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# On criteria for the regularity of Dirichlet problem.

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In I, we shall deduce some consequences from Wiener's criterion for the regularity of Dirichlet problem in space. For Newtonian potentials the following maximum principle is used frequently: Let $u_1(P)$, $u_2(P)$ be two Newtonian potentials definded by a positive mass distribution on a bounded closed set $E$. If $u_1(P) \leqq u_2(P)$ on $E$, except a set of Newtonian capacity zero, then the same relation holds in the complement of $E$. Since an analogous theorem does not hold for logarithmic potentials, the proof for Newtonian potentials must be modified for logarithmic potentials. Hence in II, we shall prove an analogue of de la Vallée-Poussin's criterion of regularity for a planar region and from it deduce Wiener's criterion and some consequences of it.

## I. Regularity for a spatial domain.

1. Let $D$ be an infinite domain in space and $\Gamma$ be its boundary, which we assume a bounded closed set and $P_0$ be a boundary point of $D$. Let $E$ be the complement of $D$ with respect to the whole space. We denote the part of $D$, which lies in a sphere $S_\rho$ of radius $\rho$ about $P_0$ by $D_\rho$ and $\Gamma_\rho$, $E_\rho$ be that of $\Gamma$, $E$ respectively, which lies in and on $S_\rho$.

We denote the Newtonian capacity of a set $M$ by $C(M)$. In I, "capacity" means "Newtonian capacity".

Let

$$W_\rho = \frac{1}{C(E_\rho)} \tag{1}$$

and

$$w_\rho(P) = \int_{E_\rho} \frac{d\omega(Q)}{r_{PQ}}, \quad \int_{E_\rho} d\omega(Q) = 1, \tag{2}$$

be the equilibrium potential of $E_\rho$, such that $w_\rho(P)\leqq W_\rho$ for any $P$ and $w_\rho(P)=W_\rho$ on $E_\rho$, except a set of capacity zero.

Let

$$u_\rho(P)=\frac{1}{W_\rho}\cdot w_\rho(P)=\int_{E_\rho}\frac{d\mu(Q)}{r_{PQ}}, \quad \int_{E_\rho}d\mu(Q)=C(E_\rho) \tag{3}$$

be the capacity potential of $E_\rho$, such that $u_\rho(P)\leqq 1$ for any $P$ and $u_\rho(P)=1$ on $E_\rho$, except a set of capacity zero.

Let $0<\sigma<\rho$ and $E_{\sigma,\rho}$ be the part of $E$, which is contained in a ring domain: $\sigma\leqq r\leqq\rho$, $r=PP_0$ and $u_{\sigma,\rho}(P)$ be its capacity potential. Then

THEOREM 1. $\lim\limits_{\sigma\to 0} u_{\sigma,\rho}(P_0)=u_\rho(P_0)$.

PROOF. By the maximum principle,

$$u_{\sigma,\rho}(P)\leqq u_\rho(P) \tag{1}$$

in the complement of $E_\rho$. Since $u_{\sigma,\rho}(P)\leqq 1$ and $u_\rho(P)=1$ on $E_\sigma$, except a set of capacity zero, (1) holds in the inside $\varDelta_\sigma$ of $S_\sigma$, except a set of capacity zero. Let $dv(P)$ be the volume element, then since $u_{\sigma,\rho}(P)$ is harmonic and $u_\rho(P)$ is superharmonic in $\varDelta_\sigma$,

$$u_{\sigma,\rho}(P_0)=\frac{1}{v(\varDelta_\sigma)}\int_{\varDelta_\sigma}u_{\sigma,\rho}(P)\,dv(P)\leqq\frac{1}{v(\varDelta_\sigma)}\int_{\varDelta_\sigma}u_\rho(P)\,dv(P)\leqq u_\rho(P_0),$$

so that

$$\overline{\lim_{\sigma\to 0}}\, u_{\sigma,\rho}(P_0)\leqq u_\rho(P_0). \tag{2}$$

Let

$$\begin{aligned} w_\rho(P)&=\int_{E_\rho}\frac{d\omega_\rho(Q)}{r_{PQ}}, \quad \int_{E_\rho}d\omega_\rho=1, \quad \operatorname*{Max}_P w_\rho(P)=W_\rho, \\ w_{\sigma,\rho}(P)&=\int_{E_{\sigma,\rho}}\frac{d\omega_{\sigma,\rho}(Q)}{r_{PQ}}, \quad \int_{E_{\sigma,\rho}}d\omega_{\sigma,\rho}=1, \quad \operatorname*{Max}_P w_{\sigma,\rho}(P)=W_{\sigma,\rho} \end{aligned} \tag{3}$$

be the equilibrium potential of $E_\rho$ and $E_{\sigma,\rho}$ respectively.

We take $\sigma_1>\sigma_2>\cdots>\sigma_n\to 0$ for $\sigma$, then we can find a partial sequence from $n$, which we denote again $n$, such that

$$\omega_{\sigma_n,\rho}\to\omega \quad (n\to\infty), \qquad \int_{E_\rho}d\omega=1. \tag{4}$$

We put

$$w(P)=\int_{E_\rho}\frac{d\omega(Q)}{r_{PQ}}. \tag{5}$$

Then from $w_{\sigma,\rho}(P)\leqq W_{\sigma,\rho}$ and Fatou's lemma, we have

$$w(P)\leqq\varliminf_{n\to\infty} w_{\sigma_n,\rho}(P)\leqq W_\rho\,, \tag{6}$$

hence

$$W_\rho\geqq\int_{E_\rho} w\,d\omega_\rho=\int_{E_\rho} w_\rho\,d\omega=W_\rho\,,$$

so that $w(P)=W_\rho$ on $E_\rho$, except a set of capacity zero, hence by the uniqueness of $\omega_\rho$, $\omega\equiv\omega_\rho$, so that $w(P)\equiv w_\rho(P)$, hence by (6),

$$w_\rho(P_0)\leqq\varliminf_{\sigma\to 0} w_{\sigma,\rho}(P_0)\,,\quad\text{or}$$

$$u_\rho(P_0)\leqq\varliminf_{\sigma\to 0} u_{\sigma,\rho}(P_0)\,. \tag{7}$$

Hence from (2), (7), we have

$$u_\rho(P_0)=\lim_{\sigma\to 0} u_{\sigma,\rho}(P_0)\,. \tag{8}$$

THEOREM 2. *The necessary and sufficient condition, that $P_0$ is a regular point, is that*

$$u_\rho(P_0)=1\,.$$

PROOF. (i) If $u_\rho(P_0)=1$, then by the lower semi-continuity of $u_\rho(P)$, we have

$$\lim_{P\to P_0} u_\rho(P)=1\,. \tag{1}$$

Hence $v(P)=1-u_\rho(P)>0$ tends to zero, when $P$ tends to $P_0$ from the inside of $D$, so that by Lebesgue-Brelot's theorem[1)], $P_0$ is a regular point.

(ii) Next suppose that $P_0$ is a regular point.

