Journal of the Mathematical Society of Japan Vol. 7, No. 3, June, 1955

On the fundamental conjecture of GLC I.

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(Received Feb. 24, 1954)

G. Gentzen has founded in his well known paper [1] a logic calculus LK, and proved the remarkable result that every provable sequence in LK is provable without cut, by means of which he could establish the consistency of the number theory [2]. The author has generalized in his former paper [4] Gentzen's LK to a logical system GLC (Generalized Logic Calculus), containing a subsystem G^1LC, which latter contains LK. The proposition "Every provable sequence in GLC (resp. G^1LC) is provable without cut" was called the fundamental conjecture of GLC (resp. G^1LC), and it was shown that from this conjecture would follow the consistency of the analysis (resp. of the theory of real numbers).

We shall prove in this paper the following theorem, which may be regarded as a special case of the fundamental conjecture of G^1LC.

THEOREM. *Let* $\mathfrak{P}$ *be a proof-figure of a sequence* $\mathfrak{S}$ *in* G^1LC. *Assume that no beginning sequence of* $\mathfrak{P}$ *contains logical symbols, and that* $\mathfrak{P}$ *has no inference-figure* $\forall$, $\exists$ *on variable of height* 1 *of the forms described below:*

$$\frac{F(V),\ \Gamma\rightarrow\Delta}{\forall\varphi F(\varphi),\ \Gamma\rightarrow\Delta}, \qquad \frac{\Gamma\rightarrow\Delta,\ F(V)}{\Gamma\rightarrow\Delta,\ \exists\varphi F(\varphi)},$$

where $F(\alpha)$ *has a proper* $\forall$ *or* $\exists$ *on variable of height* 1. *Then* $\mathfrak{S}$ *is provable without cut.*

From this theorem follows the consistency of the theory of natural numbers. In fact the mathematical induction is formalized in GL^1C by the following inference-figures. (See § 4 in our former paper [4])

$$\frac{\Gamma\rightarrow\Delta,\ \alpha[0]\wedge\forall x(\alpha[x]\vdash\alpha[x'])\vdash\alpha[T]}{\Gamma\rightarrow\Delta,\ \forall\varphi(\varphi[0]\wedge\forall x(\varphi[x]\vdash\varphi[x'])\vdash\varphi[T])},$$

$$\frac{A(0)\wedge\forall x(A(x)\vdash A(x'))\vdash A(T),\ \Gamma\rightarrow\Delta}{\forall\varphi(\varphi[0]\wedge\forall x(\varphi[x]\vdash\varphi[x'])\vdash\varphi[T]),\ \Gamma\rightarrow\Delta}.$$

The author wishes to express his thanks to Prof. T. Iwamura who has given him valuable remarks in course of this work.

§ 1. Proof-figure of G^1LC.

We begin with recapitulating the definitions and notions given in [4]. Thereby we shall notify some modifications which will simplify our expression.

1.1. Variables (called variables of type 0 in [4])

1.1.1. free variables

$a, b, c, \cdots\cdots$

1.1.2. bound variables

$x, y, z, \cdots\cdots$

1.1.3. special variables

$0, \cdots.$

1.2. f-variables with i argument-places ($i=1, 2, 3, \cdots$) (called variables of height 1 with argument-places in [4])

1.2.1. free ones

$\alpha_i[*_1, \cdots, *_i], \beta_i[*_1, \cdots, *_i], \cdots.$

1.2.2. bound ones

$\varphi_i[*_1, \cdots, *_i], \psi_i[*_1, \cdots, *_i], \cdots.$

1.2.3. special ones (for the case, when $i=2$)

$*_1 = *_2, *_1 < *_2, \cdots.$

1.3. Functions

$*_1', *_1 + *_2, \cdots.$

In [4], we considered in G^1LC free functions, bound functions and functions of height 2. To simplify the proof, we consider only the special functions of height 1, though there is no essential difficulty in proving an analogous theorem in more general case containing free functions etc..

Therefore all the logical symbols in a formula are always assumed as proper.

1.4. Logical symbols

$\neg, \vee, \forall.$

Without loss of generality, we do not use $\vee$ nor $\exists$ which is denoted by E in [4].

1.5. Terms, Formulas, and Formulas with i argument-places.

These concepts are defined in the usual manner; formulas with i argument-places were called varieties of height 1 with i argument-places in [4]. Formulas and Formulas with i argument-places are, in principle, denoted by German capital letters in [4], but we shall denote them by latin capital letters in this paper.

1.5.1. The *outermost logical symbol* of a formula containing at least one logical symbol is the logical symbol which is used at the final step of construction of the formula.

Examples of the term: $a+b, (a+b)+c$.

Example of the formula: $\alpha[a]\wedge\forall\varphi(\varphi[0]\wedge\beta[a])$.

Example of the formula with i argument-places:

$$\{x_1, \cdots, x_i\}(\forall\varphi_i(\varphi_i[x_1, \cdots, x_i]\wedge\alpha_i[x_1, \cdots, x_i]))\,.$$

1.6. Proof-figures of G^1LC.

The following terms have the same meaning as in [4]. The *sequence*, the *inference-figure* the *upper sequence* of an inference, the *lower sequence* of an inference, the *proof-figure*, the *beginning sequence*, the *end-sequence* and the expression: 'a sequence is *provable*'.

In this paper the beginning sequence is always of the form $D\rightarrow D$. And the inference-figures are figures of the following kinds.

1.6.1. Inference-schemata on structure of the sequences.

'Weakening'

$$\text{left: } \frac{\Gamma\rightarrow\Delta}{D, \Gamma\rightarrow\Delta} \qquad \text{right: } \frac{\Gamma\rightarrow\Delta}{\Gamma\rightarrow\Delta, D}\,.$$

The formulas denoted by D above are called the *weakening formula* of the weakening.

'Contraction'

$$\text{left: } \frac{D, D, \Gamma\rightarrow\Delta}{D, \Gamma\rightarrow\Delta} \qquad \text{right: } \frac{\Gamma\rightarrow\Delta, D, D}{\Gamma\rightarrow\Delta, D}\,.$$

'Exchange'

$$\text{left: } \frac{\Gamma, D, E, \Pi\rightarrow\Delta}{\Gamma, E, D, \Pi\rightarrow\Delta} \qquad \text{right: } \frac{\Gamma\rightarrow\Delta, D, E, \Lambda}{\Gamma\rightarrow\Delta, E, D, \Lambda}\,.$$

We consider the following inference as the spceial case of the exchange:

$$\frac{\Gamma \to \Delta}{\Gamma \to \Delta}.$$

1.6.2. Cut.

$$\frac{\Gamma \to \Delta, D \quad D, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}.$$

The formulas denoted by D above are called the *left* and *right cut-formula* of the cut.

