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On the multivalency of analytic functions.

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Noshiro's theorems[1] (generalizations of Dieudonné's theorem[2]) concerning the univalency of regular functions were extended to the case of p-valence by E. Sakai[3]. In the present paper we are going to generalize some of them to meromorphic functions which are defined in a multiply-connected domain. By accomplishing this task we shall also be able to extend Z. Nehari's results[4] and to make them more sharp.

LEMMA 1. *Let $\varphi(z)$ be regular in an n-ply connected domain D and let $\varphi(z) \subset T$[5] in D, where T is a given connected domain. Let us denote by $u=g(t)$ an arbitrary branch of a function mapping T conformally on $|u|<1$ and suppose that $g(\varphi(z))$ is single-valued in D, and put*

$$\frac{1-|g(\alpha)|^2}{|g'(\alpha)|} \equiv \Omega(\alpha, T) \qquad (\alpha \in T)^{6)}. \tag{1}$$

Then

$$|\varphi'(z)| \leqq 2\pi k(z, z)\Omega(\varphi(z), T) \qquad (z \in D), \tag{2}$$

where $k(z, \xi)$ denotes the Szegö kernel function[7] *of D.*

PROOF. In order that the integration be permissible we assume that the boundary Γ of D consists of smooth curves and that $\varphi(\zeta)$ is continuous on Γ; but once the result is obtained, both assumptions can easily be disposed of. Indeed, if D is not smoothly bounded, we may approximate D by a sequence of domains D_n which satisfy $D_n \subset D$, $D_n \subset D_{n+1}$, $\lim_{n\to\infty} D_n = D$ and whose boundaries Γ_n are smooth. If we replace D by D_n, the additional assumption under which we prove Lemma 1 are satisfied. The general result then follows by letting $n \to \infty$ and observing that the Szegö kernel function $k(z, z)$ is a continuous domain function.

Now by hypothesis,

$$\frac{g(\varphi(\zeta))-g(\varphi(z))}{1-g(\varphi(z))g(\varphi(\zeta))} \qquad (z, \zeta \in D)$$

is regular and single-valued in D, and by using the residue theorem, we obtain

$$\frac{g'(\varphi(z))\varphi'(z)}{1-|g(\varphi(z))|^2}=\frac{1}{2\pi i}\int_\Gamma \frac{g(\varphi(\zeta))-g(\varphi(z))}{1-g(\varphi(z))g(\varphi(\zeta))}Q(\zeta, z)\,d\zeta, \tag{3}$$

where $Q(\zeta, z)$ is an arbitrary single-valued function of $\zeta \in D$ which is regular in $D+\Gamma$ except at the point $\zeta=z$ where it has the principal part $1/(\zeta-z)^2$.

According to Garabedian[8] the particular function $Q(\zeta, z)$ can be chosen so that

$$\frac{1}{i}F(\zeta, z)Q(\zeta, z)d\zeta>0, \quad \zeta \in \Gamma, \tag{4}$$

where $F(\zeta, z)$ is the function introduced by Ahlfors[9], which maps D onto the n-times covered unit circle, and $|F(\zeta, z)|=1$ if $\zeta \in \Gamma$. By recalling that $|g(\varphi(z))|<1$ and remarking the above fact, we obtain from (3),

$$\begin{aligned}\frac{|g'(\varphi(z))||\varphi'(z)|}{1-|g(\varphi(z))|^2} &\leqq \frac{1}{2\pi}\int_\Gamma |Q(\zeta, z)d\zeta| \\ &= \frac{1}{2\pi}\int_\Gamma \left|\frac{1}{i}F(\zeta, z)Q(\zeta, z)d\zeta\right| \qquad \text{(by (4))} \\ &= \frac{1}{2\pi i}\int_\Gamma F(\zeta z)Q(\zeta, z)d\zeta \\ &= F'(z, z)\end{aligned}$$

On the other hand it was shown by Garabedian[10] that $F'(z, z)=2\pi k(z, z)$.

Therefore we obtain (2). Q. E. D.

Remark. Hereafter, for the sake of convenience we assume that the domain D in which families of functions are defined contains the origin.

THEOREM 1. *Under the assumptions of Lemma* 1, $f(z)=z^p\varphi(z)$, *where p is a positive or negative integer and $\varphi(0)\neq 0$, is $|p|$-valent and starshaped in the largest circle C about the origin all of whose points satisfy*

$$|z|\,k(z,z)\frac{\Omega(\varphi(z),T)}{|\varphi(z)|}<(2\pi)^{-1}|p|. \tag{5}$$

PROOF. By (2) and (5) we have

$$\left|z\frac{\varphi'(z)}{\varphi(z)}\right|<|p|. \tag{6}$$

Since $\Re\dfrac{zf'(z)}{f(z)}=p+\Re\dfrac{z\varphi'(z)}{\varphi(z)}$, by using (6) we obtain

$$\Re\frac{zf'(z)}{f(z)}>p-\left|z\frac{\varphi'(z)}{\varphi(z)}\right|>0 \text{ if } p>0,$$

$$\Re\frac{zf'(z)}{f(z)}<p+\left|z\frac{\varphi'(z)}{\varphi(z)}\right|<0 \text{ if } p<0.$$

Hence by Ozaki's theorem[11] $f(z)$ is $|p|$-valent and starshaped in C.

