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On the change of variables in the multiple integrals.

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The well-known formula on the change of variables in the multiple integrals

$$(*) \quad \int_{f(D)} g(y)dy=\int_{D} g(f(x))\mathrm{abs}\left|\frac{\partial f}{\partial x}\right|dx$$

has been proved by H. Rademacher and M. Tsuji under very general assumptions. They have shown that the functions f satisfying certain conditions are totally differentiable almost everywhere and consequently the Jacobian $\left|\frac{\partial f}{\partial x}\right|$ can be defined almost everywhere. They have proved further that the above formula (*) holds for integrable functions g, and f satisfying these conditions. We shall give in the following lines another proof of the last fact. Namely we suppose f as a.e. totally differentiable, g as integrable and show the validity of (*). (For the exact formulation see below.) We treat further the case where f is not necessarily univalent.

Throughout this paper, we shall concern ourselves with subsets and mappings of the euclidean n-space E^n. f represents always a mapping defined on a certain subset of E^n. Letters like x, y, a, b represent points of E^n. $\|x-y\|$ denotes the distance between x and y.

§ 1. Preliminaries.

DEFINITION 1. *A mapping $f(x)$ defined on a bounded domain $D(\subset E^n)$ is called an $\mathfrak{A}$-function on D, if it satisfies the following three conditions.*

($\mathfrak{A}_1$) *f maps D homeomorphically onto $f(D)$.*

($\mathfrak{A}_2$) *If $\mu(E)=0$ $(E\subset D)$, then $\mu(f(E))=0$.*

($\mathfrak{A}_3$) *$f(x)$ is totally differentiable almost everywhere.*

We can easily see that the set of all $\mathfrak{A}$-functions on D —we shall write this set by $\mathfrak{A}[D]$ — has the following properties.

(1) If $D \supset D'$, then $\mathfrak{A}[D] \supset \mathfrak{A}[D']$.

(2) $\mathfrak{A}[D]$ contains all non-singular linear transformations.

(3) If $\mathfrak{A}[D] \ni f(x)$, $\mathfrak{A}[f(D)] \ni g(x)$ and $\mathfrak{A}[f(D)] \ni f^{-1}(x)$, then we have $gf(x) \in \mathfrak{A}[D]$.

DEFINITION 2. *For* $f(x) \in \mathfrak{A}[D]$, *we define a measure* $\mu[f]$ *by the formula:*

$$\mu[f](E) = \mu(f(E)).$$

When E is measurable, then $f(E)$ is also measurable by ($\mathfrak{A}_1$) and ($\mathfrak{A}_2$). So $\mu(f(E))$ has a sense for every $\mathfrak{A}$-functions $f(x)$ and every measurable sets E. Furthermore, we can easily see that

(i) $\mu = \mu[x]$ (x is the identity mapping),

(ii) $\mu[f]$ is absolutely continuous with respect to $\mu = \mu[x]$.

By (ii) $\mu[f]$ has the density with respect to $\mu = \mu[x]$. Let us denote it by

$$D(f/x) = \frac{d\mu[f]}{d\mu[x]}.$$

Then we have clearly the following

THEOREM 1. *If* $\mathfrak{A}[D] \ni f(x)$, $\mathfrak{A}[f(D)] \ni g(x)$ *and the one of the next two conditions is satisfied:*

(*i*) $g(x)$ *is totally differentiable,*

(*ii*) $\mathfrak{A}[f(D)] \ni f^{-1}(x)$,

then we have

$$D(gf/x) = D(gf/f)D(f/x).$$

§ 2. Linear functions.

As we have remarked above, every non-singular linear transformation is an $\mathfrak{A}$-function for every domain D. We shall prove

THEOREM 2. *For a non-singular linear* $\mathfrak{A}$-*function:*

$$f(x) = Ax + b \qquad (|A| \neq 0),$$

$D(f/x)$ *is equal to* abs $|A|$. (abs $|A|$ *means the absolute value of the determinant* $|A|$ *of the* $n \times n$ *matrix* A).

