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On fibre spaces in the algebraic number theory.

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Recently A. Weil ([4], [5]) has introduced successfully the concept of the fibre space into the algebraic geometry, and proved the important classification theorem. In this paper we try to establish an analogous theory on the algebraic number field.

Starting from any algebraic number field k, we shall begin with constructing an analogue $S(k)$ of the algebraic curve in § 1, and then define in § 2 the " W-variety," corresponding to the algebraic variety in the algebraic geometry. After having defined the direct product of W-varieties, and " group W-variety " in § 3, we shall introduce the fibre spaces over the W-varieties, and prove the existence theorem and the classification theorem in the last two paragraphs. A certain group of fibre spaces corresponding to that of line bundles ([3], [5]) turns out to be isomorphic with the classical group of " Strahlklasse " in k, if k has a finite degree.

We shall here concern ourselves exclusively with the multiplicative structure of algebraic number fields. In order to take the additive structure of these fields also in account, it seems necessary to have recourse to the concept of " faisceaux " or some other new ideas.

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§ 1. Construction of an analogue of the algebraic curve.

Let k be any algebraic number field of a finite or an infinite degree fixed once for all. We denote by $k_\lambda (\lambda \in \Lambda)$ the fields which are subfields of k and have finite degrees over the rational number field, and we define a semi-order $\lambda < \mu$ for $k_\lambda \subset k_\mu$ in Λ, then Λ becomes a directed set.

Let $\widetilde{S}(k), \widetilde{S}(k_\lambda)$ $(\lambda \in \Lambda)$ be the set of all finite[1] prime divisors of k, k_λ respectively. (The notation $\widetilde{S}$ will have always this meaning also with respect to other fields.) The prime divisor of $\widetilde{S}(k)$, $\widetilde{S}(k_\lambda)$ will be generally denoted by $\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{p}}_\lambda$ respectively. If $\widetilde{\mathfrak{p}}_\lambda$ is induced by $\widetilde{\mathfrak{p}}$, then we write $\widetilde{\mathfrak{p}}_\lambda = \pi_\lambda(\widetilde{\mathfrak{p}})$; if $\lambda < \mu$ and $\widetilde{\mathfrak{p}}_\lambda$ is induced by $\widetilde{\mathfrak{p}}_\mu$, then we write $\widetilde{\mathfrak{p}}_\lambda = \pi_\lambda^\mu(\widetilde{\mathfrak{p}}_\mu)$. Then we have

$$\pi_\lambda^\mu \circ \pi_\mu^\nu = \pi_\lambda^\nu \ (\lambda < \mu < \nu), \quad \pi_\lambda^\mu \circ \pi_\mu = \pi_\lambda \ (\lambda < \mu),$$

and

$$\widetilde{S}(k) = \text{proj.} \lim_{\lambda} \widetilde{S}(k_\lambda).$$

Now we shall introduce a topology into $\widetilde{S}(k)$. By a topology we shall always mean a T_1-topology, namely one in which any subset consisting of only one point is closed. First introduce into $\widetilde{S}(k_\lambda)$ the weakest topology, so that a closed set in $\widetilde{S}(k_\lambda)$ is either $\widetilde{S}(k_\lambda)$ itself or a set of finite number of points. Then the topological space $\widetilde{S}(k)$ will be determined as the projective limit of the topological spaces $\widetilde{S}(k_\lambda)$.

Now, let W be the group of all roots of unity; we denote by $\overline{W}$ the set of all elements of W and the symbols 0 and ∞. We introduce the weakest topology into $\overline{W}$, and new operations in $\overline{W}$ as follows:

$$00 = 0, \ \infty\infty = \infty, \ 0^{-1} = \infty, \ \infty^{-1} = 0,$$

$$\zeta 0 = 0 \zeta = 0, \ \zeta\infty = \infty\zeta = \infty \ \text{ for all } \zeta \in W.$$

But we do not define 0∞ and $\infty 0$.

We denote by $k(W)$ the field generated by W over k. Then we can embed the multiplicative groups of the residue class fields of $k(W)$ modulo finite primes of $k(W)$ into W as follows. We denote $k(W)$ by K. Let $\widetilde{\mathfrak{p}}$ be any finite prime divisor of K, $U(\widetilde{\mathfrak{p}})$ the group of all $\widetilde{\mathfrak{p}}$-

1) "finite" means "non-archimedean". We may take all prime divisors, but it is inessential to do so for our purpose.

units of K, $H(\tilde{\mathfrak{p}})$ the group of $\tilde{\mathfrak{p}}$-units η such that $|\eta-1|_{\tilde{\mathfrak{p}}}<1$ ($|\ |_{\tilde{\mathfrak{p}}}$ means a valuation representing $\tilde{\mathfrak{p}}$), and $K(\tilde{\mathfrak{p}})^*$ the multiplicative group of the residue class fields. (* denotes always the multiplicative group of fields.) Then we have $K(\tilde{\mathfrak{p}})^*=U(\tilde{\mathfrak{p}})/H(\tilde{\mathfrak{p}})$, $W\subset U(\tilde{\mathfrak{p}})$, Moreover we have the following lemma.

LEMMA 1. *Let K be an algebraic number field containing all roots of unity and $K(\tilde{\mathfrak{p}})^*$ the multiplicative group of the residue class field modulo a finite prime divisor $\tilde{\mathfrak{p}}$ of K. Then there exists one and only one isomorphism $\iota_{\tilde{\mathfrak{p}}}$ of $K(\tilde{\mathfrak{p}})^*$ into W such that for all $c\in K(\tilde{\mathfrak{p}})^*$ the residue class of $\iota_{\tilde{\mathfrak{p}}}(c)$ mod. $\tilde{\mathfrak{p}}$ is identical with c.*

PROOF. Let ζ be any root of unity, $\mathfrak{p}$ the prime divisor induced by $\tilde{\mathfrak{p}}$ in the field generated by ζ over the rational number field, and p the prime rational number divirible by $\mathfrak{p}$. If $\zeta\equiv1$ mod. $\tilde{\mathfrak{p}}$, then $\zeta\equiv1$ mod. $\mathfrak{p}$, therefore ζ is a p^ν-th root of unity for some ν (Cf. H. Hasse [2], S. 392). Conversely if ζ is a p^ν-th root of unity, then $\zeta\equiv1$ mod. $\tilde{\mathfrak{p}}$ (Cf. H. Hasse [2], S. 391). Hence all $(p^\nu-1)$-th roots are mutually incongruent mod. $\tilde{\mathfrak{p}}$ since $p^\nu-1$ is prime to p for any fixed ν. On the other hand, any element of $K(\tilde{\mathfrak{p}})^*$ satisfies an equation $x^{p^\nu-1}=1$ for some ν. Therefore any residue class of $K(\tilde{\mathfrak{p}})^*$ can be represented by some element of W.

Now we denote by W_p the group of all p^ν-th roots of unity ($\nu=1,2,\cdots$) and by W'_p the group of all n-th roots of unity such that n are not divisible by p. Then we have $W=W_pW'_p$, $W_p\cap W'_p=\{1\}$. It follows from the above consideration that there exists one and only one element of the group W'_p in any residue class of $K(\tilde{\mathfrak{p}})^*$. We define $\iota_{\tilde{\mathfrak{p}}}(c)$ to be the element of W'_p uniquely determined in the residue class c. Then $\iota_{\tilde{\mathfrak{p}}}$ is an isomorphism of $K(\tilde{\mathfrak{p}})^*$ into W satisfying the above condition. The uniqueness is easily seen. This completes the proof.

Now we define $f(\tilde{\mathfrak{p}})$ for $f\in k^*$ and $\tilde{\mathfrak{p}}\in\tilde{S}(k(W))$ as follows:

$$f(\tilde{\mathfrak{p}})=\begin{cases}0 & \text{if } |f|_{\tilde{\mathfrak{p}}}<1,\\ \infty & \text{if } |f|_{\tilde{\mathfrak{p}}}>1,\\ \iota_{\tilde{\mathfrak{p}}}(\bar{f}) & \text{if } |f|_{\tilde{\mathfrak{p}}}=1, \text{ and } \bar{f} \text{ is the residue class of } f \text{ mod. } \tilde{\mathfrak{p}}.\end{cases}$$

Thus the assignment $\tilde{\mathfrak{p}}\to f(\tilde{\mathfrak{p}})$ is a $\overline{W}$-valued function on $\tilde{S}(k(W))$.

