Journal of the Mathematical Society of Japan Vol. 5, No. 1, April, 1953.

On a minimax theorem and its applications to functional analysis.

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(Received October, 17. 1952.)

1. Preliminaries.

In the present paper, we shall derive Mazur's theorem on convex sets and the known regularity of some Banach spaces from a minimax theorem which we shall state and prove by a procedure due, in essential, to N. Georgescu-Roegen [1], H.F. Bohnenblust, S. Karlin and L.S. Shapley [2].

In what follows, all the linear spaces to be considered are the ones on the field of real numbers.

A linear space E is said to be topological, if a separative topology is given in it so that the mappings

$$E\times E\ni(x,y)\rightarrow x+y\in E\,,$$

$$R\times E\ni(\alpha,x)\rightarrow \alpha x\in E$$

may be continuous, where $E\times E$ is the topological product of E by itself and $R\times E$ is that of E by R: namely, the set of all real numbers in the usual topology.

In case of a normed space E, the conjugate space of it will be denoted by E^*. We understand under w^*-topology the weak topology of E^* as the conjugate space of E. On the other hand, the adjective "weak" will be used for the weak topology of E or E^* by its bounded linear functionals. The unit spheres of E and E^* will be denoted by S and S^* respectively. In this paper, we shall often make use of the well-known w^*-compactness of S^*.

2. Minimax Theorems.

Let E be a topological linear space and F a (not necessarily topological) linear space. Let further X and Y be convex sets of E

and F respectively, and X is assumed to be compact.

Let $K(x,y)$ be a real valued function, which will be called later a pay-off, defined on the product space of X by Y. $K(x,y)$ is assumed to satisfy the following conditions:

(I) $K(x,y)$ is continuous in $x \in X$ for each fixed $y \in Y$.

(II) $K(\alpha_1 x_1+\alpha_2 x_2, y) \geqq \alpha_1 K(x_1, y)+\alpha_2 K(x_2, y)$ for $\alpha_1 \geqq 0$, $\alpha_2 \geqq 0$, $\alpha_1+\alpha_2 =1$, $x_1, x_2 \in X$ and $y \in Y$.

(III) $K(x, \beta_1 y_1+\beta_2 y_2) \leqq \beta_1 K(x,y_1)+\beta_2 K(x,y_2)$ for $\beta_1 \geqq 0$, $\beta_2 \geqq 0$, $\beta_1+\beta_2 =1$, $y_1, y_2 \in Y$ and $x \in X$.

Under these conditions we shall prove:

THEOREM 1. *If* $\sup_{x \in X} \inf_{y \in Y} K(x,y)$ *is finite, then we have the determinateness of the game*: *i.e.*,

$$\sup_{x \in X} \inf_{y \in Y} K(x,y)=\inf_{y \in Y} \sup_{x \in X} K(x,y).$$

PROOF. Put

$$\sup_{x \in X} \inf_{y \in Y} K(x,y)=\sigma. \tag{1}$$

Let $\varepsilon>0$ be an arbitrary positive number, then we have, by definition,

$$\sup_{x \in X} \inf_{y \in Y} K(x,y)<\sigma+\varepsilon, \tag{2}$$

which implies that for every $x \in X$ there exists some $y \in Y$ such that

$$K(x,y)<\sigma+\varepsilon. \tag{3}$$

In consequence, by virtue of the compactness of X, we can find a finite number of points $b_j \in Y$, $(j=1,2,\cdots,n)$, so that

$$\operatorname*{Min}_j K(x,b_j)<\sigma+\varepsilon \tag{4}$$

for any $x \in X$. Consider then the continuous mapping

$$X \ni x \rightarrow \varphi(x)=(\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)) \in R^n,$$

where

$$\varphi_j(x)=K(x,b_j)-(\sigma+\varepsilon).$$

Let $\varphi(x_1)$ and $\varphi(x_2)$ be any two points of $\varphi(X)$, then for $\alpha_1 \geqq 0$, $\alpha_2 \geqq 0$, and $\alpha_1+\alpha_2=1$, we have, by (II)

$$\alpha_1 \varphi_j(x_1)+\alpha_2 \varphi_j(x_2) \leqq \varphi_j(\alpha_1 x_1+\alpha_2 x_2) \tag{5}$$

and $\alpha_1 x_1+\alpha_2 x_2 \in X$ because of the convexity of X.