Let us first prove the

LEMMA. *An absolutely continuous measure $\nu(E)$ on Borel family $\mathfrak{B}$ of E^n is invariant under any translation in E^n, if and only if $\nu(E)=c\mu(E)$ for some constant c.*

PROOF OF LEMMA. "If"-part is clear. We have only to show the "only if"-part. The unit cube E_0 is measurable by ν, as $E_0 \in \mathfrak{B}$. Put $\nu(E_0)=c$. Any rational interval, for example,

$$I=\left\{(x_1, x_2, \ldots\ldots, x_n);\ \frac{n_i}{m}<x_i<\frac{n'_i}{m},\ i=1, 2, \ldots\ldots, n\right\},$$

where m, n_i, n'_i are integers, is built up by cutting E_0 in m^n equal parts, and then arranging $\prod_{i=1}^n(n'_i-n_i)$ small pieces together in a good form by translations. Thus we see easily $\nu(I)=c\mu(I)$ and so $\nu(E)=c\mu(E)$ by the absolute continuity of ν.

PROOF OF THEOREM 2. For $f(x)=Ax+b$, we shall take $f_1=x+b$ and $f_2=Ax$, so that

$$f=f_1f_2$$

$$D(f/x)=D(f_1f_2/f_2)D(f_2/x)=D(f_2/x)$$

So we can assume $b=0$ without loss of generality. Let T_a denote the translation: $x \rightarrow x+a$. Then we have

$$\mu(f(T_a(E)))=\mu(T_{Aa}(f(E)))=\mu(f(E)).$$

So $\mu[f]$ is invariant under any translation in E^n. From our lemma follows then that there exists a constant $c(A)$ depending solely on the matrix A, such that

$$\mu[f]=c(A)\mu.$$

Obviously we have

$$c(AB)=c(A)\cdot c(B),$$

and we have $c(U)=1$ for any orthogonal matrix U, since any sphere is invariant under rotation around the centre. (The invariance of Lebesgue measure under motions of rigid bodies!)

Now any matrix A can be brought into the form: $UA_0A_1\cdots A_n$, where U is an orthogonal matrix and A_i ($i=0, 1, 2, \cdots\cdots, n$) have the following forms:

$$A_0=\begin{pmatrix}1 & & 0 \\ & \ddots \; 1 & \\ 0 & & a_{nn}\end{pmatrix},$$

$$A_1=\begin{pmatrix}1 & 0\cdots\cdots 0 & a_{1n} \\ & \ddots & 0 \\ & 0 & \vdots \\ 0 & & 0 \\ & & 1\end{pmatrix}, \cdots\cdots, \quad A_{n-1}=\begin{pmatrix}1 & 0\cdots\cdots 0 & 0 \\ & \ddots \quad 0 & \vdots \\ & & 0 \\ 0 & & a_{n-1\,n} \\ & & 1\end{pmatrix},$$

$$A_n=\begin{pmatrix}a_{11} & \cdots\cdots a_{1\,n-1} & 0 \\ \cdots & \cdots\quad\cdots & \vdots \\ a_{n-1\,1} & \cdots\cdots a_{n-1\,n-1} & 0 \\ 0 & \cdots\cdots\quad 0 & 1\end{pmatrix}$$

In fact, we can find a U such that

$$U^{-1}A=\begin{pmatrix}a_{11} & \cdots\cdots a_{1\,n-1} & a_{1n} \\ \cdots & \cdots\quad\cdots & \\ a_{n-1\,1} & \cdots\cdots a_{n-1\;n-1} & a_{n-1\,n} \\ 0 & \cdots\quad 0 & a_{n\,n}\end{pmatrix} \quad (a_{nn}\neq 0).$$

A_i are then defined by these a_{jk}. Let us denote the rotation with the matrix U by ρ, and the linear transformation with the matrix A_i by f_i. Then we have $f=\rho f_0 f_1\cdots f_n$ and

$$D(f/x)=D(\rho/x)D(f_0/x)\cdots D(f_n/x).$$

Now we have $D(\rho/x)=1$, $D(f_0/x)=|a_{nn}|$ and $D(f_i/x)=1$ for $i=1, 2, \cdots, n-1$, as these f_i are essentially 1 or 2 dimensional transformations and in these cases the theorem is almost evident. We may proceed by induction with respect to n and assume

$$\mathrm{D}(f_n/x)=\text{asb}\;|A_n|.$$

Then we have

$$D(f/x)=|a_{nn}|\;\text{abs}\;|\;A_n\;| \\ =\text{abs}\;|\;A\;|.$$

This completes the proof.

