Journal of the Mathematical Society of Japan Vol. 5, No. 2, July, 1953.

On the capacity of general Cantor sets.

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(Received May 28, 1953)

1. General linear Cantor sets.

1. Let Δ be an interval on the x-axis. We take $k(\geqq 2)$ disjoint intervals Δ_{i_1} $(i_1=1,2,\cdots,k)$ in Δ and k disjoint intervals $\Delta_{i_1 i_2}$ $(i_2=1,2,\cdots,k)$ in Δ_{i_1} and proceed similarly, then after n steps, we obtain k^n intervals $\Delta_{i_1\cdots i_n}$ $(i_1,\cdots,i_n=1,2,\cdots,k)$, such that

$$\Delta_{i_1\cdots i_n}\subset\Delta_{i_1\cdots i_{n-1}}. \quad (i_n=1,2,\cdots,k). \tag{1}$$

We put

$$E=\prod_{n=1}^{\infty}\left(\sum_{i_1,\cdots,i_n}^{1,2,\cdots,k}\Delta_{i_1\cdots i_n}\right). \tag{2}$$

In § 1 and § 2, we denote the length of an interval I by $|I|$ and the logarithmic capacity of a set M by $\gamma(M)$. We assume that there exists constants $a>0$, $b>0$, such that for $n=1,2,\cdots$

$$|\Delta_{i_1\cdots i_{n-1}\nu}|\geqq a|\Delta_{i_1\cdots i_{n-1}}| \quad (\nu=1,2,\cdots,k) \tag{3_1}$$

and

the mutual distance of $\Delta_{i_1\cdots i_{n-1}\mu}$ and $\Delta_{i_1\cdots i_1\cdots i_{n-1}\nu}$ $(\mu,\nu=1,2,\cdots,k,\mu\neq\nu)$ is $\geqq b|\Delta_{i_1\cdots i_{n-1}}|$. (3_2)

Then we call E a general linear Cantor set.

THEOREM 1. *Let E be a general linear Cantor set. Then*

$$m(E)=0, \qquad \gamma(E)\geqq a^{\frac{1}{k-1}}b|\Delta|>0,$$

where $m(E)$ is the linear measure and $\gamma(E)$ the logarithmic capacity of E.

At every point of E, the upper capacity density of E is positive, so that every point of E is a regular point for Dirichlet problem.

PROOF. Let

$$E=\prod_{n=1}^{\infty}\left(\sum_{i_1,\cdots,i_n}^{1,2,\cdots,k}\Delta_{i_1\cdots i_n}\right). \tag{2}$$

By (3_1), (3_2), $\Delta_{i_1\cdots i_{n-1}}-\sum_{i_n=1}^{k}\Delta_{i_1\cdots i_{n-1}i_n}$ contains an interval δ, such that $|\delta|\geqq b\,|\Delta_{i_1\cdots i_{n-1}}|$, so that

$$|\Delta_{i_1\cdots i_{n-1}}|\geqq|\delta|+\sum_{i_n=1}^{k}|\Delta_{i_1\cdots i_{n-1}i_n}|\geqq b|\Delta_{i_1\cdots i_{n-1}}|+\sum_{i_n=1}^{k}|\Delta_{i_1\cdots i_{n-1}i_n}|,$$

$$\sum_{i_n=1}^{k}|\Delta_{i_1\cdots i_{n-1}i_n}|\leqq(1-b)\,|\Delta_{i_1\cdots i_{n-1}}|.$$

From this we have

$$\sum_{i_1,\cdots,i_n}^{1,2,\cdots,k}|\Delta_{i_1\cdots i_n}|\leqq(1-b)^n\,|\Delta|\to 0(n\to\infty),$$

hence $m(E)=0$.

From (3_1), (3_2), we have

$$|\Delta_{i_1\cdots i_n}|\geqq a^n\,|\Delta|\ \ (i_1,\cdots,i_n=1,2,\cdots,k) \tag{4_1}$$

and

the mutual distance of $\Delta_{i_1\cdots i_{n-1}}\mu$ and $\Delta_{i_1\cdots i_{n-1}}\nu$ $(\mu,\nu=1,2,\cdots,k,\ \mu\neq\nu)$

$$\text{is } \geqq a^{n-1}b\,|\Delta|. \tag{4_2}$$

Let M be a bounded closed set on the x-axis and $x_i\ (i=1,2,\cdots,n)$ be n points on M, then by Fekete-Szegö's theorem[1], if we put

$$V_n(E)=\underset{x_i\in M}{\mathrm{Max}}\sqrt[\binom{n}{2}]{\prod_{i<k}^{1,2,\cdots,n}|x_i-x_k|}, \tag{5}$$

then

$$V_n(E)\to\gamma(M)\qquad(n\to\infty). \tag{6}$$

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1) M. Fekete: Über die Verteilung der Wurzeln bei gewisser algebraischen Gleichungen mit ganzzahlingen Koeffizienten. Math. Zeits. 17 (1923).

G. Szegö: Bemerkungen zu einer Arbeit von Herrn M. Fekete „Über die Verteilung der Wurzeln bei gewisser algebraischen Gleichungen mit ganzzahligen Koeffizienten". Math. Zeits. 21 (1925).