1.6.3. Inference-schemata on logical symbols.

'$\neg$'

left: $\dfrac{\Gamma \to \Delta, A}{\neg A, \Gamma \to \Delta}$ right: $\dfrac{A, \Gamma \to \Delta}{\Gamma \to \Delta, \neg A}$.

'$\wedge$'

left (1): $\dfrac{A, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}$ right: $\dfrac{\Gamma \to \Delta, A \quad \Pi \to \Lambda, B}{\Gamma, \Pi \to \Delta, \Lambda\ A \wedge B}$

left (2): $\dfrac{B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}$.

'$\forall$ on variable'

left: $\dfrac{F(T), \Gamma \to \Delta}{\forall x F(x), \Gamma \to \Delta}$ right: $\dfrac{\Gamma \to \Delta, F(a)}{\Gamma \to \Delta, \forall x F(x)}$.

(T is an arbitrary term) (There is no a in the lower sequence)

'$\forall$ on f-variable with i argument-places'

left: $\dfrac{F(A_i), \Gamma \to \Delta}{\forall \varphi_i F(\varphi_i), \Gamma \to \Delta}$ right: $\dfrac{\Gamma \to \Delta, F(\alpha_i)}{\Gamma \to \Delta, \forall \varphi_i F(\varphi_i)}$.

(A_i is an arbitrary formula with i argument-places) (There is no α_i in the lower sequence)

In the above inference-schemata, the formulas denoted by $D, E, A,$

$B, F(T), F(a), F(A_i)$ or $F(\alpha_i)$ in the upper sequence are called the *subformulas* of the inference. And we call the formulas denoted by $D, E, \neg A, A \wedge B, \forall x F(x)$ or $\forall \varphi_i F(\varphi_i)$ in the lower sequence the *chief formulas* of the inference.

1.7. Successor.

When a formula C is contained in the upper sequence of an inference which is represented by one of the schemata 1.6.1—1.6.3, the *successor* of C is defined as follows.

1.7.1. If C is a cut formula then there is no successor of C.

1.7.2. If C is a subformula of the inference except cut and exchange, then the successor of C is the chief formula of the inference.

1.7.3. If C is a subformula denoted by D (or E) in the above schemata of exchange, then the successor of C is a chief formula denoted by D (or E) in the exchange.

1.7.4. If C is the k-th formula of Γ, Π, Δ or Λ in the upper sequence, then the successor of C is the k-th formula of Γ, Π, Δ or Λ respectively in the lower sequence.

§2. Concepts concerning a proof-figure.

In this section we define some concepts with respect to a given proof-figure.

2.1. String.

A series of sequences in the proof-figure with the following property is called a *string*. The series begins with a beginning sequence and ends with the end-sequence; every sequence of the series, except the last is the upper sequence of an inference and is followed immediately by the lower sequence.

We say 'a sequence $\mathfrak{S}_1$ is *above* another sequence $\mathfrak{S}_2$' if there is a string containing both $\mathfrak{S}_1$ and $\mathfrak{S}_2$ in which $\mathfrak{S}_1$ appears in the former order than $\mathfrak{S}_2$. If $\mathfrak{S}_1$ is above $\mathfrak{S}_2$ and $\mathfrak{S}_2$ is above $\mathfrak{S}_3$, then we say '$\mathfrak{S}_2$ is *between* $\mathfrak{S}_1$ and $\mathfrak{S}_3$' or '$\mathfrak{S}_2$ is between $\mathfrak{S}_3$ and $\mathfrak{S}_1$'.

2.2.

A formula in a beginning sequence is called *beginning formula.* A formula in the end sequence is called an *end-formula.*

2.3. Fibre.

A series of formulas in the proof-figure with the following property is called a *fibre*. The series begins with a beginning formula or a weakening formnla and ends with an end-formula or a cut-formula; every formula of the series, except the last, is followed immediately by its successor.

A formula A is called an *ancestor* of a formula B and B is called a *descendent* of A, if there is a fibre containing these formula in which A appears in the former order than B.

2.4. Predecessor.

If A is the successor of B, then B is called a *predecessor* of A; if moreover a fibre $\mathfrak{T}$ contains A and B, then $\mathfrak{T}$ contains no other predecessor of A than B, and B is called the 'predecessor of A in $\mathfrak{T}$'.

A chief formula of $\wedge$ right has two predecessors; in this case we call a predecessor B the *first* or the *second predecessor* according as B is in the left or the right upper sequence. We use analogous terminology for the unique predecessor B of a chief formula of $\wedge$ left: B is called the first or the second predecessor according as the inference is $\wedge$ left (1) or $\wedge$ left (2).

2.5. Related formulas.

This concept is defined as follows.

2.5.1. In a cut of the proof-figure, a cut-formula is related to the other cut-formula.

2.5.2. If A is related to B, then B is related to A.

2.5.3. If A_1 is the successor of A_2 and A_1 is not a chief formula of a logical inference and A_1 is related to B, then A_2 is related to B.

2.5.4. If A is related to B and both A and B are chief formulas of logical inference except $\wedge$, then the predecessor of A is related to the predecessor of B.

2.5.5. If A is related to B and each of A and B is a chief formula of an inference $\wedge$, then the first predecessor of A is related to the first predecessor of B and the second predecessor of A is related to the second predecessor of B.

We see easily that if A is related to B and each of A and B has a logical symbol, then the outermost logical symbol of A is the same as that of B.

2.6. Original formula.

Let A_i be a formula with i argument-places.

An *indication* of A_i in a formula G is given by a formula $F(\alpha_i)$ with full indication of α_i (cf. [4] for full indication), if α_i is (an f-variable with i argument-places) not contained in A_i and G is of the form $F(A_i)$; the same indication is given (only) by a formula of the form $F(\beta_i)$ where β_i is contained neither in A_i nor in $F(\alpha_i)$. If no confusion is likely to occur, we say that the indication is of the form $F(A_i)$. The indication is *void* or *non-void* according as $F(\alpha_1)$ contains no α_i or at least one α_i.

Now let F be a formula in the proof-figure considered, together with a given indication of the form $F(A_i)$. Then we determine, as follows, an indication of A_i in a predecessor of F.

2.6.1. If F is not a chief formula of a logical inference, then the predecessor of F has the same indication of A_i as the one in F.

2.6.2. If F is of the form $\neg G(A_i)$ and is the chief formula of an inference $\neg$, then the predecessor of F has the indication of A_i of the form $G(A_i)$.

2.6.3. If F is of the form $\forall x G(x, A_i)$ and is the chief formula of an inference $\forall$, then the predecessor of F has the indication of of the form $G(a, A_i)$ or $G(T, A_i)$.

2.6.4. If F is of the form $\forall \varphi_j G(\varphi_j, A_i)$ and is the chief formula of an inference $\forall$, then the predecessor of F has the indication of A_i of the form $G(\alpha_j, A_i)$ or $G(B_j, A_i)$.

