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On a direct transcendental singularity of an inverse function of a meromorphic function.

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Let Δ be an infinite domain on the z-plane, which may be infinitely multiply connected and Γ be its boundary, which consists of at most a countable number of analytic curves. We assume that Γ contains at least one curve extending to infinity. Let $w=w(z)$ be regular in Δ and on Γ, except at $z=\infty$, such that $|w(z)|<R$ in Δ and $|w(z)|=R$ on Γ and $w(z)\neq 0$ in Δ. Let Δ_r be the part of Δ, which lies in $|z|<r$. We put

$$S(r;\Delta)=\frac{1}{\pi}\iint_{\Delta_r}\frac{|w'|^2}{(1+|w|^2)^2}dxdy,\ (w=w(z),\ z=x+iy),\qquad (1)$$

$$T(r;\Delta)=\int_1^r\frac{S(r;\Delta)}{r}dr.\qquad (2)$$

Now Δ_r consists of a finite number of connected domains. Let Δ_r^0 $(r\geqq r_0)$ be the one, which contains a fixed point z_0 of Δ and θ_r be the part of $|z|=r$, which belongs to the boundary of Δ_r^0. θ_r consists of a finite number of arcs θ_r^i $(i=1,2,\cdots,\nu(r))$ and $r\theta_i(r)$ be its arc length and put $\theta(r)=\sum\theta_i(r)$. $\theta(r)$ is continuous except at most a countable number of isolated points $0<r_1<r_2<\cdots<r_\nu\to\infty$, where $\theta(r_\nu-0)=\theta(r_\nu)<\theta(r_\nu+0)$.

In the former paper,[1] I have proved the following theorem.

THEOREM. *For any* $0<\alpha<1$,

$$T(r;\Delta)\geqq\text{const.}\ e^{\pi\int_{r_0}^{\alpha r}\frac{dr}{r\theta(r)}}\ (r\geqq r_0).$$

1) M. Tsuji: On a regular function which is of constant absolute value on the boundary of an infinite domain. Tohoku Math. Journ. 3 (1951).

In the proof of the theorem on the number of direct transcendental singularities of an inverse function of a meromorphic function of finite order, Ahlfors[2] proved a similar relation:

$$T(r) \geqq \text{const.}\ e^{\pi \int_{r_0}^{\alpha r} \frac{dr}{r\theta(r)}}, \tag{3}$$

where $w(z)$ is meromorphic for $|z| < \infty$ and $T(r)$ is its characteristic function and $\theta(r)$ is defined for a simply connected domain, which is bounded by the outermost boundary curve of Δ. Our theorem is an extension of (3). In this paper, I shall give a somewhat simpler proof than the former one.

Proof. Let

$$u(z) = \log \frac{R^2 + |w|^2}{2R|w|} \geqq 0, \qquad w = w(z), \tag{4}$$

then

$$\Delta u = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} = \frac{4R^2|w'|^2}{(R^2+|w|^2)^2} \geqq 0, \quad \frac{\partial^2 u}{\partial \log r^2} + \frac{\partial^2 u}{\partial \theta^2} = r^2 \Delta u \geqq 0, \tag{5}$$

$$(z = x + iy = re^{i\theta})$$

so that $u(z)$ is subharmonic in Δ.

Since $\frac{R^2+|w|^2}{2R|w|} = 1 + \frac{(R-|w|)^2}{2R|w|}$, we see that

$$u = 0, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma, \tag{6}$$

where ν is the outer normal of Γ.

Let λ_r be the part of $|z| = r$, which lies in Δ, so that $\theta_r \subset \lambda_r$. We put

$$\mu(r) = \int_{\lambda_r} u(re^{i\theta})d\theta. \tag{7}$$

We denote Γ_r the part of Γ, which belongs to the boundary of Δ_r, so that $\Gamma_r + \lambda_r$ is the whole boundary of Δ_r.

2) L. Ahlfors: Über die asymptotischen Werte der meromorphen Funktionen endlicher Ordnung. Acta Acad. Aboensis. Math. et Phys. 6, Nr. 9 (1932).