Thus, by virtue of (4) and (5), the convex closure of $\varphi(X)$ does not intersect the positive orthant Ω of R^n: namely, the set of all points $\in R^n$ whose coordinates are non-negative. And in addition, $\varphi(X)$ is compact. Consequently, there exists a hyperplane of R^n

$$\sum_{j=1}^{n} \beta_j Z_j = 0$$

with $\beta_j \geqq 0, \sum_{j=1}^{n} \beta_j = 1$, in whose negative side $\varphi(X)$ lies. Hence we have

$$\sum_{j=1}^{n} \beta_j \varphi_j(x) < 0$$

for any $x \in X$. But, this means, by virtue of (III), that

$$K(x, \sum_{j=1}^{n} \beta_j b_j) < \sigma + \epsilon \tag{6}$$

for any $x \in X$. Therefore, since Y contains $\sum_{j=1}^{n} \beta_j b_j$ owing to its convexity, (6) implies that

$$\inf_{y \in Y} \sup_{x \in X} K(x, y) \leqq \sigma + \epsilon .$$

Thus the arbitrariness of ϵ yields

$$\inf_{y \in Y} \sup_{x \in X} K(x, y) \leqq \sigma = \sup_{x \in X} \inf_{y \in Y} K(x, y) .$$

This proves the determinateness of the game, since we have always

$$\sup_{x \in X} \inf_{y \in Y} K(x, y) \leqq \inf_{y \in Y} \sup_{x \in X} K(x, y) .$$

THEOREM 2. *Suppose next that E is a (not necessarily topological)* *linear space and F is a topological linear space, and that, in addition, $Y \subset F$ is compact. If the pay-off is continuous in $y \in Y$ for each fixed $x \in X$ and satisfies* (II) *and* (III) *as before, the finiteness of* $\inf_{y \in Y} \sup_{x \in X} K(x, y)$ *implies the determinateness of the game.*

The proof would be a mere repetition of the preceding one, therefore it should be omitted here.

3. Mazur's theorem.

Let E be a normed linear space and M a strongly closed convex subset in it. As is well-known, Mazur's theorem says that for any point $a \notin M$ there exists a bounded linear functional $f \in E^*$ such that

$$\sup_{x \in M} f(x) < f(a). \tag{7}$$

We shall give a proof of this theorem by the aid of the preceding minimax theorems.

A point $a \in E$ is said to be quasi-weakly adherent to a subset A, if for every $f \in S^*$ and every $\varepsilon > 0$ there exists a point $x \in A$ such that

$$|f(x) - f(a)| < \varepsilon. \tag{8}$$

It is easily seen that Mazur's theorem is equivalent to the following

THEOREM 3. *If a point $a \in E$ is quasi-weakly adherent to a subset A of E, then a is also strongly adherent to the convex closure $C(A)$ of A.*

In order to prove this theorem, we consider the game with the pay-off $f(x-a)$, where the maximizing player chooses his strategy f from S^* and the minimizing player chooses his strategy x from $C(A)$.

The pay-off function is w^*-continuous in the variable $f \in S^*$ for each fixed x and satisfies obviously (II) and (III), while S^* is w^*-compact.

Next, let us see the finiteness of $\sup_{f \in S^*} \inf_{x \in C(A)} f(x-a)$. First, it is obvious that $\sup_f \inf_x f(x-a) \geq 0$, because the maximizing player can choose $f=0$ as his strategy. On the other hand, let ε be an arbitrary positive number. Since, by assumption, a is quasiweakly adherent to

A, for every $f \in S^*$ there exists a point $x \in A \subseteq C(A)$ such that

$$f(x)-f(a)<\varepsilon\,,$$

which implies that

$$\sup_{f \in S^*} \inf_{x \in C(A)} f(x-a) \leqq \varepsilon\,. \tag{7}$$

Thus the arbitrariness of ε yields

$$\sup_{f \in S^*} \inf_{x \in C(A)} f(x-a)=0\,. \tag{8}$$

Therefore, all the conditions of our minimax theorem 1 are satisfied, and, in consequence, the game is determined: i.e.,

$$\sup_{f \in S^*} \inf_{x \in C(A)} f(x-a)=0=\inf_{x \in C(A)} \sup_{f \in S^*} f(x-a)\,.$$

Now, since

$$\sup_{f \in S^*} f(x-a)=\| x-a \|\,, \tag{9}$$

we obtain

$$\inf_{x \in C(A)} \| x-a \|=0\,. \tag{10}$$

This proves the theorem.

