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On some systems of regular functions.

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1. Introduction.

Recently E. Peschl and F. Erwe [1] have studied systems of n functions which are regular in a circle $|z|<r$. They called such a system a regular function vector (Funktionenspalte) and wrote it as follows

$$F(z)=\begin{pmatrix} f_1(z) \\ f_2(z) \\ \cdots \\ f_n(z) \end{pmatrix}=A_0+A_1 z+A_2 z^2+\cdots \qquad \text{for } |z|<r, \tag{1.1}$$

where $A_0, A_1, A_2, \cdots$ are n-column vectors. Their researches are confined within bounded function vectors.

The main object of the present paper is to study function vectors from the view-point of multivalence, and extend many known results on a single function.

In attempt to obtain more general results we consider first function vectors of m complex variables, and shall establish thereby an interesting generalization of Newton's interpolation formula. As for the multivalency, however, we shall obtain in this paper only few results other than those on univalency, so long as we deal with several variables. The existence of zero-factor (Nullteiler) in matrix space is an obstacle to our success along this line.

Thus in the latter half part of this paper we shall deal mostly with function vectors of one variable.

At the beginning of our study, it is useful to note that even when a function vector is univalent (or p-valent) in D, the component functions are not neccessarily univalent (or p-valent) in D.

2. Preliminaries.

[A] Differential and Integral.

DEFINITION 1. Let $W\equiv(w_1, w_2, \cdots, w_n)'$ and $Z\equiv(z_1, z_2, \cdots, z_m)'$ be

n-column and m-column vectors respectively. We shall call $W \equiv W(Z)$ an analytic function of Z, if the components w_i $(i=1, 2, \cdots, n)$ are analytic functions of all the components z_i $(i=1, 2, \cdots, m)$ of Z.

When $W \equiv W(Z)$ is an analytic function of Z, we put symbolically

$$W^{(p)} \equiv \frac{d^p W}{dZ^p} \equiv \frac{d}{dZ} \times \frac{d}{dZ} \times \cdots \times \frac{d}{dZ} \times W$$

where $\frac{d}{dZ}$ denotes $\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_m}\right)$ and the symbol $\times$ denotes the so-called " Kronecker's product." The type of $W^{(p)}$ is (n, m^p).

Then we have the following equalities:

$$\frac{d^p (AF_1(Z)+BF_2(Z))}{dZ^p} = A\frac{d^pF_1(Z)}{dZ^p} + B\frac{d^pF_2(Z)}{dZ^p} \tag{2.1}$$

where A and B are constant matrices of type (l, n),

$$\frac{d^p(F_1 \times F_2)}{dZ^p} = F_1^{(p)} \times F_2 + \binom{p}{1} F_1^{(p-1)} \times F_2^{(1)} + \binom{p}{2} F_1^{(p-2)} \times F_2^{(2)} + \cdots \\ \cdots + F_1 \times F_2^{(p)} \tag{2.2}$$

(Leibniz's theorem)

which may be proved by induction.

If the independent variable Z is a function of another variable t, then we have the following property which is important for our investigation in § 3:

$$\frac{d}{dt}\left\{\frac{d^p}{dZ^p} F\big(Z(t)\big) \cdot A\right\} = \left\{\frac{d^{p+1}}{dZ^{p+1}} F\big(Z(t)\big)\right\} \cdot \left(A \times \frac{dZ}{dt}\right), \tag{2.3}$$

A being an m^p-column constant vector.

The integral in a matrix space of type (n, m) is defined as follows

$$\int W\, dt = \left(\int w_{ij}\, dt\right),\ i=1, \cdots, n;\ j=1, \cdots, m.$$

In this paper we denote vectors by capital letters unless otherwise stated.

Then we can see immediately

$$\int AW\,dt = A\int W\,dt, \tag{2.4}$$

$$\int (F_1 \pm F_2)\,dt = \int F_1\,dt \pm \int F_2\,dt \tag{2.5}$$

where A is an n-rowed constant matrix and F_1 and F_2 are matrices of type (n, m).

[B] The norm of vectors and matrices.

We define the norm of a vector $A=(a_1, a_2, \cdots, a_n)'$ as usual

$$\| A \| = \sqrt{A^* A} = \sqrt{\textstyle\sum_{i=1}^{n} \bar{a}_i a_i}$$

where the symbol * denotes the transposed and conjugate vector.

Let $A \equiv (a_{ij})$ be a matrix of type (n, m). The norm of A is defined as follows:

$$\| A \| = \underset{\| X \| = 1}{l.u.b.} \| AX \|$$

where X ranges over all unit vectors.

Now we recall the following properties concerning the norm:

Let A and B be two matrices of type (n_1, m_1) and (n_2, m_2) respectively. Then we have

i) $$\left\{ \frac{1}{\operatorname{Max}(m, n)} \sum_{ij=1}^{m,n} | a_{ij} |^2 \right\}^{\frac{1}{2}} \leqq \| A \| \leqq \left\{ \sum_{ij=1}^{m,n} | a_{ij} |^2 \right\}^{\frac{1}{2}},$$

ii) $\| cA \| = | c | \, \| A \|$ (c: a complex number),

iii) $\| A+B \| \leqq \| A \| + \| B \|$ ($m_1 = m_2$ and $n_1 = n_2$),

iv) $\| AB \| \leqq \| A \| \, \| B \|$ ($m_1 = n_2$),

v) $\| A \times B \| = \| A \| \, \| B \|$,

vi) $\| \int_0^1 A\,dt \| \leqq \int_0^1 \| A \| \, dt$.

3. Newton's formula.

Let $Z_0, Z_1, \cdots, Z_n, \cdots$ be points (vectors) in a domain D, in which $F(Z)$ is defined, and put $[Z_j]=F(Z_j)$. If there exist matrices A of type (n, m) satisfying $A(Z_j-Z_k)=[Z_j]-[Z_k]$ we write any one of such A as $[Z_j\, Z_k]^{F(z)}$; thus we have

$$[Z_j\, Z_k]^{F(z)}(Z_j-Z_k)=[Z_j]-[Z_k].$$

Inductively we define $[Z_0\, Z_1 \cdots Z_n]^{F(z)}$ by the equality

$$[Z_0\, Z_1 \cdots Z_p]^{F(z)} \cdot \{(Z_p-Z_0)\times(Z_p-Z_1)\times\cdots\times(Z_p-Z_{p-1})\} =$$
$$\{[Z_0 \cdots Z_{p-2}\, Z_p]^{F(z)}-[Z_0 \cdots Z_{p-1}]\}^{F(z)} \cdot \{(Z_p-Z_0)\times(Z_p-Z_1)\times\cdots\times(Z_p-Z_{p-2})\}.$$

The type of $[Z_0\, Z_1 \cdots Z_p]^{F(z)}$ is (n, m^p).

From our definition we have

$$F(Z)=\sum_{i=0}^{n}[Z_0 \cdots Z_i]^{F(z)} \cdot \{(Z-Z_0)\times\cdots\times(Z-Z_{i-1})\}$$
$$+[Z_0\, Z_1 \cdots Z_n\, Z]^{F(z)} \cdot \{(Z-Z_0)\times\cdots\times(Z-Z_n)\}. \tag{3.1}$$

THEOREM 1. *Let $F(Z)$ be regular in the smallest convex polygon containing the points $Z_0, Z_1, \cdots, Z_p$. Then we can adopt*

$$Q_F(Z_0\, Z_1 \cdots Z_p)=\int_0^1 dt_1\int_0^{t_1} dt_2 \cdots \int_0^{t_{p-1}} F^{(p)}\big((1-t_1)\,Z_0+(t_1-t_2)\,Z_1+\cdots$$
$$\cdots+(t_{p-1}-t_p)\,Z_{p-1}+t_p\,Z_p\big)\,dt_p \tag{3.2}$$

as our $[Z_0\, Z_1 \cdots Z_p]^{F(z)}$ where t_i $(i=1, \cdots, p)$ are real numbers satisfying the conditions: $t_i \geqq 0$ $(i=1, 2, \cdots, p)$.