Let $\Gamma$ be contained in $S_R$. We solve the Dirichlet problem for $D_R$, with the boundary value $f(Q)=\overline{P_0Q}$, where $Q$ is a boundary point of

1) M. Brelot: Familles de Perron et problème de Dirichlet. Acta de Szeged **9** (1938).

$D_R$. Let $v(P)$ be its solution. Then $\lim_{P\to P_0} v(P)=0$ and $v(P)$ takes the boundary value $f(Q)$, except a set of capacity zero. Since $r=\overline{P_0P}$ is subharmonic,

$$v(P)\geqq\overline{PP_0} \quad \text{in } D_R. \tag{2}$$

Let $u_\rho(P)$ be the capacity potential of $E_\rho(\rho<R)$, then by (2) and the maximum principle,

$$\frac{1}{\rho}v(P)\geqq 1-u_\rho(P)>0 \text{ in } D_\rho,$$

so that

$$\lim_{P\to P_0} u_\rho(P)=1. \tag{3}$$

Let $\Delta_r$ be the inside of $S_r(r<\rho)$ and $dv(P)$ be the volume element, then since $u_\rho(P)$ is superharmonic,

$$\frac{1}{v(\Delta_r)}\int_{\Delta_r} u_\rho(P)\,dv(P)\leqq u_\rho(P_0)\leqq 1.$$

Since $u_\rho(P)=1$ on $E_r$, except a set of capacity zero, if we make $r\to 0$, we have by (3),

$$u_\rho(P_0)=1. \tag{4}$$

2. Let $0<\lambda<1$ and $E_n$ $(n=0,1,2,\cdots)$ be the part of $E$, which is contained in a ring domain: $\lambda^{n+1}\leqq r\leqq\lambda^n$, $r=\overline{PP_0}$ and $\gamma_n=C(E_n)$ be its capacity and $u_n(P)$ be its capacity potential:

$$u_n(P)=\int_{E_n}\frac{d\mu_n(Q)}{r_{PQ}}, \qquad \int_{E_n} d\mu_n=\gamma_n. \tag{1}$$

Then de la Vallée-Poussin[2] proved:

THEOREM 3. *The necessary and sufficient condition, that $P_0$ is a regular point, is that*

$$\sum_{n=0}^{\infty} u_n(P_0)=\infty.$$

Since by (1)

$$\frac{\gamma_n}{\lambda^n}\leqq u_n(P_0)\leqq\frac{\gamma_n}{\lambda^{n+1}},$$

2) C. de la Vallée-Poussin: Points irréguliers. Détermination des masses par potentiels. Bulletin de la classe des sciences. Académie royale de Belgique. 24 (1938).

we have Wiener's criterion[3]:

THEOREM 4. *The necessary and sufficient condition, that $P_0$ is a regular point, is that*

$$\sum_{n=0}^{\infty} \frac{\gamma_n}{\lambda^n} = \infty .$$

3. We shall deduce some consequences from Theorem 4. First we shall prove a lemma.

LEMMA 1. *Let $E$ be a bounded closed set in space, $\gamma = C(E)$ be its capacity and $v = v(E)$ be its spatial measure. We project $E$ on the xy-plane and let $E_z$ be the projection and $\omega = \omega(E_z)$ be its surface measure. Then*

$$\text{(i)} \quad v \leqq \frac{9\pi}{2} \gamma^3 , \qquad \text{(ii)} \quad \omega \leqq 4\pi \gamma^2 .$$

PROOF. (i) Let

$$w(P) = \int_E \frac{d\mu(Q)}{r_{PQ}} , \qquad \int_E d\mu = 1 \tag{1}$$

be the equilibrium potential of $E$, then

$$\frac{v}{\gamma} = \int_E w(P)\, dv(P) = \int_E d\mu(Q) \int_E \frac{dv(P)}{r_{PQ}} .$$

Since

$$\int_E \frac{dv(P)}{r_{PQ}} \leqq 2\pi \left( \frac{3v}{4\pi} \right)^{\frac{2}{3}} \text{[4]} ,$$

we have

$$v \leqq \frac{9\pi}{2} \gamma^3 . \tag{2}$$

(ii) Let $M$ be a bounded closed set in space. We take $n$ points $P_\mu$ $(\mu = 1, 2, \cdots, n)$ on $M$ and put

$$V_n = V_n(M) = \underset{P_\mu \in M}{\mathrm{Max.}} \binom{n}{2} \Big/ \sum_{\mu<\nu}^{1,2,\cdots,n} \frac{1}{\overline{P_\mu P_\nu}} , \tag{3}$$

3) N. Wiener: The Dirichlet problem. Journ. of Math. and Phys. Massachusetts Institute of Technology. 1924.

4) W. Sternberg: Potentialtheorie, I. Sammlung Göschen.

then as Pólya and Szegö[5] proved,

$$\lim_{n\to\infty} V_n = C(M). \tag{4}$$

We take $n$ points $P_\mu\ (\mu=1,2,\cdots,n)$ on $E_z$ and let $P_\mu$ be the projection of $P'_\mu \in E$, then $P_\mu P_\nu \leqq P'_\mu P'_\nu$, so that $V_n(E) \geqq V_n(E_z)$, hence

$$C(E) \geqq C(E_z). \tag{5}$$

Let

$$w_1(P) = \int_{E_z} \frac{d\mu_1(P)}{r_{PQ}}, \quad \int_{E_z} d\mu_1 = 1 \tag{6}$$

be the equilibrium potential of $E_z$ and $d\omega(P)$ be the surface element on the $xy$-plane, then

$$\frac{\omega}{\gamma_1} = \int_{E_z} w_1(P)\, d\omega(P) = \int_{E_z} d\mu_1(Q) \int_{E_z} \frac{d\omega(P)}{r_{PQ}}, \qquad \gamma_1 = C(E_z).$$

Since

$$\int_{E_z} \frac{d\omega(P)}{r_{PQ}} \leqq 2\pi \sqrt{\frac{\omega}{\pi}}\ {}^{6)},$$

we have by (5)

$$\omega \leqq 4\pi\gamma_1^2 \leqq 4\pi\gamma^2. \tag{7}$$

4. With the same notation as Theorem 4, let $v_n$ be the spatial measure of $E_n$.

THEOREM 5. *If*

$$\sum_{n=0}^{\infty} \frac{v_n^{\frac{1}{3}}}{\lambda^n} = \infty,$$

*then $P_0$ is a regular point.*

PROOF. By Lemma 1, $v_n^{\frac{1}{3}} \leqq \text{const.}\, \gamma_n$, so that $\sum_{n=0}^{\infty} \frac{\gamma_n}{\lambda^n} = \infty$, hence $P_0$ is a regular point, q. e. d.

Let $v(r)$ be the spatial measure of $E_r$, which is the part of $E$ contained in $S_r$.

5) G. Pólya und G. Szegö: Über die transfiniten Durchmesser (Kapazitätskonstante) von ebenen und raümlichen Mengen. Journ. f. reine und angewante Math. 165 (1931).

6) W. Sternberg: l. c. 4).