4. Regularity of Banach spaces.

Let E be a Banach space. The conjugate space of E^* will be denoted by E^{**} and its unit sphere will be denoted by S^{**}. As is known, E is called regular, if $E=E^{**}$; i.e., for every $\Phi \in E^{**}$ there exists an $x \in E$ such that (11) $\Phi(f)=f(x)$ for any $f \in E^*$.

The following is a game-theoretic approach to the well-established regularily of some Banach spaces.

First, we shall prove: *If the unit sphere S of E is weakly compact, then E is regular.* For this aim, let Φ be an arbitrary element of S^{**}, and consider the game with the pay-off

(12) $$\Psi(f, x)=\Phi(f)-f(x),$$

where the maximizing player uses strategies $f \in S^*$ and the minimizing player uses strategies $x \in S$.

This pay-off function is continuous with respect to the variable $x \in S$ in the sense of the weak topology of E, and satisfies (II) and (III). Moreover, S is, by assumption, compact in this topology.
Next, we have

$$\inf_{x \in S} \sup_{f \in S^*} \Psi(f, x) \leqq 1,$$

because

$$|\Psi(f, 0)|=|\Phi(f)| \leqq \|\Phi\| \|f\| \leqq 1.$$

And, we obtain also

$$\sup_{f \in S^*} \inf_{x \in S} \Psi(f, x) \geqq 0,$$

because $\Psi(0, x)=0$ for any $x \in S$. Hence it follows that

(13) $$0 \leqq \sup_{f \in S^*} \inf_{x \in S} \Psi(f, x) \leqq \inf_{x \in S} \sup_{f \in S^*} \Psi(f, x) \leqq 1.$$

Thus our minimax theorem 2 applies to this game. We have therefore

(14) $$\sup_{f \in S^*} \inf_{x \in S} \Psi(f, x)=\inf_{x \in S} \sup_{f \in S^*} \Psi(f, x).$$

We shall finally show that the value of (14) is just zero. Indeed, suppose that the maximizing player chooses an $f \in S^*$. Since, by definition, $\|f\|=\sup_{x \in S} f(x)$, for every $\varepsilon>0$ there exists an $x \in S$ such that $\|f\|<f(x)+\varepsilon$. Then, we have immediately

$$\Phi(f) \leqq |\Phi(f)| \leqq \|\Phi\| \|f\| \leqq \|f\|<f(x)+\varepsilon;$$

that is to say, $\Phi(f)-f(x)<\varepsilon$. Hence, taking (13) together in consideration, we have

(15) $$\sup_{f \in S^*} \inf_{x \in S} \Psi(f, x)=0.$$

Therefore the value of the game is just zero. (Notice that (15) is always true, whether the game under consideration is determined or not.) Thus from

(16) $$\sup_{f\in S^*}\inf_{x\in S}[\phi(f)-f(x)]=0=\inf_{x\in S}\sup_{f\in S^*}[\phi(f)-f(x)],$$

we obtain

$$\inf_{x\in S}\|\phi-x\|=0.$$

This proves that $S=S^{**}$ because of the completeness of E.

Another known criterion for the regularity is as follows: *a Banach space E is regular, if (and only if) the weak topology of E^* is equivalent to the w^*-topology of it.*

Let us again consider the game discussed above. If we succeed to show the determinateness of this game, we obtain at once the regularity of E by the final part of the preceding argument. This game meets all the conditions of our minimax theorem 1 except that of the compactness of S^*, provided that we consider the maximizing player's strategy space S^* from the standpoint of the weak topology of E^*. But, as, by assumption, the weak topology of E^* is equivalent to the w^*-topology of it, and as S^* is always compact in the sense of the latter, S^* is compact in the sense of the former, too. Hence we have, by virtue of our minimax theorem 1, the desired determinateness of this game.