PROOF. It is easy to see that this theorem holds for $p=1$ since

$$Q_F(Z_0\, Z_1)\,(Z_0-Z_1)=\int_0^1 F'\,\big((1-t_1)\,Z_0+t_1\,Z_1\big)\,dt_1 \cdot (Z_0-Z_1)$$
$$=-\int_0^1 \frac{d}{dt_1}\, F\big((1-t_1)\,Z_0+t_1\,Z_1\big)\,dt \quad \text{(by (2.3))}$$
$$=F(Z_0)-F(Z_1).$$

Now let us assume that the theorem holds for $p=r$, namely $Q_F(Z_0\cdots Z_r)$ are adoptable as our $[Z_0\cdots Z_r]^{F(z)}$ for any P_i, $i=0,1,\cdots,r$.
Then it follows, by making use of (2.3), that

$$Q_F(Z_0\cdots Z_{r+1})\cdot\{(Z_{r+1}-Z_0)\times(Z_{r+1}-Z_1)\times\cdots\times(Z_{r+1}-Z_r)\}$$

$$=\int_0^1 dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_r}F^{(r+1)}\Big((1-t_1)Z_0+\cdots+(t_r-t_{r+1})Z_r+(t_{r+1}Z_{r+1})$$

$$Z_r+t_{r+1}Z_{r+1}\Big)dt_{r+1}\cdot\Big\{(Z_{r+1}-Z_0)\times(Z_{r+1}-Z_1)\times\cdots\times(Z_{r+1}-Z_r)\Big\}$$

$$=\int_0^1 dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_r}\frac{d}{dt_{r+1}}F^{(r)}\Big((1-t_1)Z_0+\cdots+t_{r+1}Z_{r+1}\Big)dt_{r+1}$$

$$\cdot\Big\{(Z_{r+1}-Z_0)\times(Z_{r+1}-Z_1)\times\cdots\times(Z_{r+1}-Z_{r-1})\Big\}$$

$$=\int_0^1 dt_1\cdots\int_0^{t_{r-1}}\Big\{F^{(r)}\Big((1-t_1)Z_0+\cdots+(t_{r-1}-t_r)Z_{r-1}+t_rZ_{r+1}\Big)$$

$$-F^{(r)}\Big((1-t_1)Z_0+\cdots+(t_{r-1}-t_r)Z_{r-1}+t_rZ_r\Big)\Big\}dt_r$$

$$\cdot\Big\{(Z_{r+1}-Z_0)\times(Z_{r+1}-Z_1)\times\cdots\times(Z_{r+1}-Z_{r-1})\Big\}$$

$$=\Big\{Q_F(Z_0\cdots Z_{r-1}Z_{r+1})-Q_F(Z_0\cdots Z_r)\Big\}\cdot\Big\{(Z_{r+1}-Z_0)\times\cdots\times(Z_{r+1}-Z_{r-1})\Big\}.$$

Thus the theorem holds for any positive integral values of p. q. e. d.

Accordingly we have an explicit representation of Newton's interpolation formula as follows.

THEOREM 2. *Let $F(Z)$ be regular in the smallest convex polygon D containing the points $Z_0, Z_1, \cdots$. Then we have*

$$F(Z)=Q_F(Z_0)+Q_F(Z_0Z_1)(Z-Z_0)+Q_F(Z_0Z_1Z_2)\cdot\{(Z-Z_0)\times(Z-Z_1)\}+\cdots+Q_F(Z_0\cdots Z_n)\cdot\{(Z-Z_0)\times(Z-Z_1)\times\cdots\times(Z-Z_{n-1})\}+R_{n+1}\tag{3.3}$$

where

$$R_{n+1}=Q_F(Z_0\cdots Z_n Z)\cdot\{Z-Z_0)\times(Z-Z_1)\times\cdots\times(Z-Z_n)\}.$$

If $\| F^{(n+1)}(Z) \|$ *is bounded and if the diameter of* D *is also bounded, then* $\lim_{n\to\infty} R_{n+1}=0$.

As special cases of Theorems 1 and 2, we have

COROLLARY 1. *Let* $F(Z)$ *be regular at* $Z=0$. *Then*

$$Q_F(Z_0\cdots Z_0)=\frac{1}{p!}F^{(p)}(Z_0). \tag{3.4}$$

COROLLARY 2. (Taylor's expansion) *Let* $F(Z)$ *be regular for* $\| Z-A \| \leqq r$. *Then*

$$F(Z)=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{d^kF(A)}{dZ^k}\cdot\{\underbrace{(Z-A)\times\cdots\times(Z-A)}_{k}\}. \tag{3.5}$$

Corollary 2 has been obtained by S. Ozaki and I. Ono [14] for functions defined in a polycylinder.

4. Case $n \geqq m^p$.

In this paragraph we confine ourselves to somewhat special cases when $n \geqq m^p$, where n and m are numbers of functions and variables respectively and p is a certain positive integer.

We need the following lemma which is well-known.

LEMMA 1. *Let* $A=(a_{ij})$ *be a matrix of type* (n, q) *for which* rank$A$ $=\min(n, q)=q$ *and further let* λ *be the minimum of the characteristic values of a Hermitian matrix* A^*A, A^* *denoting the adjoint matrix of* A. *Then we have*

$$\| AX \|^2 \geqq \lambda \| X \|^2 > 0$$

for any q*-dimensional vector* $X(\neq 0)$.

Concerning $Q_F(Z_0\cdots Z_p)$ we have the following

THEOREM 3. *Let* $F(Z)$ *be regular in a convex domain* D. *If there exists a matrix* A *of type* (n, m^p) *for which* rank $A=m^p$ *where* p *is a positive integer and* $n \geqq m^p$, *and such that*

$$\left\| \frac{1}{p!} \frac{d^p F(Z)}{dZ^p} - A \right\| < \lambda^{\frac{1}{2}} \tag{4.1}$$

for all variables Z in D, where λ denotes the minimum of the characteristic values of A^*A, then for every pair of Z_i $(i=0, \cdots, p)$ satisfying $(Z_p-Z_0)\times(Z_p-Z_1)\times\cdots\times(Z_p-Z_{p-1})\neq 0$, we have

$$Q_F(Z_0\cdots Z_p)\cdot\{(Z_p-Z_0)\times(Z_p-Z_1)\times\cdots\times(Z_p-Z_{p-1})\}\neq 0\,, \tag{4.2}$$

$$Q_F(Z_0\cdots Z_p)\neq 0\,, \tag{4.3}$$

$$Q_F(Z_0\cdots Z_{p-1})\neq Q_F(Z_0\cdots Z_{p-2}\,Z_p)\,. \tag{4.4}$$

PROOF. Since $n \geqq m^p$, $\lambda > 0$ by Lemma 1.
Since D is a convex domain, we have

$$\begin{aligned}
&\| Q_F(Z_0\cdots Z_p)\cdot\{(Z_p-Z_0)\times(Z_p-Z_1)\times\cdots\times(Z_p-Z_{p-1})\}\| \\
&=\left\| A\Big\{(Z_p-Z_0)\times(Z_p-Z_1)\times\cdots\times(Z_p-Z_{p-1})\Big\}+\int_0^1 dt_1\int_0^{t_1}\cdots\int_0^{t_{p-1}}\Big\{F^{(p)}\Big((1-t_1)\right. \\
&\qquad\left. Z_0+\cdots+t_pZ_p\Big)-p\,!\,A\Big\}dt_p\cdot\Big\{(Z_p-Z_0)\times\cdots\times(Z_p-Z_{p-1})\Big\}\right\| \\
&\geqq \lambda^{\frac{1}{2}}\Big\|(Z_p-Z_0)\times\cdots\times(Z_p-Z_{p-1})\Big\|-\int_0^1 dt\int_0^{t_1}\cdots\int_0^{t_{p-1}}\Big\|F^{(p)}\Big((1-t_1)Z_0+\cdots \\
&\qquad\cdots+t_pZ_p\Big)-p\,!\,A\Big\|\,dt_p\,\Big\|(Z_p-Z_0)\times\cdots\times(Z_p-Z_{p-1})\Big\|
\end{aligned}$$

by using Lemma 1 and the properties of norm,

$$\begin{aligned}
&=\int_0^1 dt_1\int_0^{t_1}\cdots\int_0^{t_{p-1}}\Big[\lambda^{\frac{1}{2}}p\,!-\Big\|F^{(p)}\Big((1-t_1)\,Z_0+\cdots+t_pZ_p\Big)-p\,!\,A\Big\|\Big]dt_p\Big\|(Z_p-Z_0) \\
&\qquad\times\cdots\times(Z_p-Z_{p-1})\Big\|>0 \quad \text{by (4.1)}\,,
\end{aligned}$$

whence we have (4.2) and consequently we have (4.3) and (4.4) by our definition, q. e. d.