2.6.5. If F is of the form $F_1(A_i) \wedge F_2(A_i)$ and is the chief formula of an inference $\wedge$, then the first predecessor of F has the indication of A_i of the form $F_1(A_i)$ and the second predecessor of F has the indication of A_i of the form $F_2(A_i)$.

Let $F(A_i)$ be a formula in a fibre $\mathfrak{T}$ and the indication of A_i in $F(A_i)$ be not void. Now, if we start from $F(A_i)$ and go successively to the predecessors in $\mathfrak{T}$, then only the following three cases are possible.

2.6.6. All the indications determined in the prescribed manner of A_i

in the formulas in $\mathfrak{I}$ are non-void and the first formula of $\mathfrak{I}$, that is, a beginning formula or a weakening formula, is of the form $G(A_i)$, where $G(\alpha_i)$ contains a logical symbol.

2.6.7. There exists a unique formula B in $\mathfrak{I}$, with void indication of A_i, whose successor is of the form $B \wedge C$ or $C \wedge B$, where C has non-void indication of A_i.

2.6.8. There exists a unique formula of the form $A_i(T_1, \cdots, T_i)$ in $\mathfrak{I}$, where the described A_i is the indicated one, which is above all other formulas in $\mathfrak{I}$ with this property.

In the last case we call the unique formula $A_i(T_1, \cdots, T_i)$ the *original formula in* $\mathfrak{I}$ of the indication of $F(A_i)$.

'B is an original formula of the indication of $F(A_i)$' means that there exists a fibre $\mathfrak{I}$, which contains B and $F(A_i)$ and the original formula in $\mathfrak{I}$ of the indication of $F(A_i)$ is B.

2.7. Explicit and Implicit fibres, formulas etc.

2.7.1. A fibre is called *explicit*, if it ends with an end-formula, and is called *implicit*, if it ends with a cut-formula.

2.7.2. A formula in a proof-figure is called explicit or implicit according as the fibre through this formula is explicit or implicit.

2.7.3. A sequence in the proof-figure is called implicit or explicit according as this sequence contains an implicit formula or not.

2.7.4. An inference in the proof-figure is called explicit or implicit according as the chief formula of this inference is explicit or implicit.

Now we define the *end-place* ('Endstück' in Gentzen [3]) of a proof-figure.

A sequence in the proof-figure is called the sequence of the end-place of this proof-figure, if and only if there is no implicit logical inference under this sequence. An inference of a proof-figure is called an inference of the end-place of this proof-figure, if and only if the lower sequence of this inference is contained in the end-place.

2.8.

If a fibre $\mathfrak{I}$ has a chief formula of an inference $\mathfrak{J}$, then we say '$\mathfrak{I}$ is *affected* by $\mathfrak{J}$'.

The logical length of a fibre $\mathfrak{T}$ is the number of the logical inferences affecting $\mathfrak{T}$.

Let a formula A be an ancestor of a formula B. Then there exists a fibre $\mathfrak{T}$ through A and B. The logical length from A to B is the number of the logical inferences between A and B affecting $\mathfrak{T}$, and this number does not depend on the choice of $\mathfrak{T}$.

2.9. Rank of a formula.

We define the *rank* of a formula A recursively as follows.

2.9.1. If A contains no logical symbol, then the rank of A is zero.

2.9.2. If A is of the form $\neg B$, $\forall x F(x)$ or $\forall \varphi_i F(\varphi_i)$, then the rank of A is $r+1$, where r is the rank of B, $F(a)$ or $F(\alpha_i)$ respectively.

2.9.3. If A is of the form $B \wedge C$, then the rank of A is $r+1$, where r is the maximal number of the rank of B and C.

2.10.

Let $\mathfrak{J}$ be a cut in a proof-figure $\mathfrak{P}$ and $\mathfrak{S}$ be a sequence of $\mathfrak{P}$. We say that '$\mathfrak{S}$ is *loaded* on $\mathfrak{J}$' or '$\mathfrak{J}$ is the loader of $\mathfrak{S}$', if and only if $\mathfrak{S}$ is above $\mathfrak{J}$ and there exists no cut between $\mathfrak{S}$ and $\mathfrak{J}$.

§ 3. Normal proof-figure.

In this section, we define first the concept of normal proof-figure and next the concept of proof-figure with potential and correspondence of the ordinal number to a proof-figure with potential. Hereafter we follow, as a whole, Gentzen [3]. Potential is a modification of 'Höhe' in Gentzen [3].

3.1. Normal proof-figure.

A proof-figure $\mathfrak{P}$ satisfying the following conditions 3.1.1 and 3.1.2 is called *normal.*

3.1.1. If two fibres $\mathfrak{T}$, $\mathfrak{T}'$ in $\mathfrak{P}$ begin with beginning formulas A, A' respectively and end in different cut-formulas in one and same cut and if, moreover, A contains a logical symbol, then A' contains no logical symbol and $\mathfrak{T}$ is not affected by any inference $\forall$ left on f-variable.

3.1.2. The implicit logical inferences $\forall$ left on f-variable of the following forms $\mathfrak{J}$ are of the 'first stage', which means that $F(\alpha_i)$

contains no logical symbol $\forall$ on f-variable:

$$\mathfrak{J}\ \frac{F(A_i),\ \Gamma \rightarrow \Delta}{\forall \varphi_i F(\varphi),\ \Gamma \rightarrow \Delta}.$$

In the rest of this paper, we aim at proving the following theorem, which is stronger than the theorem stated in the introduction.

THEOREM. *The end-sequence of a normal proof-figure is provable without cut.*

3.2. Degree of a formula in a normal proof-figure.

We define the *degree* of a formula A in a normal proof-figure recursively as follows.

3.2.1. The degree of a beginning formula or a weakening formula is 1.

3.2.2. If A has a predecessor and is not a chief formula of an inference on logical symbol or contraction, then the degree of A is equal to the degree of the predecessor of A.

3.2.3. If A is a chief formula of a contraction, then the degree of A is the maximal number of the degrees of the predecessors of A.

3.2.4. If A is a chief formula of an inference on logical symbol except $\forall$ left on f-variable, then the degree of A is $d+1$, where d is the maximal number of the degrees of the predecessors of A.

3.2.5. Let A be a chief formula of an inference $\forall$ left on f-variable and of the form $\forall \varphi_i F(\varphi_i)$ and the predecessor of A be of the form $F(B_i)$. Then the degree of A is $a+b$, where a is the rank of $\forall \varphi_i F(\varphi_i)$ and b is the maximal number of the degrees of the original formulas of B_i (If there is no original formulas of B_i, then $b=1$).

We define the degree of a cut as the maximal number of the degrees of the cut-formulas of this cut.

3 3. Potential.

A normal proof-figure is called a proof-figure with potential, if we have attached to each sequence of this proof-figure a natural number, called its *potential*, satisfying the following conditions.