THEOREM 6. *If*

$$\int_0^1 \frac{v(r)}{r^4}\, dr = \infty ,$$

*then $P_0$ is a regular point.*

PROOF. For $\rho > 0$,

$$\int_\rho^1 \frac{v(r)}{r^4}\, dr = \left[ \frac{-v(r)}{3r^3} \right]_\rho^1 + \frac{1}{3} \int_\rho^1 \frac{dv(r)}{r^3} = O(1) + \frac{1}{3} \int_\rho^1 \frac{dv(r)}{r^3} ,$$

hence

$$\int_0^1 \frac{dv(r)}{r^3} = \infty . \tag{1}$$

Now

$$\int_{\lambda^{n+1}}^{\lambda^n} \frac{dv(r)}{r^3} \leq \frac{1}{\lambda^{3n+3}} \int_{\lambda^{n+1}}^{\lambda^n} dv(r) = \frac{v_n}{\lambda^3 \cdot \lambda^{3n}} .$$

Since $\frac{v_n}{\lambda^{3n}} \leq \frac{4\pi}{3}$, we have $\frac{v_n}{\lambda^{3n}} \leq K \frac{v_n^{\frac{1}{3}}}{\lambda^n}$, $K = \left( \frac{4\pi}{3} \right)^{\frac{1}{2}}$, so that

$$\int_{\lambda^{n+1}}^{\lambda^n} \frac{dv(r)}{r^3} \leq K \frac{v_n^{\frac{1}{3}}}{\lambda^3 \cdot \lambda^n} , \tag{2}$$

hence $\sum\limits_{n=0}^{\infty} \frac{v_n^{\frac{1}{3}}}{\lambda^n} = \infty$, so that $P_0$ is a regular point.

REMARK. If the surface measure of the part of $E$, which lies on $S_r$ is $\geq \eta r^2 (\eta > 0)$, then $v(r) \geq$ const. $r^3$, so that $P_0$ is a regular point. This is proved by Raynor.[7]

Let $E_n^{(z)}$ be the projection of $E_n$ on the $xy$-plane and $\omega_n$ be its surface measure.

THEOREM 7. *If*

$$\sum_{n=0}^{\infty} \frac{\omega_n^{\frac{1}{2}}}{\lambda^n} = \infty ,$$

*then $P_0$ is a regular point.*

7) G. E. Raynor: Dirichlet problem. Ann. of Math. 23 (1924).

PROOF. By Lemma 1, $\omega_n^{\frac{1}{2}} \leqq \text{const.}\,\gamma_n$, so that $\sum_{n=0}^{\infty} \frac{\gamma_n}{\lambda^n} = \infty$, hence $P_0$ is a regular point, q. e. d.

Let $E_r$ be the part of $E$, which lies in $S_r$ and $E_r^{(z)}$ be its projection on the $xy$-plane and $\omega(r)$ be its surface measure.

THEOREM 8. *If*

$$\int_0^1 \frac{\omega(r)}{r^3}\, dr = \infty ,$$

*then $P_0$ is a regular point.*

PROOF. Similarly as Theorem 6, we have

$$\int_0^1 \frac{d\omega(r)}{r^2} = \infty . \tag{1}$$

$$\int_{\lambda^{n+1}}^{\lambda^n} \frac{d\omega(r)}{r^2} \leqq \frac{\omega_n}{\lambda^{2n+2}} .$$

Since $\frac{\omega_n}{\lambda^{2n}} \leqq \pi$, we have $\frac{\omega_n}{\lambda^{2n}} \leqq \frac{\sqrt{\pi}\,\omega_n^{\frac{1}{2}}}{\lambda^n}$, so that

$$\int_{\lambda^{n+1}}^{\lambda^n} \frac{d\omega(r)}{r^2} \leqq \frac{\sqrt{\pi}}{\lambda^2} \cdot \frac{\omega_n^{\frac{1}{2}}}{\lambda^n} ,$$

hence $\sum_{n=0}^{\infty} \frac{\omega_n^{\frac{1}{2}}}{\lambda^n} = \infty$, so that $P_0$ is a regular point.

REMARK. If $\omega(r) \geqq \eta r^2\,(\eta > 0)$, then $P_0$ is a regular point. This is proved by Philips and Wiener.[8)]

5. Let $y = e^{-\varphi(x)}\,(0 \leqq x \leqq 1)$, where $\varphi(x) > 0$ is a continuous decreasing function of $x$, such that $\lim_{x \to +0} \varphi(x) = \infty$.

We rotate the part of the plane defined by $0 \leqq y \leqq e^{-\varphi(x)}$, $0 \leqq x \leqq 1$ about the $x$-axis, then we obtain a domain $\varDelta$. Let $D$ be the complement of $\varDelta$ with respect to the inside of a cube, whose center is the origin $O$ and whose one face is the plane $x = 1$.

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8) Phillips and Wiener: Nets and the Dirichlet problem. Journ. Math. and Phys. Massachusetts Institute of Technology. 1923.

THEOREM 9. (i) *If*

$$\int_0^1 \frac{dx}{x\varphi(x)} = \infty ,$$

*then $O$ is a regular point for $D$.*

(ii) *If*

$$\int_0^1 \frac{dx}{x\varphi(x)} < \infty ,$$

*then $O$ is an irregular point for $D$.*

A similar theorem is proved by Wiener.[9]

PROOF. The capacity $K$ of an ellipsoid $\Omega$, whose semi-axes are $a, b, c$ is given by

$$\frac{1}{K} = \frac{1}{2}\int_0^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}}\,^{10)} . \tag{1}$$

Hence if $b=c<a$, then

$$K=2ak\Big/\log\frac{1+k}{1-k} , \tag{2}$$

where

$$k=\sqrt{1-\frac{b^2}{a^2}} .$$

Since

$$\frac{1}{1-k} = \frac{a^2}{b^2}\left(1+\sqrt{1-\frac{b^2}{a^2}}\right), \tag{3}$$

we have

$$\frac{a^2}{b^2} \leqq \frac{1}{1-k} \leqq 2\frac{a^2}{b^2} ,$$

so that

$$\frac{a^2}{b^2} \leqq \frac{1+k}{1-k} \leqq 4\frac{a^2}{b^2} \leqq e^2\frac{a^2}{b^2} .$$

Hence by (2),

$$\frac{ak}{1+\log(a/b)} \leqq K \leqq \frac{ak}{\log(a/b)} \qquad (b<a). \tag{4}$$

9) N. Wiener: l. c. 3)

10) Frank v. Mises: Differentialgleichungen der Physik, I, Braunschweig. (1930), S. 611.

