of radius R with O as its center and σ be a concentric sphere of radius $\rho<R$. If $u(P)$ is harmonic in $\sum$ and $u(P)\equiv o$ in σ, then $u(P)\equiv o$ in $\sum$.*

Proof. $u(P)$ is expressed by a Poisson ingearal:

$$u(P)=\frac{1}{R_1S_n}\int_{S_1}u(Q)\,\frac{R_1^2-\overline{OP}^2}{\overline{PQ}^n}\,d\sigma_Q,$$

here S_1 is the boundary of a concentric sphere $\sum_1$ of radius $R_1<R$.

If we put $r=\overline{OP}$ and θ be the angle between OP and OQ, then $\overline{PQ}^2=R_1^2-2\,R_1\,r\cos\,\theta+r^2$, which vanishes for $r=R_1e^{\pm i\theta}$, so that if we consider r a complex variable,

$$\frac{R_1^2-\overline{OP}^2}{\overline{PQ}^n}=\frac{R_1^2-r^2}{(R_1^2-2R_1r\cos\theta+r^2)^{\frac{n}{2}}}$$

is a regular function of r for $|r|<R$. Let Q_0 be a point on S_1 and P vary on the segment OQ_0, then $u(P)$ is a regular function of r for $|r|<R_1$, which, by the hypothesis, vanishes for $o\leqq r\leqq\rho$, hence $u(P)=o$ for $o\leqq r\leqq R_1$, so that $u(P)$ vanishes on the segment OQ_0 and since Q_0 is arbitrary, $u(P)\equiv o$ in $\sum_1$. Hence for $R_1\to R$, we have $u(p)\equiv o$ in $\sum_1$ q. e. d.

From this we can deduce the following

Lemma 2 *Let $u(P)$ be harmonic in a domain D and if $u(P)\equiv o$ in a partial domain $D_1\subset D$, then $u(P)\equiv o$ in D.*

Theorem 1 *Let $\sum$ be a sphere and E be a closed set in $\sum$. Then in general $\sum-E$ consists of at most countable number of components D_i. Suppose that there exists in each componet D_i a positive harmonic function $v_i(P)$, such that $\lim v_i(P)=+\infty$, when P tends to any bouudary point of D_i, which belongs to E, then*

(i) *$\sum-E$ is connected, so that it consists of only one component.*

(ii) *Let S_E be the part of E, which lies on the boundany S of $\sum$, then S_E is of surface measure zero. Hence by Fubini'i theorem, E is of measure zero, so that E has no inner points.*

Proof. (i) Suppose that $\sum-E$ is not connected and consists of more than one component and let D_1, D_2 be any two components. The boundary point of D_1 belongs to S or E. If all its boundary points belong to E, then $v_1(P)\equiv+\infty$ in D_1, which is absurd. Hence D_1 has a boundary point on S, so that if we denote the part of the boundary of D_1, which lies on S by S_1, then S_1 has an inner point on S. Similarly the part S_2

of the boundary of D_2 on S has an inner point on S.
We put

$$u_1(P)=\frac{1}{RS_n}\int_{S_2}\frac{R^2-\overline{OP}^2}{\overline{PQ}^n}\,d\sigma_Q,$$

then $u_1(P)$ is a bounded harmonic function in $\sum$ and $u_1(P)=0$ on S_1. Let for any $\varepsilon>0$,

$$\Phi_\varepsilon(P)=u_1(P)-\varepsilon\, v_1(P),$$

then since $v_1(P)_0\geqq 0$, we have $\Phi_\varepsilon(P)\leqq 0$ on S_1 and since $u_1(P)$ is bounded and $\lim v_1(P)=+\infty$ on E, we have $\Phi_\varepsilon(P)\gtrless 0$ on E, so that by the maximum principle, $\Phi_\varepsilon(\rho)\leqq 0$ in D_1, hence for $\varepsilon\to 0$, $u_1(P)\leqq 0$ in D_1.

Simiarly considering $u_1(P)+\varepsilon v_1(P)$, we have $u_1(P)\geqq 0$ in D_1, so that $u_1(P)\equiv 0$ in D_1, hence by Lemma 2 $u_1(P)\equiv 0$ in $\sum$, which is absurd, since $u_1(P)$ tends to 1, when P tends to an inner point of S_2. Hence $\sum-E$ is connected.

(ii) Suppose that S_E is of positive surface measure and put

$$u(P)=\frac{1}{RS_n}\int_{S_E}\frac{R^2-\overline{OP}^2}{\overline{PQ}^n}\,d\sigma_Q\ ,$$

then $u(P)$ is a bounded harmonic function in $\sum$ and tends to 1 almost everywhere on S_E, when P tends to S_E non-tangentially to S. But from the argument in (i), we see that $u(P)\equiv 0$ in $\sum$, which is a contradiction. Hence S_E is of surface measure zero. Hence by Fubini's theorem, E is of measure zero. q.e.d.

Theorem 2. *Let E be a closed set in an n-dimensional space and D be its neighbourhood. Suppose that there exists a positive harmonic function $v(P)$ in $D-E$, such that $\lim v(P)=+\infty$, when P tendes to any point of E. Let $u(P)$ be a bounded harmonic function in $D-E$, then $u(P)$ is harmonic on E.*

Proof. Let O be a point of E and $\sum$ be a sphere about O of radius R, which is contained in D. Then by Theorem 1, $\sum-E$ is connected and E has no inner points. We construct a Poisson integral with $u(Q)$ as its boundary value:

$$u_1(P)=\frac{1}{RS_n}\int_{S-S_E}u(Q)\,\frac{R^2-\overline{OP}^2}{\overline{PQ}^n}\,d\sigma_Q,$$

where S_E is the part of E, which lies on the boundary S of $\sum$. Then $u_1(P)$ is a bounded harmonic function in $\sum$ and $u_1(P)=u(P)$ on $S-S_E$, so that $U(P)=u(P)-u_1(P)$ is a bounded harmonic function in $\sum-E$, which vanishes on $S-S_E$. Hence by the argument of Thecrem 1 we have $U(P)\equiv o$ in $\sum-E$, or $u(P)=u_1(P)$ in $\sum-E$. Since E has no inner points, we can continue $u(P)$ harmonically in $\sum$ by $u(P)=u_1(P)$, so that $u(P)$ is harmonic on E. q.e.d.