3.3.1. If a sequence $\mathfrak{S}_1$ is above a sequence $\mathfrak{S}_2$, then the potential of $\mathfrak{S}_1$ is not less than the potential of $\mathfrak{S}_2$.

3.3.2. If $\mathfrak{S}_1$ is an upper sequence of an inference except cut and if $\mathfrak{S}_2$ is the lower sequence of this inference, then the potential of $\mathfrak{S}_1$ is equal to the potential of $\mathfrak{S}_2$.

3.3.3. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are two upper sequences of a cut, then the potential of $\mathfrak{S}_1$ is equal to the potential of $\mathfrak{S}_2$.

3.3.4. If a sequence $\mathfrak{S}$ is an upper sequence of a cut, then the potential of $\mathfrak{S}$ is not less than the degree of this cut.

3.3.5. If a beginning sequence $D \rightarrow D$ contains logical symbols, and a fibre $\mathfrak{T}$ begins with one of two D's and ends with a cut-formula of a cut $\mathfrak{J}$, then the potential of the upper sequences of $\mathfrak{J}$ is not less than $\max(a, b+c)+1$, where

a is the degree of $\mathfrak{J}$,

b is the maximal number of the degrees of any formula related to one of two D's,

c is the logical length of $\mathfrak{T}$.

3.3.6. The potential of the end-sequence is zero.

We see easily that every normal proof-figure can be made a proof-figure with potential. Therefore, to prove the theorem in 3.1, we have only to prove that the end-sequence of a proof-figure with potential is provable without cut.

3.4. Correspondence of the ordinal number to a proof-figure with potential.

Now we make correspond an ordinal number less than the first ε-number to each sequence of the proof-figure with potential recursively as follows.

3.4.1. The ordinal number of a beginning sequence is 1.

3.4.2. If $\mathfrak{S}_1$ is the upper sequence of an inference $\mathfrak{J}$ on structure, and $\mathfrak{S}_2$ is the lower sequence of $\mathfrak{J}$, then the ordinal number of $\mathfrak{S}_2$ is equal to the ordinal number of $\mathfrak{S}_1$.

3.4.3. If $\mathfrak{S}_1$ is the upper sequence and $\mathfrak{S}_2$ is the lower sequence of an inference $\neg$, $\wedge$ left, $\forall$ on variable, $\forall$ right on f-variable, or explicit $\forall$ left on f-variable, then the ordinal number of $\mathfrak{S}_2$ is $a+1$, where a is the ordinal number of $\mathfrak{S}_1$.

3.4.4. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are two upper sequences and $\mathfrak{S}$ is the lower sequence of an inference $\wedge$ right, then the ordinal number of $\mathfrak{S}$

is $\alpha+\beta$, where α or β is the ordinal number of $\mathfrak{S}_1$ or $\mathfrak{S}_2$ respectively and $+$ is the sign of natural sum.

3.4.5. If $\mathfrak{S}_1$ is the upper sequence and $\mathfrak{S}_2$ is the lower sequence of an implicit inference $\forall$ left on f-variable, then the ordinal number of $\mathfrak{S}_2$ is $\alpha+\omega$, where α is the ordinal number of $\mathfrak{S}_1$.

3.4.6. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are two upper sequences and $\mathfrak{S}$ is the lower sequence of a cut, then the ordinal number of $\mathfrak{S}$ is

$$\left.\omega^{\cdot^{\cdot^{\cdot^{\omega^{\alpha+\beta}}}}}\right\}\sigma-\tau$$

where α or β is the ordinal number of $\mathfrak{S}_1$ or $\mathfrak{S}_2$ respectively and σ or τ is the potential of $\mathfrak{S}_1$ (and $\mathfrak{S}_2$) or $\mathfrak{S}$ respectively.

We call the *ordinal number of a proof-figure with potential* the ordinal number of its end-sequence.

§ 4. Preparations of the essential reduction.

Let $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n$ and $\mathfrak{S}$ be sequences. '$\mathfrak{S}$ is *reducible* to $\mathfrak{S}_1, \cdots, \mathfrak{S}_n$' will mean 'if $\mathfrak{S}_1, \cdots, \mathfrak{S}_n$ is provable without cut, then $\mathfrak{S}$ is provable without cut'.

Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ and $\mathfrak{P}$ be proof-figure with potential. We say that $\mathfrak{P}$ is reduced to $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$, if and only if the following conditions are satisfied.

4.1.1. For each i $(1 \leq i \leq n)$, the ordinal number of $\mathfrak{P}_i$ is less than the ordinal number of $\mathfrak{P}$.

4.1.2. The end-sequence of $\mathfrak{P}$ is reducible to the end-sequences of $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$.

Clearly our purpose is to find the reduction of every proof-figure with potential.

In this section we treat the case, that the end-place contains explicit logical inferences. Let $\mathfrak{P}$ be a proof-figure with potential and $\mathfrak{J}$ be the undermost explicit logical inference contained in the end-place of $\mathfrak{P}$. Then we divide the following cases according to the form of $\mathfrak{J}$.

4.2. The cases when $\mathfrak{J}$ is $\neg$, $\wedge$ left or $\forall$.

Since all the cases are similar, we treat only the case $\forall$ right on f-variable. Then the form of $\mathfrak{J}$ may be considered as the following one.

4.2.1.
$$\begin{array}{c} \vdots \\ \dfrac{\Gamma \to \Delta, F(\alpha_i)}{\Gamma \to \Delta, \forall \varphi_i F(\varphi_i)}\ \mathfrak{J} \\ \vdots \\ \Gamma_0 \to \Delta_0, \end{array}$$

where $\Gamma_0 \to \Delta_0$ is the end sequence.

Since there is no other logical inference under $\mathfrak{J}$, Δ_0 contains a formula $\forall \varphi_i F(\varphi_i)$. Furthermore, without the loss of generality, we can assume that $\Gamma_0 \to \Delta_0$ contains no α_i. Therefore $\Gamma_0 \to \Delta_0$ is reducible to the sequence $\Gamma_0 \to F(\alpha_i), \Delta_0$. We reduce the above proof-figure 4.2.1 with potential to the following proof-figure with potential:

4.2.2.
$$\begin{array}{c} \vdots \\ \Gamma \to \Delta, F(\alpha_i) \\ \hline \text{Some exchanges} \\ \hline \Gamma \to F(\alpha_i), \Delta \\ \hline \text{Weakening} \\ \hline \Gamma \to F(\alpha_i), \Delta, \forall \varphi_i F(\varphi_i) \\ \vdots \\ \Gamma_0 \to F(\alpha_i), \Delta_0, \end{array}$$

where every sequence of this proof-figure has the same potential as the potential of the corresponding sequence of 4.2.1. The normality and the legality of the potentials of 4.2.2 are shifted on the properties of 4.2.1. The ordinal numbers of $\Gamma \to \Delta, F(\alpha_i)$ in 4.2.1 and 4.2.2 and $\Gamma \to F(\alpha_i), \Delta$ in 4.2.2 are equal to each others, and are less than the ordinal number of $\Gamma \to \Delta, \forall \varphi_i F(\varphi_i)$ in 4.2.1. Therefore the ordinal number of the end-sequence

$\Gamma_0 \rightarrow F(\alpha_i), \Delta_0$ of 4.2.2 is less than the ordinal number of the end-sequence $\Gamma_0 \rightarrow \Delta_0$ of 4.2.1.