(i) Suppose that

$$\int_0^1 \frac{dx}{x\varphi(x)} = \infty . \tag{5}$$

First we assume that

$$e^{-\varphi(x)} \leq x, \qquad 0 \leq x \leq 1 . \tag{6}$$

Let $\varDelta_n$ be the part of $\varDelta$, which lies between two planes: $x=\lambda^{n+1}$ and $x=\lambda^n$ $(0<\lambda<1)$ and $\gamma_n$ be its capacity. Since $e^{-\varphi(x)} \leq x$, we can prove similarly as Theorem 4, that the necessary and sufficient condition, that $O$ is a regular point, is that

$$\sum_{n=0}^{\infty} \frac{\gamma_n}{\lambda^n} = \infty . \tag{7}$$

We choose $\lambda > 0$ so small that

$$1 - \frac{2\lambda}{1-\lambda} = \mu^2 > 0 . \tag{8}$$

Let $M$ be the middle point of two points $(\lambda^{n+1}, 0, 0)$, $(\lambda^n, 0, 0)$ on the $x$-axis and $\Omega$ be the ellipsoid of rotation about the $x$-axis with $M$ as its center and whose semi-axes are

$$a = \frac{1}{2}(\lambda^n - \lambda^{n+1}), \qquad b = e^{-\varphi(\lambda^{n+1})}, \tag{9}$$

then $\Omega$ is contained in $\varDelta_n$. Then by (6)

$$\frac{b}{a} = \frac{2e^{-\varphi(\lambda^{n+1})}}{\lambda^n - \lambda^{n+1}} \leq \frac{2\lambda^{n+1}}{\lambda^n - \lambda^{n+1}} = \frac{2\lambda}{1-\lambda} < 1 , \tag{10}$$

$$k = \sqrt{1 - \frac{b^2}{a^2}} > \sqrt{1 - \frac{b}{a}} > \sqrt{1 - \frac{2\lambda}{1-\lambda}} = \mu . \tag{11}$$

Since $0 < a < 1$, we have by (4),

$$\gamma_n \geq C(\Omega) \geq \frac{a\mu}{1 + \log(1/b)} > \text{const.} \frac{\lambda^n}{\varphi(\lambda^{n+1})} ,$$

so that

$$\sum_{n=0}^{\infty} \frac{\gamma_n}{\lambda^n} \geqq \text{const.} \sum_{n=0}^{\infty} \frac{1}{\varphi(\lambda^n)} . \tag{12}$$

Since

$$\int_{\lambda^{n+1}}^{\lambda^n} \frac{dx}{x\varphi(x)} \leqq \frac{1}{\varphi(\lambda^n)} \log 1/\lambda ,$$

$\sum_{n=0}^{\infty} \frac{1}{\varphi(\lambda^n)} = \infty$, so that $\sum_{n=0}^{\infty} \frac{\gamma_n}{\lambda^n} = \infty$, hence $O$ is a regular point.

If the condition $e^{-\varphi(x)} \leqq x$, $0 \leqq x \leqq 1$ is not satisfied, we define $\varphi_1(x)$ by

$$e^{-\varphi_1(x)} = \text{Min.}\,(e^{-\varphi(x)},\ x) , \tag{13}$$

then $\varphi_1(x)$ is a continuous decreasing function of $x$, such that $\lim_{n\to+0} \varphi_1(x) = \infty$ and $e^{-\varphi_1(x)} \leqq x$, $0 \leqq x \leqq 1$.

We shall prove that

$$\int_0^1 \frac{dx}{x\varphi_1(x)} = \infty . \tag{14}$$

Let $E_1$ be the set of $x$, such that $e^{-\varphi(x)} \leqq x$ and $E_2$ be that $e^{-\varphi(x)} > x$, then

$$\begin{aligned} \varphi_1(x) &= \varphi(x) \quad \text{on } E_1 , \\ &= \log \frac{1}{x} \quad \text{on } E_2 , \end{aligned} \tag{15}$$

so that

$$\int_0^1 \frac{dx}{x\varphi_1(x)} = \int_{E_1} \frac{dx}{x\varphi(x)} + \int_{E_2} \frac{dx}{x \log(1/x)} . \tag{16}$$

If $\int_{E_2} \frac{dx}{x\log(1/x)} = \infty$, then $\int_0^1 \frac{dx}{x\varphi_1(x)} = \infty$. Now $E_2$ consists of at most a countable number of open intervals. Let $(x_1, x_2)$ be one of them.

If $\int_{E_2} \frac{dx}{x\log(1/x)} < \infty$, then

$$\int_{x_1}^{x_2} \frac{dx}{x\log(1/x)} = \log\left(\frac{\log(1/x_1)}{\log(1/x_2)}\right) \to 0 , \qquad x_1, x_2 \to 0 ,$$

or

$$\frac{\log(1/x_1)}{\log(1/x_2)} \to 1, \qquad x_1, x_2 \to 0. \tag{17}$$

Since $\frac{1}{\log(1/x_1)} \leqq \frac{1}{\varphi(x)} \leqq \frac{1}{\log(1/x_2)}$, $\frac{1}{\log(1/x_1)} \leqq \frac{1}{\log(1/x)} \leqq \frac{1}{\log(1/x_2)}$ in $x_1 \leqq x \leqq x_2$,

$$\frac{\log(x_2/x_1)}{\log(1/x_1)} \leqq \int_{x_1}^{x_2} \frac{dx}{x\varphi(x)} \leqq \frac{\log(x_2/x_1)}{\log(1/x_2)},$$

$$\frac{\log(x_2/x_1)}{\log(1/x_1)} \leqq \int_{x_1}^{x_2} \frac{dx}{x\log(1/x)} \leqq \frac{\log(x_2/x_1)}{\log(1/x_2)},$$

so that by (17),

$$\int_{x_1}^{x_2} \frac{dx}{x\varphi(x)} \leqq \text{const.} \int_{x_1}^{x_2} \frac{dx}{x\log(1/x)},$$

so that $\int_{E_2} \frac{dx}{x\varphi(x)} < \infty$, consequently $\int_{E_1} \frac{dx}{x\varphi(x)} = \infty$, hence $\int_0^1 \frac{dx}{x\varphi_1(x)} = \infty$. Hence in any case (14) holds.

Let $\varDelta_1$ and $D_1$ be defined for $y = e^{-\varphi_1(x)}$ as $\varDelta$ and $D$ are defined for $y = e^{-\varphi(x)}$, then $O$ is a regular point for $D_1$. Since $D \subset D_1$, $O$ is a regular point for $D$.

(ii) Next Suppose that

$$\int_0^1 \frac{dx}{x\varphi(x)} < \infty. \tag{18}$$

Then

$$\int_x^{\sqrt{x}} \frac{dx}{x\varphi(x)} \geqq \frac{1}{2\varphi(x)} \log(1/x) \to 0, \qquad x \to 0,$$

so that for a small $x_0$,

$$e^{-\varphi(x)} \leqq x^2 \qquad (0 < x \leqq x_0), \tag{19}$$

hence $e^{-\varphi(x)} \leqq x$ in a neighbourhood of $x = 0$.