§ 3. $\mathfrak{A}$-functions.

In this section we shall prove that for any $\mathfrak{A}$-function f

$$D(f/x)=\text{abs}\left|\frac{\partial f}{\partial x}\right|$$

holds. First we shall prove some lemmas.

LEMMA 1. *The functions*

$$\varepsilon_n(x)=\sup_{\|x'-x\|<1/n} \|f(x')-f(x)-(\operatorname{grad} f(x),\ x'-x)\|/\|x'-x\|$$

$$(n=1,\ 2,\ 3,\ \cdots)$$

are all measurable, and have finite values almost everywhere.

PROOF. Since the function under sup-symbol is continuous with respect to x', we obtain the same sup-value, when we make x' vary only the points such that $x'-x$ have rational coordinates. So we have

$$\varepsilon_n(x)=\sup_a f_a^{(n)}(x),$$

where

$$f_a^{(n)}(x)=\|f(x+a)-f(x)-(\operatorname{grad} f(x), a)\|/\|a\|$$

and a is a rational point whose norm is less than $1/n$. But $f_a^{(n)}(x)$ is a measurable function and $\{f_a^{(n)}(x)\}$ is a countable set for any fixed n. So $\varepsilon_n(x)$ is a measurable function for any n. The latter assertion on $\varepsilon_n(x)$ follows from the total-differentiability of $f(x)$.

LEMMA 2. *If $f(x)$ is an $\mathfrak{A}$-function on D, i.e. $\mathfrak{A}[D]\ni f(x)$, and there exist two numbers K, k such that*

$$K\geqq \operatorname{abs}\left|\frac{\partial f(x)}{\partial x}\right|\geqq k$$

for $x\in E$, where E is a measurable set $\subset D$, then $K\geqq D(f/x)\geqq k$ almost everywhere on E.

PROOF. If $\mu(E)=0$, the lemma is trivial. So we can assume that $\mu(E)>0$. We shall consider separately the cases: (i) $k>0$ and (ii) $k>0$.

(i) $k>0$. By the absolute continuity of $\mu[f]$, there exists for any given positive number ε a positive number δ such that,

$$\mu[f](E)<\varepsilon$$

for any measurable set E whose Lebesgue measure is less than δ. As $\{\varepsilon_n(x)\}$ is a sequence of measurable functions converging to zero almost everywhere, we can find by Egoroff's theorem an open set A whose measure is less than δ such that $\{\varepsilon_n(x)\}$ converges uniformly to zero on $D-A$. On the other hand, we can also find, since the partial derivatives of f are all measurable on D, an open set B whose measure is less than δ such that these partial derivatives are all continuous on $D-B$. Furthermore, we can find a closed set F such that

(i) $E-A\cup B\supset F$,

(ii) $\mu((E-A\cup B)-F)<\varepsilon$.

For a sufficiently large natural number n_0, we obtain

$$\varepsilon_n(x)<\varepsilon \qquad (n>n_0)$$

on $D-A$. Now we cover F by a countable number of closed cubic intervals I_i with the side length s_i such that $\mu(I_i\cap I_j)=0$ $(i\neq j)$ and

$$\mu(\cup_{i=1}^{\infty} I_i-F)<\delta.$$

In each I_i we select a point x_0 of F and define

$$\bar{f}(x)=f(x_0)+(\operatorname{grad} f(x_0),\ x-x_0).$$

Construct now two intervals K_i^1 and K_i^2 for each I_i, such that

$$K_i^1\supset I_i\supset K_i^2.$$

Since $\bar{f}(x)$ is a linear function, the image of these intervals are all parallelograms with parallel faces, and we have