THEOREM 2. *Let $\varphi(z)$ be regular and single-valued in D. Further let $\Re\varphi(z)>0$ in D. Then $f(z)=z^p\varphi(z)$ is $|p|$-valent and starshaped in the largest circle about the origin all of whose points satisfy*

$$|z|\,k(z,z)<(4\pi)^{-1}|p|. \tag{7}$$

PROOF. Considering a half-plane T: $\Re t>0$ and taking a mapping function $g(t)=(1-t)/(1+t)$ in Theorem 1, we can say that $f(z)$ is $|p|$-valent and starshaped in the largest circle about the origin all of whose points satisfy

$$|z|\,k(z,z)\frac{\Re\varphi(z)}{|\varphi(z)|}<(4\pi)^{-1}|p| \tag{8}$$

or

$$|z|\,k(z,z)<(4\pi)^{-1}|p|, \tag{9}$$

since

$$1>\frac{\Re\varphi(z)}{|\varphi(z)|}>0.$$

Remark. In the case where D is the unit circle, $k(z, z)=\frac{(2\pi)^{-1}}{1-|z|^2}$, whence Theorems 1 and 2 reduce to Noshiro's and Sakai's theorems.

THEOREM 3. *Let log* $\varphi(z)$ *be regular and single-valued in* D. *Further let* $m<|\varphi(z)|<M$ *in* D. *Then* $f(z)=z^p\varphi(z)$ *is* $|p|$*-valent and starshaped in the largest circle about the origin all of whose points satisfy*

$$(10)\qquad |z|\,k(z, z)\cdot\cos\left[\frac{\pi}{2}\;\frac{\log|\varphi(z)|^2-\log m-\log M}{\log M-\log m}\right]\cdot\log\frac{M}{m}<\frac{|p|}{4}$$

or

$$(11)\qquad |z|\,k(z, z)\log\frac{M}{m}<\frac{|p|}{4}$$

PROOF. We may consider a ring-domain T: $m<|t|<M$, which can be mapped on $|u|<1$ by the function

$$(12)\qquad g(t)=\left[\exp\left(i\frac{\pi}{2}\cdot\frac{\log\frac{t}{\sqrt{mM}}}{\log\sqrt{\frac{M}{m}}}\right)-1\right]:\left[\exp\left(i\frac{\pi}{2}\cdot\frac{\log\frac{t}{\sqrt{mM}}}{\log\sqrt{\frac{M}{m}}}\right)+1\right].$$

Remark. If we put $p=1$, $m=e^{-N}$ and $M=e^{N}$, we have Z. Nehari's theorem.[4] Again in the case where D is the unit circle, we have Noshiro's or Sakai's theorem.[3]

THEOREM 4. *Let* $g(\log\psi(z))$ *be regular and single-valued in* D. *Then* $f(z)=z^p\psi(z)$ *is* $|p|$*-valent and starshaped in the largest circle* C *about the origin all of whose points satisfy*

$$(13)\qquad |z|\,k(z, z)\Omega(\log\psi(z), T)<(2\pi)^{-1}|p|,$$

where $g(\log\psi(z))$ *and* $\Omega(\log\psi(z), T)$ *are the functions defined in Lemma* 1.

PROOF. If we put in Lemma 1 $\varphi(z)=\log\psi(z)$, we have

$$\left|\frac{\psi'(z)}{\psi(z)}\right|\leqq 2\pi k(z, z)\Omega(\log\psi(z), T).$$

Hence from (13) and the above inequality we obtain

$$\left|z\frac{\psi'(z)}{\psi(z)}\right|<|p|.$$

Consequently again by Ozaki's theorem[11] $f(z)$ is $|p|$-valent and starshaped in C, if we have (13).

THEOREM 5. *Let* $\log \varphi(z)$ *be regular and single-valued in* D. *Further let* $\text{l.u.b.}\ |\log \varphi(z)|=M$. *Then* $f(z)=z^p\varphi(z)$ *is* $|p|$*-valent and starshaped in the largest circle about the origin all of whose points satisfy*

$$|z|\,k(z,z)\left(1-\frac{|\log \varphi(z)|^2}{M^2}\right)<\frac{(2\pi)^{-1}|p|}{M} \tag{14}$$

or

$$|z|\,k(z,z)<\frac{(2\pi)^{-1}|p|}{M}. \tag{15}$$

PROOF. We may take, as $g(t)$ in Theorem 4,

$$g(t)=t/M, \qquad T:\ |t|<M.$$

Remark. If we put $p=1$ and adopt (15), we have Z. Nehari's theorem[4].