Let $\omega$ be an ellipse on the $xy$-plane, whose center is $(\lambda^n, 0)$ and whose one principal axis coincides with the $x$-axis and whose semi-axis, which lies on the $x$-axis, is

$$a = 2\lambda^n \tag{20}$$

and passes through a point $(0, e^{-\varphi(\lambda^n)})$, then we can prove easily that another semi-axis is

$$b=\frac{2}{\sqrt{3}}\,e^{-\varphi(\lambda^n)}, \tag{21}$$

so that by (19),

$$b\leqq\frac{2}{\sqrt{3}}\,\lambda^{2n}=\frac{1}{2\sqrt{3}}\,a^2, \tag{22}$$

hence $b<a$ for a large $n$.

We rotate $\omega$ about the $x$-axis, then we obtain an ellipsoid of rotation $\Omega$. $\Omega$ contains $\Delta_n$ in its inside.

Hence by (4), since $k<1$, we have in virtue of (22),

$$\gamma_n\leqq C(\Omega)\leqq\frac{a}{\log(a/b)}\leqq\text{const.}\frac{a}{\log(1/\sqrt{b})}\leqq\text{const.}\frac{\lambda^n}{\varphi(\lambda^n)},$$

so that by (18)

$$\sum_{n=0}^{\infty}\frac{\gamma_n}{\lambda^n}\leqq\text{const.}\sum_{n=0}^{\infty}\frac{1}{\varphi(\lambda^n)}<\infty. \tag{23}$$

Hence $O$ is an irregular point.

REMARK. If the rotating curve is $y=Ax^{\alpha}$ $(\alpha>0)$, then $O$ is a regular point[11] and if $y=e^{-\frac{a}{x}}$ $(a>0)$, then $O$ is an irregular point[12], which is the well-known Lebesgue's example.

## II. Regularity for a planar domain.

1. Let $D$ be an infinite domain on the $z=x+iy$-plane and $\Gamma$ be its boundary, which we assume a bounded closed set and $z_0$ be a boundary point of $D$. Let $E$ be the complement of $D$ with respect to the whole $z$-plane. We denote the part of $D$, which lies in $|z-z_0|<\rho$ by $D_\rho$, and let $\Gamma_\rho$, $E_\rho$ be that of $\Gamma$, $E$, which lies in $|z-z_0|\leqq\rho$ respectively.

We denote the logarithmic capacity of a set $M$ by $C(M)$. In II, "capacity" means "logarithmic capacity".

11) E. Hopf: Bemerkungen zum ersten Randwertproblem der Potentialtheorie im Raume. Sitzungsberichte d. Berliner Math. Gesellschaft (1927).

12) Cf. R. Courant-D. Hilbert: Methoden der mathematischen Physik. II. Berlin (1937), S. 272.

Let

$$W_\rho = \log \frac{1}{C(E_\rho)} \tag{1}$$

and

$$w_\rho(z) = \int_{E_\rho} \log \frac{1}{|z-a|}\, d\omega(a), \quad \int_{E_\rho} d\omega = 1 \tag{2}$$

be the equilibrium potential of $E_\rho$, such that $w_\rho(z) \leq W_\rho$ for any $z$ and $w_\rho(z) = W_\rho$ on $E_\rho$, except a set of capacity zero.

Let

$$u_\rho(z) = \frac{1}{W_\rho} \cdot w_\rho(z) = \int_{E_\rho} \log \frac{1}{|z-a|}\, d\mu(a), \quad \int_{E_\rho} d\mu = \frac{1}{W_\rho} = 1\Big/\log \frac{1}{C(E_\rho)} \tag{3}$$

be the capacity potential of $E_\rho$, such that $u_\rho(z) = 1$ on $E_\rho$, except a set of capacity zero.

If $\rho < 1$, then $C(E_\rho) < 1$, so that $W_\rho > 0$, hence $u_\rho(z) \leq 1$. Hence if $\rho < 1$,

$$W_\rho > 0 \quad \text{and} \quad u_\rho(z) \leq 1 \quad \text{for any } z. \tag{4}$$

Let $0 < \sigma < \rho$ and $E_{\sigma,\rho}$ be the part of $E$, which lies in $\sigma \leq |z-z_0| \leq \rho$ and $u_{\sigma,\rho}(z)$ be its capacity potential. Then

THEOREM 10. $\lim_{\sigma \to 0} u_{\sigma,\rho}(z_0) = u_\rho(z_0)$.

PROOF. Let $w_\rho(z)$ be the equilibrium potential of $E_\rho$ and $w_{\sigma,\rho}(z)$ be that of $E_{\sigma,\rho}$. If we put

$$v_\rho(z) = W_\rho - w_\rho(z), \qquad v_{\sigma,\rho}(z) = W_{\sigma,\rho} - w_{\sigma,\rho}(z), \tag{1}$$

then by the maximum principle,

$$v_\rho(z) \leq v_{\sigma,\rho}(z) \tag{2}$$

in the complement of $E_\rho$. Since $v_\rho = 0$ on $E_\sigma$, except a set of capacity zero and $v_{\sigma,\rho} > 0$ on $E_\sigma$, (2) holds in a disc $\varDelta_\sigma$: $|z-z_0| \leq \sigma$, except a set of capacity zero.

Let $d\omega(z)$ be the surface element, then since $v_\rho$ is subharmonic and $v_{\sigma,\rho}$ is harmonic in $\varDelta_\sigma$,

$$v_\rho(z_0) \leqq \frac{1}{\omega(\Delta_\sigma)} \int_{\Delta_\sigma} v_\rho(z)\, d\omega(z) \leqq \frac{1}{\omega(\Delta_\sigma)} \int_{\Delta_\sigma} v_{\sigma,\rho}(z)\, d\omega(z) = v_{\sigma,\rho}(z_0),$$

or

$$0 \leqq W_\rho - w_\rho(z_0) \leqq W_{\sigma,\rho} - w_{\sigma,\rho}(z_0),$$

hence

$$\overline{\lim_{\sigma\to 0}}\, w_{\sigma,\rho}(z_0) \leqq w_\rho(z_0). \tag{3}$$

Similarly as Theorem 1, we can prove

$$w_\rho(z_0) \leqq \underline{\lim}_{\sigma\to 0}\, w_{\sigma,\rho}(z_0), \tag{4}$$

so that $\lim_{\sigma\to 0} w_{\sigma,\rho}(z_0) = w_\rho(z_0)$, hence

$$\lim_{\sigma\to 0} u_{\sigma,\rho}(z_0) = u_\rho(z_0), \qquad \text{q. e. d.}$$

Similarly as Theorem 2, we can prove

THEOREM 11. *The necessary and sufficient condition, that $z_0$ is a regular point, is that*

$$u_\rho(z_0) = 1.$$

2. Let $0 < \lambda < 1$ and $E_n\ (n=0,1,2,\cdots)$ be the part of $E$, which is contained in $\lambda^{n+1} \leqq |z-z_0| \leqq \lambda^n$ and $\gamma_n = C(E_n)$ be its capacity and $u_n(z)$ be its capacity potential:

$$u_n(z) = \int_{E_n} \log \frac{1}{|z-a|}\, d\mu_n(a), \quad \int_{E_n} d\mu_n = \mu_n = 1/\log \frac{1}{\gamma_n}. \tag{1}$$

