Journal of the Mathematical Society of Japan Vol. 5, No. 2, July, 1953.

Integration of the equation of evolution in a Banach space.

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(Received May 22, 1953)

§ 1. Introduction. Theorems.

In what follows we consider the integration of the equation of evolution[1)]

$$dx(t)/dt=A(t)x(t)+f(t), \qquad a\leqq t\leqq b, \tag{1.1}$$

and the associated homogeneous equation

$$dx(t)/dt=A(t)x(t). \tag{1.1'}$$

Here the unknown $x(t)$ is an element of a complex Banach space $\mathfrak{B}$ depending on a real variable t, while $f(t)$ is a given element of $\mathfrak{B}$ and $A(t)$ is a given, in general unbounded, linear operator in $\mathfrak{B}$, both depending on t

The solution of (1.1) is formally given by

$$x(t)=U(t,a)x+\int_a^t U(t,s)f(s)\,ds, \qquad x=x(a), \tag{1.2}$$

where $U(t,s)$ is a linear operator in $\mathfrak{B}$ depending on s,t with $s\leqq t$. The main purpose of the present paper is to give some sufficient conditions for the existence of $U(t,s)$ and to study its properties.

If $A(t)=A$ is independent of t, $U(t,s)$ is given formally by $U(t,s)=\exp[(t-s)A]$, and the rigorous definition of the exponential function has been given by Hille and Yosida in connection with the analytical theory of semi-groups.[2)] As we are going to generalize some of their results to the case in which $A(t)$ actually depends on t, it is natural to take over their assumptions on the infinitesimal generator A for our

1) The terminology after Schwarz [2].

2) Hille [1], Chap. XII, in particular Theorem 12.2.1; Yosida [3], Theorem 2.

$A(t)$ for each t. Thus it is convenient to introduce the following definition.

DEFINITION 1. An operator A in $\mathfrak{B}$ will be said to have property S if the following conditions are fulfilled: 1) A is a closed linear operator with domain dense in $\mathfrak{B}$; 2) the resolvent set of A includes all positive reals and

$$(1.3) \qquad \|(I-\alpha A)^{-1}\| \leq 1 \qquad \text{for} \qquad \alpha > 0 .$$ [3]

Then our first assumption can be expressed as

C_1. *$A(t)$ is defined for $a \leq t \leq b$ and has property* S *for each t.*

As regards the continuity property of $A(t)$, we introduce the following conditions successively.

C_2. 1) *The domain $\mathfrak{D}$ of $A(t)$ is independent of t* (then it follows from Lemma 2 below that $B(t,s)=[I-A(t)][I-A(s)]^{-1}$ [4] is a bounded operator[5] for each s, t). 2) *$B(t,s)$ is uniformly bounded, that is, there is a $M>0$ such that $\|B(t,s)\| \leq M$ for every s, t* (this is the case if $B(t,s)$ is continuous in t in the sense of the norm $\| \ \|$ at least for some s). 3) *$B(t,s)$ is of bounded variation in t in the sense that there is a $N \geq 0$ such that*

$$\sum_{j=1}^{n} \|B(t_j, s)-B(t_{j-1}, s)\| \leq N$$

for every partition $a=t_0<t_1<\cdots<t_n=b$ of the interval (a, b), at least for some s (then it follows that the same is true for every s and that N may be taken as independent of t and s).

C_3. *$B(t,s)$ is weakly continuous in t at least for some s* (then it follows that $B(t,s)$ is even continuous in t in the sense of the norm $\| \ \|$ for every s).

C_4. $B(t,s)$ is weakly differentiable in t and $\partial B(t,s)/\partial t$ is strongly continuous in t, at least for some s (then it follows that the same is true for every s).

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3) Cf. note 2). It should be noted that (1.3) implies actually that the half-plane $\mathrm{Re}\,\lambda>0$ belongs to the resolvent set of A.

4) More generally, we may take $B(t,s)=[\lambda I-A(t)][\lambda I-A(s)]^{-1}$ with an arbitrary constant $\lambda>0$, without any essential modification.

5) By a bounded operator we mean a bounded linear operator with domain $\mathfrak{B}$. Also all operators are assumed to have domain and range in $\mathfrak{B}$, unless the contrary is expressly stated.

Our main results are summarized in the following theorems.

THEOREM 1. *Let the condition* C_1 *be satisfied. Let* $x(t)$ *be defined for* $a \leqq t \leqq b$ *and satisfy the following conditions:* 1) $x(t)$ *is strongly continuous for* $a \leqq t \leqq b$; 2) $x(t)$ *has strong right-derivative* $D^+ x(t)$, $x(t) \in \mathfrak{D}[A(t)]$[6], *and* $D^+ x(t) = A(t) x(t)$ *for* $a \leqq t < b$. *Then* $\|x(t)\|$ *is a non-increasing function for* $a \leqq t \leqq b$.

COROLLARY. *Under the assumption of Theorem* 1, *the solution* $x(t)$ *of* (1.1) *is uniquely determined by the initial value* $x(a)$.

THEOREM 2. *Let the conditions* C_1, C_2 *be satisfied. Then the operators* $A(t \pm 0)$ *are defined in a sense described below, have the same domain* $\mathfrak{D}$ *as* $A(t)$, *and* $A(t \pm 0) = A(t)$ *holds except at most at a denumerable set of* t. *There exists an operator function* $U(t, s)$ *defined for* $a \leqq s \leqq t \leqq b$ *with the following properties:*

(1.4) $U(t, s)$ *is a bounded operator in* $\mathfrak{B}$ *and* $\|U(t, s)\| \leqq 1$;

(1.5) $U(t, s)$ *is strongly continuous in* s *and* t *simultaneously, and* $U(t, t) = I$;

(1.6) $U(t, r) = U(t, s)\, U(s, r)$, $\qquad r \leqq s \leqq t$;

(1.7) *if* $x \in \mathfrak{D}$, *we have for* $\varepsilon \downarrow 0$

$$\varepsilon^{-1}[U(t+\varepsilon, t) - I]\, x \to A(t+0)\, x\,, \qquad a \leqq t < b\,,$$

$$\varepsilon^{-1}[U(t, t-\varepsilon) - I]\, x \to A(t-0)\, x\,, \qquad a < t \leqq b\,,$$

in the strong sense.

If in particular $\mathfrak{B}$ *is reflexive, we have further*

(1.8) $U(t, s)\mathfrak{D} \subset \mathfrak{D}$, *and for each* $x \in \mathfrak{D}$, $x(t) = U(t, a)\, x$ *has strong right-derivative and satisfies* $D^+ x(t) = A(t+0)\, x(t)$ *for* $a \leqq t < b$. *In this case* $U(t, s)$ *is uniquely determined by these properties.*

THEOREM 3. *Let the conditions* C_1, C_2, C_3 *be satisfied. If* $\mathfrak{B}$ *is uniformly convex, we have, in addition to the assertions of Theorem* 2,

(1.9) $D^+ x(t) = A(t)\, x(t)$ *and* $A(t)\, x(t)$ *is strongly right-continuous.*

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6) We denote by $\mathfrak{D}(A)$ the domain of the operator A.

If in particular $\mathfrak{B}$ *is a Hilbert space and* $i A(t)$ *is self-adjoint, we have*

(1.10) $x(t)$ *is strongly differentiable,* $dx(t)/dt=A(t)x(t)$, *and* $A(t)x(t)$ *is strongly continuous.*

THEOREM 4. *Let* $\mathfrak{B}$ *be arbitrary and let the conditions* C_1, C_2, C_3, C_4 *be satisfied. Then the operator* $U(t,s)$ *introduced in Theorem* 2 *satisfies* $U(t,s)\mathfrak{D}\subset\mathfrak{D}$ *and* (1.10) *holds for* $x(t)=U(t,a)x$ *with* $x\in\mathfrak{D}$.

THEOREM 5. *Let the hypotheses of Theorem* 4 *be satisfied. If* $x\in\mathfrak{D}$ *and if* $t\rightarrow f(t)$ *is a function on* (a,b) *into* $\mathfrak{D}$ *such that* $[A(r)-I]f(t)$[7] *is strongly continuous in t for some fixed r, the expression* (1.2) *is well defined, belongs to* $\mathfrak{D}$ *for each t, strongly differentiable in t and satisfies the differential equation* (1.1) *with the initial condition* $x(a)=x$. *Moreover* $dx(t)/dt$ *and* $A(t)x(t)$ *as well as* $Ax(t)$ *are strongly continuous, where A is any linear operator with a closed extension and with domain containing* $\mathfrak{D}$.

REMARK. (1.2) is defined for every $x\in\mathfrak{B}$ and for every strongly continuous (or even Bochner integrable) function $f(t)$. In view of Theorem 5 and Corollary to Theorem 1 (uniqueness theorem), it may be justified to regard (1.2) as giving the solution in the generalized sense of (1.1) for such general x and $f(t)$, although it does not necessarily satisfy (1.1) in a strict sense.

In the next section we shall prove some lemmas to be used in the sequel and which may also be of some independent interest, and in § 3 we shall give the proof of the theorems. As an illustration of the use of these theorems, we shall consider in § 4 their application to the case in which $A(t)$ is an ordinary differential operator.

§ 2. Some lemmas.

First we collect some important results of the theory of semi-groups[2] which will be of frequent use in the following. If the operator A has property S (see Definition 1), there is defined an operator $\exp(tA)$ for every $t\geqq 0$ with the properties:

(2.1) $\exp(tA)$ is bounded and $\|\exp(tA)\|\leqq 1$;

7) The result is the same if we take, in place of $A(r)-I$, any closed linear operator A with domain $\mathfrak{D}$ and with A^{-1} bounded.