4.3. The case when $\mathfrak{F}$ is $\wedge$ right.

Suppose that $\mathfrak{F}$ appears in the following form.

4.3.1.
$$\dfrac{\Gamma \rightarrow \Delta, A \quad \Pi \rightarrow \Lambda, B}{\Gamma, \Pi \rightarrow \Delta, \Lambda, A \wedge B}\ \mathfrak{F}$$
$$\Gamma_0 \rightarrow \Delta_0.$$

By the similar reason as in 4.2, $\Gamma_0 \rightarrow \Delta_0$ is reducible to $\Gamma_0 \rightarrow A, \Delta_0$ and $\Gamma_0 \rightarrow B, \Delta_0$. And the proof-figure 4.3.1 is reducible to the following proof-figures.

4.3.2.
$$\dfrac{\Gamma \rightarrow \Delta, A}{\dfrac{\text{Some weakenings and exchanges}}{\Gamma, \Pi \rightarrow A, \Delta, \Lambda, A \wedge B}} \qquad \dfrac{\Pi \rightarrow \Lambda, B}{\dfrac{\text{Some weakenings and exchanges}}{\Gamma, \Pi \rightarrow B, \Delta, \Lambda, A \wedge B}}$$
$$\Gamma_0 \rightarrow A, \Delta_0, \qquad \Gamma_0 \rightarrow B, \Delta_0,$$

where every sequence of these proof-figures has the same potential as the potential of the corresponding sequence in 4.3.1.

§ 5. Reductions for the case that the end-place contains an implicit beginning sequence and no logical inferences.

Hereafter we consider only a proof-figure with potential, whose end-place contains no logical inference. In this section we consider the cases that the end-place of the proof-figure contains an implicit beginning sequence.

Let $D\to D$ be one of implicit beginning sequences in the end-place. Our consideration is divided into the following several cases.

5.1. The case, when one of two D's is explicit.

We may assume that the right D is explicit and the left D is implicit. Let the proof-figure be of the following form.

5.1.1.
$$\begin{array}{cl} \begin{array}{cc} & D\to D \\ \vdots & \vdots \\ \Gamma\to\Delta, D & D, \Pi\to\Lambda \end{array} & \\ \hline \Gamma, \Pi\to\Delta, \Lambda \quad \mathfrak{J} & \\ \vdots & \\ \Gamma_0\to\Delta_0 . & \end{array} \qquad \begin{array}{l} \text{potential } \sigma \\ \text{potential } \tau \end{array}$$

Clearly $\Gamma_0\to\Delta_0$ is reducible to $\Gamma_0\to D, \Delta_0$. We shall consider of the following proof-figure 5.1.2.

5.1.2.
$$\begin{array}{cl} \vdots & \\ \Gamma\to\Delta, D & \text{potential } \tau \\ \hline \text{Some weakenings and exchanges} & \\ \hline \Gamma, \Pi\to D, \Delta, \Lambda & \text{potential } \tau \\ \vdots & \\ \Gamma_0\to D, \Delta_0 , & \end{array}$$

where a sequence has the potential τ or the same potential as the corresponding sequence in 5.1.1, according as the corresponding sequence is loaded on $\mathfrak{J}$ or not. The normality and the legality of the potentials of 5.1.2 is shifted on the properties of 5.1.1.

Let α and β be the ordinal numbers of corresponding two sequences $\mathfrak{S}_1$ in 5.1.1 and $\mathfrak{S}_2$ in 5.1.2. If $\mathfrak{S}_1$ is above $\mathfrak{J}$ and not loaded on $\mathfrak{J}$, then α is equal to β. If $\mathfrak{S}_1$ is loaded on $\mathfrak{J}$, then

$$\beta \leq \left.\omega^{\cdot^{\cdot^{\cdot^{\omega^{\alpha}}}}}\right\}\sigma-\tau .$$

This inequality is proved by the induction on the number of sequences above $\mathfrak{S}$ and loaded on $\mathfrak{J}$, because

$$\beta_1 \leq \left.\omega^{\cdot^{\cdot^{\cdot^{\omega^{\alpha_1}}}}}\right\}\sigma-\tau \quad \text{and} \quad \beta_2 \leq \left.\omega^{\cdot^{\cdot^{\cdot^{\omega^{\alpha_2}}}}}\right\}\sigma-\tau$$

imply

$$\beta_1+\beta_2 \leq \left.\omega^{\cdot^{\cdot^{\cdot^{\omega^{\alpha_1+\alpha_2}}}}}\right\}\sigma-\tau \quad \text{and} \quad \left.\omega^{\cdot^{\cdot^{\cdot^{\omega^{\beta_1}}}}}\right\}\rho \leq \left.\omega^{\cdot^{\cdot^{\cdot^{\omega^{\left.\omega^{\cdot^{\cdot^{\cdot^{\omega^{\alpha}}}}}\right\}\rho}}}}}\right\}\sigma-\tau .$$

Therefore, in the case when $\mathfrak{S}_1$ is $\Gamma \rightarrow \Delta, D$ in 5.1.1, and $\mathfrak{S}_2$ is $\Gamma \rightarrow \Delta, D$ in 5 1.2, our inequality holds, and in the case when $\mathfrak{S}_1$ is $\Gamma, \Pi \rightarrow \Delta, \Lambda$ in 5.1.1 and $\mathfrak{S}_2$ is $\Gamma, \Pi \rightarrow D, \Delta, \Lambda$ in 5.1.2 inequality $\beta < \alpha$ holds. Hence $\beta < \alpha$ holds in the case when $\mathfrak{S}_1$ is under $\mathfrak{J}$, and after all, the ordinal number of 5.1.2 is less than the ordinal number of 5.1.1.

Therefore the proof-figure 5.1.1 is reducible to 5.1.2.

5.2. The case, when two D's are implicit and D has no logical symbol.

Let the proof-figure be of the following form.

5.2.1.