By de la Vallée-Poussin's method, we shall prove

THEOREM 12. *The necessary and sufficient condition, that $z_0$ is a regular point, is that*

$$\sum_{n=0}^{\infty} u_n(z_0) = \infty.$$

PROOF. (i) Suppose that $\sum_{n=0}^{\infty} u_n(z_0) < \infty$. We put

$$M_{n_0} = (z_0) + \sum_{n=n_0}^{\infty} E_n \tag{2}$$

and let $u(z)$ be its capacity potential:

$$u(z)=\int_{M_{n_0}} \log \frac{1}{|z-a|}\, d\sigma(a)=\sum_{n=n_0}^{\infty} v_n(z), \tag{3}$$

where

$$v_n(z)=\int_{E_n} \log \frac{1}{|z-a|}\, d\sigma(a), \quad \int_{E_n} d\sigma=\sigma_n. \tag{4}$$

Since for a large $n_0$, $M_{n_0}$ is contained in $|z-z_0|\leqq \frac{1}{2}$, $0\leqq u(z)\leqq 1$ on $M_{n_0}$, hence $0\leqq v_n(z)\leqq 1$ on $E_n$. Since $u_n(z)=1$ on $E_n$, except a set of capacity zero,

$$\sigma_n=\int_{E_n} u_n\, d\sigma=\int_{E_n} v_n\, d\mu_n \leqq \mu_n .$$

Since

$$v_n(z_0)\leqq (n+1) \log 1/\lambda \cdot \sigma_n , \qquad u_n(z_0)\geqq n \log 1/\lambda \cdot \mu_n ,$$

we have

$$v_n(z_0)\leqq \text{const.}\ u_n(z_0), \tag{5}$$

so that for a large $n_0$,

$$u(z_0)=\sum_{n=n_0}^{\infty} v_n(z_0)\leqq \text{const.} \sum_{n=n_0}^{\infty} u_n(z_0)<1, \tag{6}$$

hence by Theorem 11, $z_0$ is an irregular point.

(ii) Next suppose that $\sum_{n=0}^{\infty} u_n(z_0)=\infty$ and $z_0$ is an irregular point. We put

$$M=(z_0)+\sum_{n=0}^{\infty} E_n \tag{7}$$

and let $u(z)$ be its capacity potential:

$$u(z)=\int_M \log \frac{1}{|z-a|}\, d\mu(a), \quad \int_M d\mu=\frac{1}{W}. \tag{8}$$

Since $z_0$ is an irregular point, by Theorem 11,

$$u(z_0)=\alpha<1. \tag{9}$$

For a positive integer $k$, we put

$$E_{(\nu)}=E_{k\nu}+E_{k\nu+1}+\cdots+E_{k(\nu+1)-1}\,, \tag{10}$$

then

$$M=(z_0)+\sum_{\nu=0}^{\infty}E_{(\nu)}\,. \tag{11}$$

Let $v_\nu(z)$ be the capacity potential of $E_{(\nu)}$:

$$v_\nu(z)=\int_{F_{(\nu)}}\log\frac{1}{|z-a|}\,d\sigma_\nu(a)\,,\quad \int_{E_{(\nu)}}d\sigma_\nu=\sigma_\nu\,. \tag{12}$$

Then we shall prove tnat

$$\sum_{\nu=0}^{\infty}v_\nu(z_0)=\infty\,. \tag{13}$$

Let $k\nu\leqq n\leqq k(\nu+1)-1$, then $E_n\subset E_{(\nu)}$.

Since $v_\nu=1$ on $E_n$, except a set of capacity zero and $u_n\leqq 1$ on $E_{(\nu)}$, we have

$$\mu_n=\int_{E_n}v_\nu\,d\mu_n=\int_{E_{(\nu)}}u_n\,d\sigma_\nu\leqq\sigma_\nu\,.$$

From this we have as (5),

$$u_n(z_0)\leqq\text{const.}\,v_\nu(z_0)\,,\quad \bigl(k\nu\leqq n\leqq k(\nu+1)-1\bigr)\,.$$

Hence $\sum_{\nu=0}^{\infty}v_\nu(z_0)=\infty$, so that $\sum_{\nu=0}^{\infty}v_{2\nu}(z_0)=\infty$, or $\sum_{\nu=0}^{\infty}v_{2\nu+1}(z_0)=\infty$.

We assume that

$$\sum_{\nu=0}^{\infty}v_{2\nu}(z_0)=\infty\,, \tag{14}$$

in the other case our Theorem can be proved similarly.

We put

$$M'=(z_0)+\sum_{\nu=0}^{\infty}E_{(2\nu)} \tag{15}$$

and let $u'(z)$ be its capacity potential:

$$u'(z)=\int_{M'}\log\frac{1}{|z-a|}\,d\mu'(a)\,,\quad \int_{M'}d\mu'=\frac{1}{W'}\,, \tag{16}$$

then

$$u'(z)=\sum_{\nu=0}^{\infty}u'_{2\nu}(z)\,, \tag{17}$$

where

$$u'_{2\nu}(z)=\int_{E_{(2\nu)}}\log\frac{1}{|z-a|}\,d\mu'(a), \qquad \int_{E_{(2\nu)}}d\mu'=\mu'_{2\nu}. \tag{18}$$

Let $M^{(\rho)}$ be the part of $M$, which lies in $0<\rho\leqq|z-z_0|\leqq 1$ and $W^{(\rho)}$ be the maximum of its equiliblium potential and $u^{(\rho)}(z)$ be its capacity potential. Similarly we define $M'^{(\rho)}$, $W'^{(\rho)}$, $u'^{(\rho)}(z)$ for $M'$.

Then, by the maximum principle,

$$W^{(\rho)}\left(1-u^{(\rho)}(z_0)\right)\leqq W'^{(\rho)}\left(1-u'^{(\rho)}(z_0)\right).$$

If we make $\rho\to 0$, then by Theorem 10,

$$W\left(1-u(z_0)\right)\leqq W'\left(1-u'(z_0)\right),$$

so that by (9), since $W'>0$,

$$u'(z_0)\leqq 1-\frac{W}{W'}(1-\alpha). \tag{19}$$

We choose $k$ so large that

$$\frac{1}{W}\cdot\log\frac{1}{1-\lambda^k}<1-\alpha, \tag{20}$$

then

$$\beta=\log\frac{1}{1-\lambda^k}\cdot\frac{1}{W'}+u'(z_0)\leqq 1-\frac{W}{W'}\left(1-\alpha-\frac{1}{W}\log\frac{1}{1-\lambda^k}\right)<1. \tag{21}$$

Let $a_\mu\in E_{(2\mu)}$, $a_\nu\in E_{(2\nu)}$ $(\mu\neq\nu)$.