We put

$$E_n=\sum_{i_1,\cdots,i_n}^{1,2,\cdots,k} \Delta_{i_1\cdots i_n} \tag{7}$$

and in virtue of (6), take N points $x^{(\nu)}_{i_1\cdots i_n}$ $(\nu=1,2,\cdots,N)$ on each $\Delta_{i_1\cdots i_n}$, such that

$$\sqrt[\binom{N}{2}]{\prod_{\mu<\nu}^{1,2,\cdots,N} |x^{(\mu)}_{i_1\cdots i_n}-x^{(\nu)}_{i_1\cdots i_n}|}\to\gamma(\Delta_{i_1\cdots i_n})\ (N\to\infty). \tag{8}$$

Since there are k^nN points $x^{(\nu)}_{i_1\cdots i_n}$ on E_n, we have by (5),

$$[V_{k^nN}(E_n)]^{\binom{k^nN}{2}}\geqq\prod_{i_1,\cdots,i_n}^{1,2,\cdots,k}\prod_{j_1,\cdots,j_n}^{1,2,\cdots,k}\prod_{\mu,\nu}^{1,2,\cdots,N} |x^{(\mu)}_{i_1\cdots i_n}-x^{(\nu)}_{j_1\cdots j_n}|=\pi\,,\quad\text{say.} \tag{9}$$

Now π consists of $(n+1)$ factors:

$$\pi=\pi_0\,\pi_1\cdots\pi_n\,, \tag{10}$$

where π_0 is formed with pairs of points, which lie in the same $\Delta_{i_1\cdots i_n}$ and π_1 is formed with pairs of points, which lie in the same $\Delta_{i_1\cdots i_{n-1}}$ and belong to $\Delta_{i_1\cdots i_{n-1}j}$ and $\Delta_{i_1\cdots i_{n-1}j'}$ $(j\neq j')$ respectively and π_2 is formed with pairs of points, which lie in the same $\Delta_{i_1\cdots i_{n-2}}$ and belong to $\Delta_{i_1\cdots i_{n-2}j}$ and $\Delta_{i_1\cdots i_{n-2}j'}$ $(j\neq j')$ respectively and finally π_n is formed with pairs of points, which belong to Δ_j and $\Delta_{j'}$ $(j\neq j')$ respectively.
By (8),

$$\sqrt[\binom{N}{2}]{|\pi_0|}\to\prod_{i_1,\cdots,i_n}^{1,2,\cdots,k}\gamma(\Delta_{i_1\cdots i_n})\ (N\to\infty). \tag{11}$$

Since the logarithmic capacity of an interval I is $|I|/4$, we have by (4_1), $\gamma(\Delta_{i_1\cdots i_n})\geqq a^n|\Delta|/4$, so that

$$\lim_{N\to\infty}\sqrt[\binom{N}{2}]{|\pi_0|}\geqq\left(\frac{a^n|\Delta|}{4}\right)^{k^n}. \tag{12}$$

Since in π_1, $|x^{(\mu)}_{i_1\cdots i_n}-x^{(\nu)}_{j_1\cdots j_n}|\geqq a^{n-1}b|\Delta|$ by (4_2) and the number of such pairs is $\binom{k}{2}N^2k^{n-1}=\dfrac{(k-1)N^2}{2}k^n$,

$$|\pi_1| \geqq (a^{n-1} b |\varDelta|)^{\frac{(k-1)N^2}{2} k^n} \tag{13$_1$}$$

Since in π_2, $|x^{(\mu)}_{i_1\cdots i_n} - x^{(\nu)}_{j_1\cdots j_n}| \geqq a^{n-2} b |\varDelta|$ and the number of such pairs is $\binom{k}{2}(kN)^2 k^{n-2} = \frac{(k-1)N^2}{2} k^{n+1}$,

$$|\pi_2| \geqq (a^{n-2} b |\varDelta|)^{\frac{(k-1)N^2}{2} k^{n+1}} \tag{13$_2$}$$

and finally

$$|\pi_n| \geqq (b |\varDelta|)^{\frac{(k-1)N^2}{2} k^{2n-1}} \tag{13$_n$}$$

Since

$$S_n = (n-1) + (n-2)k + \cdots + (n-n)k^{n-1} = \frac{1}{k-1}\left(\frac{k^n-1}{k-1} - n\right),$$

we have

$$|\pi_1 \cdots \pi_n| \geqq a^{\frac{N^2}{2} k^n \left(\frac{k^n-1}{k-1} - n\right)} (b|\varDelta|)^{\frac{N^2}{2} k^n (k^n - 1)} \tag{14}$$

Since $\binom{k^n N}{2} \sim \frac{N^2 k^{2n}}{2}$ $(N \to \infty)$, we have from (9), (10), (12), (14),

$$\gamma(E_n)^{k^{2n}} \geqq \left(\frac{a^n |\varDelta|}{4}\right)^{k^n} a^{k^n\left(\frac{k^n-1}{k-1} - n\right)} (b|\varDelta|)^{k^n(k^n-1)}. \tag{15}$$

Since $\gamma(E_n) \to \gamma(E)$ $(n \to \infty)$, we have

$$\gamma(E) \geqq a^{\frac{1}{k-1}} b |\varDelta| > 0. \tag{16}$$

Let $x_0 \in E$, then there exists $i_1, i_2, \cdots$, such that

$$x_0 \in \varDelta_{i_1}, \qquad x_0 \in \varDelta_{i_1 i_2}, \cdots\cdots, x_0 \in \varDelta_{i_1 \cdots i_n}, \cdots.$$

Let $E_{i_1 \cdots i_n}$ be the part of E, which is contained in $\varDelta_{i_1 \cdots i_n}$, then by (16),

$$\gamma(E_{i_1 \cdots i_n}) \geqq a^{\frac{1}{k-1}} b |\varDelta_{i_1 \cdots i_n}|.$$

From this we see that the upper capacity density of E at x_0 is positive, so that x_0 is a regular point for Dirichlet problem.[2] This can be proved simply as follows. By Frostman's theorem,[3] the equiliblium potential of E,

$$u(z)=\int_E \log\frac{1}{|z-a|}\,d\mu(a)\,,\quad \int_E d\mu(a)=1 \quad (z=x+iy)$$

attains its maximum value V at x_0. From the lower semi-continuity of $u(z)$, $\lim_{z\to x_0} u(z)=V$, when z tends to x_0 from the outside of E. Hence

$$w(z)=V-u(z)>0$$

is a barrier at x_0, so that x_0 is a regular point for Dirichlet problem.