It is evident that in the special case when $p=1$, (4.2) yields the univalency of $F(Z)$, which is a generalization of S. Takahashi's theorem [2] given by S. Ozaki and others [3]. Moreover it is easily seen that when $n \geqq 1$ and $m=1$, (4.2) yields the p-valency of $F(z)$. This special case will be studied in detail in the following paragraphs. For general $p, m, n(n \geqq m^p)$, however, we can say nothing about the number of the valency of $F(Z)$.

As for the univalency, however, we have more generally the following

THEOREM 4. *Let K be a convex domain which contatns the origin $Z=0$, and let*

$$F(Z)=A\frac{1}{Z}+G(Z)$$

where A is a constant matrix of type (n, m), $\frac{1}{Z}=\left(\frac{1}{z_1}, \frac{1}{z_2}, \cdots, \frac{1}{z_m}\right)'$ and $G(Z)$ is regular in K. Suppose that there exists a matrix B for which rank $B=m$ *and such that*

$$\left\| \frac{dG(Z)}{dZ}-B \right\| < \sqrt{\lambda}-\frac{\sqrt{\mu}}{\rho^2} \tag{4.5}$$

where λ is the minimum of the characteristic values of B^*B and μ is the maximum of the characteristic values of A^*A, then $F(Z)$ is univalent in the common part of K and $|z_i|>\rho$ $(i=1, 2, \cdots, m)$.

This theorem is a generalizaiion of T. Sato's theorem [4] and can be proved analogously to [3] referring to the method of [4] together with Lemma 1. The detail may be omitted here.

Using the above theorem we obtain the following

THEOREM 5. *Let*

$$F(Z)=A\frac{1}{Z}+G(Z)=A\frac{1}{Z}+A_0+A_1Z+\cdots+A_n\cdot\overbrace{Z\times\cdots\times Z}^{n}+\cdots, \tag{4.6}$$

where $G(Z)$ is regular for $\|Z\| \leqq r$. Suppose that there exists the relation

$$\sqrt{\lambda}-\frac{\sqrt{\mu}}{\rho^2} \geqq 2\|A_2\|r+3\|A_3\|r^2+\cdots+n\|A_n\|r^{n-1}+\cdots \tag{4.7}$$

between its coefficient matrices, where λ is the minimum of the charac-

teristic values of $A_1^* A_1$ *and* μ *the maximum of the characteristic values of* A^*A. *Then* $F(Z)$ *is univalent in the common part of* $\|Z\| < r$ *and* $|z_i| > \rho$ $(i=1, 2, \cdots, m)$.

PROOF. By (2.1) and (2.2) we have

$$G'(Z) = A_1 + A_2 (E \times Z + Z \times E) + \cdots + A_n (E \times Z \times \cdots \\ \cdots \times Z + Z \times E \times Z \times \cdots Z + \cdots + Z \times \cdots \times Z \times E) + \cdots,$$

where E is the m-rowed unit matrix.
Hence by the property of norm

$$\| G'(\mathrm{P}) - A_1 \| < 2 \| A_2 \| r + 3 \| A_3 \| r^2 + \cdots + n \| A_n \| r^{n-1} + \cdots .$$

for $\|Z\| < r$. By virtue of (4.7), we have

$$\| G'(Z) - A_1 \| < \sqrt{\lambda} - \frac{\sqrt{\mu}}{\rho^2} \quad \text{for } \|Z\| < r,$$

which shows that $F(Z)$ is univalent in the common part of $\|Z\| < r$ and $|z_i| > \rho$ $(i=1, 2, \cdots m)$ by Theorem 4.

5. Case $n \geqq 2$, $m=1$.

Now let us consider function vectors of one variable. In this case there is no obstacle due to zero-factor anymore and we obtain directly from Theorem 3 the following

THEOREM 6. *Let* $F(z)$ *be regular in a convex domain* D. *If there exists a vector* A *such that*

$$\| F^{(p)}(z) - A \| < \| A \| \tag{5.1}$$

for all z *in* D, *then* $F(z)$ *is* p-*valent in* D.

COROLLARY 3. *Let* $F(z) = \sum_{n=0}^{\infty} A_n z^n$ *be a function vector regular for* $|z| < r$ *and suppose that*

$$\| A_k \| \geqq \sum_{n=k+1}^{\infty} \binom{n}{k} \| A_n \| r^{n-k} \quad \text{in } |z| < r \tag{5.2}$$

then $F(z)$ *is absolutely* k*-valent in* $|z|<r$.

REMARK. The concept of absolute k-valence may be understood analogously as in the case $n=1$. Cf. S. Ozaki [5]. Namely, $F(z)$ is absolutely p-valent if not only $F(z)$ but also

$$C_0+C_1 z+\cdots+C_{p-1} z^{p-1}+F(z)$$

are p-valent in D for arbitrary constant vectors $C_0, C_1, \cdots, C_{p-1}$.

Now Corollary 3 is a generalization of Itihara's theorem and moreover can be extended as follows.

THEOREM 7. *Let*

$$F(z)=A_{-k}\frac{1}{z^k}+A_{-k+1}\frac{1}{z^{k-1}}+\cdots+A_0+A_1 z+\cdots+A_n z^n+\cdots \tag{5.3}$$

be a function vector regular for $0<|z|<r$ *and suppose that*

$$\|A_{l-k}\|\geqq\sum_{n=p+1}^{\infty}\|A_n\|\frac{n-p}{n+k-l}\binom{n+k}{p+k}\binom{p+k}{l}r^{n+k-l} \tag{5.4}$$

$$p+k\geqq l\geqq 0,\ l\neq k.$$

Then $F(z)$ *is at most* $(k+p)$*-valent in* $0<|z|<r$.

PROOF. If $F(z)$ is at least $(k+p+1)$-valent in $0<|z|<r$, then we have $p+k+1$ points which satisfy the following conditions

$$F(z_0)=F(z_1)=\cdots=F(z_{p+k})=C,$$

$$z_i\neq z_j,\ 0<|z_j|<r,$$

$$i,j=0,1,\cdots,p+k.$$

Namely

$$A_{-k}z_i^{-k}+A_{-k+1}z_i^{-k+1}+\cdots+A_0-C+\sum_{n=1}^{\infty}A_n z_i^n=0,$$

$$i=0,1,\cdots,p+k.$$

Hence

$$(C)\begin{cases} A_{-k}+A_{-k+1}z_0+\cdots+(A_0-C)z_0^k+\sum_{n=1}^{\infty}A_n z_0^{n+k}=0 \cdots\cdots\cdots\cdots\cdots(0), \\ A_{-k}+A_{-k+1}z_1+\cdots+(A_0-C)z_1^k+\sum_{n=1}^{\infty}A_n z_1^{n+k}=0 \cdots\cdots\cdots\cdots\cdots(1), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ A_{-k}+A_{-k+1}z_{p+k}+\cdots+(A_0-C)z_{p+k}^k+\sum_{n=1}^{\infty}A_n z_{p+k}^{n+k}=0\cdots\cdots(p+k). \end{cases}$$

Let us employ the notation

$$(5.5)\qquad V(k_0\cdots k_{p+k})\equiv\begin{vmatrix} z_0^{k_0} & z_0^{k_1} & \cdots\cdots\cdots\cdots & z_0^{k_{p+k}} \\ z_1^{k_0} & z_1^{k_1} & \cdots\cdots\cdots\cdots & z_1^{k_{p+k}} \\ \cdots & \cdots & \cdots\cdots\cdots\cdots & \cdots \\ z_{p+k}^{k_0} & z_{p+k}^{k_1} & \cdots\cdots & z_{p+h}^{k_{p+k}} \end{vmatrix}$$

and let $\varDelta(z_j^i)\,(i,j=0,\cdots,p+k)$ be the cofactor of z_j^i $(i,j=0,\cdots,p+k)$ in $V(0,1,\cdots,p+k)$.