(2.2) $\exp(tA)$ is strongly continuous in t and $\exp(0A)=I$;

(2.3) $\exp[(t+s)A]=\exp(tA)\exp(sA)$;

(2.4) $\exp(tA)\mathfrak{D}(A)\subset\mathfrak{D}(A)$, and for each $x\in\mathfrak{D}(A)$, $\exp(tA)x$ is strongly differentiable with $(d/dt)\exp(tA)x=A\exp(tA)x=\exp(tA)Ax$.

(2.5) $\exp(tA)$ is permutable with the resolvent $(\lambda I-A)^{-1}$.

LEMMA 1. *If the operator A has property* S, *we have for each* $x\in\mathfrak{D}(A)$

$$\|(I+\varepsilon A)x\|\leqq\|x\|+o(\varepsilon), \qquad \varepsilon\downarrow 0.$$

PROOF. We have $(I+\varepsilon A)x=(I-\varepsilon^2A^2)(I-\varepsilon A)^{-1}x=(I-\varepsilon A)^{-1}x-\varepsilon^2A(I-\varepsilon A)^{-1}Ax$ and hence $\|(I+\varepsilon A)x\|\leqq\|x\|+\varepsilon\|B_\varepsilon Ax\|$ by (1.3), where we have set $B_\varepsilon=\varepsilon A(I-\varepsilon A)^{-1}=-I+(I-\varepsilon A)^{-1}$. Hence $\|B_\varepsilon\|\leqq 2$ by (1.3) and B_ε is uniformly bounded. On the other hand if $y\in\mathfrak{D}(A)$, $B_\varepsilon y=\varepsilon(I-\varepsilon A)^{-1}Ay$ and $\|B_\varepsilon y\|\leqq\varepsilon\|Ay\|\to 0$ for $\varepsilon\downarrow 0$. Thus we have $B_\varepsilon\to 0$ strongly, for $\mathfrak{D}(A)$ is dense in $\mathfrak{B}$, and the above inequality proves the lemma.

LEMMA 2. *Let A be a closed linear operator, and let B be a linear operator with a closed extension and such that $\mathfrak{D}(B)\supset\mathfrak{D}(A)$. If λ belongs to the resolvent set of A, then $B(\lambda I-A)^{-1}$ is bounded.*

PROOF. Set $C=B(\lambda I-A)^{-1}$. C is defined everywhere in $\mathfrak{B}$ so that it is sufficient to show that C is closed. Let $x_n\to x$ and $Cx_n\to y$. Then $(\lambda I-A)^{-1}x_n\equiv z_n\to z\equiv(\lambda I-A)^{-1}x$ and $Bz_n=Cx_n\to y$. This shows that $y=B_0z$, where B_0 is a closed extension of B. But since $z\in\mathfrak{D}(A)\subset\mathfrak{D}(B)$, we must have $y=Bz=Cx$, showing that C is closed.

LEMMA 3. *Let A, B have property* S *and let $\mathfrak{D}(A)\subset\mathfrak{D}(B)$. Then we have for $t\geqq 0$, $\lambda>0$,*

$$\|[\exp(tA)-\exp(tB)](\lambda I-A)^{-1}\|\leqq t\|(A-B)(\lambda I-A)^{-1}\|,$$

where $(A-B)(\lambda I-A)^{-1}=A(\lambda I-A)^{-1}-B(\lambda I-A)^{-1}$ is bounded by Lemma 2.

PROOF. For each $x\in\mathfrak{D}(A)$, $\exp[(t-s)B]\exp(sA)x$ is differentiable in s for $0\leqq s\leqq t$, for

$$h^{-1}\{\exp[(t-s-h)B]\exp[(s+h)A]x-\exp[(t-s)B]\exp(sA)x\}$$

$$=\exp[(t-s-h)B]h^{-1}\{\exp[(s+h)A]x-\exp(sA)x\}$$

$$+h^{-1}\{\exp[(t-s-h)B]-\exp[(t-s)B]\}\exp(sA)x$$

and, since $x\in\mathfrak{D}(A)$, $\exp(sA)x\in\mathfrak{D}(A)\subset\mathfrak{D}(B)$ by (2.4) and hypothesis, we obtain by making $h\to 0$

$$(d/ds)\{\exp[(t-s)B]\exp(sA)x\}=\exp[(t-s)B]A\exp(sA)x$$

$$-\exp[(t-s)B]B\exp(sA)x=\exp[(t-s)B](A-B)\exp(sA)x$$

by (2.2) and (2.4). Now let $y\in\mathfrak{B}$ be arbitrary and set $x=(\lambda I-A)^{-1}y \in\mathfrak{D}(A)$ in the above result. Since $\exp(sA)$ and $(\lambda I-A)^{-1}$ are permutable, the right-hand side is strongly continuous in s, for $(A-B)(\lambda I-A)^{-1}$ is bounded as stated in the lemma. Thus the above identity can be integrated from 0 to t, yielding

$$[\exp(tA)-\exp(tB)](\lambda I-A)^{-1}y$$

$$=\int_0^t \exp[(t-s)B](A-B)(\lambda I-A)^{-1}\exp(sA)y\,ds\,.$$

This gives immediately the desired inequality by (2.1).

LEMMA 4. *Let $\{A_n(s)\}$, $\{B_n(s)\}$ be two sequences of bounded linear operators depending on a variable s in a compact set S of a euclidean space. Let $A_n(s)\to A(s)$, $B_n(s)\to B(s)$, $n\to\infty$, strongly and uniformly with respect to s, where $B(s)$ is strongly continuous in s. Further let there exist a $M\geqq 0$ such that $\|A_n(s)\|\leqq M$ for all s and n. Then $A_n(s)B_n(s)\to A(s)B(s)$ holds strongly and uniformly with respect to s.*

PROOF. For each $x\in\mathfrak{B}$ we have

$$\|A_n(s)B_n(s)x-A(s)B(s)x\|$$

$$\leqq\|A_n(s)[B_n(s)-B(s)]x\|+\|[A_n(s)-A(s)]B(s)x\|\,.$$

The first term can be made arbitrarily small independently of s by making n sufficiently large, for $\|A_n(s)\|\leqq M$ and $B_n(s)\to B(s)$ uniformly in s. The second term is

$$\leqq\|[A_n(s)-A(s)]B(s_0)x\|+\|[A_n(s)-A(s)][B(s)-B(s_0)]x\|$$

$$\leqq\|[A_n(s)-A(s)]B(s_0)x\|+2M\|[B(s)-B(s_0)]x\|$$

for a fixed s_0. Since $B(s)$ is strongly continuous, to each $\varepsilon>0$ there is a $\delta>0$ such that $\|[B(s)-B(s_0)]x\|<\varepsilon/2M$ for $|s-s_0|<\delta$. Then we can take n so large that $\|[A_n(s)-A(s)]B(s_0)x\|<\varepsilon$ for every $s\in S$. Thus, for each $\varepsilon>0$ and $s_0\in S$, there are $\delta=\delta(s_0,\varepsilon)>0$ and $n_0=n_0(s_0,\varepsilon)$ such that

$$\|A_n(s)B_n(s)x-A(s)B(s)x\|<3\varepsilon \quad \text{for} \quad n>n_0, \quad |s-s_0|<\delta.$$

For fixed ε, there is thus defined a neighborhood $|s-s_0|<\delta$ for each $s_0\in S$ with the above property. Since S is compact, it can be covered by a finite number of these neighborhoods. Let N be the largest among the corresponding n_0. Then the above inequality holds for all $s\in S$ provided $n>N$, and this completes the proof.

LEMMA 5. *Let $\mathfrak{B}$ be reflexive and let A be a closed linear operator with non-empty resolvent set. Let $\{x_n\}$ be a sequence such that $x_n\in\mathfrak{D}(A)$,* weak $\lim_{n\to\infty} x_n=x$, *and $\{\|Ax_n\|\}$ is bounded. Then $x\in\mathfrak{D}(A)$, and* $Ax=$ weak $\lim_{n\to\infty} Ax_n$.

PROOF. We may assume without loss of generality that $\lambda=0$ belongs to the resolvent set of A. Thus A^{-1} exists and bounded so that $(A^{-1})^*$ also exists and we have for every $x^*\in\mathfrak{B}^*$

$$(Ax_n,(A^{-1})^*x^*)=(A^{-1}Ax_n,x^*)=(x_n,x)\to(x,x^*)\,.$$ [8]

But $(A^{-1})^*\mathfrak{B}^*$ is dense in $\mathfrak{B}^*$, for otherwise there would exist a $z\in\mathfrak{B}$, $z\neq0$, orthogonal to $(A^{-1})^*\mathfrak{B}^*$, for $\mathfrak{B}$ is reflexive by hypothesis, which leads to the contradiction $0=(z,(A^{-1})^*x^*)=(A^{-1}z,x^*)$, $A^{-1}z=0$, $z=0$. Since $\{Ax_n\}$ is bounded, it follows that $\{Ax_n\}$ is weakly convergent. Since a reflexive space is weakly complete, there is a $y\in\mathfrak{B}$ such that $y=$ weak lim Ax_n. Thus we have

$$(Ax_n,(A^{-1})^*x^*)\to(y,(A^{-1})^*x^*)=(A^{-1}y,x^*)\,.$$

A comparison of these two relations gives $(x,x^*)=(A^{-1}y,x^*)$, $x=A^{-1}y$. This shows that $x\in\mathfrak{D}(A)$, $Ax=y=$ weak lim Ax_n.