REMARK. The ordinary Cantor set E is obtained as follows. Let $\varDelta: 0\leqq x\leqq 1$ and $\varDelta_1: 0\leqq x\leqq x_1$, $\varDelta_2: y_1\leqq x\leqq 1$, $(x_1<y_1)$, such that

$$|\varDelta_1|=|\varDelta_2|=\frac{1}{2p}|\varDelta|\,,\quad y_1-x_1=\left(1-\frac{1}{p}\right)|\varDelta|\,,\quad |\varDelta|=1,\ (p>1)\,. \tag{17}$$

We perform the similar operations on $\varDelta_1$, $\varDelta_2$ and proceed similarly, then after n steps, we obtain 2^n intervals $\varDelta_{i_1\cdots i_n}$ $(i_1,\cdots,i_n=1,2)$. Then

$$E=\prod_{n=1}^{\infty}\Big(\sum_{i_1,\cdots,i_n}^{1,2}\varDelta_{i_1\cdots i_n}\Big) \tag{18}$$

is an ordinary Cantor set. In this case, in (3_1), (3_2),

$$k=2\,,\qquad a=\frac{1}{2p}\,,\qquad b=1-\frac{1}{p}\,,$$

so that $a^{\frac{1}{k-1}}b=\frac{1}{2p}-\frac{1}{2p^2}$, hence

$$\gamma(E)\geqq\frac{1}{2p}-\frac{1}{2p^2}>0\,, \tag{19}$$

which is proved by R. Nevanlinna.[4]

2) G. C. Evans: Potentials of positive mass, II. Trans. of the Amer. Math. Soc. 38 (1935).

3) O. Frostman: Poteniel d'eqilibre et capacité des ensembles. Lund. (1935). Frostman proved the case of Newtonian potential, the case of logarithmic potential can be proved by the same method.

4) R. Nevanlinna: Eindeutige analytische Funktionen. Berlin (1936). p. 148.

2 General planar Cantor sets.

1. Let $\varDelta$ be a circular disc of radius R. We take $k\,(\geqq 2)$ disjoint circular discs $\varDelta_{i_1}\,(i_1=1,2,\cdots,k)$ of radius R_{i_1} in $\varDelta$ and proceed similarly, then after n steps, we obtain k^n circular discs $\varDelta_{i_1\cdots i_n}\,(i_1,\cdots,i_n=1,2,\cdots,k)$ of radius $R_{i_1\cdots i_n}$, such that

$$\varDelta_{i_1\cdots i_{n-1}i_n}\subset\varDelta_{i_1\cdots i_{n-1}}\quad(i_n=1,2,\cdots,k)\,. \tag{1}$$

We put

$$E=\prod_{n=1}^{\infty}\Big(\sum_{i_1,\cdots,i_n}^{1,2,\cdots,k}\varDelta_{i_1\cdots i_n}\Big). \tag{2}$$

We assume that there exists constants $a>0$, $b>0$, such that for $n=1,2,\cdots$,

$$R_{i_1\cdots i_{n-1}\nu}\geqq a\,R_{i_1\cdots i_{n-1}}\quad(\nu=1,2,\cdots,k) \tag{3_1}$$

and

the mutual distance of $\varDelta_{i_1\cdots i_{n-1}\mu}$ and $\varDelta_{i_1\cdots i_{n-1}\nu}$ $(\mu,\nu=1,2,\cdots,k,\ \mu\neq\nu)$

$$\text{is}\ \geqq b\,R_{i_1\cdots i_{n-1}}\,. \tag{3_2}$$

Then we call E a general planar Cantor set.

THEOREM 2. *Let E be a general planar Cantor set. Then*

$$m(E)=0\,,\qquad \gamma(E)\geqq a^{\frac{1}{k-1}}\,b\,R>0\,,$$

where $m(E)$ is the plane measure and $\gamma(E)$ the logarithmic capacity of E.

At every point of E, the upper capacity density of E is positive, so that every point of E is a regular point for Dirichlet problem.

PROOF. We have only to prove that $m(E)=0$, for the other part can be proved similarly as Theorem 1. Let C_i be the bounding circle of $\varDelta_i$ and O_i be its center and R_i be its radius. Let O be the center of $\varDelta$. We draw a circle C_i' of radius $R_i+\dfrac{bR}{2}$ about O_i, then $C_i'\,(i=1,2,\cdots,k)$ are disjoint. Hence if we denote the points of intersection of the segment $O_i\,O$ with C_i and C_i' by A_i, A_i' respectively, then a circle Γ_i, with the diameter $A_i A_i'$ lies in Δ and Γ_i are disjoint each other. Let D_i be the inside of Γ_i and $|D_i|$ be its area, then

$$|D_i|=\pi\left(\frac{bR}{4}\right)^2=\left(\frac{b}{4}\right)^2\pi R^2=\left(\frac{b}{4}\right)^2|\Delta|.$$

From this, we have $\sum_{i=1}^{k}|\Delta_i|\leqq\alpha|\Delta|$, where $0<\alpha<1$ is a constant, which depends on b only. Hence

$$m(E)\leqq\sum_{i_1,\cdots,i_n}^{1,2,\cdots,k}|\Delta_{i_1\cdots i_n}|\leqq\alpha^n|\Delta|\rightarrow 0 \quad (n\rightarrow\infty),$$

so that $m(E)=0$.

2. To generalize Theorem 2, we shall use the following lemma.

LEMMA 1.[5] *Let Γ be an analytic Jordan curve on a plane, then*

$$\gamma(\Gamma)\geqq\frac{d(\Gamma)}{4},$$

where $\gamma(\Gamma)$ is the logarithmic capacity and $d(\Gamma)$ the diameter of Γ.

PROOF. Let $P\in\Gamma$, $Q\in\Gamma$, such that $\overline{PQ}=d(\Gamma)$. We take the line PQ as the x-axis on the xy-plane and let $P=(a,0)$, $Q=(b,0)$ $(a<b)$. We take points $a\leqq x_1<x_2<\cdots<x_n\leqq b$ and let P_ν be a point on Γ, whose projection on the x-axis is x_ν, then $\overline{P_\mu P_\nu}\geqq|x_\mu-x_\nu|$. From this we have easily $\gamma(\Gamma)\geqq\gamma(PQ)=\dfrac{\overline{PQ}}{4}=\dfrac{d(\Gamma)}{4}$. q. e. d.