Since $u=0$ at the end points of λ_r we have by (6) and Green's formula,

$$r\mu'(r)=\int_{\lambda_r}\frac{\partial u}{\partial r}rd\theta=\int_{\Gamma_r+\lambda_r}\frac{\partial u}{\partial \nu}ds=\iint_{\Delta_r}\Delta u\,dxdy=$$

$$4R^2\iint_{\Delta_r}\frac{|w'|^2}{(R^2+|w|^2)^2}dxdy>0,$$

so that

$$\int_{\lambda_r}u(re^{i\theta})d\theta=\mu(r)=4R^2\int_{r_0}^{r}\frac{dr}{r}\iint_{\Delta_r}\frac{|w'|^2}{(R^2+|w|^2)^2}dxdy+\text{const.}\leq$$

$$\text{const.}\int_{r_0}^{r}\frac{dr}{r}\iint_{\Delta_r}\frac{|w'|^2}{(1+|w|^2)^2}dxdy+\text{const.}=\text{const. }T(r;\,\Delta)+\text{const.}\quad (9)$$

Following Carleman[3], we put

$$m(r)=\frac{1}{2\pi}\int_{\theta_r}[u(re^{i\theta})]^2d\theta.\quad (10)$$

We denote Γ_r^0 the part of Γ_r which belongs to the boundary of Δ_r^0, so that $\Gamma_r^0+\theta_r$ is the whole boundary of Δ_r^0. Since $u=0$ at the end points of θ_r by (6) and Green's formula, we have for $r_\nu<r<r_{\nu+1}$,

$$\frac{dm(r)}{d\log r}=\frac{1}{\pi}\int_{\theta_r}u\frac{\partial u}{\partial\log r}d\theta=\frac{1}{\pi}\int_{\theta_r}u\frac{\partial u}{\partial r}rd\theta=\frac{1}{\pi}\int_{\Gamma_r^0+\theta_r}u\frac{\partial u}{\partial\nu}ds=$$

$$\frac{1}{\pi}\iint_{\Delta_r^0}\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)dxdy+\frac{1}{\pi}\iint_{\Delta_r^0}u\Delta u\,dxdy>0,\quad (11)$$

so that $m(r)$ increases at $r\,(\neq r_\nu)$. At r_ν, we see that $m(r_\nu-0)=m(r_\nu)$ $<m(r_\nu+0)$. Hence $m(r)$ is an increasing function of r.

By (5), we have

$$\frac{d^2m(r)}{d\log r^2}=\frac{1}{\pi}\int_{\theta_r}\left(\left(\frac{\partial u}{\partial\log r}\right)^2+u\frac{\partial^2 u}{\partial\log r^2}\right)d\theta=$$

$$\frac{1}{\pi}\int_{\theta_r}\left(\left(\frac{\partial u}{\partial\log r}\right)^2+ur^2\Delta u-u\frac{\partial^2 u}{\partial\theta^2}\right)d\theta\geq$$

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3) T. Carleman: Sur une inegalité différentielle dans la théorie des fonctions analytiques. C. R. 196 (1936).

$$\frac{1}{\pi}\int_{\theta_r}\left(\left(\frac{\partial u}{\partial \log r}\right)^2-u\frac{\partial^2 u}{\partial \theta^2}\right)d\theta=\frac{1}{\pi}\int_{\theta_r}\left(\left(\frac{\partial u}{\partial \log r}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2\right)d\theta, \tag{12}$$

by the integration by parts.

Now

$$\left(\frac{dm(r)}{d\log r}\right)^2\leqq\frac{1}{\pi^2}\int_{\theta_r}u^2\,d\theta\int_{\theta_r}\left(\frac{\partial u}{\partial \log r}\right)^2d\theta=\frac{2m(r)}{\pi}\int_{\theta_r}\left(\frac{\partial u}{\partial \log r}\right)^2d\theta,$$

so that

$$\frac{1}{\pi}\int_{\theta_r}\left(\frac{\partial u}{\partial \log r}\right)^2d\theta\geqq\frac{1}{2\,m(r)}\left(\frac{dm(r)}{d\log r}\right)^2. \tag{13}$$

Since $u=0$ at the end pints of θ_r^i, we have by Wirtinger's inequality,

$$\int_{\theta_r^i}\left(\frac{\partial u}{\partial \theta}\right)^2d\theta\geqq\frac{\pi^2}{(\theta_i(r))^2}\int_{\theta_r^i}u^2\,d\theta\geqq\frac{\pi^2}{(\theta(r))^2}\int_{\theta_r^i}u^2\,d\theta.$$

Summing up for i,

$$\frac{1}{\pi}\int_{\theta_r}\left(\frac{\partial u}{\partial \theta}\right)^2d\theta\geqq\frac{\pi}{(\theta(r))^2}\int_{\theta_r}u^2\,d\theta=\frac{1}{2}\left(\frac{2\pi}{\theta(r)}\right)^2m(r). \tag{14}$$

Hence by (12), (13), (14),

$$\begin{aligned}\frac{d^2m(r)}{d\log r^2}&\geqq\frac{1}{2m(r)}\left(\frac{dm(r)}{d\log r}\right)^2+\frac{1}{2}\left(\frac{2\pi}{\theta(r)}\right)^2m(r)\\&\geqq\frac{dm(r)}{d\log r}\cdot\frac{2\pi}{\theta(r)}\quad(r_\nu<r<r_{\nu+1}),\end{aligned} \tag{15}$$

since $\dfrac{dm(r)}{d\log r}>0$ by (11).