As an application of Theorem 2, we have

Theorem 3. *Let $g(z_1,\dots\dots z_n)$ be a regular function of n complex variables in a $2n$-dimensional domain D and E be the manifold defined by $g(z_1,\dots\dots,z_n)=o$. Let $f(z_1,\dots\dots,z_n)$ be a bounded regular function in $D-E$, then $f(z_1,\dots\dots,z_n)$ is regular on E.*[3]

Proof. It is evident that E has no inner points. If we put $f=u+iv$, then u, v are bounded harmonic functions in $D-E$. Let $|g(z_1,\dots\dots,z_n)|\leqq M$ in D, then

$$V(P)=\log\frac{M}{|g(P)|},\qquad P=(z_1,\dots\dots,z_n)$$

is a positive harmonic function in $D-E$, such that $\lim V(P)=+\infty$, when P tends to any point of E, hence by Theorem 2, u and v are harmonic on E, so that f is regular on E. q.e.d

From Theorem 1, we have

Theorem 4. *Let $g(z_1,\dots\dots,z_n)$ be regular in a domain D and E be the manifold defined by $g(z_1,\dots\dots,z_n)=o$ and $\sum$ be a sphere which is contained in D and contains points of E, then $\sum-E$ is connected*[4] *and S_E is of surface measure zero, where S_E is the part of E, which lies on the boundary S of $\sum$.*

2. Let E be an $(n-2)$-dimensional manifold in an n-dimensional $(x_1,\dots\dots,x_n)$-space, which is defined by

$$x_i=\varphi_i(t_1,\dots\dots,t_{n-2})=\varphi_i(t)\quad(i=1,2,\dots\dots,n),\tag{1}$$

where $\varphi_i(t)$ are defined in an $(n-2)$-dimensional domain Δ in $(t_1,\dots\dots t_{n-2})$-space and satisfy the Lipschitz's condition:

$$|\varphi_i(t)-\varphi_i(t')|=|\varphi(t_1,\dots\dots,t_{n-2})-\varphi_i(t_1',\dots\dots,t'_{n-2})|\leqq K\sum_{\nu=1}|t_\nu-t'_\nu|,\tag{2}$$

3) Osgood: Lehrbuch d. Funktionentheorie II_1. p. 191

4) Bochner: Functions of several complex variables (1936) p. 194. Lemma.

where K is a constant and (t), (t') are any two points of Δ.
Let

$$v(P)=\int_{\Delta}\cdots\int\frac{dt_1\ldots dt_{n-2}}{\overline{PQ}^{n-2}}, \tag{3}$$

where $P=(x_1,\ldots,x_n)$ is a variable point in the space and $Q=(t)=(\varphi_1(t),\ldots$ $\ldots,\varphi_n(t))$ is a point of E, then $v(P)$ is harmonic outside of E. We will prove:

Lemma 3 *$\lim v(P)=+\infty$, when P tends to any point of E.*

Proof. In the proof, we denote constants by K_1, K_2,.......
Let $Q_0=(t^0)$ be any point of E, then by (2),

$$\overline{PQ}^2\geqq(\overline{PQ_0}+\overline{Q_0Q})^2\geqq=K_1(r^2+\sum_{\nu=1}^{n-2}(t_\nu-t_\nu^0)^2)$$

where $r=\overline{PQ_0}$.

By putting $r\tau_\nu=t_\nu-t_\nu^0$, we have

$$v(P)\geqq K_2\int_{\sum_{\nu=1}^{n-2}\tau_\nu^2\leqq\frac{\delta^2}{r^2}}\cdots\int\frac{d\tau_1\ldots\ldots d\tau_{n-2}}{(1+\sum_{\nu=1}^{n-2}\tau_\nu^2)^{\frac{n-2}{2}}},$$

where we take δ so small that $\sum_{\nu=1}^{n-2}(t_\nu-t_\nu^0)^2\leqq\delta^2$ is contained in Δ.
Let $\sum_{\nu=1}^{n-2}\tau_\nu^2=\rho^2$, $\tau_\nu=a_\nu\rho$, then it is easily seen that

$$v(P)\geqq K_3\int_1^{\frac{\delta}{r}}\frac{\rho^{n-3}d\rho}{(1+\rho^2)^{\frac{n-2}{2}}}\geqq K_4\int_1^{\frac{\delta}{r}}\frac{d\rho}{\rho}=K_4\log\frac{\delta}{r},$$

so that $v(P)\to+\infty$, for $P\to Q_0$, q. e. d.
From Theorem 2 we have

Theorem 5. *Let E satisfy the condition of Lemma 3 and $u(P)$ be a bounded harmonic function in a neighbourhood of E, then $u(P)$ is harmonic on E.*

Theorem 6. *Let $f(z_1,\ldots,z_n)$ be a bounded regular function of n complex variables in a neighbourrhood of a $(2n-2)$-dimensional manifold E, which satisfies the condition of Lemma 3 (with $2n$ instead of n), then $f(z_1,\ldots,z_n)$ is regular on E.*

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