In the following, a Jordan domain is a domain, which is bounded by an analytic Jordan curve and $d(M)$ is the diameter of a set M. Let Δ be a Jordan domain. We take $k\,(\geqq 2)$ disjoint Jordan domains $\Delta_{i_1}\,(i_1=1,2,\cdots,k)$ in Δ and proceed similarly, then after n steps, we obtain k^n Jordan domains $\Delta_{i_1\cdots i_n}\,(i_1,\cdots,i_n=1,2,\cdots,k)$ and put

$$E=\prod_{n=1}^{\infty}\left(\sum_{i_1,\cdots,i_n}^{1,2,\cdots,k}\Delta_{i_1\cdots i_n}\right). \tag{1}$$

We assume that

$$d(\Delta_{i_1\cdots i_{n-1}\nu})\geqq a\,d(\Delta_{i_1\cdots i_{n-1}})\;(\nu=1,2,\cdots,k) \tag{2_1}$$

5) G. Pôlya und G. Szegö: Aufgaben und Lehrsätze aus der Analysis, II. Berlin (1925). p. 25.

and the mutual distance of $\Delta_{i_1\cdots i_{n-1}\mu}$ and $\Delta_{i_1\cdots i_{n-1}\nu}$ $(\mu, \nu=1, 2, \cdots, k, \mu\neq\nu)$ is $\geqq b\ d(\Delta_{i_1\cdots i_{n-1}})$, $\qquad (2_2)$

where $a>0$, $b>0$ are constants. Then by means of Lemma 1, we can prove similarly as Theorem 1, the following generalization of Theorem 2.

THEOREM 3. $\gamma(E)\geqq a^{\frac{1}{k-1}}\ b\ d(\Delta)>0$. *At every point of E, the upper capacity density of E is positive, so that every point of E is a regular point for Dirichlet problem.*

3. We shall prove a lemma.

LEMMA 2. *Let Δ be a ring domain on the z-plane, which is bounded by two analytic Jordan curves C_1, C_2, where C_1 lies inside of C_2. Let C be an analytic Jordan curve, which lies between C_1 and C_2 and contains C_1 in its inside. Let $w=f(z)$ be regular and schlicht in Δ and Γ be the image of C on the w-plane, L be its length and D be its diameter. Then*

$$L\leqq\kappa D,$$

where $\kappa>0$ is a constant, which depends on C and Δ only.

PROOF. Let $\overline{\Delta_1}\subset\Delta$ be a closed ring domain, which contains C and z_0 be a point of $\overline{\Delta_1}$. Then by Koebe's distortion theorem,

$$A|f'(z_0)|\leqq|f'(z)|\leqq B|f'(z_0)|, \qquad z\in\overline{\Delta_1}, \tag{1}$$

where $A>0$, $B>0$ are constants. Hence

$$L=\int_C |f'(z)|\ |dz|\leqq\text{const. } |f'(z_0)|. \tag{2}$$

Let C' be an analytic Jordan curve in $\overline{\Delta_1}$, which is contained inside of C and contains C_1 in its inside and Γ' be its image on the w-plane. We take two points $w\in\Gamma$, $w'\in\Gamma'$, such that $|w-w'|$ is equal to the shortest distance of Γ and Γ'. Then the segment $w\,w'$ is contained in a ring domain, which is bounded by Γ and Γ'. Hence its image γ on the z-plane is contained in $\overline{\Delta_1}$. Hence by (1),

$$D\geqq|w-w'|=\int_\gamma |f'(z)|\ |dz|\geqq\text{const. } |f'(z_0)|. \tag{3}$$

From (2), (3), we have

$L \leqq \text{const.}\, D$.   q. e. d.

4. Let Δ be a domain on the z-plane, which contains $z=\infty$ and G be a group of Schottky type, whose elements are schlicht meromorphic functions $f(z)$ in Δ, which transform Δ into itself. We assume that the fundamental domain Δ_0 of G is bounded by $p(2\leqq p\leqq\infty)$ pairs of disjoint equivalent analytic Jordan curves C_i, C_i' $(i=1,2,\cdots,p)$ and a bounded closed set M_0.
Let Δ_ν be equivalents of Δ_0 by G, then Δ_ν cluster to a non-dense perfect set E, which is called the singular set of G.

If we denote the boundary of Δ by Γ, then

$$\Gamma=E+\sum_{\nu=0}^{\infty}M_\nu\,,$$

where M_ν are equivalents of M_0.
We shall prove the following theorem, which is a precise form of Myrberg's theorem[6], who proved that $\gamma(\Gamma)>0$.

THEOREM 4. (i) $\gamma(E)>0$.

(ii) *If $2\leqq p<\infty$, then at every point of E, the upper capacity density of E is positive, so that every point of E is a regular point for Dirichlet problem.*

PROOF. First we suppose that $2\leqq p<\infty$. Then the totality of equivalents of C_i, C_i' $(i=1,2,\cdots,p)$, which lie inside of C_1 can be written in the form: $C_{1\,i_1\cdots i_n}$ $(i_1,\cdots,i_n=1,2,\cdots,q,\ q=2p-1)$, such that

$$D_{1\,i_1\cdots i_{n-1}i_n}\subset D_{1\,i_1\cdots i_{n-1}}\ (i_n=1,2,\cdots,q)\,,$$

where $D_{1\,i_1\cdots i_n}$ is the inside of $C_{1\,i_1\cdots i_n}$.
We denote the part of E, which lies in C_1 by E_1.