If $\mu<\nu$, then $\left|\dfrac{a_\nu-z_0}{a_\mu-z_0}\right|\leqq\lambda^k$, hence

$$\left|\frac{a_\mu-a_\nu}{a_\mu-z_0}\right|\geqq\frac{|a_\mu-z_0|-|a_\nu-z_0|}{|a_\mu-z_0|}\geqq 1-\lambda^k.$$

If $\mu>\nu$, then $\left|\dfrac{a_\mu-z_0}{a_\nu-z_0}\right|\leqq\lambda^k$, $\left|\dfrac{a_\nu-z_0}{a_\mu-z_0}\right|\geqq\dfrac{1}{\lambda^k}$, so that

$$\frac{|a_\mu-a_\nu|}{|a_\mu-z_0|}\geqq\frac{|a_\nu-z_0|-|a_\mu-z_0|}{|a_\mu-z_0|}\geqq\frac{1}{\lambda^k}-1\geqq 1-\lambda^k.$$

Hence in any case,

$$\frac{1}{1-\lambda^k}\cdot\frac{1}{|a_\mu-z_0|}\geqq\frac{1}{|a_\mu-a_\nu|},$$

so that

$$\log\frac{1}{1-\lambda^k}\int_{F_{(2\mu)}}d\mu'(a)+\int_{F_{(2\mu)}}\log\frac{1}{|a-z_0|}\,d\mu'(a)\geqq\int_{E_{(2\mu)}}\log\frac{1}{|a-a_\nu|}\,d\mu'(a).$$

Summing up for $\mu=0,1,2,\cdots,\mu\neq\nu$, we have since $\int_{M'}d\mu'=1/W'$,

$$\beta=\log\frac{1}{1-\lambda^k}\cdot\frac{1}{W'}+u'(z_0)\geqq u'(a_\nu)-u'_{2\nu}(a_\nu).$$

Since $u'(a_\nu)=1$ on $E_{(2\nu)}$, except a set of capacity zero, we have

$$u'_{2\nu}(z)\geqq 1-\beta>0 \text{ on } E_{(2\nu)}, \tag{22}$$

except a set of capacity zero. Since $v_{2\nu}=1$ on $E_{(2\nu)}$, except a set of capacity zero,

$$\mu'_{2\nu}=\int_{E_{(2\nu)}}v_{2\nu}\,d\mu'=\int_{E_{(2\nu)}}u'_{2\nu}\,d\sigma_{2\nu}\geqq(1-\beta)\,\sigma_{2\nu}.$$

From this we have as (5),

$$u'_{2\nu}(z_0)\geqq\text{const. }v_{2\nu}(z_0), \tag{23}$$

so that

$$1\geqq u'(z_0)=\sum_{\nu=0}^{\infty}u'_{2\nu}(z_0)\geqq\text{const.}\sum_{\nu=0}^{\infty}v_{2\nu}(z_0)=\infty,$$

which is absurd. Hence $z_0$ is a regular point.

3. Since by (1) of the proof of the last theorem,

$$\frac{n\log 1/\lambda}{\log 1/\gamma_n}\leqq u_n(z_0)\leqq\frac{(n+1)\log 1/\lambda}{\log 1/\gamma_n},$$

we have

THEOREM 13. *The necessary and sufficient condition, that $z_0$ is a regular point, is that*

$$\sum_{n=1}^{\infty} \frac{n}{\log 1/\gamma_n} = \infty .$$

Wiener's criterion[13] is somewhat different from this.

Since $x \log 1/x$ is an increasing function of $x$ in a neighbourhood of $x=0$ and $\gamma_n \leqq \lambda^n$, we have

$$\gamma_n \log 1/\gamma_n \leqq \lambda^n \log 1/\lambda^n , \quad \text{or} \quad \frac{\gamma_n}{\lambda^n} \leqq \frac{n \log 1/\lambda}{\log 1/\gamma_n} ,$$

hence

THEOREM 14. *If*

$$\sum_{n=0}^{\infty} \frac{\gamma_n}{\lambda^n} = \infty ,$$

*then $z_0$ is a regular point.*

Let $E_r$ be the part of $E$, which is contained in $|z-z_0| \leqq r$ and $\gamma(r)$ be its capacity and $u_r(z)$ be its capacity potential:

$$u_r(z) = \int_{E_r} \log \frac{1}{|z-a|} d\mu(a), \quad \int_{E_r} d\mu = \frac{1}{\log 1/\gamma(r)} . \tag{1}$$

Suppose that $z_0$ is an irregular point and $r=\lambda^N$. Then as (6) of the proof of Theorem 12, we have $u_r(z_0) \leqq \text{const.} \sum_{n=N}^{\infty} u_n(z_0)$, hence $\lim_{r\to 0} u_r(z_0)=0$. Since by (1),

$$u_r(z_0) \geqq \frac{\log 1/r}{\log 1/\gamma(r)} ,$$

we have

$$\lim_{r\to 0} \frac{\log 1/r}{\log 1/\gamma(r)} = 0 . \tag{2}$$

Hence

THEOREM 15. *If* $\overline{\lim_{r\to 0}} \frac{\log 1/r}{\log 1/\gamma(r)} > 0$, *especially, if* $\overline{\lim_{r\to 0}} \frac{\gamma(r)}{r} > 0$, *then $z_0$ is a regular point.*

4. We shall prove a lemma.

LEMMA 2. *Let $E$ be a bounded closed set on a plane, $\gamma=C(E)$ be its capacity and $\omega=\omega(E)$ be its surface measure.*

13) N. Wiener: l. c. 3).

*Let* $z\in E$ *and* $E^{(x)}$ *be the set of* $|z|$ $(z\in E)$ *on the positive real axis and* $L=L(E^{(x)})$ *be its linear measure. Then*

$$\text{(i)}\quad \omega\leqq e\pi\gamma^2,\quad \text{(ii)}\quad L\leqq 4\gamma.$$

PROOF. (i). Let

$$w(z)=\int_E \log\frac{1}{|z-a|}\,d\mu(a),\qquad \int_E d\mu=1 \tag{1}$$

be the equilibrium potential of $E$ and $d\omega(z)$ be the surface element, then

$$\omega\log 1/\gamma=\int_E w(z)\,d\omega(z)=\int_E d\mu(a)\int_E \log\frac{1}{|z-a|}\,d\omega(z).$$

We define $R$ by $\pi R^2=\omega$, then

$$\int_E \log\frac{1}{|z-a|}\,d\omega(z)\leqq\int_0^{2\pi}\int_0^R \log\frac{1}{r}\,rdrd\theta=\frac{\omega}{2}\log\frac{\pi}{\omega}+\frac{\omega}{2},$$

so that $\log 1/\gamma\leqq\frac{1}{2}\log\frac{\pi}{\omega}+\frac{1}{2}$, or

$$\omega\leqq e\pi\gamma^2. \tag{2}$$

(ii) By comparing transfinite diameters, we have $C(E)\geqq C(E^{(x)})$. First suppose that $E^{(x)}$ consists of a finite number of closed intervals, which is contained in $[a, b]$. Let $t=L(x)$ be the linear measure of the part of $E^{(x)}$, which is contained in the interval $[a, x]$ and $M$ be the interval $0\leqq t\leqq L$ on the $t$-axis. Let $0\leqq t_1<t_2<\cdots<t_n\leqq L$ and $t_\nu=L(x_\nu)$, then $|t_\mu-t_\nu|\leqq|x_\mu-x_\nu|$, so that $C(E^{(x)})\geqq C(M)=L/4$. Hence $C(E)\geqq L/4$. In the general case, we approximate $E^{(x)}$ by a finite sum of closed intervals and we obtain the same relation.