From (11), we see that at r_ν,

$$\left(\frac{dm(r)}{d\log r}\right)_{r_\nu-0}=\left(\frac{dm(r)}{d\log r}\right)_{r_\nu}<\left(\frac{dm(r)}{d\log r}\right)_{r_\nu+0},$$

so that integrating (15), we have

$$\log\frac{dm(r)}{d\log r}-\log\frac{dm(r_0)}{d\log r_0}\geqq 2\pi\int_{r_0}^{r}\frac{dr}{r\theta(r)},$$

or

$$r\, m'(r) = \frac{dm(r)}{d\log r} \geq \text{const. } e^{2\pi\int_{r_0}^{r} \frac{dr}{r\theta(r)}}. \tag{16}$$

Since $m(r)$ is an increasing function of r, we have for any $0 < \beta < 1$,

$$m(r) > m(r) - m(r_0) \geq \int_{r_0}^{r} m'(r)dr \geq \text{const.}\int_{r_0}^{r} \frac{dr}{r}\, e^{2\pi\int_{r_0}^{r} \frac{dt}{t\theta(t)}} \geq$$

$$\text{const.}\int_{\beta r}^{r} \frac{dr}{r}\, e^{2\pi\int_{r_0}^{r} \frac{dt}{t\theta(t)}} \geq \text{const.}\int_{\beta r}^{r} \frac{dr}{r}\, e^{2\pi\int_{r_0}^{\beta r} \frac{dt}{t\theta(t)}} =$$

$$\text{const. } e^{2\pi\int_{r_0}^{\beta r} \frac{dr}{r\theta(r)}} \tag{17}$$

Let $u(z)$ attain its maximum at $z = re^{i\theta_0}$ on θ_r, then

$$\left(u(re^{i\theta_0})\right)^2 \geq \frac{1}{2\pi}\int_{\theta_r}\left[u(re^{i\theta})\right]^2 d\theta = m(r) \geq \text{const. } e^{2\pi\int_{r_0}^{\beta r} \frac{dr}{r\theta(r)}},$$

so that

$$u(re^{i\theta_0}) \geq \text{const. } e^{\pi\int_{r_0}^{\beta r} \frac{dr}{r\theta(r)}}. \tag{18}$$

Let

$$U(z) = U(re^{i\theta}) = \frac{1}{2\pi}\int_{\lambda_\rho} u(\rho e^{i\varphi}) \frac{\rho^2 - r^2}{\rho^2 - 2\rho r\cos(\varphi - \theta) + r^2}\, d\varphi, \tag{19}$$

$$(r < \rho)$$

then $U(z) = u(z)$ on λ_ρ and $U(z) > 0 = u(z)$ on Γ_ρ and since $u(z)$ is subharmonic in Δ_ρ, we have $u(z) < U(z)$ in Δ_ρ. Hence if we put $\rho = kr\,(k > 1)$, then by (19) and (9),

$$u(re^{i\theta_0}) < U(re^{i\theta_0}) \leq \frac{\rho + r}{\rho - r}\cdot\frac{1}{2\pi}\int_{\lambda_\rho} u(\rho e^{i\varphi})d\varphi = \frac{k+1}{k-1}\cdot\frac{1}{2\pi}\int_{\lambda_{kr}} u(kre^{i\varphi})d\varphi$$

$$\leq \text{const. } T(kr; \Delta) + \text{const.},$$

so that by (18),

$$T(kr;\Delta)\geqq \text{const.}\, e^{\pi\int_{r_0}^{\beta r}\frac{dr}{r\theta(r)}}-\text{const.}$$

Hence if we put r instead of kr and $\alpha=\frac{\beta}{k}$, then

$$T(r;\Delta)\geqq \text{const.}\, e^{\pi\int_{r_0}^{\alpha r}\frac{dr}{r\theta(r)}}-\text{const.}$$

From this we have easily,

$$T(r;\Delta)\geqq \text{const.}\, e^{\pi\int_{r_0}^{\alpha r}\frac{dr}{r\theta(r)}}\quad(r\geqq r_0).$$

Since α is any number, such that $0<\alpha<1$, our theorem is proved.

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