Now we introduce an equivalence relation on $\tilde{S}(k(W))$ as follows: Let $\tilde{\mathfrak{p}}$ be equivalent to $\tilde{\mathfrak{q}}$ if and only if $f(\tilde{\mathfrak{p}})=f(\tilde{\mathfrak{q}})$ for all $f\in k^*$. We denote by $S(k)$ the quotient set of $\tilde{S}(k(W))$ with respect to this equivalence relation. Then the assignment $\tilde{\mathfrak{p}}\to f(\tilde{\mathfrak{p}})$ induces a $\overline{W}$-valued function on $S(k)$: we denote by $\mathfrak{p}, \mathfrak{q}, \cdots$ the elements of $S(k)$, and $f(\tilde{\mathfrak{p}})$ by $f(\mathfrak{p})$ for $\mathfrak{p}$ containing $\tilde{\mathfrak{p}}$. So we need not distinguish $f\in k^*$ and the function $\mathfrak{p}\to f(\mathfrak{p})$.

Let $\tilde{\mathfrak{p}}$ be in $\tilde{S}(k(W))$, and $\mathfrak{p}'\in \tilde{S}(k)$ induced by $\tilde{\mathfrak{p}}$. Then $\mathfrak{p}'$ depends only upon the equivalence class $\mathfrak{p}$ of $\tilde{\mathfrak{p}}$, and $\mathfrak{p}$ will be said to *lie above* $\mathfrak{p}'$. Now the canonical mappings of $\tilde{S}(k(W))$, $\tilde{S}(k_\mu(W))$ onto $\tilde{S}(k_\lambda(W))$ $(\lambda<\mu)$ induce mappings of $\tilde{S}(k), \tilde{S}(k_\mu)$ onto $\tilde{S}(k_\lambda)$ respectively, and we have

$$S(k)=\text{proj.}\lim_{\lambda} S(k_\lambda).$$

LEMMA 2. *Let $\mathfrak{p}'$ be a finite prime divisor of k. If $\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{q}}$ are two prolongations of $\mathfrak{p}'$ to $k(W)$ and $\iota_{\tilde{\mathfrak{p}}}$ and $\iota_{\tilde{\mathfrak{q}}}$ are as above, then*

$$\iota_{\tilde{\mathfrak{p}}}(k(\mathfrak{p}')^*)=\iota_{\tilde{\mathfrak{q}}}(k(\mathfrak{p}')^*).$$

We denote by Z this subgroup of W and by $k(Z)$ the field generated by Z over k. Then $f(\tilde{\mathfrak{p}})=f(\tilde{\mathfrak{q}})$ for all $f\in k^*$ if and only if the prime divisors induced by $\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{q}}$ in $k(Z)$ coincide with each other.

PROOF. The first part follows from well-known properties of algebraic extensions of a finite field. Next, if we denote by $\bar{\mathfrak{p}}$ and $\bar{\mathfrak{q}}$ the prime divisors induced by $\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{q}}$ in $k(Z)$, then we have

$$k(Z)(\bar{\mathfrak{p}})^*=k(Z)(\bar{\mathfrak{q}})^*=k(\mathfrak{p}')^*.$$

If $f(\tilde{\mathfrak{p}})=f(\tilde{\mathfrak{q}})$ for all $f\in k^*$, then $\iota_{\tilde{\mathfrak{p}}}$ and $\iota_{\tilde{\mathfrak{q}}}$ coincide on $k(\mathfrak{p}')^*$. Therefore the residue classes of any $\zeta\in Z$ mod. $\bar{\mathfrak{p}}$ and $\bar{\mathfrak{q}}$ are equal, since their images by $\iota_{\tilde{\mathfrak{p}}}$ and $\iota_{\tilde{\mathfrak{q}}}$ coincide with each other. Hence $\bar{\mathfrak{p}}=\bar{\mathfrak{q}}$ in $k(Z)$. This reasoning admits the converse.

COROLLARY. *Let k be of a finite degree, and $\mathfrak{p}'\in \tilde{S}(k)$. Let ζ be a primitive $(N\mathfrak{p}'-1)$-th root of unity. Then the number of points in $S(k)$ lying above $\mathfrak{p}'$ is equal to the degree of $k(\zeta)$ over k, and is in particular finite.*

Now, introduce into $S(k_\lambda)$ the quotient topology, and the topological space $S(k)$ will be determined as the projective limit of the topological sprces $S(k_\lambda)$. Then $S(k)$ is provided with a T_1-topology. It suffices to prove this in the case of a finite degree. In this case, it follows that the canonical mapping of $\tilde{S}(k(W))$ onto $S(k)$ is open. In fact, let U be any non-empty open subset of $\tilde{S}(k(W))$, $\mathfrak{p}$ any class of $S(k)$, and then it suffices to prove that $U \cup \mathfrak{p}$ is open. Let k_λ be subfields of $k(W)$ which have finite degrees, $\pi_\lambda, \pi_\lambda^\mu$ the canonical mappings of $\tilde{S}(k(W)), \tilde{S}(k_\mu)$ onto $\tilde{S}(k_\lambda)$ $(\lambda < \mu)$ respectively, and U_λ non-empty open subsets of $\tilde{S}(k_\lambda)$ such that $U = \bigcup_\lambda \pi_\lambda^{-1}(U_\lambda)$ where λ runs over some subset of the indexing set. Let $\tilde{\mathfrak{p}} \in \mathfrak{p}$, Z, and $\bar{\mathfrak{p}}$ as in the lemma 2. Since $k(Z) = k_\nu$ has a finite degree, $k(Z)k_\lambda = k_{\lambda'}$ has also a finite degree. It follows that $U'_{\lambda'} = (\pi_\lambda^{\lambda'})^{-1}(U_\lambda) \cup (\pi_\nu^{\lambda'})^{-1}(\{\bar{\mathfrak{p}}\})$ is open in $\tilde{S}(k_{\lambda'})$, because the topology of $\tilde{S}(k_{\lambda'})$ is the weakest one, and U_λ is non-empty and open in $\tilde{S}(k_\lambda)$. Then we have by the lemma 2 $U \cup \mathfrak{p} = \bigcup_{\lambda'} \pi_{\lambda'}^{-1}(U'_{\lambda'})$. Therefore $U \cup \mathfrak{p}$ is open. Now let $\mathfrak{p}$ and $\mathfrak{q}$ be any two distinct points in $S(k)$, and $\tilde{\mathfrak{p}}, Z, \bar{\mathfrak{p}}, \bar{\mathfrak{q}}$ as above. Then we have $\bar{\mathfrak{p}} \neq \bar{\mathfrak{q}}$. Let $U \subset \tilde{S}(k(W))$ be the set of all $\tilde{\mathfrak{p}}_1 \in \tilde{S}(k(W))$ which are not prolongations of $\bar{\mathfrak{q}}$. Then U is open, therefore the image of U by the canonical mapping of $\tilde{S}(k(W))$ onto $S(k)$ is open. Since this subset of $S(k)$ contains $\mathfrak{p}$ and does not contain $\mathfrak{q}$ by the lemma 2, it follows that $S(k)$ is provided with a T_1-topology. Moreover, it follows from the corollary of the lemma 2 that the topology of $S(k)$ is the weakest one, if k has a finite degree.

Now we have the following obvious analogy with the case of the algebraic geometry. W corresponds to an algebraically closed field, and $S(k)$ to a curve with the Zariski-topology. The construction of $S(k)$ corresponds to the extension of a field of definition to the algebraically closed field, and k^* (considered as the set of functions on $S(k)$) to the field of rational functions on a curve.

§ 2. *W*-varieties and rational mappings.

Now we shall state a general definition which corresponds to the definition of algebraic varieties.