$$\bar{f}(K_i^1)\supset \bar{f}(I_i)\supset \bar{f}(K_i^2).$$

We adjust the size of the intervals so that the distances between the corresponding faces of $\bar{f}(K_i^1)$ and $\bar{f}(I_i)$, resp. of $\bar{f}(I_i)$ and $\bar{f}(K_i^2)$ are all equal to $2\varepsilon s_i$ (supposing ε not too large).

Then there exists a constant M such that

$$\mu(\bar{f}(K_i^1)-\bar{f}(K_i^1))\leqq \varepsilon s_i^n M.$$

Honceforth we obtain

$$\begin{aligned}\mu(f(I_i))&\geqq \mu(\bar{f}(K_i^1))=\mu(\bar{f}(I_i-(I_i-K_i^2)))\\ &=\mu(\bar{f}(I_i))-\mu(\bar{f}(I_i-K_i^1))\\ &\geqq \mu(\bar{f}(I_i))-\varepsilon s^n M\\ &\geqq k\mu(I_i)-\varepsilon s^n M,\\ \mu(f(I_i))&\leqq K\mu(I_i)+\varepsilon s^n M.\end{aligned}$$

So we obtain the inequalities

$$k\mu(\bigcup_{i=1}^{\infty}I_i)-\varepsilon M\mu(\bigcup_{i=1}^{\infty}I_i)\leqq\mu(f(\bigcup_{i=1}^{\infty}I_i))$$
$$\leqq K\mu(\bigcup_{i=1}^{\infty}I_i)+\varepsilon M\mu(\bigcup_{i=1}^{\infty}I_i).$$

On the other hand, from our assumption follows

$$|\mu(f(\bigcup_{i=1}^{\infty}I_i))-\mu(f(E))|\leqq|\mu(f(E))-\mu(f(E-A\cup B))|$$
$$+|\mu(f(E-A\cup B))-\mu(f(F))|+|\mu(f(F))-\mu(f(\bigcup_{i=1}^{\infty}I_i))|<4\varepsilon$$

and

$$|\mu(\bigcup_{i=1}^{\infty}I_i)-\mu(E)|\leqq|\mu(E)-\mu(E-A\cup B)|$$
$$+|\mu(E-A\cup B)-\mu(F)|+|\mu(F)-\mu(\bigcup_{i=1}^{\infty}I_i)|<4\delta.$$

Therefore we have

$$k\mu(E)-4\cdot(\varepsilon M+\varepsilon+\delta)\leqq\mu(f(E))$$
$$\leqq k\mu(E)+4\cdot(\varepsilon M+\varepsilon+\delta).$$

So we obtain

$$k\mu(E)\leqq\mu(f(E))\leqq K\mu(E),$$

since ε and δ are arbitrary. Similarly we have

$$k\mu(F)\leqq\mu(f(F))\leqq K\mu(F)$$

for any measurable subset F in E. This is the required result.

(ii) $k=0$. First we shall assume that $K=0$. We form I_i by the same construction as above. Then $\bar{f}(I_i)$ is mapped into a hyperplane in this case, and there exists a constant M such that

$$\mu(\bar{f}(K_i^1))<\varepsilon s_i^n M.$$

So we obtain

$$\mu(f(I_i))\leqq\mu(f(K_i^1))<\varepsilon s_i^n M,$$

whence follows

$$\mu(f(\bigcup_{i=1}^{\infty}I_i))<\varepsilon\mu(\bigcup_{i=1}^{\infty}I_i)M,$$

and finally

$$\mu[f](E)=0$$

by the same argument as above. Similarly we have $\mu[f](F)=0$ for any subset F in E, so $D(f/x)=0$ $(x\in E)$ almost everywhere. When

$K > 0$, we subdivide the interval $[K, 0]$ into a countable number of intervals as follows:

$$[K, 0] = \cup_{i=1}^{\infty}[K/2^{i-1}, K/2^i] \cup \{0\}.$$

As we have already proved the lemma for any of the subintervals $[K/2^{i-1}, K/2^i]$ or $\{0\}$, we see that the lemma is true also in this case.