LEMMA 6. *Let $\mathfrak{B}$ be uniformly convex and let $\{x_n\}$ be a sequence such that $x_n\in\mathfrak{B}$,* weak $\lim_{n\to\infty} x_n=x$, *and* $\limsup_{n\to\infty}\|x_n\|\leq\|x\|$. *Then $x_n\to x$ holds strongly.*

PROOF. There is a $x^*\in\mathfrak{B}^*$ such that $(x,x^*)=\|x\|$, $\|x^*\|=1$. Then

8) We denote by (x,x^*) the scalar product of $x\in\mathfrak{B}$ and $x^*\in\mathfrak{B}^*$.

the relations $(x_n, x^*) \to (x, x^*) = \|x\|$, $|(x_n, x^*)| \leq \|x_n\|$ show that lim inf $\|x_n\| \geq \|x\|$. Hence the hypothesis implies $\lim \|x_n\| = \|x\|$. Since the lemma is trivial if $x=0$, we may assume $x \neq 0$ and $x_n \neq 0$. Setting $y_n = \|x_n\|^{-1} x_n$ and $y = \|x\|^{-1} x$, we have $\|y_n\| = \|y\| = 1$ and $(y_n, x^*) = \|x_n\|^{-1} (x_n, x^*) \to \|x\|^{-1}(x, x^*) = 1$, $(y, x^*) = \|x\|^{-1}(x, x^*) = 1$ so that $((y_n + y)/2, x^*) \to 1$. On the other hand $|((y_n+y)/2, x^*)| \leq \|(y_n+y)/2\| \leq (\|y_n\| + \|y\|)/2 = 1$. Hence we must have $\|(y_n+y)/2\| \to 1$, $\|y_n\| = \|y\| = 1$. Since $\mathfrak{B}$ is uniformly convex, it follows that $\|y_n - y\| \to 0$, that is, $y_n \to y$ strongly. Hence we have $x_n = \|x_n\| y_n \to \|x\| y = x$ by $\|x_n\| \to \|x\|$ already proved.

§ 3. Proof of the theorems.

1. Proof of theorem 1. For fixed t, we have by hypothesis $\|x(t+\varepsilon) - [I + \varepsilon A(t)] x(t)\| = o(\varepsilon)$ for $\varepsilon \downarrow 0$. It follows by Lemma 1 that $\|x(t+\varepsilon)\| \leq \|[I+\varepsilon A(t)] x(t)\| + o(\varepsilon) \leq \|x(t)\| + o(\varepsilon)$. This shows that the upper right derivative of $\|x(t)\|$ is not positive. Since $\|x(t)\|$ is by hypothesis a continuous function of t, it follows that $\|x(t)\|$ is non-increasing.

2. Before proceeding further, we make the following convention in order to simplify the notation. We shall write $A(t)$ instead of $A(t) - I$ so far considered. This is equivalent to assuming

$$(3.1) \qquad \|A(t)^{-1}\| \leq 1, \qquad \|[I - \alpha A(t)]^{-1}\| \leq (1+\alpha)^{-1}, \qquad \alpha > -1,$$

from the beginning and there is no loss of generality, for we have only to multiply $U(t,s)$ and $x(t)$ by the numerical factor $\exp(t-s)$ in the theorems stated above. Thus the only change in the results to be proved is

$$(3.2) \qquad \|U(t,s)\| \leq \exp[-(t-s)]$$

in place of (1.4).

With this convention, the conditions of C_2 can be written as

$$(3.3) \qquad B(t,s) = A(t) A(s)^{-1} = B(s,t)^{-1}, \qquad \|A(t) A(s)^{-1}\| \leq M.$$

3. First we prove various assertions stated in parentheses in C_2 etc. If $B(t,s)$ is continuous in t in the sense of the norm $\|\ \|$ for some s, the same is true for its inverse[9] $B(s,t)$, and both are bounded

9) See e.g. Hille [1], p, 92.

as $\|B(t,s)\| \leqq M'$, $\|B(s,t)\| \leqq M''$ with M', M'' independent of t. Then we have for arbitrary t_1 and t_2 $\|B(t_1,t_2)\| = \|B(t_1,s)B(s,t_2)\| \leqq M'M''$. Next, by virtue of the inequality $\|B(t_j,r)-B(t_{j-1},r)\| \leqq \|B(t_j,s)-B(t_{j-1},s)\| \ \|B(s,r)\|$ and $\|B(s,r)\| \leqq M$, it is clear that $B(t,s)$ is of bounded variation in the sense of C_2 for each s if this is the case for some s, and that the total variation of $B(t,s)$ has a finite upper bound N independent of s. The assertions in C_3 and C_4 can be proved in a similar fashion (note the next paragraph for C_3).

4. Next we give the definition of $A(t\pm 0)$ mentioned in Theorem 2. Set $B(t)=B(t,s)$ for a fixed s. Since $B(t)$ is of bounded variation in the strong sense of C_2, $B(t\pm 0)=\lim B(t\pm\varepsilon)$ exists in the sense of convergence by the norm $\| \ \|$. Since

$$B(t')^{-1}-B(t)^{-1}=B(t')^{-1}[B(t)-B(t')]B(t)^{-1}$$

and $B(t)^{-1}=B(s,t)$ is uniformly bounded, it follows that $B(t\pm 0)^{-1}$ exists, is bounded, and is equal to the limit of $B(t\pm\varepsilon)^{-1}$ for $\varepsilon\downarrow 0$. We now define the operator $A(t\pm 0)$ with domain $\mathfrak{D}$ by $A(t\pm 0)=B(t\pm 0)A(s)$. Then $A(t\pm 0)^{-1}=A(s)^{-1}B(t\pm 0)^{-1}$ is bounded, so that $A(t\pm 0)$ is closed. Since it is clear that $A(t\pm\varepsilon)x\to A(t\pm 0)x$ for each $x\in\mathfrak{D}$, $A(t\pm 0)$ is actually independent of s used in defining it. Also it is easily seen that $A(t\pm 0)^{-1}=\lim A(t\pm\varepsilon)^{-1}$ in the sense of convergence by the norm, so that we have $\|A(t\pm 0)^{-1}\| \leqq 1$. Since similar results are obtained by considering $[\lambda I-A(t)][\lambda I-A(s)]^{-1}$ instead of $B(t)$, it follows that $A(t\pm 0)$ has property S. Also we see easily that $A(t)A(s\pm 0)$ is of bounded variation with the total variation not larger than N, and that $B(t\pm 0, s\pm 0)=A(t\pm 0)A(s\pm 0)^{-1}$ is also bounded by M. Finally we note that $B(t\pm 0)=B(t)$ holds except at most at a denumerable set of t. Hence the same holds for $A(t)$.

5. We shall now give the proof of Theorem 2 in several steps. First let us construct the operator $U(t,s)$. We consider a partition Δ of the interval (a,b)

(3.4) $$a=t_0<t_1<\cdots<t_n=b, \qquad t_{j-1}\leqq\tau_j\leqq t_j \qquad (j=1,2,\cdots,n)$$

and set

(3.5) $$U(\Delta)=X_nX_{n-1}\cdots X_1,$$

$$X_j=\exp[(t_j-t_{j-1})A_j], \qquad A_j=A(\tau_j) \qquad (j=1,2,\cdots n).$$

It is our purpose to show that $U(\Delta)$ converges strongly to a limit when the maximum chord $|\Delta|=\mathrm{Max}(t_j-t_{j-1})$ tends to zero. For this purpose we further set

$$(3.6)\qquad U_{j,k}=X_jX_{j-1}\cdots X_k, \qquad W_j=A_jU_{j,1}A_0^{-1}, \qquad A_0=A(a).$$

We note that

$$(3.7)\qquad \|X_j\|\leqq\exp[-(t_j-t_{j-1})]\leqq 1, \qquad \|U_{j,k}\|\leqq\exp[-(t_j-t_{k-1})]\leqq 1$$

by (2.1) and the fact that $A(t)+I$ has property S on account of our convention. Thus X_j and $U_{j,k}$ are uniformly bounded. The same is true for W_j, for we have $A_jX_j\supset X_jA_j$[10] by (2.4) and hence

$$W_j=A_jX_j\cdots X_1A_0^{-1}=X_jA_jA_{j-1}^{-1}X_{j-1}A_{j-1}A_{j-2}^{-1}X_{j-2}\cdots X_1A_1A_0^{-1},$$

which shows that W_j is bounded and that

$$\|W_j\|\leqq\|A_jA_{j-1}^{-1}\|\;\|A_{j-1}A_{j-2}^{-1}\|\cdots\|A_1A_0^{-1}\|$$

by (3.7). But we have

$$\|A_kA_{k-1}^{-1}\|=\|I+(A_k-A_{k-1})A_{k-1}^{-1}\|\leqq 1+\|(A_k-A_{k-1})A_{k-1}^{-1}\|$$
$$\leqq 1+\|(A_k-A_{k-1})A_0^{-1}\|\;\|A_0A_{k-1}^{-1}\|\leqq 1+M\|(A_k-A_{k-1})A_0^{-1}\|,$$

so that

$$(3.8)\quad \|W_j\|\leqq\exp[\sum_{k=1}^{j}M\|(A_k-A_{k-1})A_0^{-1}\|]\leqq\exp(MN)\quad (j=1,2,\cdots n),$$

since the total variation of $A(t)A_0^{-1}$ is $\leqq N$.