DEFINITION 1. *Let S be a topological space and $\mathfrak{R}(S)$ a set of*

$\overline{W}$-valued functions defined on some non-empty open subsets of S; we denote by $\mathfrak{D}(f)$ the domain of definition of $f \in \mathfrak{R}(S)$. We call the pair $(S, \mathfrak{R}(S))$ a W-variety,[2] *if the following conditions are satisfied.*

1) *Any two non-empty open subsets of S always intersect with each other.*

2) *Any $f \in \mathfrak{R}(S)$ does not take identically the value 0 or ∞. If $f \in \mathfrak{R}(S)$ is not a continuous function, then f takes a constant value on some non-empty open subset of S, and $f(x) \neq 0, \infty$ for all $x \in \mathfrak{D}(f)$.*

3) *For any two functions f and g in $\mathfrak{R}(S)$, there exists one and only one function h in $\mathfrak{R}(S)$ such that $f(x)g(x)$ and $h(x)$ are defined and coincide with each other for all x in some non-empty open subset of S. We denote by fg this function h. Whenever $f(x)g(x)$ is defined, $(fg)(x)$ is defined and $f(x)g(x) = (fg)(x)$.*

4) *There exists a function e in $\mathfrak{R}(S)$ such that $\mathfrak{D}(e) = S$ and $e(x) = 1$ for all $x \in S$.*

5) *For any $f \in \mathfrak{R}(S)$, there exists a function f' in $\mathfrak{R}(S)$ such that $\mathfrak{D}(f') \subset \mathfrak{D}(f)$ and $f'(x) = f(x)^{-1}$ for all $x \in \mathfrak{D}(f')$.*

We call any function in $\mathfrak{R}(S)$ a *rational function* on S. If a function in $\mathfrak{R}(S)$ takes a constant value on some non-empty open subset of S, then we call this a *constant function.*[3] We can easily see that $\mathfrak{R}(S)$ forms an abelian group with respect to the product defined as above; and if f and $g \in \mathfrak{R}(S)$ take the same values on some non-empty open subset of S, then $f = g$.

Examples of W-varieties:

1) $S = W$, $\mathfrak{R}(W)$ = the set of all assignments of ζ^n to $\zeta \in W$ for all integers n. It is easily seen that $(W, \mathfrak{R}(W))$ is a W-variety.

2) $S = \overline{W}$, $\mathfrak{R}(\overline{W})$ = the set of all assignments of ζ^n to $\zeta \in \overline{W}$ for all integers n. For $n > 0$, 0^n and ∞^{-n} mean 0, 0^{-n} and ∞^n mean ∞, and ζ^0 mean 1 for all $\zeta \in \overline{W}$. Then $(\overline{W}, \mathfrak{R}(\overline{W}))$ forms a W-variety.

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2) We may take any infinite abelian group instead of W as far as the general theory is concerned. We use here the word "variety", although only the multiplicative structure is considered. A W-variety $(S, \mathfrak{R}(S))$ will be sometimes denoted simply by S.

3) According to this definition, there may exist constant functions which are not continuous. This would seen unnatural, but the later examples will show the convenience of our definition. The constant function in the usual sense will be specified as the "everywhere defined constant valued function".

3) Let k be any algebraic number field and $S(k)$ as in § 1. $S=S(k)$, $\mathfrak{R}(S(k))=k^*$ (considered as the set of functions on $S(k)$ as explained in § 1.). We shall prove that $(S(k), k^*)$ is a W-variety. The condition 1) of the definition of W-variety easily follows. Let f be in k^*. Then $\mathfrak{D}(f)=S(k)$ and f is continuous if and only if $f=1$ or $f\neq$ a root of unity other than 1. In fact, let f be not a root of unity, and $\zeta \in W$. Assume that f and ζ are contained in a subfield k_λ of k of a finite degree. Then there exists only a finite number of points $\mathfrak{p}_\lambda \in S(k_\lambda)$ such that $f(\mathfrak{p}_\lambda)=\zeta$. (This fact is also valid when $\zeta=0$ or ∞.) It follows that f is continuous as the function on $S(k)$. Conversely, let f be a primitive n-th root of unity ($n>1$) and contained in some k_λ as above. Then the set of all $\mathfrak{p}_\lambda \in S(k_\lambda)$ such that $f(\mathfrak{p}_\lambda)\neq f$ is non-empty and contains only a finite number of points. It follows that f is not continuous as the function on $S(k)$. Let f and g be in k^* and contained in some k_λ as above. If $f\neq g$, then the set of all $\mathfrak{p}_\lambda \in S(k_\lambda)$ such that $f(\mathfrak{p}_\lambda)=g(\mathfrak{p}_\lambda)$ is finite. It follows that two functions in $\mathfrak{R}(S(k))$ which take the same values on some non-empty open subset of $S(k)$ coincide with each other. From these facts it follows that all conditions in the definition of W-variety hold.

4) Let k be an algebraic number field of a finite degree and $\mathfrak{m}$ an integral divisor of k which may contain real primes; we denote by $|\mathfrak{m}|$ the set of all points lying above finite primes contained in $\mathfrak{m}$. $S=S(k)-|\mathfrak{m}|$, $\mathfrak{R}(S)=$the set of all elements $f\in k^*$ such that $f\equiv 1$ mod. m. Then $(S, \mathfrak{R}(S))$ forms a W-variety. The proof runs similarly as above.

Now we shall define some general concepts. Let S and S' be W-varieties and ρ a mapping of S into S'. We call ρ a *rational mapping* if the following condition is satisfied: If $\rho(x)$ is contained in $\mathfrak{D}(f')$ for $x\in S$ and $f'\in \mathfrak{R}(S')$, then there exists f in $\mathfrak{R}(S)$ such that $x\in\mathfrak{D}(f)$ and $f(y)=f'(\rho(y))$ for all y in some neighborhood of x. We easily see that f depends only upon f', and we have $\mathfrak{D}(f)\supset\rho^{-1}(\mathfrak{D}(f'))$ and $f(y)=f'(\rho(y))$ for all $y\in\rho^{-1}(\mathfrak{D}(f'))$. Let S, S' and S'' be W-varieties, ρ a rational mapping of S into S', and ρ' a rational mapping of S' into S''. Then we easily see that the composite mapping $\rho'\circ\rho$ is a rational mapping of S into S''. We easily see also that a mapping ρ of S into W is a rational mapping if and only if ρ is a rational function on S. Let S and S' be W-varieties and ρ a rational mapping of S

onto S'. We call ρ a *birational mapping* of S onto S' if ρ is a homeomorphism of S onto S' and ρ^{-1} is rational.

Let U be any non-empty open subset of a W-variety S, and $\mathfrak{R}(U)$ the set of all functions obtained from functions in $\mathfrak{R}(S)$ by the restrictions of the domains of definition to U. Then $(U, \mathfrak{R}(U))$ is a W-variety. We call $(U, \mathfrak{R}(U))$ an *open subvariety* of $(S, \mathfrak{R}(S))$. We can easily verify the following lemma.

LEMMA 3. *Let S and S' be W-varieties, $U_i (i \in J)$ open subvarieties of S such that $S = \bigcup_{i \in J} U_i$, U'_i open subvarieties of S', and ρ a mapping of S into S' such that $\rho(U_i) \subset U'_i$ for all $i \in J$. Then ρ is a rational mapping of S into S' if and only if the restriction of ρ to U_i is a rational mapping of U_i into U'_i for every $i \in J$.*

LEMMA 4. *Let $\{U_i\}_{i \in J}$ be a covering of a given non-empty set S. Let each U_i be provided with a topology, and $\mathfrak{R}(U_i)$ a set of $\overline{W}$-valued functions on some subsets of U_i such that $(U_i, \mathfrak{R}(U_i))$ forms a W-variety. Suppose moreover that for every pair i, j in J, $U_i \cap U_j$ is non-empty and is open in U_i and U_j, and that two structures of $U_i \cap U_j$ as the open subvarieties of U_i and U_j coincide with each other. Then there exists one and only one structure of W-variety of S such that $(U_i, \mathfrak{R}(U_i))$ is the open subvariety of $(S, \mathfrak{R}(S))$ for every i.*

PROOF. We denote by σ_{ij} the isomorphism of $\mathfrak{R}(U_i)$ onto $\mathfrak{R}(U_i \cap U_j)$ induced by the natural injection of $U_i \cap U_j$ into U_i. Let $\mathfrak{R}(S) \subset \prod_{i \in J} \mathfrak{R}(U_i)$ be the set of those elements $(f_i)_{i \in J}$ such that $\sigma_{ij}(f_i) = \sigma_{ji}(f_j)$ for every pair i, j in J. We may regard $f = (f_i)_{i \in J} \in \mathfrak{R}(S)$ as a $\overline{W}$-valued function defined on $\bigcup_{i \in J} \mathfrak{D}(f_i)$ as follows:

$$f(x) = f_i(x) \text{ for } x \in \mathfrak{D}(f_i).$$

This definition of $f(x)$ does not depend upon a choice of i such that $x \in \mathfrak{D}(f_i)$ because of properties of the σ_{ij}. We introduce a topology into S such that the U_i are open subspaces with the relative topologies. So we can easily verify that $(S, \mathfrak{R}(S))$ is a W-variety and $(U_i, \mathfrak{R}(U_i))$ are the open subvarieties of $(S, \mathfrak{R}(S))$. The uniqueness of the structure of W-variety in S is easily seen.