To eliminate the values $A_{-k},\cdots,A_p$ from (C), except A_{-k+l} where $0\leqq l\leqq p+k,\ l\neq k,$ we calculate

$$(0)\times\varDelta(z_0^l)+(1)\times\varDelta(z_1^l)+(2)\times\varDelta(z_2^l)+\cdots+(p+k)\times\varDelta(z_{p+k}^l).$$

Then we obtain

$$(-1)^{p+k-l}A_{-k+l}V(0,1,\cdots,p+k)+\sum_{n=p+1}^{\infty}A_nV(0,1,\cdots,\\ l-1,l+1,\cdots,p+k,n+k)=0$$

and hence

$$(-1)^{p+k-l}A_{-k+l}+\sum_{n=p+1}^{\infty}A_nV(0,1,\cdots,l-1,l+1,\cdots,\\ p+k,n+k,)/V(0,1,\cdots,p+k)=0.$$

Now it has long been known that when the non-negative integers k_j are arranged in an increasing sequence, $V(k_1, k_2, \cdots, k_p)/V(0, 1, \cdots, p-1)$ is a positive polynomial (i. e. the coefficient of every term of the polynomial has the sign $+$), [6], [13].

From this fact it follows immediately that

$$V(0,1,\cdots,l-1,l+1,\cdots,p+k,n+k)\,/\,V(0,1,\cdots,p+k),\quad n\geqq p+1,$$

takes its maximum values for $|z|\leqq r$ at $z_j=r$ $(j=1,\cdots,p+k)$.

So we have by Mitchell's theorem [6],

$$|\,V(0,1,\cdots,l-1,l+1,\cdots,p+k,n+k)\,/\,V(0,1,\cdots,p+k)\,|$$
$$\leqq \frac{n-p}{n+k-l}\binom{n+k}{p+k}\binom{p+k}{l}r^{n+k-l}.$$

Therefore we obtain

$$\|A_{-k+l}\|<\sum_{n=p+1}^{\infty}\|A_n\|\frac{n-p}{n+k-l}\binom{n+k}{p+k}\binom{p+k}{l}r^{n+k-l}.$$

And hence $F(z)$ is at most $(k+p)$-valent in $|z|<r$ if we have (5.4).

REMARK. The above theorem is a generalization of the result in [7].

6. Radii of p-valence of some function vectors.

In the case $m=n=1$ the radii of univalence have been obtained under certain conditions, for example, $|f(z)|<M$ or $|f'(z)|<M$. We are going to extend these results by determining the radii of p valence for the analogous families of function vectors of one variable.

For this purpose we need the following results due to [1]:

Let A be a vector with $\|A\|<1$, and define a matrix $\Gamma(A)$ by

$$\Gamma(A)=\frac{1}{1+v(A)}AA^*+v(A)E \tag{6.1}$$

where E is the n-rowed unit matrix and

$$v(A)=\sqrt{1-\|A\|^2}\,. \tag{6.2}$$

$\Gamma(A)$ has the following properties

$$\Gamma(A)^*=\Gamma(A)\,,\ \Gamma(A)A=A\,,\ A^*\Gamma(A)=A^*\,, \tag{6.3}$$

$$v(A)\,\|B\|\leqq\|\Gamma(A)B\|\leqq\|B\|\,. \tag{6.4}$$

An analytical transformation of $\|Z\|<1$ onto $\|W\|<1$ such that $W(A)=0$ are written as follows

$$W=\Gamma(A)\frac{Z-A}{1-A^*Z} \tag{6.5}$$

and inversely

$$Z=\Gamma(A)\frac{W+A}{1+A^*W}\,. \tag{6.6}$$

For this transformation holds the inequality

$$\frac{\|A\|-\|Z\|}{1-\|A\|\ \|Z\|}\leqq\|W\|\leqq\frac{\|A\|+\|Z\|}{1+\|A\|\ \|Z\|}\,. \tag{6.7}$$

Let $F(z)$ be regular and $\|F(z)\|<1$ for $|z|<1$. Suppose that $F(z)$ takes a value W_0 at $z_0, z_1, \cdots, z_m$ in $|z|<1$. Then according to [1], we have the inequality

$$\left\|\Gamma(W_0)\frac{F(z)-W_0}{1-W_0^*F(z)}\right\|\leqq|h(z)| \tag{6.8}$$

where
$$h(z)=\prod_{\nu=0}^{m}\frac{z-z_\nu}{1-\bar{z}_\nu z}$$

and

$$\frac{\|W_0\|-|h(z)|}{1-\|W_0\|\ |h(z)|}\leqq\|F(z)\|\leqq\frac{\|W_0\|+|h(z)|}{1+\|W_0\|\ |h(z)|} \tag{6.9}$$

Now we extend a theorem due to K. Noshiro [8].

THEOREM 8. *Let $F(z)$ be regular for $|z|<1$ and let $F^{(k)}(0)=A_k\neq 0$, $F^{(k+1)}(0)=A_{k+1}$ be given. If $\|F^{(k)}(z)\|<1$ for $|z|<1$, then $F(z)$ is absolutely k-valent for*

$$|z|<R=\frac{1}{2}\left\{-\|B\|\,(1-\|C\|)+\sqrt{\|B\|^2(1-\|C\|)^2+4\|C\|}\right\}$$

where

$$B=\Gamma(A_k)\frac{A_{k+1}}{1-\|A_k\|^2} \quad \text{and} \quad C=\frac{A_k}{\|A_k\|^2+\|\Gamma(A_k)(E-A_k A_k^*)\|}.$$

PROOF. Since $\|F^{(k)}(z)\|<1$, $\|F_{(k)}(0)\|=\|A_k\|<1$.

Let us put

$$(6.10)\qquad W(z)=\Gamma(A_k)\frac{F^{(k)}(z)-A_k}{1-A_k^*F^{(k)}(z)}=\Gamma(A_k)\frac{A_{k+1}}{1-\|A_k\|^2}z+\cdots=Bz+\cdots.$$

Then $W(z)$ is regular and $\|W(z)\|<1$ for $|z|<1$ by (6.5), and by (6.6)

$$F^{(k)}(z)=\Gamma(A_k)\frac{W(z)+A_k}{1+A_k^*W(z)}.$$

Hence

$$\|F^{(k)}(z)-A_k\|=\left\|\Gamma(A_k)\frac{W(z)+A_k}{1+A_k^*W(z)}-\Gamma(A_k)A_k\right\|=\left\|\Gamma(A_k)\frac{(E-A_kA_k^*)W(z)}{1+A_k^*W(z)}\right\|\leqq\|\Gamma(A_k)(E-A_kA_k^*)\|\frac{\|W(z)\|}{1-\|A_k\|\,\|W(z)\|}.$$

Hence, by Theorem 6, $F(z)$ is absolutely k-valent for $|z|<R$, provided that

$$(6.11)\qquad \|W(z)\|<\frac{\|A_k\|}{\|A_k\|+\|\Gamma(A_k)(E-A_kA_k^*)\|}=\|C\| \text{ for } |z|<R.$$

On the other hand, putting $G(z)=\dfrac{W(z)}{z}$ we have $G(0)=B$ and $\|G(z)\|\leqq 1$ for $|z|<1$.

By (6.9) $$\|G(z)\| \leqq \frac{\|B\|+|z|}{1+\|B\|\ |z|}$$

and hence

$$\|W(z)\| \leqq r\frac{\|B\|+r}{1+\|B\|r} \text{ for } |z| \leqq r.$$

It is seen that the inequality (6.11) holds good, because R is the root between 0 and 1 of the equation

$$r\frac{\|B\|+r}{1+\|B\|r} = \|C\|. \qquad \text{q. e. d.}$$

REMARK. If we put $k=1$ and $n=1$ in the above theorem referring to (6.3), we have Noshiro's theorem.

LEMMA 2. *Let* $$F(z) = \sum_{-\infty}^{\infty} A_n z^n$$

be a functihn vector regular for $0 < |z| \leqq 1$ *and let*

$$\left[\frac{1}{2\pi}\int_0^{2\pi} \|F(e^{i\theta})\|^2 d\theta\right]^{\frac{1}{2}} \leqq M.$$

Then we have the inequality

$$\sum_{n=-\infty}^{\infty} \|A_n\|^2 \leqq M. \tag{6.12}$$

Proof of this lemma is analogous to the case where $n=1$ and may be omitted.