Let C be an analytic Jordan curve in Δ_0, which contains C_i, C_i' $(i=1,2,\cdots,p)$ in its inside and M_0 lies outside of C and Δ_0' be the domain, which is bounded by C_i, C_i' $(i=1,2,\cdots,p)$ and C. Let $z_0\in\Delta_0'$, then by Koebe's distortion theorem, for any $f(z)\in G$,

$$A\,|f'(z_0)|\leqq|f'(z)|\leqq B\,|f'(z_0)|\,,\qquad z\in\Delta_0'\,, \tag{1}$$

where $A>0$, $B>0$ are constants. Hence if we denote the length of $C_{1\,i_1\cdots i_n}$ by $L_{1\,i_1\cdots i_n}$, then

6) P. J. Myrberg: Die Kapazität der singulären Menge der linearen Gruppe. Ann. Acad. Fenn. Ser. A. 10 (1941).

$$\text{const.}\ |f'(z_0)| \leqq L_{1i_1\cdots i_n} \leqq \text{const.}\ |f'(z_0)|. \tag{2}$$

Hence by Lemma 2,

$$d(D_{1\,i_1\cdots i_{n-1}\nu}) \geqq \text{const.}\ L_{1\,i_1\cdots i_{n-1}\nu} \geqq \text{const.}\ |f'(z_0)|$$
$$\geqq \text{const.}\ L_{1\,i_1\cdots i_{n-1}} \geqq \text{const.}\ d(D_{1\,i_1\cdots i_{n-1}})\ (\nu=1,2,\cdots,q). \tag{3}$$

Let d be the shortest distance of $D_{1\,i_1\cdots i_{n-1}\mu}$ and $D_{1\,i_1\cdots i_{n-1}\nu}$, then there exists $z_\mu \in C_{1i_1\cdots i_{n-1}\mu}$, $z_\nu \in C_{1i_1\cdots i_{n-1}\nu}$, such that

$$|z_\mu - z_\nu| = d. \tag{4}$$

The segment $z_\mu z_\nu$ meets the boundary of $\Delta'_{1\,i_1\cdots i_{n-1}}$ in general, where $\Delta'_{1\,i_1\cdots i_{n-1}}$ is an equivalent of Δ'_0, whose outermost boundary is $C_{1\,i_1\cdots i_{n-1}}$. Let z' be a point on the segment $z_\mu z_\nu$, which lies on the boundary of $\Delta'_{1i_1\cdots i_{n-1}}$, such that the segment $z_\mu z'$ lies in $\Delta'_{1\,i_1\cdots i_{n-1}}$ and γ be its image in Δ'_0, where if the segment $z_\mu z_\nu$ does not meet the boundary of $\Delta'_{1i_1\cdots i_{n-1}}$, then we take $z'=z_\nu$. Then

$$d \geqq |z_\mu - z'| = \int_\gamma |f'(z)|\,|dz| \geqq \text{const.}\ |f'(z_0)|$$
$$\geqq \text{const.}\ L_{1\,i_1\cdots i_{n-1}} \geqq \text{const.}\ d(D_{1\,i_1\cdots i_{n-1}}). \tag{5}$$

By (3), (5), the condition of Theorem 3 is satisfied, so that $\gamma(E_1)>0$ and at every point of E_1, the upper capacity density of E_1 is positive, so that every point of E_1 is a regular point for Dirichlet problem. A similar relation holds for the part of E, which is contained in C_i, C'_i $(i=1,2,\cdots,p)$. Hence (ii) is proved. (i) (where $p=\infty$) can be deduced from the case $2\leqq p<\infty$. Hence our theorem is proved.

5. We shall prove a lemma.

LEMMA 3. *Let x_1, x_2, x_3, x_4 be four points on the x-axis and by a linear transformation, x_i be transformed into ξ_i $(i=1,2,3,4)$ on the x-axis, such that $\xi_1<\xi_2<\xi_3<\xi_4$ and put*

$$\delta=\xi_2-\xi_1, \qquad \Delta=\xi_3-\xi_2, \qquad \delta'=\xi_4-\xi_3.$$

Then

$$\Delta \leqq \kappa\delta, \qquad \Delta \leqq \kappa\delta',$$

where $\kappa>0$ is a constant, which depends on x_1, x_2, x_3, x_4 only.

PROOF. Let

$$\frac{x_2-x_1}{x_4-x_1} : \frac{x_2-x_3}{x_4-x_3} = -\frac{1}{\kappa}.$$

Then since the anharmonic ratio is invariant by a linear transformation,

$$\frac{\xi_2-\xi_1}{\xi_4-\xi_1} : \frac{\xi_2-\xi_3}{\xi_4-\xi_3} = -\frac{1}{\kappa},$$

or

$$\frac{\delta}{\varDelta+\delta+\delta'} = \frac{1}{\kappa} \cdot \frac{\varDelta}{\delta'},$$

$$\kappa\delta\delta' = \varDelta(\varDelta+\delta+\delta') \geqq \delta'\varDelta,$$

hence

$$\varDelta \leqq \kappa\delta.$$

Similarly

$$\varDelta \leqq \kappa\delta'.$$

6. Let F be a Riemann surface spread over the w-plane, whose genus p is $2 \leqq p \leqq \infty$. Let $F^{(\infty)}$ be an unramified covering surface of F, which is of planar character. By $w=F(z)$, we map $F^{(\infty)}$ on a schlicht domain $\varDelta$ on the z-plane conformally. Then $F(z)$ is automorphic with respect to a group G of Schottky type, whose elements are meromorphic schlicht functions $f(z)$ in $\varDelta$, which transform $\varDelta$ into itself. Let $\varDelta_0$ be the fundamental domain of G, then $\varDelta_0$ is bounded by p pairs of disjoint analytic Jordan curves C_i, C_i' $(i=1, 2, \cdots, p)$ and a bounded closed set M_0, where C_i, C_i' are equivalent by G. Let Γ be the boundary of $\varDelta$ and E be the singular set of G. By $z=\varphi(\zeta)$, we map the universal covering surface of $\varDelta$ on $|\zeta|<1$ conformally. Then $\varphi(\zeta)$ is automorphic with respect to a Fuchsian group $\mathfrak{G}$ in $|\zeta|<1$. Let D_0 be its fundamental domain and e_0 be the image of E on $|\zeta|=1$, with lies on the boundary of D_0. Then

THEOREM 5. (*i*) $\gamma(e_0)>0$, *where* $\gamma(e_0)$ *is the logarithmic capacity of* e_0.