§ 3. Direct products and group W-varieties.

Now we shall define the direct product of two W-varieties. Let

S and S' be W-varieties and assume that if $f \in \mathfrak{R}(S)$ and $f' \in \mathfrak{R}(S')$ take the same constant value on some non-empty open subsets of S and S' respectively, then f and f' are everywhere defined constant valued functions. Under this assumption we shall define the product W-variety of S and S'. (Whenever we say in the following "the product of W-varieties S, S' can be defined", we imply that this assumption is satisfied.) This assumption is satisfied if e is the only constant function in $\mathfrak{R}(S)$ or $\mathfrak{R}(S')$, e.g. when S or S' is one of the W-varieties given as 1), 2) above.

Let $f \in \mathfrak{R}(S)$ and $f' \in \mathfrak{R}(S')$. We denote by $\mathfrak{D}(f, f')$ the set of all pairs $(x, x') \in S \times S'$ such that $x \in \mathfrak{D}(f)$, $x' \in \mathfrak{D}(f')$, and $f(x)f'(x')$ are defined; we denote by (f, f') the function defined on $\mathfrak{D}(f, f')$ whose value at (x, x') is equal to $f(x)f'(x')$. Then we define $\mathfrak{R}(S \times S')$ to be the set of functions (f, f') defined as above for all $f \in \mathfrak{R}(S)$ and $f' \in \mathfrak{R}(S')$. Next we introduce the weakest topology in $S \times S'$ such that the projections $S \times S' \to S$ and S' are continuous, and that $(f, f') \in \mathfrak{R}(S \times S')$ is continuous whenever one of f and f' is not a constant function. This topology in $S \times S'$ is not always the same as the product topology.

Then we can prove that $(S \times S', \mathfrak{R}(S \times S'))$ is a W-variety; we call this W-variety the *product W-variety* of S and S'. The conditions 2), 4) and 5) of the definition of W-variety are easily seen, and we shall prove the conditions 1) and 3).

PROOF of 1). Let U and U' be non-empty open subsets of S and S' respectively, $(f_i, f_i') \in \mathfrak{R}(S \times S')$ and $\zeta_i \in W (i = 1, 2, \cdots, n)$. We denote by G_i the set of all pairs (x, x') such that $(f_i, f_i')(x, x')$ are defined and not equal to ζ_i. We assume that one of f_i and f_i' is not a constant function and G_i is not empty. Now it suffices to prove that $(U \times U') \cap G_1 \cap \cdots \cap G_n$ is not empty. Let U_i be the set of all points $x \in \mathfrak{D}(f_i)$ such that if f_i' takes a constant value, say ζ_i', on some non-empty subset of S' then

$$f_i(x) \neq \zeta_i'^{-1} \zeta_i .$$

Then from the condition on (f_i, f_i') it follows that U_i is non-empty and open for every i. If V is the set of all points x such that $f_i(x)$ are defined and in W for every i, then V is non-empty and open. Hence $V \cap U \cap U_1 \cap \cdots \cap U_n$ contains at least one point a. Next, let U_i' be the set of all points $x' \in S'$ such that $f_i'(x')$ are defined and

$$f_i'(x') \neq f_i(a)^{-1} \zeta_i .$$

Then U_i' is non-empty and open because of the choice of a for every i. Hence $U' \cap U_1' \cap \cdots \cap U_n'$ contains at least one point a'. Thus $(U \times U') \cap G_1 \cap \cdots \cap G_n$ contains (a, a'), since we easily see that $G_i \supset (V \cap U_i) \times U_i'$ for each i, so it is not empty. This completes the proof of 1).

PROOF of 3). Let $(f, f'), (g, g')$ be two functions in $\mathfrak{R}(S \times S')$. We denote by G_0 the set of all pairs $(x, x') \in S \times S'$ such that $(f, f')(x, x')$, $(g, g')(x, x')$ and their products are defined. Then G_0 is non-empty and open in $S \times S'$ and $f(x), g(x), f'(x'), g'(x'), f(x)\,g(x), f'(x')\,g'(x')$ and $(f(x)\,g(x))\,(f'(x')\,g'(x')) = (fg)(x)\,(f'g')(x') = (fg, f'g')(x, x')$ are defined for every $(x, x') \in G_0$. It remains to show that if (f, f') and (g, g') take the same values on some non-empty open subset of $S \times S'$ then $(f, f') = (g, g')$ as functions in $\mathfrak{R}(S \times S')$. Since $f(x), f'(x'), g(x)$, and $g'(x')$ are defined and not equal to 0 or ∞ for every (x, x') in some non-empty open subset G of $S \times S'$, we have

$$f(x)\,g^{-1}(x) = f'^{-1}(x')\,g'(x') \text{ for every } (x, x') \in G .$$

So it suffices to show that the set of all x such that $(x, a') \in G$ for any fixed $a' \in S'$ is an open subset of S. In fact, if that is so, fg^{-1} and $f'^{-1}g'$ are constant functions taking the same constant value, say ζ, on some open subsets of S and S' respectively and it follows from our assumption that these functions are everywhere defined constant valued functions. Therefore $f(x) = g(x)\,\zeta$ for all $x \in \mathfrak{D}(f) = \mathfrak{D}(g)$, $f'(x')\,\zeta = g'(x')$ for all $x' \in \mathfrak{D}(f') = \mathfrak{D}(g')$ and $f(x) f'(x') = g(x)\,g'(x')$ for all $(x, x') \in \mathfrak{D}(f, f') = \mathfrak{D}(g, g')$, so $(f, f') = (g, g')$,

Now we shall prove the above assertion. Let a be any point of S such that $(a, a') \in G$. Since G is open, there exist open subsets U of S and U' of S', some functions (f_i, f_i') in $\mathfrak{R}(S \times S')$ and ζ_i in $\overline{W}$ $(i = 1, 2, \cdots, n)$ such that $(a, a') \in (U \times U') \cap G_1 \cap \cdots G_n \subset G$ where G_i is the set of all pairs (x, x') such that $f_i(x) f_i'(x')$ are defined and not equal to ζ_i. Let U_i be the set of all $x \in S$ such that

$$f_i(x) \neq \begin{cases} f_i'(a')^{-1}\,\zeta_i & \text{if } f_i'(a') \neq 0 , \\ \zeta_i & \text{if } f_i'(a') = 0 \end{cases}$$

for each i. Then $U_i \ni a$ is open and $U_i \times \{a'\} \subset G_i$ for every i. So $U \cap U_1 \cap \cdots \cap U_n$ containing a is open, and for any point x in this set, $(x,$

$a') \in G$. This completes the proof.

If $S \times S'$ is the product W-variety of W-varieties S and S', then it follows easily that the projections $S \times S' \to S$ and S' are rational. Let U and U' be any open subvarieties of S and S' respectively. Then the product W-variety $U \times U'$ can be defined and coincides with the open subvariety of $S \times S'$. Let G be a W-variety which is a group, and assume that the function e satisfying the condition 4) of the definition 1 is the only constant function in $\mathfrak{R}(G)$. If the group operations $(\xi, \eta) \to \xi\eta$ and $\xi \to \xi^{-1}$ are continuous and rational, then we call G a group W-variety.(4) The following lemmas will be used later.