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \Downarrow\\ \Gamma\to\Delta, D\end{array}\quad\begin{array}{c}D\to D\\ \Downarrow\\ D,\Pi\to\Lambda_1, D, \Lambda_2\end{array}}{\Gamma,\Pi\to\Delta,\Lambda_1, D,\Lambda_2}\qquad\begin{array}{l}\text{potential }\sigma\\ \text{potential }\tau\end{array}\\ \Downarrow\end{array}$$

$$\begin{array}{c}\dfrac{\tilde\Gamma\to\tilde\Delta, D\qquad\qquad\begin{array}{c}\Downarrow\\ D,\Phi\to\Psi\end{array}}{\tilde\Gamma,\Phi\to\tilde\Delta,\Psi}\\ \Downarrow\\ \Gamma_0\to\Delta_0\,.\end{array}$$

5.2.1 is reduced to the following.

5.2.2.

$$\begin{array}{c}\Downarrow\\ \cfrac{\Gamma\to\Delta, D}{\cfrac{\text{Some weakenings and exchanges}}{\Gamma,\Pi\to\Delta,\Lambda_1, D,\Lambda_2}}\qquad\begin{array}{l}\text{potential }\tau\\ \\ \text{potential }\tau\end{array}\\ \Downarrow\end{array}$$

$$\begin{array}{c}\dfrac{\tilde\Gamma\to\tilde\Delta, D\qquad\qquad\begin{array}{c}\Downarrow\\ D,\Phi\to\Psi\end{array}}{\tilde\Gamma,\Phi\to\tilde\Delta,\Psi}\ \mathfrak{J}\\ \Downarrow\\ \Gamma_0\to\Delta_0\,.\end{array}$$

The correspondence of potentials and ordinal numbers are the same as in the 5.1. And since D has no logical symbol, the condition of the normal proof-figure and the condition 3.3.5 of the potential pose no restriction on the cut $\mathfrak{J}$.

5.3. The case, when two D's are implicit and D has logical symbols.

Let the proof-figure be of the following form.

5.3.1.

$$
\begin{array}{cc}
\begin{array}{c}
\begin{array}{cc} & D \to D \\ \Downarrow & \Downarrow \end{array} \\
\dfrac{\Gamma \to \Delta, D \quad D, \Pi \to \Lambda_1, D, \Lambda_2}{\Gamma, \Pi \to \Delta, \Lambda_1, D, \Lambda_2}\ \mathfrak{J}_1 \\
\begin{array}{cc} \Downarrow & \qquad\qquad \Downarrow \end{array} \\
\dfrac{\tilde{\Gamma} \to \tilde{\Delta}, D \qquad\qquad D, \Phi \to \Psi}{\tilde{\Gamma}, \Phi \to \tilde{\Delta}, \Psi}\ \mathfrak{J}_2 \\
\Downarrow \\
\dfrac{\qquad}{\Gamma_1 \to \Delta_1} \\
\Downarrow \\
\Gamma_0 \to \Delta_0,
\end{array}
&
\begin{array}{l}
\\ \\
\text{potential } \sigma_1 \\
\text{potential } \sigma_2 \\
\\
\text{potential } \tau_1 \\
\\
\text{potential } \tau_1 \\
\text{potential } \tau_2 < \tau_1 \\
\\
\end{array}
\end{array}
$$

where $\Gamma_1 \to \Delta_1$ is the uppermost sequence under $\mathfrak{J}_2$, whose potential is less than τ_1.

We reduce 5.3.1 to the following 5.3.2.

5.3.2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \dfrac{\Gamma \to \Delta, D}{\Gamma, \Pi \to \Delta, \Lambda_1, D, \Lambda_2}\ \text{Some weakenings and exchanges}\\ \vdots \\ \dfrac{\tilde{\Gamma} \to \tilde{\Delta}, D}{\tilde{\Gamma}, \Phi \to D, \tilde{\Delta}, \Psi}\ \text{Some weakenings and exchanges}\\ \vdots \\ \Gamma_1 \to D, \Delta_1\end{array}}{\Gamma_1 \to \Delta_1, D}\ \text{Some exchanges} \qquad \dfrac{\begin{array}{c}\vdots\\ \dfrac{D, \Phi \to \Psi}{\tilde{\Gamma}, \Phi, D \to \tilde{\Delta}, \Psi}\ \text{Some weakenings and exchanges}\\ \vdots \\ \Gamma_1, D \to \Delta_1\end{array}}{D, \Gamma_1 \to \Delta_1}\ \text{Some exchanges}}{\Gamma_1, \Gamma_1 \to \Delta_1, \Delta_1}\ \mathfrak{J}_3}{\Gamma_1 \to \Delta_1}\ \text{Some exchanges and contractions}}{}$$

$$\begin{array}{c}\vdots\\ \Gamma_0 \to \Delta_0\,,\end{array}$$

potential $\tau_1 - 1$
potential $\tau_2 < \tau_1$

where the potential of the sequences loaded on $\mathfrak{J}_3$ is $\tau_1 - 1$, and the potential of the sequences with the same loader as $\Gamma, \Pi \to \Delta, \Lambda_1, D, \Lambda_2$ in 5.3.2 is $\tau_1 - 1$ or σ_2, according as $\Gamma, \Pi \to \Delta, \Lambda_1, D, \Lambda_2$ is loaded on $\mathfrak{J}_3$ or not, and moreover the other sequences has the same potential as the corresponding sequence in 5.3.1.

By the condition 3.1.1 in 5.3.1, an arbitrary fibre which ends with the cut-formula of $\mathfrak{J}_3$, begins with a weakening formula or a beginning formula without logical symbol, so the conditions 3.1.1 and 3.3.5 are trivial on $\mathfrak{J}_3$. By the condition 3.3.5 on $\mathfrak{J}_2$, we see that the potential τ_1 of upper sequence of $\mathfrak{J}_2$ is not less then $a+1$, where a is the maximal number of the degrees of the left cut-formula of $\mathfrak{J}_1$ and of the right cut-formula of $\mathfrak{J}_2$. Hence the potential $\tau_1 - 1$ of upper sequences of $\mathfrak{J}_3$ is not less than the degree a of $\mathfrak{J}_3$, that is, the condition 3.3.4.

on $\mathfrak{J}_3$ is satisfied.

Comparison of the ordinal numbers of 5.3.1 and 5.3.2 are similar as in the case 5.1.

§ 6. Essential reduction.

In this section we assume that the end-place contains none of the implicit beginning sequence and the logical inference.

First we define several concepts.

6.1. Boundary.

An inference $\mathfrak{J}$ belongs to the *boundary* of the end-place, if and only if the lower sequence of $\mathfrak{J}$ belongs to the end-place and the upper sequence of $\mathfrak{J}$ does not belong to the end-place. If $\mathfrak{J}$ belongs to the boundary of the end-place, then $\mathfrak{J}$ is an implicit logical inference.

6.2.

A cut in the end-place is called *suitable*, if and only if each cut-formula of this cut has a fibre ending with this cut-formula, which contains the chief formula of an inference at the boundary.

6.3.

A cut-formula is called *weakening*, if and only if all the fibres ending with this cut-formula begin with weakening formulas.

6.4. The case, when there is a cut in the end-place, one of whose cut-formulas is weakening.

Let the proof-figure be of the following form.