(ii) *If* $2 \leqq p < \infty$, *then* $m(e_0)=0$ *and at every point of* e_0, *the upper capacity density of* e_0 *is positive, so that every point of* e_0 *is a regular point for Dirichlet problem, where* $m(e_0)$ *is the linear measure of* e_0.

PROOF. By a linear transformation, we map $|\zeta|<1$ on the upper half $\Im x>0$ of the x-plane. Then $\mathfrak{G}=\mathfrak{G}_\zeta$ and $G=G_z$ correspond to linear groups $\mathfrak{G}_x$ and G_x in $\Im x>0$ respectively. Let $D_0^{(x)}$ be the image of $D_0=D_0^{(\zeta)}$ in $\Im x>0$, then $D_0^{(x)}$ is the fundamental domain of $\mathfrak{G}_x$ and $e_0^{(x)}$ be the image of $e_0=e_0^{(\zeta)}$ on the boundary of $D_0^{(x)}$. Let $\Delta_0^{(x)}$ be the image of Δ_0 on the x-plane, then $\Delta_0^{(x)}$ is the fundamental domain of G_x.
We may assume that $\Delta_0^{(x)}\subset D_0^{(x)}$. $\Delta_0^{(x)}$ can be constructed as follows. We take $z_i\,(i=1,2,\cdots,p)$ in Δ_0, where if $p=\infty$, then we choose z_i, such that the cluster points of $\{z_i\}$ lie on M_0. Let a_i, a_i' be equivalent points on C_i, C_i' respectively. We connect $a_i\,(a_i')$ to z_i by a Jordan arc $\gamma_i\,(\gamma_i')$ in Δ_0 and connect z_i, z_{i+1} by a Jordan arc λ_i in Δ_0, such that γ_i, γ_i', λ_i $(i=1,2,\cdots,p)$ have no common points, except the end points. We take off $\sum_{i=1}^{p}(\gamma_i+\gamma_i'+\lambda_i)$ and at most a countable number of suitable cross cuts, whose end points lie on M_0 from Δ_0, then we obtain a simply connected domain Δ_0'. Let $\Delta_0^{(x)}$ be one of the images of Δ_0' in $\Im x>0$, then $\Delta_0^{(x)}$ is the fundamental domain of G_x.
First we assume that $2\leqq p<\infty$. The totality of equivalents of C_i, C_i' $(i=1,2,\cdots,p)$ by G, which lie inside of C_1 can be written in the form $C_{1\,i_1\cdots i_n}\,(i_1,\cdots,i_n=1,2,\cdots,q,\,q=2p-1)$, such that

$$\delta_{1\,i_1\cdots i_{n-1}i_n}\subset\delta_{1\,i_1\cdots i_{n-1}}\quad(i_n=1,2,\cdots,q)\,, \tag{1}$$

where $\delta_{1\,i_1\cdots i_n}$ is the inside of $C_{1\,i_1\cdots i_n}$.

Let $C_{1\,i_1\cdots i_n}^{(x)}$ be one of the images of $C_{1 i_1\cdots i_n}$ on the x-plane, which has common points with $D_0^{(x)}$, then $C_{1\,i_1\cdots i_n}^{(x)}$ is a Jordan arc, whose two end points lie on the x-axis. Let $E_{1\,i_1\cdots i_n}$ be the segment, which is bounded by these two end points. We may assume the $\delta_{1\,i_1\cdots i_n}$ is mapped on a finite domain on the x-plane, then

$$E_{1\,i_1\cdots i_{n-1}i_n}\subset E_{1\,i_1\cdots i_{n-1}}\;(i_n=1,2,\cdots,q)\,. \tag{2}$$

If we put

$$E^{(1)}=\prod_{n=1}^{\infty}\Big(\sum_{i_1,\cdots,i_n}^{1,2,\cdots,q}E_{1\,i_1\cdots i_n}\Big)\,, \tag{3}$$

then $E^{(1)}$ is a sub-set of $e_0^{(x)}$.
We shall prove that

$$|E_{1i_1\cdots i_{n-1}\nu}| \geqq a|E_{1i_1\cdots i_{n-1}}| \quad (\nu=1,2,\cdots,q) \tag{4_1}$$

and

the mutual distance of $E_{1i_1\cdots i_{n-1}\mu}$ and $E_{1i_1\cdots i_{n-1}\nu}$ $(\mu, \nu=1,2,\cdots,q,$ $\mu\neq\nu)$ is $\geqq b|E_{1i_1\cdots i_{n-1}}|$, (4₂)

where $a>0$, $b>0$ are constants.
We shall prove (4_1), (4_2) for the case $i_{n-1}=1$, the other case can be proved similarly.

Let

$$E_{1i_1\cdots i_{n-2}}=[x_1, y_1],$$

$$E_{1i_1\cdots i_{n-2}1}=[x_{11}, y_{11}], \qquad E_{1i_1\cdots i_{n-2}2}=[x_{12}, y_{12}],$$

$$E_{1i_1\cdots i_{n-2}11}=[x_{111}, y_{111}],\cdots\cdots, \quad E_{1i_1\cdots i_{n-2}1q}=[x_{11q}, y_{11q}],$$

where we may asume that

$$x_1<\overbrace{x_{11}<x_{111}<y_{111}<x_{112}<y_{112}<\cdots<x_{11q}<y_{11q}<y_{11}}^{M}<x_{12}<y_{12}<y_1.$$
$$\lfloor\delta_0\rfloor \quad \lfloor\delta_1\rfloor \quad \lfloor M_1\rfloor \quad \lfloor\delta_2\rfloor \quad \lfloor M_2\rfloor \qquad \lfloor M_q\rfloor \quad \lfloor\delta_{q+1}\rfloor \quad \lfloor\delta_0'\rfloor$$