LEMMA 2. *A necessary and sufficient condition for a function* $f(z)=z^p\varphi(z)$, $\varphi(0)\neq 0$, *regular for* $|z|<r$, *to be* p*-valently convex*[12] *in* $|z|<\rho$ *for every* $\rho<r$ *is that*

$$1+\Re\frac{zf''(z)}{f'(z)}>0 \qquad \textit{for } |z|<r. \tag{16}$$

PROOF. Evidently (16) is necessary.

If (16) is satisfied, then $f(z)$ maps $|z|<\rho$ onto a locally convex[12] region and by Ozaki's theorem[13] $f(z)$ is p-valent in $|z|<\rho$ for every $\rho<r$. Hence (16) is sufficient. Q. E. D.

Using the above lemma and noticing the relation

$$1+\Re\frac{zf''(z)}{f'(z)}=\Re\frac{z\,[zf'(z)]'}{zf'(z)},$$

we obtain the following theorems by a slight modification of the methods of proof used above.

THEOREM 6. *Let* $g(z^{1-p}f'(z))$ *be regular and single-valued in* D *and let* $z^{1-p}f'(z)\neq 0$ *at the origin* (p *: positive integer*). *Let further* r *denote the radius of the largest circle about the origin all of whose points satisfy*

(17) $$|z|^p\, k(z,z)\frac{\Omega(z^{1-p}f'(z),T)}{|f'(z)|}<(2\pi)^{-1}p,$$

where $g(z^{1-p}f'(z))$ *and* $\Omega(z^{1-p}f'(z),T)$ *are the functions defined in Lemma 1. Then the circle* $|z|<\rho$ *is mapped by* $f(z)$ *onto a p-valently convex region for every* $\rho<r$.

THEOREM 7. *Let* $g(\log[z^{1-p}f'(z)])$ *be regular and single-valued in* D (p : *positive integer*). *Let further r denote the radius of the largest circle about the origin all of whose points satisfy*

(18) $$|z|\, k(z,z)\Omega(\log[z^{1-p}f'(z)],T)<(2\pi)^{-1}p,$$

where $g(\log[z^{1-p}f'(z)])$ *and* $\Omega(\log[z^{1-p}f'(z)],T)$ *are the functions defined in Lemma* 1. *Then the circle* $|z|<\rho$ *is mapped by* $f(z)$ *onto a p-valently convex region for every* $\rho<r$.

Remark. If we add, to the assumptions of the above Theorems 6 and 7, $m<|z^{1-p}f'(z)|<M$ and $|\log[z^{1-p}f'(z)]|<M$ respectively, and take the mapping function (12) and $g(t)=t/M$ respectively, then we have a generalization of Z. Nehari's theorem[4] concerning the radius of convexity to the case of p-valence.

Analogously we have the following

THEOREM 8. *Let* $f'(z)$ *be regular and single-valued in* D. *Further let* $\Re f'(z)>0$. *Then* $f(z)$ *is univalent and convex in the largest circle about the origin all of whose points satisfy*

(19) $$|z|\, k(z,z)<(4\pi)^{-1}.$$

COROLLARY. *Let* $f'(z)$ *be regular and* $\Re f'(z)>0$ *for* $|z|<1$. *Then* $f(z)$ *is univalent in* $|z|<1$ *and convex for* $|z|<\sqrt{2}-1$.

PROOF. As for the univalency in $|z|<1$, Noshiro's theorem[14] can be used. For the convexity we may use Theorem 8.

Remark. Recently the present author has obtained many sufficient conditions for $f(z)$ to be convex in one direction in generalized forms,[15] which are also sufficient for $f(z)$ to be p-valent in $|z|<r$. By using those conditions we can give many theorems analogous to these in the present paper. But we refrain from describing those results.

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Notes.

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[3] E. Sakai, On the multivalency of analytic functions, J. Math. Soc. Japan. vol. **2**, Nos. 1-2 (1950), pp 105-113.

[4] Z. Nehari, The radius of univalence of analytic functions, Amer. J. Math. vol. **LXXI**, No. 4 (1949).

[5] We mean by $\varphi(z)\subset T$ that the set of values taken by $\varphi(z)$ in D belongs to the domain T.

[6] The positive quantity $\Omega(\alpha, T)$ depends only on α and T, and neither on the selection of the mapping function nor on that of the branch $g(t)$ of the mapping function. See [1], foot-notes at p. 90.

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