In the same way it can easily be shown that

$$(3.9)\qquad \begin{aligned}&\|A_jA_{j-1}^{-1}W_{j-1}\|\leqq\exp(MN) \qquad (j=2,\cdots n),\\ &\|AU(\Delta)A_0^{-1}\|=\|AA_n^{-1}W_n\|\leqq M\exp(MN).\end{aligned}$$

where A is any one of $A(s)$ or $A(s\pm 0)$.

6. Let Δ and Δ' be two partitions of the interval (a, b) such that Δ is a subpartition of Δ', and let us construct $U(\Delta)$ and $U(\Delta')$ as above. As Δ is a subpartition of Δ', each factor X_j' of $U(\Delta')$ corresponding to

10) $A\supset B$ means that A is an extension of B.

(3.5) can further be decomposed by (2.3) into the product of several factors in such a way that we can write

$$U(\Delta)=X_nX_{n-1}\cdots X_1, \qquad X_j=\exp[(t_j-t_{j-1})A_j], \qquad A_j=A(\tau_j),$$

$$U(\Delta')=X'_nX'_{n-1}\cdots X'_1, \qquad X'_j=\exp[(t_j-t_{j-1})A'_j], \qquad A'_j=A(\tau'_j),$$

with the same n and t_j. The only point to be noted is that, whereas $t_{j-1}\leqq\tau_j\leqq t_j$, τ'_j does not in general satisfy this inequality, although τ'_j and τ_j belong to the same subinterval of the partition Δ'.

If we define $U'_{j,k}$ and W'_j by replacing τ_i by τ'_i in $U_{j,k}$ and W_j of (3.6), it is clear that they satisfy the same inequalities (3.7) to (3.9). We have now

$$U(\Delta')-U(\Delta)=\sum_{j=1}^{n}U'_{n,j+1}(X'_j-X_j)U_{j-1,1},$$

where we set $U'_{n,n+1}=U_{0,1}=I$. On multiplying from right by A_0^{-1} and noting that $U_{j-1,1}A_0^{-1}=A_j^{-1}A_jA_{j-1}^{-1}W_{j-1}$ by (3.6), we obtain

$$\|[U(\Delta')-U(\Delta)]A_0^{-1}\|\leqq\sum\|(X'_j-X_j)A_j^{-1}\|\ \|A_jA_{j-1}^{-1}W_{j-1}\|$$
$$\leqq\exp(MN)\sum\|(X'_j-X_j)A_j^{-1}\|,$$

where we have used (3.7) and (3.9). But we have by Lemma 3

$$\|(X'_j-X_j)A_j^{-1}\|=\|\{\exp[(t_j-t_{j-1})A'_j]-\exp[(t_j-t_{j-1})A_j]\}A_j^{-1}\|$$
$$\leqq(t_j-t_{j-1})\|(A'_j-A_j)A_j^{-1}\|\leqq(t_j-t_{j-1})\|(A'_j-A_j)A_0^{-1}\|\ \|A_0A_j^{-1}\|.$$

Hence it follows by $\|A_0A_j^{-1}\|\leqq M$ that

$$\|[U(\Delta')-U(\Delta)]A_0^{-1}\| \leqq M\exp(MN)\sum_{j=1}^{n}(t_j-t_{j-1})\|(A'_j-A_j)A_0^{-1}\|. \tag{3.10}$$

To calculate the right-hand side, we first take the sum for those subintervals (t_{j-1},t_j) contained in a fixed subinterval, say I'_k, of the partition Δ'. Then both τ'_j and τ_j belong to I'_k, and $\|(A'_j-A_j)A_0^{-1}\|$ is not larger than the oscillation, and a fortiori, than the variation $v(I'_k)$, of $A(t)A_0^{-1}$ in the interval I'_k. Thus the sum under consideration is not larger than $v(I'_k)\sum(t_j-t_{j-1})=v(I'_k)\ |I'_k|\leqq v(I'_k)\,|\Delta'|$, where $|\Delta'|$

is the maximum chord of Δ'. On adding these for all I'_k, we obtain

$$\|[U(\Delta')-U(\Delta)]A_0^{-1}\| \leq L|\Delta'|, \qquad L=MN\exp(MN), \tag{3.11}$$

since $\sum v(I'_k)$ is not larger than the total variation of $A(t)A_0^{-1}$ and hence $\leq N$.

7. Let Δ' and Δ'' be arbitrary two partitions of (a,b). If we denote by Δ their common subpartition, we have (3.11) and a similar inequality for Δ''. Hence

$$\|[U(\Delta')-U(\Delta'')]A_0^{-1}\| \leq L(|\Delta'|+|\Delta''|). \tag{3.12}$$

It follows that $U(\Delta')A_0^{-1}$ has a uniform limit for $|\Delta'|\to 0$. For each $x\in\mathfrak{D}$, we can write $x=A_0^{-1}y$, $y=A_0x$, so that $U(\Delta')x=U(\Delta')A_0^{-1}y$ has a limit for $|\Delta'|\to 0$. On the other hand $U(\Delta')$ are uniformly bounded by $\|U(\Delta')\|=\|U'_{n,1}\|\leq\exp[-(b-a)]\leq 1$ by (3.7). Since $\mathfrak{D}$ is dense in $\mathfrak{B}$, it follows that $U(\Delta')$ converges strongly to a limit for $|\Delta'|\to 0$. Let us denote the limit by $U(b,a)$. Clearly $U(b,a)$ is a bounded operator and $\|U(b,a)\|\leq\exp[-(b-a)]\leq 1$. Incidentally we note the following inequality obtained from (3.12) by letting $|\Delta''|\to 0$:

$$\|[U(\Delta')-U(b,a)]A_0^{-1}\| \leq L|\Delta'|. \tag{3.13}$$

Clearly the above construction can be applied to any subinterval (s,t) of (a,b) and leads to an operator $U(t,s)$ with $\|U(t,s)\|\leq\exp[-(t-s)]$, which proves (3.2). Also we set $U(t,t)=I$ by definition. (1.6) can now be proved easily. Let Δ' and Δ'' be partitions of the intervals (r,s) and (s,t) respectively, and let Δ be the partition of (r,t) obtained by combining Δ' and Δ''. Then we have $U(\Delta)=U(\Delta'')U(\Delta')$ by (3.5). If we let $|\Delta'|\to 0$ and $|\Delta''|\to 0$, then also $|\Delta|\to 0$ so that we have $U(\Delta')\to U(s,r)$, $U(\Delta'')\to U(t,s)$, and $U(\Delta)\to U(t,r)$ strongly, and we obtain (1.6) for $r<s<t$. Of course (1.6) is trivial if $s=r$ or $s=t$.

Incidentally we note that $U(t,s)=\exp[(t-s)A]$ if $A(t)=\text{const.}=A$.

8. Suppose that there is another operator function $A'(t)$ satisfying the conditions C_1, C_2, and such that its domain $\mathfrak{D}'$ contains $\mathfrak{D}$. Then we can define the operator $U'(t,s)$ for $A'(t)$ in the same way as we have defined $U(t,s)$ for $A(t)$. Now the following generalization of Lemma 3 holds:

$$\|[U'(t,s)-U(t,s)]A_0^{-1}\| \leq M\exp(MN)\int_s^t \|[A'(\tau)-A(\tau)]A_0^{-1}\|\,d\tau. \tag{3.14}$$

This can be proved by considering a partition Δ of (s,t) common to $A(t)$ and $A'(t)$ and proceeding in the same way as in paragraph 6; then we can deduce an inequality similar to (3.10), and on passing to the limit $|\Delta|\to 0$, we obtain the desired result (3.14). The constants M, N and the operator A_0^{-1} are those belonging to $A(t)$ and not to $A'(t)$.

We use (3.14) to generalize the inequality (3.13). Let Δ be again the partition of (a,b) given by (3.4), and let $A'(t)$ be the step function defined by

$$A'(t)=A_j=A(\tau_j), \qquad t_{j-1}\leqq t<t_j \qquad (j=1,2,\cdots,n).$$

Then the corresponding operator $U'(b,a)$ is identical with $U(\Delta)$ given by (3.5). In general let us write $U(t,s;\Delta)$ for $U'(t,s)$. Clearly $U(t,s;\Delta)$ has the form

$$U(t,s;\Delta)=\exp[(t-t_{j-1})A_j]\exp[(t_{j-1}-t_{j-2})A_{j-1}]\cdots\exp[(t_k-s)A_k] \tag{3.15}$$
$$\text{for} \qquad t_{j-1}\leqq t\leqq t_j, \qquad t_{k-1}\leqq s\leqq t_k.$$

In this case (3.14) becomes

$$\|[U(t,s;\Delta)-U(t,s)]A_0^{-1}\|\leqq M\exp(MN)N|\Delta|=L|\Delta| \tag{3.16}$$

quite in the same way as in deducing (3.11). (3.16) shows that $U(t,s;\Delta)\to U(t,s)$, $|\Delta|\to 0$, holds strongly and uniformly with respect to s,t.