THEOREM 3. *If $f(x)$ is an $\mathfrak{A}$-function on D, then we have*

$$D(f/x) = \text{abs}\left|\frac{\partial f}{\partial x}\right|$$

almost everywhere.

PROOF. If the proposition is false, then either

$$\mu\left\{x;\ D(f/x) > \text{abs}\left|\frac{\partial f}{\partial x}\right|\right\} > 0$$

or

$$\mu\left\{x;\ D(f/x) < \text{abs}\left|\frac{\partial f}{\partial x}\right|\right\} > 0$$

should hold. Since we may proceed in a similar way in either case, we shall assume that

$$\mu\left\{x;\ D(f/x) > \text{abs}\left|\frac{\partial f}{\partial x}\right|\right\} > 0.$$

We can further assume that

$$\mu\left\{x;\ \text{abs}\left|\frac{\partial f}{\partial x}\right| > 0\right\} > 0,$$

as the theorem is trivial when $\left|\frac{\partial f}{\partial x}\right| = 0$. In fact, we have then $D(f/x) = 0$ almost everywhere from the above lemma. Under these assumptions, there exists some natural number m such that the measure of the set

$$E_1 = \left\{x;\ D(f/x) - \text{abs}\left|\frac{\partial f}{\partial x}\right| > \frac{1}{m}\right\}$$

is positive. Then we can take an interval $[a_1, a_2]$ such that

(1) a_1 and a_2 are rational numbers,

(2) $a_2-a_1<\frac{1}{m}$,

(3) $\mu\left\{x;\ \text{abs}\left|\frac{\partial f}{\partial x}\right|\in[a_1,\ a_2],\ x\in E_1\right\}>0.$

From the above lemma follows

$$D(f/x)\in[a_1,\ a_2]$$

almost everywhere in

$$E_2=\left\{x;\ \text{abs}\left|\frac{\partial f}{\partial x}\right|\in[a_1,\ a_2],\ x\in E_1\right\},$$

in contradiction to our assumption. So the theorem is proved.

COROLLARY. *If $f(x)$ is an $\mathfrak{A}$-function on D and $g(x)$ is integrable on $f(D)$, then we have*

$$\int_{f(E)} g(x)dx=\int_E g(f(x))\cdot\text{abs}\left|\frac{\partial f}{\partial x}\right|dx,$$

where E is any measurable set.

§ 4. Generalized $\mathfrak{A}$-functions.

Now we shall examine the case, where transformation is not necessarily a homeomorphism.

DEFINITION 3. *A transformation $f(x)$ from a compact domain D into E^n is called a generalized $\mathfrak{A}$-function on D, if the following conditions are satisfied:*

($\mathfrak{A}_1'$) *$f(x)$ is a continuous mapping, locally homeomorphic in D except on a null-set E.*

($\mathfrak{A}_2$) *If $\mu(F)=0$ $(F\subset D)$, then $\mu(f(F))=0$.*

($\mathfrak{A}_3$) *$f(x)$ is totally differentiable in D almost everywhere.*

We shall consider in the following a fixed generalized $\mathfrak{A}$-function $f(x)$ on a compact set D, with a possible exceptional null-set E. We shall now proceed to evaluate the integral

$$\int_D\left|\frac{\partial f}{\partial x}\right|dx.$$

First, we shall prove some lemmas.

LEMMA 1. Denote with D^r the boundary of D. The inverse image of x_0 contains at most a finite number of points, if

(1) $f(E)\not\ni x_0$ (see $(\mathfrak{A}_1')$),

(2) $f(D^r)\ni x_0$.

We shall denote with $m(x_0)$ the number of points contained in the inverse image of such x_0.