THEOREM 9. *Let* $F(z) = A_0 + A_1 z + \cdots + A_n z^n + \cdots$ *be regular for* $|z| < 1$ *and*

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} \|F(e^{i\theta})\|^2 d\theta\right\}^{\frac{1}{2}} < M.$$

Then $F(z)$ *is at most* p*-valent in* $|z| < R$ *where* R *is a root between* 0 *and* 1 *of the equation, putting* $\zeta = Re^{i\theta}$

$$(E)\qquad \frac{1}{2\pi}\int_{|\zeta|=R}\left|\frac{\zeta^{p-k+1}}{k!\,(p-k)!}\frac{d^{p-k}}{d\zeta^{p-k}}\frac{1}{\zeta(1-\zeta)^{k+1}}\right|^2 d\theta$$
$$=\frac{M^2-(\|A_0\|^2+\cdots+\|A_{k-1}\|^2+\|A_{k+1}\|^2+\cdots+\|A_p\|^2)}{M^2-(\|A_0\|^2+\cdots\cdots+\|A_p\|^2)}.$$

PROOF.

$$\sum_{n=p+1}^{\infty}\|A_n\|\frac{n-p}{n-k}\binom{n}{p}\binom{p}{k}r^{n-k}$$
$$\leqq\sqrt{\sum_{n=p+1}^{\infty}\|A_n\|^2}\sqrt{\sum_{u=p+1}^{\infty}\left(\frac{n-p}{n-k}\right)^2\binom{n}{p}^2\binom{p}{k}^2 r^{2(n-k)}}$$
$$\leqq\sqrt{M^2-\sum_{n=0}^{p}\|A_n\|^2}\sqrt{\sum_{n=p+1}^{\infty}\left(\frac{n-p}{n-k}\right)^2\binom{n}{p}^2\binom{p}{k}^2 r^{2(n-k)}}<\|A_k\|$$

for $r<R$. Here R satisfies

$$\sum_{n=p+1}^{\infty}\left(\frac{n-p}{n-k}\right)^2\binom{n}{p}^2\binom{p}{k}^2 R^{2(n-k}=\frac{\|A_k\|^2}{M^2-(\|A_0\|^2+\cdots+\|A_p\|^2)},$$

that is the equation (E). Hence $F(z)$ is at most p-valent in $|z|<R$ by Theorem 7 in the case $k=0$.

In the same manner we can derive the following

THEOREM 10. *Let* $F(z)=\sum_{n=0}^{\infty}A_n z^n$ *be regular in* $|z|\leqq 1$ *and let*

$$\left\{\frac{1}{2\pi}\int_{|z|=1}\left\|\frac{z^{p-k+1}}{k!\,(p-k)!}\frac{d^{p-k}}{dz^{p-k}}\frac{F^{(k)}(z)}{z}\right\|^2 d\theta\right\}^{\frac{1}{2}}<M,\quad z=e^{i\theta}.$$

Then $F(z)$ *is at most* p*-valent in* $|z|<r$ *where* r *is a root between* 0 *and* 1 *of the equation*

$$(M^2-\|A_k\|^2)r^{2(p-k+1)}+\|A_k\|^2 r^2-\|A_k\|^2=0.$$

REMARK. If we put in Theorems 9 and 10, $k=p$ and $n=1$ we have K. Nabetani's theorems [9]. If we consider a meromorphic function vector in the form (5.3) and use Theorem 7, we can obtain more general results, which may be omitted here.

7. On absolutely p-valent function vectors.

The main object of this paragraph is to obtain the generalization of the results given by S. Ozaki [10].

THEOREM 11. *In order that a function* $F(z)$ *be absolutely* p*-valent in* D*, it is necessary and sufficient that the function*

$$\frac{C_0+C_1 z+\cdots+C_{p-1} z^{p-1}+F(z)}{d_0+d_1 z+\cdots+d_{p-1} z^{p-1}} \tag{7.1}$$

should be absolutely p*-valent in* D*, where* $C_0, C_1, \cdots, C_{p-1}$ *are arbitrary constant vectors and* $d_0, d_1, \cdots, d_{p-1}$ *are arbitrary constant numbers.*

PROOF. For a given vector W_0, the equation

$$\frac{C_0+C_1 z+\cdots+C_{p-1} z^{p-1}+F(z)}{d_0+d_1 z+\cdots+d_{p-1} z^{p-1}}=W_0$$

can be written as follows: $(C_0-d_0 W_0)+(C_1-d_1 W_0) z+\cdots+(C_{p-1}-d_{p-1} W_0) z^{p-1}+F(z)=0$,

$$C_0'+C_1' z+\cdots+C_{p-1}' z^{p-1}+F(z)=0\,. \tag{7.2}$$

Hence, if $F(z)$ is absolutely p-valent in D the equation (7.2) has at most p roots in D, namely the function (7.1) is at most p-valent, and inversely, if the function (7.1) is at most p-valent in D we know the p-valency of $C_0+C_1 z+\cdots+C_{p-1} z^{p-1}+F(z)$ by putting $d_0=1, d_1=d_2=\cdots=d_{p-1}=0$ in (7.1), namely, $F(z)$ is absolutely p-valent in D.

For the special case $C_0=C_1=\cdots=C_{p-1}=0$ we obtain the following

COROLLARY. *If* $F(z)$ *is absolutely* p*-valent in* D*,*

$$\frac{F(z)}{z-\alpha_1},\ \frac{F(z)}{(z-\alpha_1)(z-\alpha_2)},\ \cdots,\ \frac{F(z)}{\prod_{i=1}^{p-1}(z-\alpha_i)}$$

are also at most p*-valent in* D*, where* $\alpha_1, \alpha_2, \cdots, \alpha_{p-1}$ *are arbitrary constant numbers.*

The special case $\alpha_1=\alpha_2=\cdots=\alpha_{p-1}=0$ in the above corollary is important, namely, if $F(z)$ is absolutely p-valent in D,

$$\frac{F(z)}{z},\ \frac{F(z)}{z^2},\ \cdots,\ \frac{F(z)}{z^{p-1}}$$

are also at most p-valent.

By the above Theorem 11 we know that a sufficient condition for absolutely p-valency of $F(z)$ is also a sufficient condition for at most p-valency of infinite numbers of function vectors of the form (7.1). Making use of the above theorem we obtain the following

THEOREM 12. *Let $F(z)$ be meremorphic in a convex domain D and suppose that there exists a vector A for which*

$$\left\| \frac{d^k}{dz^k}\left\{z^p F(z)\right\} - A \right\| < \left\| A \right\|, \quad k-1 \geqq p \geqq 0, \tag{7.3}$$

for any z in D. Then $F(z)$ is absolutely k-valent of the class p in D.

REMARK. We say, after S. Ozaki, that $F(z)$ is absolutely k-valent of the class p in D if not only $F(z)$ but also $F(z)+P(z)$ is meromorphic and at most k-valent in D, where

$$P(z) = \sum_{n=-p}^{k-p-1} C_n z^n \quad (C_{-p}, C_{-p+1}, \cdots, C_{k-p-1}: \textit{arbitrary vectors}).$$

COROLLARY 1. *Let $F(z)=\sum\limits_{n=-p}^{\infty} A_n z^n$ be meromorphic in $|z|<r$ and suppose that*

$$\left\| \frac{\frac{d^k}{dz^k}\left\{z^p F(z)\right\}}{k!} - A_{k-p} \right\| < \left\| A_{k-p} \right\|, \quad k-1 \geqq p \geqq 0, \tag{7.4}$$

for any z in $|z|<r$. Then $F(z)$ is absolutely k-valent of the class p in $|z|<r$.

COROLLARY 2. *Let $F(z)=\sum\limits_{n=-p}^{\infty} A_n z^n$ and suppose that*

$$\left\| A_{k-p} \right\| \geqq \sum_{n=1}^{\infty} \binom{k+n}{k} \left\| A_{k-p+n} \right\| r^n, \quad k-1 \geqq p \geqq 0. \tag{7.5}$$

Then $F(z)$ is meromorphic and absolutely k-valent of the class p in D.

LEMMA. 3. *The necessary and sufficient condition that the function*

$f(z)=\sum_{n=-p}^{\infty} a_n z^n$ *where* $a_{k-p}<0$, $(k-1\geqq p\geqq 0)$, $a_{k-p+n}\geqq 0$ $(n=1,2,\cdots)$ *should be meromorphic and absolutely k-valent of the class p in* $|z|<r$ *is that there should exist the relation*

$$|a_{k-p}|\geqq\sum_{n=1}^{\infty}\binom{k+n}{k}\left|a_{k-p+n}\right|r^n \tag{7.6}$$

between its coefficients.