9. By virtue of (2.2) it follows from (3.15) that $U(t,s;\Delta)$ is strongly continuous in s,t simultaneously. Furthermore, since each factor of (3.15) takes $\mathfrak{D}$ into $\mathfrak{D}$ by (2.4), $U(t,s;\Delta)x$ is strongly differentiable to the right in t provided $x\in\mathfrak{D}$. Thus we have

$$D_t^+ U(t,s;\Delta)x=A'(t)U(t,s;\Delta)x, \qquad x\in\mathfrak{D}.$$

The right-hand side is strongly piecewise continuous in t, so that we can integrate both sides and obtain

$$[U(t',s;\Delta)-U(t,s;\Delta)]x=\int_t^{t'} A'(\tau)U(\tau,s;\Delta)x\,d\tau, \qquad s\leqq t\leqq t'.$$

But it can easily be shown that $\|A'(t)U(t,s;\Delta)A_0^{-1}\|\leqq\exp(MN)$ in the same way as (3.8). Hence we have

(3.17) $\|[U(t',s;\Delta)-U(t,s;\Delta)]x\| \leqq (t'-t)\exp(MN)\|A_0x\|, \quad s\leqq t\leqq t'.$

Similarly we have

$$[U(t,s';\Delta)-U(t,s;\Delta)]x=-\int_s^{s'}U(t,\sigma;\Delta)A'(\sigma)x\,d\sigma$$

and

(3.18) $\|[U(t,s';\Delta)-U(t,s;\Delta)]x\| \leqq (s'-s)M\|A_0x\|, \quad s\leqq s'\leqq t.$

These inequalities show that $\|U(t,s;\Delta)x\|$ are equicontinuous with respect to the partitions Δ, at least in s or t separately, provided $x\in\mathfrak{D}$. We note that (3.17) and (3.18) can also be written as

(3.19)
$$\|[U(t',s;\Delta)-U(t,s;\Delta)]A_0^{-1}\| \leqq (t'-t)\exp(MN),$$
$$\|[U(t,s';\Delta)-U(t,s;\Delta)]A_0^{-1}\| \leqq (s'-s)M.$$

On making $|\Delta|\to 0$ in (3.17) and (3.18) and noting that $U(t,s;\Delta)x \to U(t,s)x$ by paragraph 8, we obtain

(3.20)
$$\|[U(t',s)-U(t,s)]x\| \leqq (t'-t)\exp(MN)\|A_0x\|,$$
$$\|[U(t,s')-U(t,s)]x\| \leqq (s'-s)M\|A_0x\|.$$

These inequalities show that $U(t,s)x$ is continuous in s or t provided $x\in\mathfrak{D}$. But as $U(t,s)$ is uniformly bounded by (3.2) already proved, it follows that $U(t,s)$ is strongly continuous in s or t. By making use of the relation $U(t,s)=U(t,r)U(r,s)$, $s\leqq r\leqq t$, we then conclude that $U(t,s)$ is strongly continuous in s and t simultaneously (see the note at tee end of the paper). Thus we have proved (1.5).

10. Next we set $t=s+\varepsilon$, $A'(\tau)=A(s+0)$, $s\leqq\tau\leqq t$, in (3.14). Then we have $U'(t,s)=\exp[\varepsilon A(s+0)]$ and the right-hand side of (3.14) is $o(\varepsilon)$ by the definition of $A(s+0)$ given in paragraph 4. Thus we have

$$\|\{\exp[\varepsilon A(s+0)]-U(s+\varepsilon,s)\}A_0^{-1}\|=o(\varepsilon).$$

It follows that for each $x\in\mathfrak{D}$

$$\varepsilon^{-1}\{\exp[\varepsilon A(s+0)]x-U(s+\varepsilon,s)x\}\to 0, \qquad \varepsilon\downarrow 0,$$

for x may be written as $x=A_0^{-1}y$, $y=A_0x$. But as

$$\varepsilon^{-1}\{\exp[\varepsilon A(s+0)]x-x\}\to A(s+0)x$$

by (2.4), we obtain the first relation of (1.7). The second relation can similarly be proved.

It should be noted that (1.7) does not imply that $D_t^+(t, s)x = A(t+0) U(t, s)x$, for it is not proved that $x \in \mathfrak{D}$ implies $U(t, s)x \in \mathfrak{D}$. This can be proved, however, at least if $\mathfrak{B}$ is reflexive. We have shown in paragraph 5 that $AU(\Delta)A_0^{-1}$ is uniformly bounded by $\|AU(\Delta)A_0^{-1}\| \leqq M\exp(MN)$ (see (3.9)). Let $\{\Delta_n\}$ be a sequence of partitions of (a, b) such that $|\Delta_n| \to 0$, and set $x_n = U(\Delta_n)x$ for a fixed $x \in \mathfrak{D}$. Then we have $x_n \to U(b, a)x$ and $\|Ax_n\| \leqq \|AU(\Delta_n)A_0^{-1}\| \, \|A_0 x\|$ is bounded for $n \to \infty$. It follows from Lemma 5 that $U(b, a)x \in \mathfrak{D}$ and $\|AU(b, a)A_0^{-1}\| \leqq M\exp(MN)$.

In the same way we can show that $U(t, s)x \in \mathfrak{D}$ whenever $x \in \mathfrak{D}$. Hence we have

$$\varepsilon^{-1}[U(t+\varepsilon, s) - U(t, s)]x = \varepsilon^{-1}[U(t+\varepsilon, t) - I]U(t, s)x$$

and (1.8) follows from (1.7). That $U(t, s)$ is uniquely determined is then a consequence of Theorem 1. With these results, the proof of Theorem 2 is complete.

11. We now proceed to the proof of Theorem 3. We recall that the condition C_3, together with C_1 and C_2, implies that $B(t) = A(t)A(s)^{-1}$ and $B(t)^{-1}$ are continuous in the sense of the norm $\| \quad \|$ (paragraphs 3 and 4).

We have shown in the preceding paragraph that $\|AU(b, a)A_0^{-1}\| \leqq M\exp(MN)$, provided $\mathfrak{B}$ is reflexive. Hence we have $\|AU(b, a)A^{-1}\| \leqq M^2\exp(MN)$. We recall that here A may be any $A(r)$ for a fixed r, M is an upper bound of $\|A(t)A(s)^{-1}\|$ for $a \leqq s, t \leqq b$, and N is an upper bound of the total variation of $A(t)A(s)^{-1}$ as a function of t for $a \leqq t \leqq b$. Of course a similar inequality holds if we replace $U(b, a)$ by $U(t, s)$, (s, t) being any subinterval of the small interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ for a fixed t_0, A by $A(t_0)$ and M, N by the corresponding quantities M_ε, N_ε defined for the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$. But as $A(t)A(s)^{-1}$ and its inverse are continuous and of bounded variation in t in the sense of the norm $\| \quad \|$, we have $M_\varepsilon \to 1$ and $N_\varepsilon \to 0$ for $\varepsilon \downarrow 0$. Thus we have

$$\limsup_{s \to t_0,\ t \to t_0} \|AU(t, s)A^{-1}\| \leqq 1, \qquad A = A(t_0). \tag{3.21}$$

On the other hand we know that strong $\lim U(t, s) = I$ by (1.5) already proved. It follows that

$$\bigl(A\,U(t,s)A^{-1}x,\,(A^{-1})^*x^*\bigr)$$
$$=\bigl(U(t,s)A^{-1}x,\,x_*\bigr)\to(A^{-1}x,x^*)=\bigl(x,(A^{-1})^*x^*\bigr)$$

for every $x\in\mathfrak{B}$ and $x^*\in\mathfrak{B}^*$. But as $(A^{-1})^*\mathfrak{B}^*$ is dense in $\mathfrak{B}^*$ as was shown in the proof of Lemma 5, it follows that $A\,U(t,s)A^{-1}x$ converges weakly to x. If $\mathfrak{B}$ is uniformly convex, we conclude by (3.21) and Lemma 6 that $A\,U(t,s)A^{-1}x\to x$ even strongly.

If we let $t'\downarrow t$ for a fixed t in the identity

$$A\,U(t',s)A^{-1}=A\,U(t',t)A^{-1}A\,U(t,s)A^{-1},\quad s\leqq t\leqq t',\quad A=A(t),$$

and note that $A\,U(t',t)A^{-1}\to I$ strongly as we have just proved, we have $A\,U(t',s)A^{-1}\to A\,U(t,s)A^{-1}$ strongly. Thus $A\,U(t',s)A^{-1}$ is strongly continuous to the right with respect to t'. Then the same is true for $A(t')U(t',s)A_0^{-1}=A(t')A^{-1}A\,U(t',s)A^{-1}AA_0^{-1}$, and this completes the proof of (1.9).

If $\mathfrak{B}$ is a Hilbert space and $iA(t)$ is self-adjoint, there is no distinction between the positive and negative sense of t, so that $A(t)x(t)$ is continuous to the left as well as to the right. This proves the last part of Theorem 3.