6.4.1.
$$\begin{array}{c} \vdots \qquad\qquad \vdots \\ \dfrac{\Gamma \to \Delta, D \qquad D, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}\,\mathfrak{J} \quad \begin{array}{l}\text{potential } \sigma \\ \text{potential } \tau\end{array} \\ \vdots \\ \Gamma_0 \to \Delta_0, \end{array}$$

where the right cut-formula of $\mathfrak{J}$ is weakening.

Now we eliminate all the ancestors of D in the proof-figure to

$D, \Pi \rightarrow \Lambda$. Clearly we have a normal proof-figure to $\Pi \rightarrow \Lambda$. Now we reduce 6.4.1 to the following proof-figure 6.4.2.

$$
6.4.2. \qquad
\begin{array}{c}
\vdots \\
\Downarrow \\
\Pi \rightarrow \Lambda \\
\hline
\text{Some weakenings and exchanges} \\
\hline
\Gamma, \Pi \rightarrow \Delta, \Lambda \\
\vdots \\
\Downarrow \\
\Gamma_0 \rightarrow \Delta_0 .
\end{array}
\qquad
\begin{array}{l}
\text{potential } \tau \\
\\
\text{potential } \tau
\end{array}
$$

All the circumstance is the same as in the case 5.1.

Now, in addition to the condition at the beginning of this section, the proof-figure may be considered to contain no weakening cut-formula.

We prove by the induction on the number of the cuts in the end-place that there is a suitable cut.

Let $\mathfrak{J}$ be the lowest cut and of the following form.

$$\frac{\Gamma \rightarrow \Delta, D \qquad D, \Pi \rightarrow \Lambda}{\Gamma, \Pi \rightarrow \Delta, \Lambda}\ \mathfrak{J}.$$

Let $\mathfrak{P}_1$ be the proof-figure to $\Gamma \rightarrow \Delta, D$ and $\mathfrak{P}_2$ be the proof-figure to $D, \Pi \rightarrow \Lambda$. We see clearly that the end-place of $\mathfrak{P}_i$ has a sequence not contained in the end-place of the proof-figure $\mathfrak{P}$ to $\Gamma, \Pi \rightarrow \Delta, \Lambda$, if and only if there is an inference at the boundary of $\mathfrak{P}$, whose chief formula is an ancestor of the cut-formula of $\mathfrak{J}$. Therefore, if $\mathfrak{J}$ is not suitable, then the end-place $\mathfrak{E}$ of $\mathfrak{P}_1$ or $\mathfrak{P}_2$ is a subset of the end-place of $\mathfrak{P}$. Then, by the hypothesis of the induction, there exists a suitable cut $\mathfrak{J}_0$ in $\mathfrak{E}$. Clearly $\mathfrak{J}_0$ is a suitable cut of $\mathfrak{P}$.

Now we assume that the end-place contains a suitable cut. Let $\mathfrak{J}$ be a suitable cut. The case is divided into the following ones according to the outermost logical symbol of the cut-formula of $\mathfrak{J}$.

6.5. The case, when the outermost logical symbol of the cut-formula of $\mathfrak{J}$ is $\neg$, $\wedge$ or $\forall$ on variable.

We assume that the proof-figure is of the following form.

6.5.1.

$$
\begin{array}{c}
\dfrac{
\begin{array}{c}
\vdots \\
\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \to \Delta_1, A\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma_2 \to \Delta_2, B\end{array}}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, A \wedge B} \\
\vdots \\
\Gamma_3 \to \Delta_3, A \wedge B
\end{array}
\qquad
\begin{array}{c}
\vdots \\
\dfrac{\begin{array}{c}\vdots\\ A, \Pi_1 \to \Lambda_1\end{array}}{A \wedge B, \Pi_1 \to \Lambda_1} \\
\vdots \\
A \wedge B, \Pi_2 \to \Lambda_2
\end{array}
}{\Gamma_3, \Pi_2 \to \Delta_3, \Lambda_2} \, \mathfrak{J} \quad \text{potential } \sigma \\
\vdots \\
\dfrac{\qquad\qquad}{\Gamma_4 \to \Delta_4} \quad \begin{array}{r}\text{potential } \sigma \\ \text{potential } \tau < \sigma\end{array} \\
\vdots \\
\Gamma_0 \to \Delta_0 ,
\end{array}
$$

where $\Gamma_4 \to \Delta_4$ is the uppermost sequence under $\mathfrak{J}$, whose potential is less than σ.

We reduce 6.5.1 to the following proof-figure 6.5.2.

6.5.2.

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c}\vdots\\ \Downarrow\\ \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Downarrow\\ \Gamma_1\to\Delta_1, A\end{array}}{\text{Some weakenings and exchanges}}}{\Gamma_1, \Gamma_2\to A, \Delta_1, \Delta_2, A\wedge B}\\ \vdots\\ \Downarrow\\ \Gamma_3\to A, \Delta_3, A\wedge B\end{array}
\quad
\begin{array}{c}\vdots\\ \Downarrow\\ A\wedge B, \Pi_2\to\Lambda_2\end{array}
}{
\begin{array}{c}\Gamma_3, \Pi_2\to A, \Delta_3, \Lambda_2\\ \vdots\\ \Downarrow\end{array}
}
}{\Gamma_4\to A, \Delta_4}
}{\text{Some exchanges}}
}{\Gamma_4\to\Delta_4, A}
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c}\vdots\\ \Downarrow\\ \Gamma_3\to\Delta_3, A\wedge B\end{array}
\quad
\begin{array}{c}\vdots\\ \Downarrow\\ \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Downarrow\\ A, \Pi_1\to\Lambda_1\end{array}}{\text{Some weakenings and exchanges}}}{A\wedge B, \Pi_1, A\to\Lambda_1}\\ \vdots\\ \Downarrow\\ A\wedge B, \Pi_2, A\to\Lambda_2\end{array}
}{
\begin{array}{c}\Gamma_3, \Pi_2, A\to\Delta_3, \Lambda_2\\ \vdots\\ \Downarrow\end{array}
}
}{\Gamma_4, A\to\Delta_4}
}{\text{Some exchanges}}
}{A, \Gamma_4\to\Delta_4}
\ \mathfrak{J}_0
}{
\dfrac{\dfrac{\Gamma_4, \Gamma_4\to\Delta_4, \Delta_4}{\text{Some exchanges and contractions}}}{\begin{array}{c}\Gamma_4\to\Delta_4\\ \vdots\\ \Downarrow\\ \Gamma_0\to\Delta_0,\end{array}}
}
$$

potential σ

potential σ
potential $\sigma-1$

potential $\sigma-1$
potential τ

where potentials of sequences are, except the indicated ones in the figure 6.5.2, equal to potentials corresponding sequences in 6.5.1. Clearly the proof-figure 6.5.2 is normal, and the ordinal number of 6.5.2 is less than the ordinal number of 6.5.1, by the similar reason as in the case 5.1, for $\beta_1<\alpha$ and $\beta_2<\alpha$ imply

$$\omega^{\beta_1}+\omega^{\beta_2}<\omega^{\alpha}, \quad \text{that is} \quad \left.\omega^{\cdot^{\cdot^{\cdot^{\omega^{\omega^{\beta_1}+\omega^{\beta_2}}}}}}\right\}\sigma-1-\tau < \left.\omega^{\cdot^{\cdot^{\cdot^{\omega^{\alpha}}}}}\right\}\sigma-\tau .$$