PROOF. Since D is compact, so $f^{-1}(x_0)$ must have a cluster point x_1, if it contains an infinite number of points. But as x_1 does not belong to $E\cup D^r$, so $f(x)$ is homeomorphic on some neighbourhood of x_1,—which is clearly a contradiction.

LEMMA 2. *For a point x_0 such that*

(i) $f(E)\not\ni x_0$,

(ii) $f(D^r)\not\ni x_0$,

(iii) $f(D)\ni x_0$,

there exists a neighbourhood U of x_0 which satisfies:

(1) *$f^{-1}(U)$ is a direct sum of neighbourhoods $V_i (i=1, 2, \cdots, m)$ of points $x_1, \cdots, x_m$, where $f^{-1}(x_0)=\{x_1, \cdots, x_m\}$ and $m=m(x_0)$,*

(2) *V_i $(i=1, 2, \cdots, m)$ is mapped onto U by $f(x)$ homeomorphically.*

PROOF. If we take a sufficiently small neighbourhood U of x_0, there exist clearly the open sets $V_1, \cdots, V_m$ in D, each of which is mapped onto U by $f(x)$ homeomorphically.

If $f^{-1}(U)\neq\sum_{i=1}^m V_i$, then we can find points x' in U, such that $f^{-1}(x')\not\subset\sum_{i=1}^m V_i$. We shall call such points "exceptional". If there are only a finite number of exceptional points, then we may substitute U by a small neighbourhood U', not containing these exceptional points, and obtain a neighbourhood of required nature. Even if there are an infinite number of exceptional points, we can attain our purpose in the same way, if they do not accumulate around x_0. Assume now there exists a sequence of exceptional points $x_1', x_2', \cdots, x_n', \cdots$ converging to x_0. Let a_i be a point in D, such that $a_i\not\in\sum_{i=1}^m V_i$ and $f(a_i)=x_i'$. $\{a_i\}$ has a cluster point a in the compact set $(D-\sum_{i=1}^m V_i)$. Then we must have $f(a)=x_0$ and $a\in f^{-1}(x_0)$, which is a contradiction.

LEMMA 3. *If we put $f^{-1}f(E)=\overline{E}$, then*

$$\int_{\overline{E}}\left|\frac{\partial f}{\partial x}\right|dx=0.$$

PROOF. We shall prove that

$$\int_{\bar{E}} \text{abs}\left|\frac{\partial f}{\partial x}\right| dx = 0 .$$

Since $\mu(E)=0$, this equation is equivalent to

$$\int_{\bar{E}-E} \text{abs}\left|\frac{\partial f}{\partial x}\right| dx = 0 .$$

If $\mu(\bar{E}-E)=0$, then our proposition is trivial. So we assume $\mu(\bar{E}-E)>0$. Under this assumption, we have only to prove that for any closed subset F of $\bar{E}-E$

$$\int_{F} \text{abs}\left|\frac{\partial f}{\partial x}\right| dx = 0 .$$

Now, every point of F has a neighbourhood, on which $f(x)$ is a homeomorphism, and as F is compact, F is covered by a finite number of such neighbourhoods as follows:

$$\cup_{i=1}^{k} U_i \supset F .$$

In each U_i, we have from the result of the last section,

$$0=\mu(f(F \cap U_i)) = \int_{F \cap U_i} \text{abs}\left|\frac{\partial f}{\partial x}\right| dx .$$

Thus we obtain

$$0 \leqq \int_{F} \text{abs}\left|\frac{\partial f}{\partial x}\right| dx \leqq \sum_{i=1}^{k} \int_{F \cap U_i} \text{abs}\left|\frac{\partial f}{\partial x}\right| dx = 0 .$$

This completes the proof.