1°. Next we prove Theorem 4. We again consider the partition Δ and the operators $U_{j,k}$, W_j, etc. introduced in paragraph 5. We note that by virtue of the relation $A_kX_k\supset X_kA_k$,

$$U_{j,k}A_kA_{k-1}^{-1}W_{k-1}=U_{j,k+1}X_kA_kU_{k-1,1}A_0^{-1}$$
$$=U_{j,k+1}A_kX_kU_{k-1,1}A_0^{-1}=U_{j,k+1}A_kU_{k,1}A_0^{-1}=U_{j,k+1}W_k,$$

where $k=1,2,\cdots,j$ and we set $U_{j,j+1}=W_0=I$. It follows that

$$W_j-U_{j,1}=\sum_{k=1}^{j}U_{j,k}(A_kA_{k-1}^{-1}-I)\,W_{k-1}$$
$$=\sum_{k=1}^{j}U_{j,k}(A_k-A_{k-1})A_0^{-1}A_0A_{k-1}^{-1}W_{k-1}.$$

From now on we take $\tau_k=t_k$ and set

(3.22)
$$B_k=A_kA_0^{-1}=A(t_k)A_0^{-1}=B(t_k)\qquad\bigl(B(t)=A(t)A_0^{-1}\bigr),$$
$$C_k=B_k^{-1}=A_0A_k^{-1},\qquad k=0,1,2,\cdots,n.$$

Then all B_k and C_k are bounded operators, and the above identity can be written as

$$W_j - U_{j,1} = \sum_{k=1}^{j} U_{j,k}(B_k - B_{k-1}) C_{k-1} W_{k-1}, \quad j=1,2,\cdots,n. \tag{3,23}$$

This can be regarded as a recurrence equation of Volterra type for W_j. Hence it can be solved for W_j by the method of successive approximation, yielding

$$W_j = \sum_{p=0}^{\infty} W_j^{(p)}, \tag{3.24}$$

$$\begin{aligned} &W_j^{(0)} = U_{j,1} \qquad (U_{0,1} = I), \\ &W_j^{(p)} = \sum_{k=1}^{j} U_{j,k}(B_k - B_{k-1}) C_{k-1} W_{k-1}^{(p-1)}, \quad j=0,1,\cdots,n,\ p=1,2,\cdots. \end{aligned} \tag{3.25}$$

Of course (3.24) is a finite series, for it is easily seen that $W_j^{(p)}=0$ for $j<p$. However, as we shall ultimately let $n\to\infty$ and $|\Delta|\to 0$, we need an estimate of the magnitude of each term of the series. We shall show by induction that

$$\| W_j^{(p)} \| \leq \frac{M^p}{p!}\left(\int_a^{t_j} \| dB(t) \|\right)^p \leq \frac{(MN)^p}{p!}, \qquad \begin{aligned} &j=1,2,\cdots,n, \\ &p=0,1,2,\cdots. \end{aligned} \tag{3.26}$$

Since this is clear for $p=0$ by $\| W_j^{(0)} \| = \| U_{j,1} \| \leq 1$, we assume that it is already proved for $p-1$. Then, noting that $\| U_{j,k} \| \leq 1$ and $\| C_k \| \leq M$, it follows from (3.25) that

$$\begin{aligned} \| W_j^{(p)} \| &\leq \sum_{k=1}^{j} \| B_k - B_{k-1} \| M \frac{M^{p-1}}{(p-1)!}\left(\int_a^{t_{k-1}} \| dB(t) \|\right)^{p-1} \\ &\leq \frac{M^p}{(p-1)!} \int_a^{t_j} \| dB(t) \| \left[\int_a^{t} \| dB(t) \|\right]^{p-1} = \frac{M^p}{p!}\left(\int_a^{t_j} \| dB(t) \|\right)^p, \end{aligned}$$

as we wished to show. Thus we see that the series (3.24) converges not less slowly than the series of $\exp(MN)$, uniformly with respect to j and independently of the partition Δ.

13. Before letting $|\Delta|\to 0$, it is convenient to change the notation. We introduce the following step functions:

$$
(3.27)\qquad \left.\begin{aligned} &C(t;\Delta)=C_j\\ &W(t,a;\Delta)=W_j\\ &W^{(p)}(t,a;\Delta)=W_j^{(p)}\end{aligned}\right\}\ \text{for}\ t_j\leqq t<t_{j+1},\qquad j=0,1,\cdots,n;
$$

$$
V(t,s;\Delta)=U_{j,k}\qquad \text{for}\begin{cases} t_j\leqq t<t_{j+1}, & j=0,1,\cdots,n-1,\\ t_{k-1}\leqq s<t_k, & k=1,2,\cdots,n,\end{cases}
$$

where $s\leqq t$ and it should be recalled that $U_{j,j+1}=U_{0,1}=I$. Then (3.24) and (3.26) can be written as

$$
(3.28)\qquad W(t,a;\Delta)=\sum_{p=0}^{\infty}W^{(p)}(t,a;\Delta),\quad \|W^{(p)}(t,a;\Delta)\|\leqq (MN)^p/p!.
$$

So far we have been assuming only the conditions C_1, C_2, C_3. Now let us introduce C_4. Since $B(t)=A(t)A_0^{-1}$ is weakly differentiable and $\dot{B}(t)=dB(t)/dt$ is strongly continuous, it follows easily that for each $x\in\mathfrak{B}$

$$
(B_k-B_{k-1})x=[B(t_k)-B(t_{k-1})]x=\int_{t_{k-1}}^{t_k}\dot{B}(s)x\,ds.
$$

(Incidentally this implies that $B(t)$ is actually strongly differentiable.) Thus we have by (3.27)

$$
\begin{aligned} &U_{j,k}(B_k-B_{k-1})C_{k-1}W_{k-1}^{(p-1)}x\\ &\qquad=\int_{t_{k-1}}^{t_k}V(t,s;\Delta)\dot{B}(s)\,C(s;\Delta)W^{(p-1)}(s,a;\Delta)x\,ds,\end{aligned}
$$

where t may be any value such that $t_j\leqq t<t_{j+1}$, and (3.25) may be written as

$$
(3.29)\qquad \begin{aligned} &W^{(0)}(t,a;\Delta)=V(t,a;\Delta),\\ &W^{(p)}(t,a;\Delta)x=\int_a^{t_j}V(t,s;\Delta)\dot{B}(s)C(s;\Delta)W^{(p-1)}(s,a;\Delta)x\,ds,\\ &\qquad\qquad t_j\leqq t<t_{j+1}.\end{aligned}
$$

It should be noted that the integrand is a step function so that there is no difficulty in the meaning of the integral. Also the integrand has an upper bound independent of s,t and Δ, for we have the inequalities $\|V(t,s;\Delta)\|\leqq 1$, $\|C(s;\Delta)\|\leqq M$, $\|W^{(p)}(s,a;\Delta)\|\leqq \exp(MN)$ by (3.27) and

(3.26), and there is a $K \geqq 0$ such that $\|\dot{B}(t)\| \leqq K$ by the strong continuity of $\dot{B}(t)$. Therefore the value of the above integral is changed only slightly if we replace t_j by t; more precisely, we have

$$(3.30) \qquad \| W^{(p)}(t,a;\Delta)x - \int_a^t V(t,s;\Delta)\dot{B}(s)C(s;\Delta)W^{(p-1)}(s,a;\Delta)x\,ds \| \leqq KM\exp(MN)|\Delta|\ \|x\|$$

since $|t-t_j| \leqq |\Delta|$.

14. We shall now show that

$$(3.31) \qquad V(t,s;\Delta) \to U(t,s), \qquad |\Delta| \to 0,$$

strongly and uniformly in t, s. For this purpose it is convenient to compare $V(t,s;\Delta)$ with $U(t,s;\Delta)$ introduced in paragraph 8. It is clear that $V(t_j,t_{k-1};\Delta) = U(t_j,t_{k-1};\Delta) = U_{j,k}$. Hence we have for $t_{k-1} \leqq s < t_k$, $t_j \leqq t < t_{j+1}$ and $s \leqq t$

$$U(t,s;\Delta) - V(t,s;\Delta) = U(t,s;\Delta) - U(t_j,t_{k-1};\Delta)$$

$$= \begin{cases} [U(t,s;\Delta) - U(t_j,s;\Delta)] + [U(t_j,s;\Delta) - U(t_j,t_{k-1};\Delta)], & t_{k-1} < t_j, \\ [U(t,s;\Delta) - U(t,t_j;\Delta)] + [U(t,t_j;\Delta) - I], & t_{k-1} = t_j, \end{cases}$$

and hence by (3.17) and (3.18) it follows that

$$\| U(t,s;\Delta)x - V(t,s;\Delta)x \| \leqq |\Delta| [\exp(MN) + M] \| A_0 x \|$$

provided $x \in \mathfrak{D}$. Thus the left-hand side tends to zero uniformly in s, t when $|\Delta| \to 0$ for $x \in \mathfrak{D}$. But as $U(t,s;\Delta) - V(t,s;\Delta)$ is uniformly bounded, it converges strongly to zero. Since $U(t,s;\Delta) \to U(t,s)$ holds strongly and uniformly in s,t as we have shown in paragraph 8, this proves (3.31).

(3.31) shows in particular that $W^{(0)}(t,a;\Delta) \to U(t,a)$ strongly and uniformly in t. We shall prove in general that

$$(3.32) \qquad W^{(p)}(t,a;\Delta) \to W^{(p)}(t,a), \qquad |\Delta| \to 0, \qquad p = 0,1,2,\cdots,$$

strongly and uniformly in t, where $W^{(p)}(t,a)$ are operators defined by the recurrence formulas:

$$\begin{aligned} &W^{(0)}(t,a)=U(t,a), \\ (3.33) \quad &W^{(p)}(t,a)x=\int_a^t U(t,s)\,\dot{B}(s)C(s)\,W^{(p-1)}(s,a)x\,ds, \\ &\qquad C(t)=A_0A(t)^{-1}, \quad p=1,2,\cdots. \end{aligned}$$

First we note that $W^{(p)}(t,a)$ are well defined and strongly continuous in t. This is easily shown by induction, for $U(t,s)$, $\dot{B}(s)$, and $C(s)$ are all strongly continuous in t,s. Next we prove (3.32) also by induction. As it is already proved for $p=0$, we assume that it is already proved for $p-1$ and prove it for p. It is clear that $C(t;\varDelta)\to C(t)$, $|\varDelta|\to 0$, holds strongly and uniformly in t, for $C(t)$ is continuous even in the sense of the norm $\|\ \|$ as was shown in paragraph 4 and $C(t;\varDelta)=C(t_j)$ for $t_j\leqq t<t_{j+1}$. Thus we have $V(t,s;\varDelta)\to U(t,s)$, $C(s;\varDelta)\to C(s)$, $W^{(p-1)}(s,a;\varDelta)\to W^{(p-1)}(s,a)$ strongly and uniformly in s,t. Noting that these operator functions are uniformly bounded and that their limits are strongly continuous in s,t, we conclude by Lemma 4 applied successively that

$$V(t,s;\varDelta)\dot{B}(s)C(s;\varDelta)\,W^{(p-1)}(s,a;\varDelta)\to U(t,s)\dot{B}(s)C(s)\,W^{(p-1)}(s,a)$$

strongly and uniformly in s,t. On making $|\varDelta|\to 0$ in (3.30) and noting (3.33) we therefore obtain (3.32) and complete the induction.