LEMMA 5. *Let A, B and B' be W-varieties, ρ and ρ' be continuous rational mappings of A into B and B' respectively. Assume that the product W-variety of B and B' can be defined. Then the mapping (ρ, ρ') of A into $B \times B'$ such that $(\rho, \rho')(x) = (\rho(x), \rho'(x))$ for all $x \in A$ is continuous and rational.*

PROOF. Let x be in A, and (f, f') in $\mathfrak{R}(B \times B')$ such that $(f, f')(\rho(x), \rho'(x))$ is defined. Then $f(\rho(x))$ and $f'(\rho'(x))$ are defined, therefore there exist g and g' in $\mathfrak{R}(A)$ such that $g(y) = f(\rho(y)), g'(y) = f'(\rho'(y))$ for all y in some neighborhood of x. Hence $(gg')(y) = (f, f')(\rho(y), \rho'(y))$ for all y in some neighborhood of x. It follows from this that (ρ, ρ') is continuous and rational. This completes the proof.

The following two lemmas are easily verified by the lemma 5.

LEMMA 6. *Let A, A', B and B' be W-varieties, ρ and ρ' be continuous rational mappings of A into B and A' into B' respectively. Assume that the product W-varieties $A \times A'$ and $B \times B'$ can be defined. Then the mapping $\rho \times \rho'$ of $A \times A'$ into $B \times B'$ such that $(\rho \times \rho')(x, x') = (\rho(x), \rho'(x))$ for all $(x, x') \in A \times A'$ is continuous and rational.*

LEMMA 7. *Let S and F be W-varieties, G a group W-variety and ρ a continuous rational mapping of S into G. Assume that the product W-variety $S \times F$ can be defined and G operates on F, that is, a continuous rational mapping ψ of $G \times F$ onto F such that $\psi(\xi, \psi(\eta, \alpha)) = \psi(\xi\eta, \alpha)$ for all $\xi, \eta \in G$ and $\alpha \in F$, is given. Then the mapping: $(x, \alpha) \to (x, \psi(\rho(x), \alpha))$ is a birational mapping of $S \times F$ onto itself.*

4) The mapping $\xi \to \xi\eta$ is not always rational for fixed $\eta \in G$.

§ 4. Fibre spaces and existence theorem.

Now we shall proceed to define fibre spaces over a W-variety. First, we shall define principal fibre W-spaces over a W-variety. (Cf. Bourbaki [1]).

DEFINITION 2. *A principal fibre W-space is a collection as follows:*

1) *A W-variety B called the principal fibre W-variety,*
2) *a W-variety S called the base W-variety,*
3) *a rational mapping π of B onto S called the projection,*
4) *a group W-variety G called the structural group,*
5) *a continuous rational mapping φ of $B\times G$ onto B called the law of transformation such that $\varphi(\varphi(u, \xi), \eta)=\varphi(u, \xi\eta)$ for all $u\in B$ and $\xi, \eta\in G$. We shall denote $\varphi(u, \xi)$ simply by $u\cdot\xi$.*

Assume that there exist an open covering $\{U_i\}_{i\in J}$ of S and a birational mapping Φ_i of $U_i\times G$ onto $\pi^{-1}(U_i)$ for each i such that

$$(\sharp)\qquad \begin{cases}\pi(\Phi_i(x, \xi))=x, \\ \Phi_i(x, \xi)\cdot\eta=\Phi_i(x, \xi\eta) \text{ for all } x\in U_i \text{ and } \xi, \eta\in G.\end{cases}$$

If W-varieties S and G are fixed, we shall denote a principal fibre W-space (B, S, π, G, φ) by (B, π) or more simply by B. Let B, π, G, $\{U_i, \Phi_i\}$ be as above. Then there exists[5] a continuous rational mapping s_{ij} of $U_i\cap U_j$ into G for each pair i, j such that

$$\Phi_i^{-1}\Phi_j(x, \xi)=(x, s_{ij}(x)\xi) \text{ for all } (x, \xi)\in(U_i\cap U_j)\times G.$$

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5) In order to prove this, we need the following lemma. (The corresponding fact is trivial in the case of the topology.)

LEMMA. *Let S be a W-variety, and G a group W-variety. Let Φ be a continuous rational mapping of $S\times G$ into G such that $\Phi(x, \xi)\ \eta=\Phi(x, \xi\eta)$ for all $x\in S$, $\xi, \eta\in G$. Then there exists a continuous rational mapping t of S into G such that $\Phi(x, \xi)=t(x)\ \xi$.*

PROOF. Let $f\in\mathfrak{R}(G)$ and $1\in\mathfrak{D}(f)$ where 1 means the identity of G. Since $\xi\to(\xi, \xi^{-1})$ and $(\xi, \eta)\to\xi\eta$ are rational, there exists $g\in\mathfrak{R}(G)$ such that $g(\xi)=f(1)$ on some open subset of G. Hence g is a constant function, therefore $1=g(\xi)=f(1)$ for some $\xi\in G$. From this fact it follows that $x\to(x, 1)$ is a rational mapping of S into $S\times G$, and this mapping is continuous (Cf. § 3, Proof of 3)). Hence $x\to\Phi(x, 1)=t(x)$ is continuous and rational, and $\Phi(x, \xi)=t(x)\xi$.

The system $\{s_{ij}\}$ is uniquely determined by $\{U_i, \Phi_i\}$.

Now let $\{U'_{i'}\}_{i' \in J'}$ be any refinement of $\{U_i\}_{i \in J}$ and i', j' any pair of elements in J'. Let Φ'_i and $s'_{i'j'}$ be the restrictions of Φ_i and s_{ij} to $U'_{i'} \times G$ and $U'_{i'} \cap U'_{j'}$ respectively for some pair i, j such that $U'_{i'} \subset U_i$ and $U'_{j'} \subset U_j$. Then it is easily seen that the covering $\{U'_{i'}\}$ and the system of birational mappings $\{\Phi'_{i'}\}$ satisfy the above condition ($\sharp$) and $\{s'_{i'j'}\}$ is determined from $\{\Phi'_{i'}\}$ as above. From this fact it follows easily that for two principal fibre W-spaces (B, π) and (B', π') with the same base and group, there exists a birational mapping Ψ of B onto B' such that $\pi' \circ \Psi = \pi$ and $\Psi(u) \cdot \xi = \Psi(u \cdot \xi)$ for all $u \in B$ and $\xi \in G$ if and only if these fibre W-spaces satisfy the condition ($\sharp$) for some $\{U_i, \Phi_i\}$ and $\{U_i, \Phi'_i\}$ determining the same $\{s_{ij}\}$. Such two principal fibre W-spaces are said to be isomorphic. Then the following proposition is easily proved as in usual topology.[5]

PROPOSITION. *Let $\{U_i\}_{i \in J}$ be an open covering of a W-variety S, (B, π) and (B', π') be two principal fibre W-spaces with the same base S and group G. Assume that $\{\Phi_i\}$ and $\{\Phi'_i\}$ satisfy ($\sharp$) for (B, π) and (B', π') respectively, and determine $\{s_{ij}\}$ and $\{s'_{ij}\}$ respectively. Then (B, π) and (B', π') are isomorphic if and only if there exists a continuous rational mapping t_i of U_i into G for each $i \in J$ such that for each pair i, j*

$$s'_{ij}(x) = t_i(x) s_{ij}(x)\, t_j(x)^{-1} \textit{ for all } x \in U_i \cap U_j\,.$$

THEOREM 1. *Let S be a W-variety, G a group W-variety. Let $\{U_i\}_{i \in J}$ be an open covering of S and s_{ij} a continuous rational mapping of $U_i \cap U_j$ into G for each pair i, j in J. Then there exists a principal fibre W-space (B, π) such that there exists a system $\{\Phi_i\}$ satisfying ($\sharp$) and determining $\{s_{ij}\}$ if and only if the following condition is satisfied:*

$$s_{ij}(x) s_{jl}(x) = s_{il}(x) \textit{ for all } x \in U_i \cap U_j \cap U_l\,,$$
$$\textit{for each triple } i,\ j,\ l \textit{ in } J.$$