This lemma is due to S. Ozaki [10]. Using this lemma we obtain the following

THEOREM 13. *If*

$$G(z)=-\|A_{k-p}\|z^{k-p}+\sum_{n=k-p+1}^{\infty}\|A_n\|z^n,\ k-1\geqq p\geqq 0$$

is meromorphic and absolutely k-valent of the class p in $|z|<r$*, then* $F(z)=\sum_{n=-p}^{\infty} A_n z^n$ *is also absolutely k-valent of the class p in* $|z|<r$.

COROLLARY. *If*

$$G(z)=-|a_{k-p}|z^{k-p}+\sum_{n-k-p+1}^{\infty}|a_n|z^n,\ k-1\geqq p\geqq 0,$$

is meromorphic and absolutely k-valent of the class p in $|z|<r$*, then* $H(z)=\sum_{n=-p}^{\infty} A_n z^n$ *where* $\|A_{k-p}|\geqq|a_{k-p}|$, $\|A_{k-p+n}\|\geqq|a_{k-p+n}|$ $(n=1,2\cdots)$, *is also absolutely k-valent of the class p in* $|z|<r$.

8. Another extension of Theorem 6 and its application.

THEOREM 14. *Suppose that* $\phi(z)$ *is a convex function in D. If* $F(z)$ *is regular in D and if there exists a vector A such that*

$$\left\|\frac{d^kF(z)}{d\phi(z)^k}-A\right\|<\left\|A\right\|$$

for any z in D, then $F(z)$ *is at most k-valent in D.*

PROOF. By our hypothesis $\zeta=\phi(z)$ maps D onto a convex domain T on the ζ-plane one-to-one and conformally. Accordingly $F(z)$

$=F\{\phi^{-1}(\zeta)\}$ is a regular function of ζ in T. Now since

$$\left\| \frac{d^k F(z)}{d\zeta^k} - A \right\| = \left\| \frac{d^k F}{d\phi^k} - A \right\| < \| A \|$$

$F\{\phi^{-1}(\zeta)\}$ is k-valent in T. Namely $F(z)$ is k-valent in D.

Making use of the following convex functions for $\phi(z)$ in Theorem 14,

$$\text{(i) } -\log(1-z), \quad \text{(ii) } (1-z)^{-1}, \quad \text{(iii) } \frac{1}{2}\log\frac{1+z}{1-z},$$

we obtain the following theorems:

THEOREM 15. *Let* $F(z)=A_1 z+A_2 z^2+\cdots+A_n z^n+\cdots$ *be regular for* $|z|<1$ *and put*

$$A_n=A_n^{(1)}, \quad nA_n^{(1)}-(n-1)A_{n-1}^{(1)}=A_n^{(2)},$$
$$(n-1)A_n^{(2)}-(n-2)A_{n-1}^{(2)}=A_n^{(3)},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(n-k+1)A_n^{(k)}-(n-k)A_{n-1}^{(k)}=A_n^{(k+1)}.$$

If $$\| A_k^{(k)} \| > \sum_{n=k+1}^{\infty} \| (n-k)A_{n-1}^{(k)}-(n-k+1)A_n^{(k)} \|,$$

then $F(z)$ *is* k*-valent in* $|z|<1$.

THEOREM 16. *Let* $F(z)=A_1 z+A_2 z^2+\cdots+A_n z^n+\cdots$ *be regular for* $|z|<1$ *and put*

$$A_n=B_n^{(1)}, \quad nB_n^{(1)}-2(n-1)B_{n-1}^{(1)}+(n-2)B_{n-2}^{(1)}=B_n^{(2)},$$
$$(n-1)B_n^{(2)}-2(n-2)B_{n-1}^{(2)}+(n-3)B_{n-2}^{(2)}=B_n^{(3)},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$(n-k+1)B_n^{(k)}-2(n-k)B_{n-1}^{(k)}+(n-k-1)B_{n-2}^{(k)}=B_n^{(k+1)}.$$

If $\| B_k^{(k)} \| > \sum_{n=k+1}^{\infty} \| (n-k+1)B_n^{(k)}-2(n-k)B_{n-1}^{(k)}+(n-k-1)B_{n-2}^{(k)} \|$, *then* $F(z)$ *is* k*-valent in* $|z|<1$.

THEOREM 17. *Let* $F(z)=A_1 z+A_2 z^2+\cdots\cdots$
be regular for $|z|<1$ *and put*

$$A_n=C_n^{(1)},\quad nC_n^{(1)}-(n-2)\,C_{n-2}^{(1)}=C_n^{(2)},$$

$$(n-1)\,C_n^{(2)}-(n-3)\,C_{n-2}^{(2)}=C_n^{(3)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$(n-k+1)\,C_n^{(k)}-(n-k-1)\,C_{n-2}^{(k)}=C_n^{(k+1)}.$$

If $\|C_k^{(k)}\|>\sum_{n=k+1}^{\infty}\|(n-k+1)\,C_n^{(k)}-(n-k-1)\,C_{n-2}^{(k)}\|$,
then $F(z)$ *is k-valent in* $|z|<1$.

REMARK. These theorems are generalizations of the results obtained in author's previous paper [11].

9. Some inequalities.

As the consequence of Theorem 1, we can easily obtain the following

LEMMA 4. *If* $H(z)$ *is regutar in a convex domain* D, *we obtain*

(i)
$$\operatorname*{Max}_{z\in D}\|H^{(p)}(z)\|\geqq\|p!\,Q_{H(z)}(z_0\,z_1\cdots z_p)\|$$
$$\geqq\|A\|-\operatorname*{Max}_{z\in D}\|H^{(p)}(z)-A\|$$

for arbitrary constant vector A. *In particular, if* $H(z)=\sum_{n=0}^{\infty}A_n z^n$ *is regular in* $|z|\leqq R$, *for arbitrary values of* z *in* $|z|\leqq R$ *we obtain*

(ii)
$$\sum_{n=p}^{\infty}\binom{n}{p}\|A_n\|\,R^{n-p}\geqq\|Q_{H(z)}(z_0\,z_1\cdots z_p)\|$$

$$\geqq\|A_p\|-\operatorname*{Max}_{|z|=R}\left\|\frac{H^{(p)}(z)}{p!}-A_p\right\|\geqq\left\|A_p\right\|-\sum_{n=p+1}^{\infty}\binom{n}{p}\left\|A_n\right\|R^{n-p}.$$

LEMMA 5. *If* $G(z)$ *is regular in a reciprocally convex domain**

* If the image of a domain E mapped by the function $w=1/z$ is a convex domain, then we shall call, after S. Osaki [10], the original domain E reciprocally convex.

E, then for arbitrary valuss of z_i in the common part of E and $r \leqq |z| \leqq R$ we obtain

$$\text{(i)}\qquad \operatorname*{Max}_{z\in E}\left\|\left(\frac{z}{r}\right)^{p+1} G^{(p)}(z)\right\| \geqq \left\| p!\,[z_0 z_1 \cdots z_p]^{G(z)}\right\|$$

$$\geqq \frac{1}{R^{p+1}}\left\{\| A \| - \operatorname*{Max}_{z\in E} \| z^{p+1} G^{(p)}(z) - A \|\right\}$$

for arbitrary constant vector A. In particular, if $G(z)=\sum_{n=1}^{\infty}\frac{A_{-n}}{z^n}$ is regular in $|z| \geqq r$, for arbitrary values of z_i in $r \leqq |z| \leqq R$ we obtain

$$\text{(ii)}\qquad \sum_{n=1}^{\infty}\left\|\binom{-n}{p}\frac{A_{-n}}{r^{n+p}}\right\| \geqq \left\|[z\, z_1 \cdots z_p]^{G(z)}\right\| \geqq \frac{1}{R^{p+1}}\left\{\left\|A_{-1}\right\| - \operatorname*{Max}_{|z|=R}\right.$$

$$\left.\left\|\frac{z^{p+1}G^{(p)}(z)}{p!} - A_{-1}\right\|\right\} \geqq \frac{1}{R^{p+1}}\left\{\left\|A_{-1}\right\| - \sum_{n=2}^{\infty}\left\|\binom{-n}{p}\frac{A_{-n}}{r^{n-1}}\right\|\right\}.$$

PROOF. In order to prove this lemma we need the following two equalities

$$(9.1)\qquad [z_0 z_1 \cdots z_p]^{G(z)} = \frac{(-1)^p}{z_0 z_1 \cdots z_p} Q_{\zeta^{p-1}G(\frac{1}{\zeta})}\left(\frac{1}{z_0}, \frac{1}{z_1}, \cdots, \frac{1}{z_p}\right), \quad \left(\zeta = \frac{1}{z}\right),$$

$$(9.2)\qquad \frac{d^p}{d\left(\frac{1}{z}\right)^p}\left(\frac{G(z)}{z^{p-1}}\right) = (-1)^p z^{p+1} G^{(p)}(z)$$

which are the generalizations of S. Ozaki's results [10] and can be proved analogously.