15. (3.32) shows that each term on the right-hand side of (3.28), being majorized by the corresponding term of the series of $\exp(MN)$ independent of $\varDelta$, converges strongly to $W^{(p)}(t,a)$. It follows that

$$(3.34) \qquad W(t,a;\varDelta)\to W(t,a)=\sum_{p=0}^{\infty} W^{(p)}(t,a), \qquad |\varDelta|\to 0,$$

where

$$(3.35) \quad \|W(t,a)\|\leqq\exp(MN), \quad \|W^{(p)}(t,a)\|\leqq(MN)^p/p!, \quad p=0,1,2,\cdots,$$

and $W(t,a)$ is strongly continuous in t since this is the case for all $W^{(p)}(t,a)$.

Since $W(b,a;\varDelta)=W_n=A_nU_{n,1}A_0^{-1}=A(b)\,U(\varDelta)\,A_0^{-1}$ and $U(\varDelta)\to U(b,a)$, $W(b,a;\varDelta)\to W(b,a)$ for $|\varDelta|\to 0$, we have for each $y\in\mathfrak{B}$

$$U(\varDelta)A_0^{-1}y\to U(b,a)A_0^{-1}y, \qquad A(b)U(\varDelta)A_0^{-1}y\to W(b,a)y.$$

This shows that $U(b,a)A_0^{-1}y$ belongs to the domain $\mathfrak{D}$ of the closed operator $A(b)$ and that $A(b)U(b,a)A_0^{-1}y=W(b,a)y$. In other words, we have shown that $U(b,a)\mathfrak{D}\subset\mathfrak{D}$ and $A(b)U(b,a)A_0^{-1}=W(b,a)$. It is clear that the same is true if we replace b by any t, $a\leqq t\leqq b$. Then an argument similar to that given in paragraph 10 shows that $x(t)=U(t,a)x$ for $x\in\mathfrak{D}$ is strongly differentiable to the right. But as $D^+x(t)=A(t)x(t)=W(t,a)A_0x$ is continuous in t, we must have

$$x(t')-x(t)=\int_t^{t'}A(t)x(t)\,dt$$

and, since the integrand is continuous, it follows that $x(t)$ is actually strongly differentiable and $dx(t)/dt=A(t)x(t)$. This completes the proof of Theorem 4.

16. Finally we prove Theorem 5. We note that $W(t,s)$ can be defined for $s\leqq t$ quite similarly as $W(t,a)$ defined in (3.34). We have only to replace $W^{(p)}(t,a)$ by $W^{(p)}(t,s)$, which are to be defined by recurrence formulas similar to (3.33). It is easily shown that $W^{(p)}(t,s)$ and hence $W(t,s)$ is strongly continuous in t,s. Also we have the relation

$$A(t)U(t,s)A(s)^{-1}=W(t,s) \tag{3.36}$$

as in the preceding paragraph where this was proved for $s=a$.

Let now $f(s)$ be as in Theorem 5. According to our convention to replace $A(t)-I$ by $A(t)$, the hypothesis implies that $A(r)f(s)$ is continuous in s for some r, and hence $A(s)f(s)=A(s)A(r)^{-1}A(r)f(s)$ is also continuous in s as well as $f(s)=A(r)^{-1}A(r)f(s)$. Then (3.36) shows that $A(t)U(t,s)f(s)=W(t,s)A(s)f(s)$ is continuous in t,s as well as $U(t,s)f(s)$. It follows that the right-hand side of (1.2) can be differentiated in the usual way and we have

$$dx(t)/dt=A(t)U(t,a)x+f(t)+\int_a^t A(t)U(t,s)f(s)\,ds \tag{3.37}$$

by virtue of $U(t,t)=I$ and the result of Theorem 4. But as $A(t)^{-1}$ is bounded, we have

$$A(t)^{-1}\int_a^t A(t)U(t,s)f(s)\,ds=\int_a^t A(t)^{-1}A(t)\cdots=\int_a^t U(t,s)f(s)\,ds.$$

This shows that $\int_a^t U(t,s)f(s)\,ds$ belongs to $\mathfrak{D}$ and that

(3.38)
$$A(t)\int_a^t U(t,s)f(s)\,ds=\int_a^t A(t)\,U(t,s)f(s)\,ds.$$

Thus (3.37) shows that $x(t)$ satisfies the differential equation (1.1). It is clear that $x(a)=x$. The continuity of $dx(t)/dt$ and of $A(t)x(t)$ follows from (3.37) and (3.38), together with the above remarks. Finally, if A is an operator stated in the theorem, $AA(r)^{-1}$ is bounded by Lemma 2, so that $Ax(t)=AA(r)^{-1}A(r)A(t)^{-1}A(t)x(t)$ is continuous with $A(t)x(t)$.

§ 4. Application to differential operators.

1. As an illustration how our theorems are applied to concrete problems, let us consider the case in which $A(t)$ is an ordinary differential operator. Since we do not aim at the generality, we restrict ourselves to the case of a second order linear differential operator defined on a circle S. It is convenient to introduce on S the coordinate ξ, $-\infty<\xi<+\infty$, and identify the points ξ and $\xi+2n\pi$ $(n=0,\pm 1, \pm 2,\cdots)$. *In what follows all functions of* ξ *are assumed to be periodic with period* 2π *unless the contrary is expressly stated.*

We consider the formal differential operator T given by

(4.1)
$$T[x]=p(\xi)x''+q(\xi)x'+r(\xi)x, \qquad x=x(\xi),$$

where $'$ means $d/d\xi$ and $p(\xi)$, $q(\xi)$, $r(\xi)$ are real-valued functions with continuous derivatives of second, first, and zeroth order respectively. Further we assume that $p(\xi)>0$. The formal adjoint T^* of T is defined by

(4.2)
$$T^*[y]=p(\xi)y''+q^*(\xi)y'+r^*(\xi)y,$$
$$q^*=2p'-q, \qquad r^*=p''-q'+r.$$

In what follows we also assume that

(4.3)
$$r(\xi)\leqq 0, \qquad r^*(\xi)\leqq 0,$$

which does not affect the generality, for otherwise we have only to add a negative constant to $r(\xi)$.

Then the Green function $G_\lambda(\xi,\eta)$ for the differential equation $T_\lambda[x]=T[x]-\lambda x=0$ (with the periodic boundary conditions) exists if $\lambda>0$.

To show this, we have only to show that there is no non-trivial (periodic) solution of $T_\lambda[x]=0$, and this follows from the fact that the solution of $T_\lambda[x]=0$ has neither a positive maximum nor a negative minimum. Similarly the Green function $G^*_\lambda(\xi,\eta)$ for $T^*_\lambda[y]=T^*[y]-\lambda y=0$ exists for $\lambda>0$ and we have $G^*_\lambda(\xi,\eta)=G_\lambda(\eta,\xi)$. The Green function has the important property

$$G_\lambda(\xi,\eta)\geqq 0, \qquad \lambda>0. \tag{4.4}$$

To show this, we note that $G_\lambda(\xi,\eta)$, as a function of ξ, satisfies the differential equation $T_\lambda[G_\lambda]=0$ except at $\xi=\eta+2n\pi$. It follows as above that G_λ has no negative minimum for $\xi\neq\eta+2n\pi$, and it is clear by the characteristic singularity

$$G'_\lambda(\eta+0,\eta)-G'_\lambda(\eta-0,\eta)=-1/p(\eta) \tag{4.5}$$

that G_λ has no minimum at $\xi=\eta+2n\pi$ too, thus proving (4.4).

We next show that

$$\int_0^{2\pi} G_\lambda(\xi,\eta)\,d\xi\leqq\lambda^{-1}, \qquad \int_0^{2\pi} G_\lambda(\xi,\eta)\,d\eta\leqq\lambda^{-1}. \tag{4.6}$$

For this purpose we take the formula

$$\int_0^{2\pi}\left(T_\lambda[x]y-xT^*_\lambda[y]\right)d\xi=[px'y-x(py)'+qxy]_{\eta-0}^{\eta+0}$$

and set $x(\xi)=G_\lambda(\xi,\eta)$, $y(\xi)=1$. Then we have by (4.5)

$$\int_0^{2\pi}[\lambda-r^*(\xi)]G_\lambda(\xi,\eta)\,d\xi=1,$$

and the first inequality of (4.6) follows by $r^*(\xi)\leqq 0$. The second inequality can be proved by exchanging T and T^*.