We put

$$M=[x_{11}, y_{11}], \qquad \delta_0=[x_1, x_{11}], \qquad \delta_1=[x_{11}, x_{111}],$$
$$M_1==[x_{111}, y_{111}], \qquad \delta_2=[y_{111}, x_{112}],$$
$$M_2=[x_{112}, y_{112}],\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \quad \delta_q=[y_{11q-1}, x_{11q}],$$
$$M_q=[x_{11q}, y_{11q}], \qquad \delta_{q+1}=[y_{11q}, y_{11}], \qquad \delta_0'=[y_{11}, x_{12}].$$

Now $C_{1i_1\cdots i_{n-2}}^{(x)}$ is obtained from one of $C_i^{(x)}$, $C_i'^{(x)}$ $(i=1,2,\cdots,p)$ by a transformation $S\in G_x$, where $C_i^{(x)}$ $(C_i'^{(x)})$ is the image of C_i (C_i') on the x-plane, which has common points with $D_0^{(x)}$.

Let

$$\xi_1=S^{-1}(x_1), \qquad \xi_2=S^{-1}(x_{11}), \qquad \xi_3=S^{-1}(x_{111}), \qquad \xi_4=S^{-1}(y_{111}),$$

then ξ_1, ξ_2, ξ_3, ξ_4 belong to a certain set A, which consists of a finite number of points and is independent of S.

Hence by Lemma 3,

$$|\delta_1| \leqq \kappa |\delta_0|, \qquad |\delta_1| \leqq \kappa |M_1|,$$

where $\kappa > 0$ is a constant independent of S. Similarly we have

$$|M_1| \leqq \kappa |\delta_1|, \qquad |M_1| \leqq \kappa |\delta_2|,$$

$$|\delta_2| \leqq \kappa |M_1|, \qquad |\delta_2| \leqq \kappa |M_2|,$$

$$\cdots\cdots\cdots\cdots$$

$$|M_q| \leqq \kappa |\delta_q|, \qquad |M_q| \leqq \kappa |\delta_{q+1}|,$$

$$|\delta_{q+1}| \leqq \kappa |M_q|, \qquad |\delta_{q+1}| \leqq \kappa |\delta_0'|.$$

It follows that

$$|M_\nu| \geqq a|M| \quad (\nu = 1, 2, \cdots, q),$$

$$|\delta_j| \geqq b|M| \quad (j = 1, 2, \cdots, q+1),$$

where $a > 0$, $b > 0$ are constants independent of S.

Hence

$$|E_{1i_1\cdots i_{n-2}1\nu}| \geqq a|E_{1i_1\cdots i_{n-2}1}| \quad (\nu = 1, 2, \cdots, q)$$

and the mutual distance of $E_{1i_1\cdots i_{n-2}1\mu}$ and $E_{1i_1\cdots i_{n-2}1\nu}$ is $\geqq b|E_{1i_1\cdots i_{n-2}1}|$.

Hence (4_1), (4_2) are proved, so that by Theorem 1, $m(E^{(1')}) = 0$ and $\gamma(E^{(1')}) > 0$ and at every point of $E^{(1)}$, the upper capacity density of $E^{(1)}$ is positive, so that every point of $E^{(1)}$ is a regular point for Dirichlet problem. From this we see that every point of $e_0^{(x)}$ has the same property and hence its image e_0 on $|\zeta| = 1$ has the same property. Hence (ii) is proved. (i) can be deduced from (ii).

3. General spatial Cantor sets.

Let Δ be a spherical domain, which is bounded by a sphere of radius R. We take $k (\geqq 2)$ disjoint spherical domains Δ_{i_1} $(i_1 = 1, 2, \cdots, k)$ of radius R_{i_1} in Δ and proceed similarly, then after n steps, we obtain k^n spherical domains $\Delta_{i_1\cdots i_n}$ $(i_1, \cdots, i_n = 1, 2, \cdots, k)$ of radius $R_{i_1\cdots i_n}$, such that

$$\Delta_{i_1\cdots i_{n-1}i_n} \subset \Delta_{i_1\cdots i_{n-1}} \quad (i_n = 1, 2, \cdots, k). \tag{1}$$

We put

$$E=\prod_{n=1}^{\infty}\Big(\sum_{i_1,\cdots,i_n}^{1,2,\cdots,k}\Delta_{i_1\cdots i_n}\Big). \tag{2}$$

We assume that there exists constants $a>0$, $b>0$, such that for $n=1,2,\cdots$

$$R_{i_1\cdots i_{n-1}\nu}\geqq aR_{i_1\cdots i_{n-1}}\ (\nu=1,2,\cdots,k) \tag{3_1}$$

and

the mutual distance of $\Delta_{i_1\cdots i_{n-1}\mu}$ and $\Delta_{i_1\cdots i_{n-1}\nu}$ $(\mu,\nu=1,2,\cdots k,\mu\neq\nu)$

$$\text{is } \geqq b\,R_{i_1\cdots i_{n-1}}. \tag{3_2}$$

Then we call E a general spatial Cantor set.

THEOREM 6. *Let E be a general spatial Cantor set. Then*

$$m(E)=0,$$

where $m(E)$ is the spacial measure of E.

(ii) *If $ak>1$, then*

$$\gamma(E)\geqq\frac{b(ak-1)R}{a(k-1)}>0,$$

where $\gamma(E)$ is the Newtonian capacity of E and at every point of E, the upper capacity density of E is positive, so that every point of E is a regular point for Dirichlet problem.

PROOF. Since the first part can be proved similarly as Theorem 2, we shall prove the second part.