5. With the same notation as Theorem 13, let $\omega_n$ be the surface measure of $E_n$.

THEOREM 16. *If*

$$\sum_{n=1}^{\infty}\frac{n}{\log 1/\omega_n}=\infty,$$

*then* $z_0$ *is a regular point.*

PROOF. By Lemma 2, $\log 1/\gamma_n\leqq$const. $\log 1/\omega_n$, so that $\sum_{n=1}^{\infty}\frac{n}{\log 1/\gamma_n}=\infty$, hence $z_0$ is a regular point.

Let $E_r$ be the part of $E$, which is contained in $|z-z_0| \leqq r$ and $\omega(r)$ be its surface measure.

THEOREM 17. *If*

$$\int_0^1 \frac{\omega(r)}{r^3}\, dr = \infty ,$$

*then* $z_0$ *is a regular point.*

PROOF. Similarly as Theorem 6, we have

$$\int_0^1 \frac{d\omega(r)}{r^2} = \infty , \tag{1}$$

so that there exists $n_\nu$ $(\nu=1, 2, \cdots)$, such that

$$\int_{\lambda^{n_\nu+1}}^{\lambda^{n_\nu}} \frac{d\omega(r)}{r^2} \geqq \frac{1}{n_\nu^2} .$$

Since $\displaystyle\int_{\lambda^{n_\nu+1}}^{\lambda^{n_\nu}} \frac{d\omega(r)}{r^2} \leqq \frac{\omega_{n_\nu}}{\lambda^{2n_\nu+2}}$, we have

$$\frac{n_\nu}{\log 1/\omega_{n_\nu}} \geqq \text{const.} > 0 \ (\nu=1, 2, \cdots) , \tag{2}$$

so that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\log 1/\omega_n} = \infty$, hence $z_0$ is a regular point, q. e. d.

Let $z_0=0$ be a boundary point of a domain $D$. We define $E_r^{(x)}$ for $E_r$ as Lemma 2 and $\mu(r)$ be linear measure.

THEOREM 18[14]. *If*

$$\int_0^1 \frac{d\mu(r)}{r} = \infty ,$$

*then* $z_0=0$ *is a regular point.*

PROOF. If we put $L_n = \displaystyle\int_{\lambda^{n+1}}^{\lambda^n} d\mu(r)$, then

$$\frac{L_n}{\lambda^n} \leqq \int_{\lambda^{n+1}}^{\lambda^n} \frac{d\mu(r)}{r} \leqq \frac{L_n}{\lambda^{n+1}} ,$$

14) A. Beurling: Étude sur un problème de majoration. Thèse Uppsala (1933).

so that $\sum_{n=0}^{\infty} \frac{L_n}{\lambda^n}=\infty$. Since by Lemma 2, $L_n \leq 4\gamma_n$, we have $\sum_{n=0}^{\infty} \frac{\gamma_n}{\lambda^n}=\infty$, so that by Theorem 14, $z_0=0$ is a regular point.

6. To prove an extension of Theorem 18, we need a lemma. Let $D$ be an infinite domain on the $z$-plane and $\Gamma$ be its boundary and $z=0$ be an inner point or a boundary point of $D$.

We define $\overline{\theta}(r)$ as follows. If a circle $|z|=r$ is contained entirely in $D$, then we put $\overline{\theta}(r)=\infty$. If $|z|=r$ meets $\Gamma$, then the part of $|z|=r$, which lies in $D$ consists of at most a countable number of circular arcs $\theta_i$ $(i=1, 2, \cdots)$ and let $r\theta_i(r)$ be its arc length. Then we put

$$\overline{\theta}(r)=\sup_i \theta_i(r).$$

Let $z=0$ be an inner point of $D$. The part of $D$, which lies in $|z|<R$ consists of connected domains. Let $D_R$ be such one, which contains $z=0$ and $\Gamma_R$, $\gamma_R$ be the part of $\Gamma$ and $|z|=R$, which belong to the boundary of $D_R$ respectively, so that $\Gamma_R+\gamma_R$ is the whole boundary of $D_R$. Let $v_R(z)$ be the harmonic measure of $\gamma_R$ with respect to $D_R$, such that $v_R(z)$ is harmonic in $D_R$, $v_R(z)=1$ on $\gamma_R$ and $v_R(z)=0$ on $\Gamma_R$, except a set of capacity zero. Then I have proved in the former paper[15]:

LEMMA 3. $v_R(0) \leq C e^{-\int_0^{\alpha R} \frac{dr}{r\theta(r)}}$, $\qquad (0<\alpha<1)$,

*where* $C=\sqrt{\frac{2e}{1-\alpha}}$,

*e being the base of the natural logarithm.*

Let $z_0=0$ be a boundary point of $D$ and we define $\mu(r)$ as Theorem 18.

THEOREM 19.[16] *If*

$$\int_0^1 \frac{dr}{r\theta(r)}=\infty, \quad or \quad \int_0^1 \frac{d\mu(r)}{r\theta(r)}=\infty, \quad \theta(r)=\mathrm{Min}\left(2\pi, \overline{\theta}(r)\right),$$

*then $z_0=0$ is a regular point for $D$.*

PROOF. Let $E$ be the complement of $D$ with respect to the whole

15) M. Tsuji: A theorem on the majoration of harmonic measure and its applications. Tôhoku Math. Journ. 3 (1951).

16) Tsuji: l. c. 15).

$z$-plane and $E_{\rho,R}$ be its part, which is contained in $0<\rho\leqq|z|\leqq R$. The complement of $E_{\rho,R}$ with respect to $|z|<R$ consists of connected domains. Let $D_{\rho,R}$ be such one, that contains $z=0$ and $\gamma_R$ be the part of $|z|=R$, which belongs to the boundary of $D_{\rho,R}$.

Let $v_{\rho,R}(z)$ be the harmonic measure of $\gamma_R$ with respect to $D_{\rho,R}$, then by Lemma 3,

$$v_{\rho,R}(0)\leqq Ce^{-\int_{\rho}^{\alpha R}\frac{dr}{r\theta(r)}} \quad (0<\alpha<1),$$

so that

$$\lim_{\rho\to 0} v_{\rho,R}(0)=0.$$

Let $u_{\rho,R}(z)$ be the capacity potential of $E_{\rho,R}$, then if we take $R\leqq 1/2$, then $u_{\rho,R}(z)>0$ in $D_{\rho,R}$, so that by the maximum principle,

$$1-v_{\rho,R}(0)\leqq u_{\rho,R}(0)\leqq 1,$$

hence $\lim_{\rho\to 0} u_{\rho,R}(0)=1$, so that by Theorem 10 and 11, $z_0=0$ is a regular point.

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