By (9.1) and Theorem 1 we obtain

$$\left\|[z_0, z_1, \cdots, z_p]^{G(z)}\right\| \leqq \operatorname*{Max}_{z\in E}\left|\frac{1}{z^{p+1}}\right| \operatorname*{Max}_{z\in E}\left\|\frac{1}{p!}\frac{d^p}{d\left(\frac{1}{z}\right)^p}\left(\frac{G(z)}{z^{p-1}}\right)\right\|$$

$$= \operatorname*{Max}_{z\in E}\left|\frac{1}{z^{p+1}}\right| \operatorname*{Max}_{z\in E}\left\|\frac{1}{p!}(-1)^p z^{p+1} G^{(p)}(z)\right\|$$

$$\leqq \operatorname*{Max}_{z\in E}\left\|\frac{1}{p!}\left(\frac{z}{r}\right)^{p+1} G^{(p)}(z)\right\|$$

for arbitrary values of z_i in the common part of E and $|z| \geqq r$. Analogously we obtain for arbitrary values of z_i in the common part of E and $|z| < R$,

$$\| [z_0, z_1, \cdots, z_p]^{G(z)} \| \geqq \frac{1}{R^{p+1}} \left\{ \left\| A \right\| - \operatorname*{Max}_{z \in E} \left\| \frac{z^{p+1} G^{(p)}(z)}{p\,!} - A \right\| \right\}.$$

Let $F(z)$ be the sum of $G(z)$ and $H(z)$, namely,

$$F(z) = G(z) + H(z)$$

then evidently

$$[z_0, z_1, \cdots, z_p]^{F(z)} = [z_0, z_1, \cdots, z_p]^{G(z)} + [z_0, z_1, \cdots, z_p]^{H(z)}.$$

Hence

$$\| [z_0, z_1, \cdots, z_p]^{F(z)} \| \geqq \| [z_0, z_1, \cdots, z_p]^{G(z)} \| \sim \| [z_{01} z_1, \cdots, z_p]^{H(z)} \|$$

and, as the consequence of the above lemma, we obtain the following result:

LEMMA 6. *Let*

$$F(z) = G(z) + H(z) = \sum_{-\infty}^{\infty} A_n z^n$$

be regular for $r \leqq |z| \leqq R$, *where*

$$G(z) = \sum_{n=1}^{\infty} \frac{A_{-n}}{z^n} \quad \textit{and} \quad H(z) = \sum_{n=0}^{\infty} A_n z^n.$$

Then, for arbitrary values of z_i *in* $r \leqq |z| \leqq R$ *we obtain for arbitrary constant vector* A

(i) $\| p\,! [z_0, z_1, \cdots, z_p]^{F(z)} \| \geqq \| A \| - \operatorname*{Max}_{|z|=R} \| H^{(p)}(z) - A \| - \operatorname*{Max}_{|z|=r} \| G^{(p)}(z) \|,$

$$\| [z_0, z, \cdots, z_p]^{F(z)} \| \geqq \| A_p \| - \operatorname*{Max}_{|z|=R} \left\| \frac{H^{(p)}(z)}{p\,!} - A_p \right\| - \operatorname*{Max}_{|z|=r} \left\| \frac{G^{(p)}(z)}{p\,!} \right\|$$

$$\geqq \left\| A_p \right\| - \sum_{n=p+1}^{\infty} \binom{n}{p} \left\| A_n \right\| R^{n-p} - \sum_{n=1}^{\infty} \left\| \binom{-n}{p} A_{-n} \right\| \frac{1}{r^{n+p}};$$

(ii) $$\|p![z_0, z_1, \cdots, z_p]^{F(z)}\| \geqq \frac{1}{R^{p+1}}\Big\{\|A\| - \underset{|z|=r}{\mathrm{Max}}\|z^{p+1}G^{(p)}(z) - A\| - \underset{|z|=R}{\mathrm{Max}}\|z^{p+1}H^{(p)}(z)\|\Big\},$$

$$\|[z_0, z_1, \cdots, z_p]^{F(z)}\| \geqq \frac{1}{R^{p+1}}\Big\{\|A_{-1}\| - \underset{|z|=R}{\mathrm{Max}}\Big\|\frac{z^{p+1}H^{(p)}(z)}{p!}\Big\|$$

$$- \underset{|z|=r}{\mathrm{Max}}\Big\|\frac{z^{p+1}G^{(p)}(z)}{p!} - (-1)^p A_{-1}\Big\|\Big\} \geqq \frac{1}{R^{p+1}}\Big\{\|A_{-1}\| - \sum_{n=p}^{\infty}\binom{n}{p}\|A_n\|R^{n+1}$$

$$- \sum_{n=2}^{\infty}\Big|\binom{-n}{p}\Big|\|A_{-n}\|\frac{1}{r^{n-1}}\Big\}.$$

10. Sufficient conditions for absolutely p-valency in a multiply connected domain.

THEOREM 18. *Let*

$$F(z) = G(z) + H(z) = \sum_{n=-\infty}^{\infty} A_n z^n$$

where $G(z) = \sum_{n=1}^{\infty} \frac{A_{-n}}{z^n}$ *and* $H(z) = \sum_{n=0}^{\infty} A_n z^n$.

If $F(z)$ *satisfies one of the following conditions*

(i) $$\|A\| \geqq \underset{|z|=R}{\mathrm{Max}}\|H^{(p)}(z) - A\| + \underset{|z|=r}{\mathrm{Max}}\|G^{(p)}(z)\|$$
(A: a certain constant vector),

(ii) $$\|A_p\| \geqq \underset{|z|=R}{\mathrm{Max}}\Big\|\frac{H^{(p)}(z)}{p!} - A_p\Big\| + \underset{|z|=r}{\mathrm{Max}}\Big\|\frac{G^{(p)}(z)}{p!}\Big\|,$$

(iii) $$\|A_p\| \geqq \sum_{n=p+1}^{\infty}\binom{n}{p}\|A_n\|R^{n-p} + \sum_{n=1}^{\infty}\Big|\binom{-n}{p}\Big|\|A_{-n}\|\frac{1}{r^{n+p}},$$

(iv) $$\|A\| \geqq \underset{|z|=R}{\mathrm{Max}}\|z^{p+1}H^{(p)}(z)\| + \underset{|z|=r}{\mathrm{Max}}\|z^{p+1}G^{(p)}(z) - A\|,$$

(v) $$\|A_{-1}\| \geqq \operatorname*{Max}_{|z|=R}\left\|\frac{z^{(p+1}H^{(p)}(z)}{p!}\right\| + \operatorname*{Max}_{|z|=r}\left\|\frac{z^{p+1}G^{(p)}(z)}{p!}-(-1)^p A_{-1}\right\|,$$

(vi) $$\|A_{-1}\| \geqq \sum_{n=p}^{\infty}\binom{n}{p}\|A_n\|R^{n+1} + \sum_{n=1}^{\infty}\left\|\binom{-n}{p}A_{-n}\right\|\frac{1}{r^{n+1}},$$

then $F(z)$ is regular and absolutely p-valent in $r<|z|<R$.

PROOF. The regularity of $F(z)$ in $r<|z|<R$ is evident from the inequality of hypothesis. In these cases it is also evident that

$$\|[z_0, z_1, \cdots, z_p]^{F(z)}\| > 0,$$

namely $$[z_0, z_1, \cdots, z_p]^{F(z)} \neq 0$$

by lemma 6, hence $F(z)$ is absolutely p-valent in $r<|z|<R$.