5. Regularity of uniformly convex Banach spaces.

We end this work by presenting a game theoretic approach to the regularity of uniformly convex Banach spaces; this might be ragarded as a modification of the usual approach to this fact.
For this aim, we begin with

LEMMA. *Let E be a normed linear space. For any $f_1, f_2\in S^*$, $\phi\in S^{**}$ and $\epsilon>0$ there exists an $x\in S$ such that* $\phi(f_i)-f_i(x)\leq\epsilon$ $(i=1,2)$.

PROOF. If $\operatorname{Max}_i[\phi(f_i)-f_i(x)]>\epsilon$ for any $x\in S$, the image of S, which is convex and bounded, by the continuous mapping

$$S \ni x \rightarrow (\Phi(f_1)-f_1(x)-\varepsilon, \Phi(f_2)-f_2(x)-\varepsilon) \in R^2$$

would lie in the positive side of a straight line $\alpha_1 Z_1+\alpha_2 Z_2=0$ with $\alpha_1 \geqq 0, \alpha_2 \geqq 0, \alpha_1+\alpha_2=1$. Hence we have

$$\| \alpha_1 f_1+\alpha_2 f_2 \| \geqq \Phi (\alpha_1 f_1+\alpha_2 f_2) > \alpha_1 f_1(x)+\alpha_2 f_2(x)+\varepsilon$$

for any $x \in S$, which is a contradiction.

Let E be now a uniformly convex Banach space; namely its norm satisfies the following condition: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $x, y \in S$ and $\| x+y \| > 2(1-\delta)$ yield $\| x-y \| < \varepsilon$.

Let Φ be a point of S^{**} such that $\| \Phi \|=1$. We shall show the determinateness of the game with the pay-off $\Phi(f)-f(x)$, which player 1 maximizes by choosing $f \in S^*$ and player 2 minimizes by choosing $f \in S$.

In fact, let ε be an arbitrary positive number, and choose a $\delta > 0$ such that $x, y \in S$ and $\| x+y \| > 2(1-\delta)$ imply $\| x-y \| < \varepsilon/2$. Next let $\gamma > 0$ be chosen in such a way that $\gamma < \mathrm{Min}(\varepsilon/2, \delta/2)$. Since $\| \Phi \|=1$, there exists an $h \in S^*$ which meets the condition: $1-\gamma \leqq \Phi(h)$. For this h, put

$$H_\varepsilon=\{x ; x \in S, \Phi(h)-h(x) \leqq \gamma\}.$$

The set H_ε is clearly non-empty. A point in H_ε will be called an ε-optimal strategy of the minimizing player.

Let x and y be any two points in H_ε. Then, we have

$$\begin{aligned} \| x+y \| &\geqq \| h \| \| x+y \| \\ &\geqq h(x)+h(y) \geqq 2(\Phi(h)-\gamma) \\ &\geqq 2(1-2\gamma) > 2(1-\delta). \end{aligned}$$

Hence it follows that $\| x-y \| < \varepsilon/2$ for $x, y \in H_\varepsilon$.

Suppose now that the minimizing player chooses an ε-optimal strategy a. Then, for every $f \in S^*$, by virtue of the preceding lemma, there exists a point x in H_ε such that $\Phi(f)-f(x) \leqq \gamma$. Thus

$$\Phi(f)-f(a)=\Phi(f)-f(x)+f(x)-f(a).$$
$$\leq \gamma+\| x-a \| < \varepsilon/2+\varepsilon/2=\varepsilon$$

for any $f \in S^*$. This proves

$$\inf_{x\in S} \sup_{f\in S^*} [\Phi(f)-f(x)]=0.$$

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Notes and References

As is well-known, Helly's theorem plays an important rôle in discussions on the regularity of Banach spaces. It may be said that our approach to the regularity is in essential a game-theoretic rearrangement of Helly's theorem. As to Mazur's theorem, our approach is clearly roundabout, since we are based upon the w^*-compactness of S^* and, in addition, Hahn-Banach's extension theorem, while the usual one need not be aided by the former fact.

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