PROOF. The "only if" part is easily seen. To prove the "if" part, we shall construct a fibre W-space as follows. Let $\tilde{B} \subset S \times G \times J$ be the set of those triples (x, ξ, i) such that $x \in U_i$. Then $\tilde{B}$ is the union of the disjoint subsets $U_i \times G \times \{i\}$ $(i \in J)$. We define in $\tilde{B}$ an equivalence relation: $(x, \xi, i) \sim (y, \eta, j)$ if $x=y$, $\xi = s_{ij}(x)\eta$. We define B to be the set of equivalence classes of this relation in $\tilde{B}$. We define a mapping Φ_i of $U_i \times G$ into B to be the assignment of the equivalence class of (x, ξ, i) to $(x, \xi) \in U_i \times G$ for each i. We can define a mapping π of B onto S by $\pi(\Phi_i(x, \xi)) = x$ for any (x, ξ, i) without contradictions. Then Φ_i is a one-to-one mapping of $U_i \times G$ onto $\pi^{-1}(U_i)$ for every i and $\Phi_i^{-1}\Phi_j(x, \xi) = (x, s_{ij}(x)\xi)$ for all $x \in U_i \cap U_j$ for any i, j. Next we can define a mapping φ of $B \times G$ onto B to be

$$\varphi(\Phi_i(x, \xi), \eta) = \Phi_i(x, \xi\eta) \text{ for every } i.$$

It follows from the lemma 7 that the mapping $\Phi_i^{-1}\Phi_j$ defined on $(U_i \cap U_j) \times G$ is a birational mapping of $(U_i \cap U_j) \times G$ onto itself. So if we introduce the structure of W-variety into $\pi^{-1}(U_i)$ by the mapping Φ_i of $U_i \times G$ onto $\pi^{-1}(U_i)$ for every i, then we have by the lemma 4 a W-variety $(B, \mathfrak{R}(B))$ such that the W-varieties $\pi^{-1}(U_i)$ are open subvarieties of B.

It follows from the lemma 3 that π is a rational mapping of B onto S, since the restriction of π to $\pi^{-1}(U_i)$ is the composite mapping of Φ_i^{-1} and the projection of $U_i \times G$ onto U_i for every i. It follows similarly that φ is a continuous rational mapping satisfying the equality in the definition 2. This completes the proof.

Next we shall define a fibre W-space associated to a principal fibre W-space and a given fibre.

DEFINITION 3. *Let (B, S, π, G, φ) be a principal fibre W-space, F a W-variety and assume that the product W-variety $S \times F$ can be defined. Let ψ be a continuous rational mapping of $G \times F$ onto F such that $\psi(\xi, \psi(\eta, \alpha)) = \psi(\xi\eta, \alpha)$ for all $\xi, \eta \in G$ and all $\alpha \in F$. ψ will be called the law of transformation. Then a fibre W-space of the fibre (F, ψ) associated to the principal fibre W-space (B, S, π, G, φ) is a collection as follows:*

1) *A W-variety $\bar{B}$,*

2) *the principal fibre W-space (B, S, π, G, φ),*

3) *the W-variety F and the law of transformation $\psi: G \times F \to F$,*

4) *a continuous rational mapping $\bar{\pi}$ of $\bar{B}$ onto S called the projection,*

5) *a mapping* θ *of* $B \times F$ *onto* $\bar{B}$ *such that*

$\bar{\pi}(\theta(u, \alpha)) = \pi(u)$ *for all* $(u, \alpha) \in B \times F$,

$\theta(\varphi(u, \xi), \alpha) = \theta(u, \psi(\xi, \alpha))$ *for all* $(u, \xi, \alpha) \in B \times G \times F$,

and that for a system $\{U_i, \Phi_i\}$ *corresponding to* (B, S, π, G, φ) *in the sense of the definition* 2, *the mapping* $(x, \alpha) \to \theta(\Phi_i(x, 1), \alpha)$ *supplies a birational mapping of* $U_i \times F$ onto $\bar{\pi}^{-1}(U_i)$. *We shall denote by* $\bar{\Phi}_i$ *this mapping.*

We can prove that for any principal fibre W-space (B, S, π, G) and any W-variety F, on which G operates, there exists a fibre W-space of the fibre F associated to (B, S, π, G) provided that $S \times F$ can be defined. In fact, we define in $B \times F$ an equivaelnce relation: $(u, \alpha) \sim (v, \beta)$ if there exists an element ξ in G such that $u = \varphi(v, \xi)$ and $\beta = \psi(\xi, \alpha)$. We define $\bar{B}$ to be the set of equivalence classes of this relation in $B \times F$. We denote by $\theta(u, \alpha)$ the equivalence class of $(u, \alpha) \in B \times F$. We define $\bar{\pi}$: $\theta(u, \alpha) \to \pi(u)$. Let $\{U_i, \Phi_i\}$ be a system corresponding to (B, S, π, G, φ) in the sense of the definition 2. Then the mappings $\bar{\Phi}_i$: $(x, \alpha) \to \theta(\Phi_i(x, 1), \alpha)$ are one-to-one mappings of the $U_i \times F$ onto the $\bar{\pi}^{-1}(U_i)$. If we introduce the structures of W-varieties by $\bar{\Phi}_i$ and the W-varieties $U_i \times F$ into $\bar{\pi}^{-1}(U_i)$, then it follows from the lemma 4 that we have the W-variety $(\bar{B}, \mathfrak{R}(\bar{B}))$ such that the W-varieties $\bar{\pi}^{-1}(U_i)$ are the open subvarieties. Then we can easily verify by the previous lemmas that all conditions in the definition of the fibre W-space for $(\bar{B}, (B, S, \pi, G), F, \bar{\pi}, \theta)$ hold.

If $(\bar{B}, (B, S, \pi, G), F, \bar{\pi}, \theta)$ and $(\bar{B}', (B, S, \bar{\pi}, G), F, \pi', \theta')$ are two fibre W-spaces of the fibre F associated to (B, S, π, G), there exists a birational mapping $\bar{\Psi}$ of $\bar{B}$ onto $\bar{B}'$, such that $\bar{\Psi} \circ \theta = \theta'$ and therefore $\bar{\pi}^{-1} \circ \bar{\Psi} = \bar{\pi}$. Since an associated fibre W-space is essentially unique in this sense, we shall fix an associated fibre W-space denoted by $(\bar{B}, \bar{\pi})$ for each principal fibre W-space (B, π), for a fixed fibre F.

§ 5. Classification theorem.

Hereafter we shall confine ourselves to the case of $G = W$ and $F = \bar{W}$. In this case rational mappings s_{ij} of $U_i \cap U_j$ into G are rational functions on $U_i \cap U_j$. So we may identify s_{ij} with the function in $\mathfrak{R}(S)$ uniquely determined by s_{ij} for every pair i, j. The law of transformation: $G \times F \to F$ is defined by $(\xi, \alpha) \to \xi\alpha$ for all $\xi \in G = W$ and $\alpha \in F = \bar{W}$.

A W-variety $(S, \mathfrak{R}(S))$ is said to be of a *finite type*, if the topology of S is the weakest one, any function in $\mathfrak{R}(S)$ is everywhere defined and the following additional condition is satisfied:

6) *For any $x \in S$, there exists a function $t_x \in \mathfrak{R}(S)$ such that $t_x(x)=0$ and for any $f \in \mathfrak{R}(S)$ there exists an integer n satisfying the condition: $(t_x^{-n} f)(x) \neq 0$ and $\neq \infty$.*

We call such a function as t_x a uniformizing variable at x, we can define at any $x \in S$ an order function ν_x such that ν_x is a homomorphism of $\mathfrak{R}(S)$ onto the additive group of all integers and $\nu_x(f)>0$ if and only if $f(x)=0$ for every $f \in \mathfrak{R}(S)$. ν_x is uniquely determined for each $x \in S$. Hereafter we shall denote by $\mathfrak{p}, \mathfrak{q}, \cdots$ points in S.