2. We now introduce a Banach space $\mathfrak{B}$. In the following we take as $\mathfrak{B}$ one of the complex function spaces $L_p(S)$, $1\leqq p<\infty$, and $C(S)$. Let $\mathfrak{D}_1$ be the totality of complex-valued functions $x(\xi)$ with continuous derivative of the second order. Then $\mathfrak{D}_1$ is dense in $\mathfrak{B}$ and the operator A_1 defined by $A_1x=T[x]$ for $x\in\mathfrak{D}_1$ is a linear operator in $\mathfrak{B}$, and we have the following

THEOREM 6. *The closure A of A_1 exists and has property S*

(introduced in Definition 1). *The domain $\mathfrak{D}(A)$ of A is the totality of $x \in \mathfrak{B}$ such that x and x' are absolutely continuous and $x'' \in \mathfrak{B}$.*

PROOF. If $x \in \mathfrak{D}_1$, $(\lambda I - A_1)x = -T_\lambda[x] = y$, we have

$$x(\xi) = \int_0^{2\pi} G_\lambda(\xi, \eta) y(\eta) d\eta \tag{4.7}$$

by the fundamental property of the Green function. Conversely, to each continuous function $y(\xi)$, (4.7) defines a function $x(\xi)$ belonging to $\mathfrak{D}_1$ and $(\lambda I - A_1)x = y$, so that the range of $\lambda I - A_1$ is dense in $\mathfrak{B}$. Moreover, it is well known that the integral operator G_λ with the kernel $G_\lambda(\xi, \eta)$ is defined everywhere in $\mathfrak{B}$ and bounded. Hence $(\lambda I - A_1)^{-1}$ exists and is a contraction of G_λ, and it is easily seen that the closure A of A_1 exists and $(\lambda I - A)^{-1} = G_\lambda$. Thus $\mathfrak{D}(A)$ is the totality of x expressed by (4.7) with $y \in \mathfrak{B}$. It follows easily that $\mathfrak{D}(A)$ is characterized by the properties stated in the theorem. Since $G_\lambda = (\lambda I - A)^{-1}$ is bounded for $\lambda > 0$, the resolvent set of A contains the positive real axis. In order to prove that A has property S, it only remains to show that $\|G_\lambda\| \leqq \lambda^{-1}$. In the case $\mathfrak{B} = L_p$, $1 < p < \infty$, this is shown by the following inequalities: for x, y of (4.7)

$$|x(\xi)| \leqq \left[\int_0^{2\pi} G_\lambda(\xi, \eta) d\eta\right]^{\frac{p-1}{p}} \left[\int_0^{2\pi} G_\lambda(\xi, \eta) |y(\eta)|^p d\eta\right]^{\frac{1}{p}},$$

$$\|x\|^p = \int_0^{2\pi} |x(\xi)|^p d\xi \leqq \lambda^{-(p-1)} \int_0^{2\pi} G_\lambda(\xi, \eta) |y(\eta)|^p d\eta d\xi = \lambda^{-(p-1)} \lambda^{-1} \|y\|^p,$$

where use is made of (4.4) and (4.6). In the case $\mathfrak{B} = L_1$ or C, the proof is similar and even simpler.

REMARK 1. Theorem 6 shows, in particular, that the domain $\mathfrak{D}(A)$ is independent of $p(\xi)$, $q(\xi)$, $r(\xi)$, as long as these functions satisfy the general conditions stated above.

REMARK 2. If $\mathfrak{B} = C(S)$, A coincides with A_1 and $\mathfrak{D}(A)$ with $\mathfrak{D}_1$.

3. With these preparations, we proceed to the consideration of a formal differential operator

$$T_t[x] = p(\xi, t)x'' + q(\xi, t)x' + r(\xi, t)x \tag{4.8}$$

depending on a parameter t. For simplicity we assume that (p, q, r are periodic in ξ and) $\partial^2 p/\partial \xi^2$, $\partial q/\partial \xi$, r, $\partial p/\partial t$, $\partial q/\partial t$, $\partial r/\partial t$ all exist and are continuous for $-\infty<\xi<+\infty$, $a\leqq t\leqq b$, and that $p>0$, $r\leqq 0$, $r^*=\partial^2 p/\partial\xi^2-\partial q/\partial\xi+r\leqq 0$.

Then the results obtained above are valid for each t. Thus we can determine from T_t a closed linear operator $A(t)$ in $\mathfrak{B}$ for each t, *which has property* S *and has domain* $\mathfrak{D}$ *independent of* t, as is seen from Remark 1 above. The condition C_1 and the first part of C_2 are obviously satisfied. To see if other conditions are also fulfilled, we note that $A(t)$ can be written as

$$A(t)=P(t)D_2+Q(t)D_1+R(t),$$

where the operators $D_1=d/d\xi$, $D_2=d^2/d\xi^2$ are assumed to have domain $\mathfrak{D}$, and $P(t)$ etc. are multiplicative operators defined by $[P(t)x](\xi)=p(\xi,t)x(\xi)$ etc. It is clear that D_2 is identical with A of the preceding paragraph for the special case $p(\xi)=1$, $q(\xi)=r(\xi)=0$ so that D_2 is closed. It is easily seen that D_1 has a closed extension. Thus $D_2[A(s)-I]^{-1}$ and $D_1[A(s)-I]^{-1}$ are bounded by Lemma 2. Furthermore, it is easily seen that $P(t)$, $Q(t)$, $R(t)$ are bounded operators, differentiable in t in the sense of the norm $\|\ \|$, and that $dP(t)/dt$ etc. are continuous in t in the sense of $\|\ \|$. Noting that

$$A(t)[A(s)-I]^{-1}=P(t)D_2[A(s)-I]^{-1}$$
$$+Q(t)D_1[A(s)-I]^{-1}+R(t)[A(s)-I]^{-1},$$

it follows easily that the rest of C_2 and C_3, C_4 are also satisfied.

Thus Theorems 1, 4, and 5 are applicable; we see that the differential equation

$$\frac{\partial x}{\partial t}=p(\xi,t)\frac{\partial^2 x}{\partial\xi^2}+q(\xi,t)\frac{\partial x}{\partial\xi}+r(\xi,t)x+f(\xi,t) \tag{4.9}$$

has a unique solution $x(\xi,t)$ for any initial value $x(\xi)=x(\xi,a)$ belonging to $\mathfrak{D}$ at least in a generalized sense, provided that f and $\partial^2 f/\partial\xi^2$ are strongly continuous in t (note that we may take D_2 instead of $A(r)$ of Theorem 5). In general it is difficult to decide how far $x(\xi,t)$ satisfies the concrete differential equation (4.9). However, the situation is rather simple if $\mathfrak{B}=C(S)$, for in this case the existence of the derivative $dx(t)/dt$ in the strong sense implies that of $\partial x(\xi,t)/\partial t$ in the usual sense, and the strong continuity of $x(t)$ and $dx(t)/dt$ implies that $x(\xi,t)$ and $\partial x(\xi,t)/\partial t$ are continuous in ξ and t simultaneously.

We therefore consider the case $\mathfrak{B}=C(S)$ and assume that the initial value $x=x(\xi)$ has continuous derivative of second order so that $x\in\mathfrak{D}$. Further let $\partial^2 f/\partial\xi^2$ exist and be continuous in ξ, t with f itself. Then $D_2 f$ is strongly continuous in t, and the hypotheses of Theorem 5 are fulfilled. It follows that there exists a unique solution of the differential equation (4.9) in the ordinary sense. Moreover, Theorem 5 shows that 1) $dx(t)/dt$ is strongly continuous in t and that 2) $x(t)\in\mathfrak{D}$ for each t and $D_1 x(t)$, $D_2 x(t)$ are strongly continuous in t, for D_1 and D_2 satisfy the condition for the operator A of that theorem. Translating these properties into ordinary language, we see that 1) $\partial x(\xi,t)/\partial t$ is continuous in ξ, t and 2) $\partial x(\xi,t)/\partial\xi$ and $\partial^2 x(\xi,t)/\partial\xi^2$ are continuous in ξ, t. In this way we have proved the following theorem as an application of our general theory.

THEOREM 7. *Let $p(\xi,t)$, $q(\xi,t)$, $r(\xi,t)$, $f(\xi,t)$ be defined for $-\infty<\xi<+\infty$, $a\leqq t\leqq b$ and periodic in ξ with period L. Let p, $\partial p/\partial\xi$, $\partial^2 p/\partial\xi^2$, q, $\partial q/\partial\xi$, r, f, $\partial^2 f/\partial\xi^2$, $\partial p/\partial t$, $\partial q/\partial t$, $\partial r/\partial t$ exist and be continuous in ξ and t simultaneously. If $x(\xi)$ is periodic with period L and has continuous derivative of second order, the differential equation* (4.9) *has one and only one periodic solution such that $x(\xi,a)=x(\xi)$ and $x(\xi,t)$, $\partial x(\xi,t)/\partial\xi$, $\partial^2 x(\xi,t)/\partial\xi^2$, $\partial x(\xi,t)/dt$ are continuous in ξ and t simultaneously.*

4. In this way we see that our general theorems give fairly satisfactory results when applied to ordinary differential operators. It seems, however, that matters are not so simple when $A(t)$ is a partial differential operator. The method stated above may perhaps be used to show that, under certain general conditions, that $A(t)$ has property S for each t. But the characterization of the domain of $A(t)$ is not so simple as above, at least for the spaces $\mathfrak{B}=L_1$ and C. Thus we do not know at present whether the domain of $A(t)$ is independent of t or not, although it can be shown that this is the case at least for $\mathfrak{B}=L_2$. The writer wishes to discuss these problems elsewhere.

In conclusion the writer wishes to express his hearty thanks to Professor K. Yosida for his interest in this work and valuable suggestions.

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Note (added in proof). The proof of the continuity of $U(t, s)$ given at the end of § 3.9 was incomplete, since it covers only the case $s < t$. However, a complete proof can easily be derived directly from (3.20).