It is evident that the above theorem can be extended as follows;

THEOREM 19. *Let $F(z)=G(z)+H(z)$, where $H(z)$ is regular in a convex domain D and $G(z)$ is regular in a reciprocally convex domain E. If $F(z)$ satisfies the condition*

(i) $$\|A\| \geqq \operatorname*{Max}_{z\in D}\|H^{(p)}(z)-A\| + \operatorname*{Max}_{z\in E}\left\|\left(\frac{z}{r}\right)^{p+1}G^{(p)}(z)\right\|,$$

A : a constant vector,

then $F(z)$ is absolutely p-valent in the common part of three domains D, E and $|z|>r$. And if

(ii) $$\|A\| \geqq \operatorname*{Max}_{z\in E}\|z^{p+1}G^{(p)}(z)-A\| + \operatorname*{Max}_{z\in D}\|R^{p+1}H^{(p)}(z)\|$$

then $F(z)$ is absolutely p-valent in the common part of D, E and $|z|<R$.

COROLLARY. *Let $F(z)=G(z)+H(z)$ where $H(z)$ is regular in a convex domain D and $G(z)=\sum_{n=1}^{\infty}\frac{A_{-n}}{z^n}$ is regular in $|z|\geqq r$. If*

(i) $$\operatorname*{Max}_{z\in D}\|H^{(p)}(z)-A\| \leqq \|A\| - p!\sum_{n=1}^{\infty}\left\{\left\|\binom{-n}{p}A_{-n}\right\|\frac{1}{r^{n+p}}\right\}$$

then $F(z)$ is absolutely p-valent in the common part of D and $|z|>r$. And if

(ii) $$\operatorname*{Max}_{z\in D} |H^{(p)}(z)| \leqq \frac{p!}{R^{p+1}}\left\{ |A_{-1}| - \sum_{n=2}^{\infty}\left[\left|\binom{-n}{p}A_{-n}\right| \frac{1}{r^{n-1}}\right]\right\}$$

then $(F(z)$ is absolutely p-valent in the common part of D and $r<|z|<R$.

REMARK. These results are considered to be generalizations of S. Ozaki's results [12].

11. On the multivalency of function vectors with known zero-points.

Let $p(z)=d_1+d_2z+\cdots+d_pz^{p-1}$ be an arbitrary polynomial of degree at most $p-1$. If $F(z)$ is regular in a convex domain D and if

$$|F^{(p)}(z)-A|<|A|, \quad A: \text{a certain constant vector,}$$

for all z in D, then $F(z)/p(z)$ is also at most p-valent in D by Theorems 6 and 11. In particular, however, if $F(z)$ and $p(z)$ have k common roots in D, we can prove that $F(z)/p(z)$ is at most $p-k$ valent in D. In this paragraph the proof of this fact and its generalization are stated which are again the extensions of S. Ozaki's results [12]. We need the following

LEMMA 7. *Let $\alpha_1, \alpha_2, \cdots, \alpha_k$ be the zero-points of $F(z)$. Put*

$$F(z)=(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_k)\,\Phi(z)$$

then

$$[\alpha_1, \alpha_2, \cdots, \alpha_k, z_0, z_1, \cdots, z_{p-k}]^{F(z)}=[z_0, z_1, \cdots, z_{p-k}]^{\Phi(z)}. \tag{11.1}$$

PROOF. By a simple calculation, we know that $[z, \alpha_1, \alpha_2, \cdots, \alpha_k]^{F(z)}=\Phi(z)$. Hence we easily obtain (11.1).

By Theorem 6 and Lemma 7 we obtain the following

THEOREM 20. *Let $F(z)$ be regular in a convex domain D and let $\alpha_1, \alpha_2, \cdots, \alpha_k$ be k zero-points in D. If in D*

$$\| F^{(p)}(z) - A \| < \| A \|, \ A: \text{a certain constant vector,}$$

then $F(z) / \prod_{i=1}^{k} (z-\alpha_i)$ *is absolutely* $(p-k)$*-valent in* D.

COROLLARY. *Let* $F(z)$ *be regular in a convex domain* D *and let*

$$\| F^{(p)}(z) - A \| < \| A \|, \ A: \text{a certain constant vector.}$$

Suppose that $p(z) = d_1 + d_2 z + \cdots + d_p z^{p-1}$ *be an arbitrary polynomial of degree at most* $p-1$ *and that* $F(z)$ *and* $p(z)$ *have* k *common roots in* D. *Then* $F(z)/p(z)$ *is at most* $(p-k)$*-valent in* D.

The above corollary can be proved by making use of Theorems 11 and 20.

Theorem 20 can be extended in the form of Theorem 19. Namely we obtain the following

THEOREM 21. *Let* $F(z) = G(z) + H(z)$ *where* $H(z)$ *is regular in a convex domain* D *and* $G(z)$ *is regular in a domain* E *whose image* E_1 *mapped by the function* $w = 1/z$ *is a convex domain. Further let* $p(z) = \prod_{i=1}^{k} (z-\alpha_i)$ *and* $q(z) = \prod_{i=1}^{l} (z-\beta_i)$ *be polynomials of degree* k *and* l *respectively and whose zero-points are all in* D *and* E_1 *respectively. If* $F(z)$ *satisfies the condition,* A *being a certain constant vector,*

$$\text{(i)} \qquad \| A \| \geqq \operatorname*{Max}_{z \in D} \left\| \frac{1}{(p+k)!} \frac{d^{p+k}}{dz^{p+k}} \left\{ p(z) H(z) \right\} - A \right\| + \operatorname*{Max}_{z \in E_1} \left\| \frac{1}{(p+l)!\, r^{p+1}} \frac{d^{p+l}}{dz^{p+l}} z^{p-1} q(z) G\left(\frac{1}{z}\right) \right\|$$

then $F(z)$ *is absolutely* p*-valent in the common part of three domain* D, E *and* $|z| > r$. *And if*

$$\text{(ii)} \qquad \| A \| \geqq \operatorname*{Max}_{z \in E_1} \left\| \frac{1}{(p+l)!\, R^{p+1}} \frac{d^{p+l}}{dz^{p+l}} \left\{ z^{p-1} q(z) G\left(\frac{1}{z}\right) \right\} - A \right\| + \operatorname*{Max}_{z \in D} \left\| \frac{1}{(p+k)!} \frac{d^{p+k}}{dz^{p+k}} \left\{ p(z) H(z) \right\} \right\|,$$

then $F(z)$ *is absolutely* p*-valent in the common part of* D, E *and* $|z| < R$.

To prove this theorem we may use Lemmas 6 and 7.

COROLLARY. *Let* $F(z)=\sum_{-\infty}^{\infty} A_n z^n$ *be regular in* $r \leqq |z| \leqq R$. *Let* $p(z)=\prod_{i=1}^{k}(z-\alpha_i)$ *and* $q(z)=\prod_{i=1}^{l}(z-\beta_i)$ *be polynomials of degree* k *and* l *respectively, whose zero-points are all in* $|z|<R$ *and* $|z|<1/r$ *respectively. If* $F(z)$ *satisfies one of the following conditions:*

$$\text{(i)} \quad \|A_p\| \geqq \underset{|z|=R}{\mathrm{Max}} \left\| \frac{1}{(p+k)!} \frac{d^{p+k}}{dz^{p+k}} \left[p(z) \sum_{n=p+1}^{\infty} A_n z^n \right] \right\| + \underset{|z|=1/r}{\mathrm{Max}} \left\| \frac{z^{p+1}}{(p+l)!} \frac{d^{p+l}}{dz^{p+l}} \left[q(z) \sum_{n=1}^{\infty} A_{-n} z^{n+p-1} \right] \right\|,$$

$$\text{(ii)} \quad \|A_{-1}\| \geqq \underset{|z|=R}{\mathrm{Max}} \left\| \frac{z^{p+1}}{(p+k)!} \frac{d^{p+k}}{dz^{p+k}} \left[p(z) \sum_{n=0}^{\infty} A_n z^n \right] \right\| + \underset{|z|=1/r}{\mathrm{Max}} \left\| \frac{1}{(p+l)!} \frac{d^{p+l}}{dz^{p+l}} \left[q(z) \sum_{n=2}^{\infty} A_{-n} z^{n+p-1} \right] \right\|$$

then $F(z)$ *is absolutely* p*-valent in* $r<|z|<R$.

REMARK. If we put $n=1$ in the above theorems, namely if we consider one function of one variable, we have S. Ozaki's results [12].

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