Now we shall define some concepts in order to state the classification theorem. $\mathfrak{p}$ is said to be a *conjugate* of $\mathfrak{q}$, if $\nu_{\mathfrak{p}}(f)=\nu_{\mathfrak{q}}(f)$ for all $f \in \mathfrak{R}(S)$. This is an equivalence relation. A principal fibre W-space (B, π, S) is said to be *rational* if there exists an open covering $\{U_i\}_{i \in J}$ satisfying the condition of the definition 2 such that any conjugate point of every $\mathfrak{p} \in U_i$ is contained in U_i for each i (Such a covering is said to be *rational.*).

By a *divisor* on S, we understand an element of the free abelian group $D(S)$, the generators of which are points of S. The divisor on S will be generally denoted by $\sum n_{\mathfrak{p}} \mathfrak{p}$. By a *rational divisor* on S, we understand a divisor $\sum n_{\mathfrak{p}} \mathfrak{p}$ such that $n_{\mathfrak{p}}=n_{\mathfrak{q}}$ whenever $\mathfrak{p}$ is a conjugate of $\mathfrak{q}$. And by a *prime rational divisor* on S, we understand $\sum \mathfrak{p}$ (finite sum) where $\mathfrak{p}$ runs over all conjugates of some fixed $\mathfrak{q}$. Then the group $D_0(S)$ of all rational divisors is a free abelian group. the generators of which are prime rational divisors on S. Since the set of all $\mathfrak{p}$ such that $\nu_{\mathfrak{p}}(f) \neq 0$ is finite for each $f \in \mathfrak{R}(S)$, we can define a divisor $\sum \nu_{\mathfrak{p}}(f)\, \mathfrak{p}$. This divisor is said to be a *principal divisor,* the divisor of f, and denoted by (f). We shall denote by $P(S)$ the group of all principal divisors.

Next we shall define a group structure in a set of principal fibre W-spaces over S. Here we need not confine ourselves to W-varieties of a finite type. We denote by $\mathfrak{B}(S)$ a set of principal fibre W-spaces over S such that for any principal fibre W-space (B, π) over S there exists one and only one $(B', \pi') \in \mathfrak{B}(S)$ isomorphic to (B, π). If $\{U_i, s_{ij}\}$ and $\{U_i, s'_{ij}\}$ correspond to (B, π) and $(B', \pi') \in \mathfrak{B}(S)$ respectively, then there exists one and only one $(B'', \pi'') \in \mathfrak{B}(S)$ corresponding to $\{U_i, s_{ij} s'_{ij}\}$

by the theorem 1. We shall call (B'', π'') the product of (B, π) and (B', π'), then it follows from the theorem 1 that $\mathfrak{B}(S)$ forms a group by this multiplication. The trivial principal fibre W-space in $\mathfrak{B}(S)$ which is identified with $S\times W$, is the identity of the group $\mathfrak{B}(S)$. The set of all rational principal fibre W-spaces in $\mathfrak{B}(S)$ forms a subgroup $\mathfrak{B}_0(S)$ of $\mathfrak{B}(S)$.

THEOREM 2. *If S is a W-variety of a finite type, then there exists an isomorphism of $\mathfrak{B}(S)$ onto the factor group $D(S)/P(S)$ which induces an isomorphism of $\mathfrak{B}_0(S)$ onto $D_0(S)/P(S)$.*

This isomorphism is obtained as follows: Let (B, π) be any principal fibre W-space in $\mathfrak{B}(S)$ and ρ be any cross section of the associated fibre W-space $(\bar{B}, \bar{\pi})$, namely, a rational mapping of S into $\bar{B}$ such that $\bar{\pi}\circ\rho=1$. Let $\{U_i, \Phi_i\}$ be a system satisfying the condition ($\sharp$) in the definition 2, and let, for each i, f_i be the rational function in $\mathfrak{R}(S)$ such that $\rho(\mathfrak{p})=\bar{\Phi}_i(\mathfrak{p}, f_i(\mathfrak{p}))$ for all $\mathfrak{p}\in U_i$. We define a divisor (ρ) to be $\sum n_{\mathfrak{p}}\mathfrak{p}$ where $n_{\mathfrak{p}}=\nu_{\mathfrak{p}}(f_i)$ for $\mathfrak{p}\in U_i$. ($\nu_{\mathfrak{p}}(f_i)$ does not depend upon a choice of i nor (ρ) upon a choice of $\{U_i, \Phi_i\}$.) If ρ runs over all cross sections of $(\bar{B}, \bar{\pi})$, then (ρ) runs over all divisors in a divisor class modulo $P(S)$ which depends only upon (B, π). The assignment $\rho\to(\rho)$ supplies the above isomorphism. (Cf. Kodaira and Spencer [3], Weil [4], [5])

PROOF. First we shall prove the existence of a cross section ρ and the possibility of the definition of (ρ). Let $\{s_{ij}\}$ be a system of rational functions in $\mathfrak{R}(S)$ determined by Φ_i for (B, π) as in § 3. Let us fix an index i. Since for every $\mathfrak{p}$ in S, $\bar{\Phi}_j(\mathfrak{p}, s_{ji}(\mathfrak{p}))$ does not depend upon a choice of j such that $\mathfrak{p}\in U_j$, we define $\rho(\mathfrak{p})$ to be $\bar{\Phi}_j(\mathfrak{p}, s_{ji}(\mathfrak{p}))$ for U_j containing $\mathfrak{p}$. Then ρ is a cross section of $(\bar{B}, \bar{\pi})$. Let ρ be any cross section of $(\bar{B}, \bar{\pi})$. For all $\mathfrak{p}\in U_i\cap U_j$, $\rho(\mathfrak{p})=\bar{\Phi}_i(\mathfrak{p}, f_i(\mathfrak{p}))=\bar{\Phi}_j(\mathfrak{p}, s_{ji}(\mathfrak{p})f_i(\mathfrak{p}))=\bar{\Phi}_j(\mathfrak{p}, f_j(\mathfrak{p}))$, therefore $\nu_{\mathfrak{p}}(f_i)=\nu_{\mathfrak{p}}(f_j)$ since $\nu_{\mathfrak{p}}(s_{ij})=0$.

Next let ρ' be also any cross section of $(\bar{B}, \bar{\pi})$ and put

$$\rho'(\mathfrak{p})=\bar{\Phi}_i(\mathfrak{p}, f'_i(\mathfrak{p})) \text{ for all } \mathfrak{p}\in U_i\ (f'_i\in\mathfrak{R}(S)).$$

Then $f_i f_j^{-1}=s_{ij}=f'_i f'^{-1}_j$ for any i, j, therefore $f'_i f_i^{-1}\in\mathfrak{R}(S)$ does not depend upon a choice of i. So we denote by t this function in $\mathfrak{R}(S)$, then we have $\nu_{\mathfrak{p}}(f')=\nu_{\mathfrak{p}}(f_i)+\nu_{\mathfrak{p}}(t)$ and $(\rho')=(\rho)+(t)$. (ρ') runs over all divisors of a divisor class. It follows easily from the proposition in

§ 4 that the divisor class of (ρ) does not depend upon a choice of $\{U_i, \Phi_i\}$.

From the definition of the group operation in $\mathfrak{B}(S)$ and the theorem 1, we can easily see that the mapping defined by $\rho \to (\rho)$ is a homomorphism of $\mathfrak{B}(S)$ into $D(S)/P(S)$. Now if (ρ) is principal, then for some cross section ρ' and each i, j

$$\rho'(\mathfrak{p}) = \bar{\Phi}_i(\mathfrak{p}, f_i(\mathfrak{p})),\ f_i^{-1} s_{ij} f_j = 1 \text{ and } \nu_{\mathfrak{p}}(f_i) = 0 \text{ for all } \mathfrak{p} \in U_i\,.$$

Hence (B, π) is isomorphic to $S \times W$. Therefore the above homomorphism is an isomorphism. In order to show that this isomorphism is onto, it suffices to prove that for any $\mathfrak{p} \in S$ there exist (B, π) and a cross section ρ of $(\bar{B}, \bar{\pi})$ such that $(\rho) = \mathfrak{p}$. Let t be a uniformizing variable at $\mathfrak{p}$. We define $U_0 = S - \{\mathfrak{p}\}$ and $U_1 =$ the set consisting of $\mathfrak{p}$ and of all $\mathfrak{q} \in S$ such that $\nu_{\mathfrak{p}}(t) = 0$. Then $\{U_0, U_1\}$ is an open covering of S. We define: $s_{00} = s_{11} = 1$ and $s_{10} = s_{01}^{-1} = t$. Then there exists a principal fibre W-space (B, π) corresponding to $\{U_i, \Phi_i\}$ where $\Phi_i^{-1}\Phi_j(\mathfrak{q}, \xi) = (\mathfrak{q}, s_{ij}(\mathfrak{q})\,\xi)$ for all $(\mathfrak{q}, \xi) \in (U_i \cap U_j) \times W$. We easily see that $(\rho) = \mathfrak{p}$ for the cross section ρ of $(\bar{B}, \bar{\pi})$ defined by $\rho(\mathfrak{q}) = \bar{\Phi}_i(\mathfrak{q}, s_{i0}(\mathfrak{q}))$ for all $\mathfrak{q} \in S$. This completes the proof.