By (3_1), (3_2),

$$R_{i_1\cdots i_n}\geqq a^n R \tag{4_1}$$

and

$$\text{the mutual distance of } \Delta_{i_1\cdots i_{n-1}\mu} \text{ and } \Delta_{i_1\cdots i_{n-1}\nu} \text{ is } \geqq a^{n-1}bR \tag{4_2}$$

Let M be a bounded closed set in space, we denote its Newtonian capacity by $\gamma(M)$. We take n points p_ν $(\nu=1,2,\cdots,n)$ on M and put

$$V_n(M)=\operatorname*{Max}_{p_\nu\in M}\binom{n}{2}\Big/\sum_{\mu<\nu}^{1,2,\cdots,n}\frac{1}{p_\mu p_\nu}, \tag{5}$$

then by Pólya-Szegö's theorem[7],

$$V_n(M)\to\gamma(M)\quad(n\to\infty). \tag{6}$$

We put

$$E_n=\sum_{i_1,\cdots,i_n}^{1,2,\cdots,k}\Delta_{i_1\cdots i_n} \tag{7}$$

and take N point $p^{(\nu)}_{i_1\cdots i_n}$ $(\nu=1,2,\cdots,N)$ on each $\Delta_{i_1\cdots i_n}$, such that by (6),

$$\binom{N}{2}\Big/\sum_{\mu<\nu}^{1,2,\cdots,N}\frac{1}{\overline{p^{(\mu)}_{i_1\cdots i_n}p^{(\nu)}_{i_1\cdots i_n}}}\to\gamma(\Delta_{i_1\cdots i_n})\,(N\to\infty). \tag{8}$$

Since there are k^nN points $p^{(\nu)}_{i_1\cdots i_n}$ on E_n, we have

$$V_{k^nN}(E_n)\geqq\binom{k^nN}{2}\Big/\sum_{i_1,\cdots,i_n}^{1,2,\cdots,k}\sum_{j_1,\cdots,j_n}^{1,2,\cdots,k}\sum_{\mu,\nu}^{1,2,\cdots,N}\frac{1}{\overline{p^{(\mu)}_{i_1\cdots i_n}p^{(\nu)}_{j_1\cdots j_n}}}$$

$$=\binom{k^nN}{2}\Big/\sum,\ \text{say}. \tag{9}$$

Now $\sum$ consists of $(n+1)$ parts:

$$\sum=\sum_0+\sum_1+\cdots+\sum_n, \tag{10}$$

where $\sum_0$ is formed with pairs of points, which belong to the same $\Delta_{i_1\cdots i_n}$ and $\sum_1$ is formed with pairs of points, which lie in the same $\Delta_{i_1\cdots i_{n-1}}$ and belong to $\Delta_{i_1\cdots i_{n-1}j}$ and $\Delta_{i_1\cdots i_{n-1}j'}$ $(j\neq j')$ respectively and finally, $\sum_n$ is formed with pairs of points, which belong to Δ_j and $\Delta_{j'}$ $(j\neq j')$ respectively.

Since the Newtonian capacity of a sphere of radius r is r, we have by (4_1), (8),

$$\sum_0\leqq(1+\varepsilon_N)\binom{N}{2}\sum_{i_1,\cdots,i_n}^{1,2,\cdots,k}\frac{1}{\gamma(\Delta_{i_1\cdots i_n})}\leqq\frac{(1+\varepsilon_N)N^2k^n}{2a^nR}, \tag{11}$$

where $\varepsilon_N\to0$ $(N\to\infty)$ and by (4_2), similarly as the proof of Theorem 1, we have

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7) G. Pólya uns G. Szegö: Über die transfiniten Durchmesser (Kapazitätskonstante) von ebenen und raümlichen Mengen. Jour. f. reine u. angewandte Math. 165 (1931).

$$\sum_1 \leqq \binom{k}{2} \frac{N^2 k^{n-1}}{a^{n-1} b R} = \frac{(k-1) N^2 k^n}{2 a^{n-1} b R},$$

$$\sum_2 \leqq \frac{(k-1) N^2 k^{n-1}}{2 a^{n-2} b R},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\sum_n \leqq \frac{(k-1) N^2 k^{2n-1}}{2 b R},$$

hence

$$\sum_1 + \cdots + \sum_n \leqq \frac{(k-1) N^2 k^n ((ak)^n - 1)}{2 a^{n-1} b\ R(ak-1)}, \tag{12}$$

so that by (9), (10), (11), (12),

$$V_{k^n N}(E_n) \geqq \binom{k^n N}{2} \left[\frac{(1+\varepsilon_N) N^2 k^n}{2 a^n R} + \frac{(k-1) N^2 k^n ((ak)^n - 1)}{2 a^{n-1} b\ R(ak-1)} \right].$$

If we make $N \to \infty$,

$$\gamma(E_n) \geqq \frac{b(ak)^n (ak-1) R}{b(ak-1) + a(k-1)((ak)^n - 1)}$$

and $n \to \infty$, then we have

$$\gamma(E) \geqq \frac{b(ak-1) R}{a(k-1)} > 0. \tag{13}$$

The other part of the theorem can be proved similarly as Theorem 1.

REMARK. Suppose that for $n = 1, 2, \cdots$

$$R_{i_1 \cdots i_{n-1} i_n} = a R_{i_1 \cdots i_{n-1}} \ (i_n = 1, 2, \cdots, k), \tag{14}$$

where $a > 0$ is a constant, such that $ak < 1$. Then $R_{i_1 \cdots i_n} = a^n R$.

Since for n bounded closed sets $M_1, \cdots, M_n$,

$$\gamma(M_1 + \cdots + M_n) \leqq \sum_{\nu=1}^{n} \gamma(M_\nu),$$

we have

$$\gamma(E_n) \leqq \sum_{i_1,\cdots,i_n}^{1,2,\cdots,k} \gamma(\Delta_{i_1\cdots i_n}) \leqq (ak)^n\ R \to 0\ (n \to \infty),$$

so that $\gamma(E)=0$.

Hence if in Theorem 6, $\gamma(E)$ may be zero, if $ak<1$.

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