Now denote with A (x, f, D) the degree of mapping on D at x, and put

$$f(D)_m = \{x ;\ \mathrm{A}\ (x, f, D) = m\} \qquad (m=0, \pm 1, \pm 2, \cdots) .$$

Then we have

$$\cup_{m=-\infty}^{+\infty} f(D)_m = f(D) - f(D^r) .$$

THEOREM 4. *If* $\mu(D^r)=0$, *and* abs$\left|\dfrac{\partial f}{\partial x}\right|$ *is integrable on D, then we have*

$$\int_{D} \left|\frac{\partial f}{\partial x}\right| dx = \sum_{m=-\infty}^{+\infty} m \int_{f(D)_m} dx .$$

PROOF. Let us represent $f(D)_m - f(E)$ as the union of closed cubes I_i $(i=1, 2, \cdots)$ such that

(i) $\mu(I_i \cap I_j)=0$ if $i \neq j$,

(ii) $f^{-1}(I_i)$ is the union of a finite number of disjoint closed domains J^i_j $(j=1, 2, \cdots, \alpha_i)$,

(iii) J^i_j 's are mapped homeomorphically onto I_i by $f(x)$.

The existence of such J^i_j's is assured by lemma 2. For $x \in I_i$ we have obviously

$$A(x, f, J^i_j)=\operatorname{sgn}\left|\frac{\partial f}{\partial x}\right|$$

and

$$A(x, f, (D-\textstyle\sum_{j=1}^{\alpha_i} J^i_j))=0\,.$$

Furthermore, since we have

$$m=A(x, f, D)=\textstyle\sum_{j=1}^{\alpha_i} A(x, f, J^i_j)+A(x, f, (D-\sum_{j=1}^{\alpha_i} J^i_j)),$$

we can easily see that

$$m\int_{I_i} dx=\sum_{j=1}^{\alpha_i}\int_{J^i_j}\left|\frac{\partial f}{\partial x}\right| dx$$

by theorem 3 of the last section. Thus

$$m\int_{f(D)_m} dx=m\int_{f(D)_m-f(E)} dx=m\int_{\sum_{i=1}^{\infty} I_i} dx$$

$$=\sum_{i=1}^{\infty} m\int_{I_i} dx=\sum_{i=1}^{\infty}\sum_{j=1}^{\alpha_i}\int_{J^i_j}\left|\frac{\partial f}{\partial x}\right| dx$$

$$=\int_{f^{-1}(f(D)_m)-f^{-1}f(E)}\left|\frac{\partial f}{\partial x}\right| dx\,.$$

Now, as abs$\left|\frac{\partial f}{\partial x}\right|$ is integrable,

$$S=\sum_{m=-\infty}^{+\infty}\int_{f^{-1}(f(D)_m)-f^{-1}f(E)}\left|\frac{\partial f}{\partial x}\right| dx$$

is finite. From the equation just proved follows then that

$$\sum_{m=-\infty}^{+\infty} m\int_{f(D)_m} dx$$

is also finite and equal to S. The last sum is equal to

$$\int_{D-f^{-1}f(E)-f^{-1}f(Dr)}\left|\frac{\partial f}{\partial x}\right|dx=\int_D\left|\frac{\partial f}{\partial x}\right|dx\,.$$

Thus our proposition is proved.

The following theorem can be proved in the same way.

THEOREM 5. *If one of the integrals:*

$$\int_D g(f(x))\left|\frac{\partial f}{\partial x}\right|dx \quad \text{and} \quad \sum_{m=-\infty}^{+\infty} m\int_{f(D)_m} g(y)dx$$

is finite, then the other is also finite and they are equal to each other.

COROLLARY. *Let $f(x)$ be a generalized $\mathfrak{A}$-function on D and S a hypersphere in D. If $f(x)$ maps S homeomorphically onto $f(S)$ and* abs $\left|\frac{\partial f}{\partial x}\right|$ *is integrable on D, then we have*

$$\int_{[f(S)]} g(y)dy=\operatorname{sgn} A\,[x,\,f,\,D]\int_{[S]} g(f(x))\left|\frac{\partial f}{\partial x}\right|dx,$$

where $[*]$ represents the interior of $*$.

This corollary may be regarded as a direct generalization of the well-known formula in the integral calculus:

$$\int_{f(a)}^{f(b)} g(y)dy=\int_a^b g(f(x))f'(x)\,dx\,.$$

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