If k is an algebraic number field of a finite degree, the example 3) in § 2 is a special case of the example 4). In the case of 4), it is easily seen that $D_0(S)$ may be regarded as the group $A_{\mathfrak{m}}$ of all ideals in k prime to $\mathfrak{m}$, and $P(S)$ the group $S_{\mathfrak{m}}$ of all principal ideals generated by elements $f \in k^*$ such that $f \equiv 1$ (mod. $\mathfrak{m}$).

Hence we have

$$\mathfrak{B}_0(S) \cong A_{\mathfrak{m}}/S_{\mathfrak{m}} \text{ (so called “Strahlklassengruppe mod. } \mathfrak{m}\text{”)}$$

Thus the concept of the rational fibre W-space (of the fibre $\overline{W}$ and the group W) just corresponds to the concept of the ideal class mod. $\mathfrak{m}$ in the classical number theory.

In order to treat the case of an infinite degree, we shall consider the projective limit of W-varieties of a finite type as follows. Let Λ be a directed set. Let S_λ $(\lambda \in \Lambda)$ be W-varieties of a finite type, and π_λ^μ $(\lambda < \mu)$ be continuous rational mappings of S_μ onto S_λ such that $\pi_\lambda^\mu \cdot \pi_\mu^\nu = \pi_\lambda^\nu$ $(\lambda < \mu < \nu)$. Now let S be the projective limit of the topological spaces S_λ, and π_λ continuous mappings of S onto S_λ such that $\pi_\lambda^\mu \cdot \pi_\mu = \pi_\lambda (\lambda < \mu)$. We define $\mathfrak{R}(S)$ to be the set of all functions $f_\lambda \circ \pi_\lambda$ defined on S for all $\lambda \in \Lambda$ and $f_\lambda \in \mathfrak{R}(S_\lambda)$. Then it follows easily that $(S, \mathfrak{R}(S))$ is a W-variety, and the π_λ are rational. We may identify $f_\lambda \circ \pi_\lambda$ with f_λ for all $f_\lambda \circ \pi_\lambda \in \mathfrak{R}(S)$, and then may regard all $\mathfrak{R}(S_\lambda)$ as subgroups of $\mathfrak{R}(S)$, we have $\mathfrak{R}(S) = \bigcup_{\lambda \in \Lambda} \mathfrak{R}(S_\lambda)$. This fact holds also without the assumption that S_λ are of a finite type whenever all rational functions on S_λ are everywhere defined.

We define $D(S)$ to be $\lim_{\lambda} D(S_\lambda)$ which is defined by the mappings $j_\lambda^\mu : \mathfrak{p}_\lambda \to \sum_{\mathfrak{p}_\mu} \nu_{\mathfrak{p}_\mu}(t_{\mathfrak{p}_\lambda}) \mathfrak{p}_\mu$ (finite sum) for $\mathfrak{p}_\lambda \in S_\lambda$ and $\mathfrak{p}_\mu \in S_\mu (\lambda < \mu)$. So we may regard $D(S_\lambda) \subset D(S_\mu) \subset D(S) (\lambda < \mu)$; we have $D(S=) \bigcup_{\lambda \in \Lambda} D(S_\lambda)$. Then we easily see that for any $f_\lambda \in \mathfrak{R}(S_\lambda)$ the divisor of f in S coincides with the divisor of f in S_λ for each λ. So we define as follows: $P(S) = \lim P(S_\lambda)$, $D_0(S) = \lim D_0(S_\lambda)$. Then we have $D(S) \supset D_0(S) \supset P(S)$. A principal fibre W-space (B, π) over S is said to be of a finite type, if there exists an open covering $\{U_i^{(\lambda)}\}$ of S_λ for some λ such that the open covering $\{\pi_\lambda^{-1}(U_i^{(\lambda)})\}$ satisfies the condition of the definition 2; moreover (B, π) is said to be rational, provided that $\{U_i^{(\lambda)}\}$ is rational in the above. We define groups $\mathfrak{B}(S)$ and $\mathfrak{B}_0(S)$ as in the case of a finite type, and define also $\mathfrak{B}_1(S)$ as the subgroup of all (B, π) of a finite type in $\mathfrak{B}(S)$. Then we have also in this case

$$\mathfrak{B}_1(S) \cong D(S) / P(S), \quad \mathfrak{B}_0(S) \cong D_0(S) / P(S).$$

PROOF. Let (B, π) be any principal fibre W-space in $\mathfrak{B}_1(S)$. Then the existence of a cross section of $(\bar{B}, \bar{\pi})$ follows in the same way as in the proof of the theorem 2. We shall use the same notations as used there. Let ρ be a cross section of $\bar{B}$. Then there exist $\lambda \in \Lambda$, $B_\lambda \in \mathfrak{B}(S_\lambda)$, an open covering $\{U_i^{(\lambda)}\}$ of S_λ, birational mappings $\Phi_i^{(\lambda)}$, and functions $f_i \in \mathfrak{R}(S_\lambda)$ such that ($\sharp$) is satisfied for $\{U_i^{(\lambda)}, \Phi_i^{(\lambda)}\}$, and also for the open covering $\{\pi_\lambda^{-1}(U_i^{(\lambda)})\}$ of S and some system $\{\Phi_i\}$, and that $\rho(\mathfrak{p}) = \bar{\Phi}_i(\mathfrak{p}, f_i(\mathfrak{p}))$ whenever $\mathfrak{p} \in \pi_\lambda^{-1}(U_i^{(\lambda)})$. Then we can define a cross section ρ_λ of $\bar{B}_\lambda$ as follows: $\rho_\lambda(\mathfrak{p}_\lambda) = \bar{\Phi}_i^{(\lambda)}(\mathfrak{p}_\lambda, f_i(\mathfrak{p}_\lambda))$ whenever $\mathfrak{p}_\lambda \in U_i^{(\lambda)}$. It is easily seen that the divisor class of (ρ_λ) modulo $P(S)$ depends only upon B. Thus we obtain a mapping of $\mathfrak{B}_1(S)$ into $D(S) / P(S)$ by $\rho \to (\rho_\lambda)$.

Conversely, let B_λ be in $\mathfrak{B}(S_\lambda)$ for any λ, and ρ_λ a cross section of $\bar{B}_\lambda$. Then there exist $B \in \mathfrak{B}_1(S)$ and a cross section ρ of $\bar{B}$ such that B_λ, ρ_λ correspond to B, ρ in the above consideration. If B_λ is the identity of $\mathfrak{B}(S_\lambda)$, then B is the identity of $\mathfrak{B}_1(S)$. Hence it follows from the theorem 2 that the mapping defined as above by $\rho \to (\rho_\lambda)$ is an isomorphism of $\mathfrak{B}_1(S)$ onto $D(S)/P(S)$. This isomorphism induces an isomorphism of $\mathfrak{B}_0(S)$ onto $D_0(S)/P(S)$. This completes the proof.

This result can be applied for the W-variety $(S(k), k^*)$ where k is of an infinite degree. If k contains all roots of unity, then $S(k)$ may be regarded as the set of all finite primes in k, and we have

$$\mathfrak{B}_1(S) = \mathfrak{B}_0(S), \quad D(S) = D_0(S).$$

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