Therefore, only the conditions of potentials for 6.5.2 are to be examined. Since any other cut in 6.5.2 satisfies the conditions in accordence with corresponding cut in 6.5.1, we consider only the cut $\mathfrak{J}_0$. Though the potential of the upper sequences is less by 1 in $\mathfrak{J}_0$ than in $\mathfrak{J}$, the degree of the cut and the logical length of each fibre ending with a cut-formula is less by 1 at least in $\mathfrak{J}_0$ than in $\mathfrak{J}$. Hence the conditions of potentials for 6.5.2 follow from the conditions for 6.5.1.

6.6. The case, when the outermost logical symbol of the cut-formula of $\mathfrak{J}$ is $\forall$ on f-variable.

Before the reduction we prove the following proposition.

6.6.1. Let $\mathfrak{S}$ be a provable sequence which has the form $F(A_i), \Gamma\rightarrow\Delta$ or the form $\Gamma\rightarrow\Delta, F(A_i)$, and $F(\alpha_i)$ have no logical symbol $\forall$ on f-variable. And let

a be the degree of $F(A_i)$ in an arbitrary proof-figure $\mathfrak{P}$ to the sequence $\mathfrak{S}$,

b be the maximal number of 1 and the degrees of the original formulas $A_i(T_1, \cdots, T_i)$ of A_i in $\mathfrak{P}$,

c be the maximal number of the logical lengths from original formulas $A_i(T_1, \cdots, T_i)$ to $F(A_i)$, and

d be the rank of $F(\alpha_i)$.

Then $$a\leq b+d \quad \text{and} \quad c\leq d.$$

PROOF. If $F(\alpha_i)$ has no logical symbol, then proposition is clear, for $a=b$ and $c=d=0$. Therefore the proposition is easily proved by the induction on d.

Now we consider the reduction. We assume that the proof-figure is the following.

6 6.2.

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \to \Delta_1, F(\alpha_i)\end{array}}{\begin{array}{c}\Gamma_1 \to \Delta_1, \forall\varphi_i F(\varphi_i)\\ \vdots\\ \Gamma_2 \to \Delta_2, \forall\varphi_i F(\varphi_i)\end{array}} \qquad \dfrac{\begin{array}{c}\vdots\\ F(A_i), \Pi_1 \to \Lambda_1\end{array}}{\begin{array}{c}\forall\varphi_i F(\varphi_i), \Pi_1 \to \Lambda_1\\ \vdots\\ \forall\varphi_i F(\varphi_i), \Pi_2 \to \Lambda_2\end{array}}\;\mathfrak{F}_1}{\begin{array}{c}\Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2\\ \vdots\\ \Gamma_0 \to \Delta_0 .\end{array}}\;\mathfrak{F} \quad \text{potential } \sigma
$$

We divide the reduction into two stages. First we consider the following proof-figure.

6.6.3.

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \to \Delta_1, F(\alpha_i)\end{array}}{\text{Some weakenings and exchanges}}}{\begin{array}{c}\Gamma_1 \to F(\alpha_i), \Delta_1, \forall\varphi_i F(\varphi_i)\\ \vdots\\ \Gamma_2 \to F(\alpha_i), \Delta_2, \forall\varphi_i F(\varphi_i)\end{array}} \qquad \begin{array}{c}\vdots\\ \forall\varphi_i F(\varphi_i), \Pi_2 \to \Lambda_2\end{array}}{\begin{array}{c}\Gamma_2, \Pi_2 \to F(\alpha_i), \Delta_2, \Lambda_2\\ \vdots\\ \Gamma_0 \to F(\alpha_i), \Delta_0 ,\end{array}}
$$

where the potential of a sequence of 6.6.3 is equal to the potential of the corresponding sequence of 6.6.2. Clearly this proof-figure is normal and satisfies the conditions of potentials, and the ordinal number of 6.6.3. is less than the ordinal number of 6.6.2. Therefore, by the transfinite induction on the ordinal number of the proof-figure, we see that

$$\Gamma_0 \to F(\alpha_i), \Delta_0$$

is provable without cut.

Let $\mathfrak{P}_1$ be a proof-figure to $\Gamma_0 \rightarrow \Delta_0, F(\alpha_i)$ without cut, whose beginning sequence has no logical symbol. And let $\mathfrak{P}_2$ be the proof-figure obtained from $\mathfrak{P}_1$ by substituting A_i for α_i in $\mathfrak{P}_1$. Now we reduce 6.6.2 to the following.

6.6.4.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathfrak{P}_2 \\ \vdots \\ \Gamma_0 \rightarrow \Delta_0, F(A_i)\end{array} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots \\ \Gamma_2 \rightarrow \Delta_2, \forall \varphi_i F(\varphi_i)\end{array} \qquad \begin{array}{c}\vdots \\ \dfrac{F(A_i), \Pi_1 \rightarrow \Lambda_1}{\forall \varphi_i F(\varphi_i), \Pi_1, F(A_i) \rightarrow \Lambda_1}\text{ Some weakenings and exchanges} \\ \vdots \\ \forall \varphi_i F(\varphi_i), \Pi_2, F(A_i) \rightarrow \Lambda_2 \end{array}}{\Gamma_2, \Pi_2, F(A_i) \rightarrow \Delta_2, \Lambda_2}\ \text{potential } \sigma}{F(A_i), \Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2}\text{ Some exchanges}}{}\ \text{potential } \sigma}{\Gamma_0, \Gamma_2, \Pi_2 \rightarrow \Delta_0, \Delta_2, \Lambda_2}\ \mathfrak{J}_0}{\Gamma_2, \Pi_2, \Gamma_0 \rightarrow \Delta_0, \Delta_2, \Lambda_2}\text{ Some exchanges}}{\begin{array}{c}\vdots \\ \Gamma_0, \Gamma_0 \rightarrow \Delta_0, \Delta_0\end{array}}}{\Gamma_0 \rightarrow \Delta_0\,,}\text{ Some exchanges and contractions}
$$

where the potential of the sequence loaded over $\mathfrak{J}_0$ is σ and the potential of the other sequence is equal to the potential of the corresponding sequence in 6.6.2.

Let α be the ordinal number of $\forall \varphi_i F(\varphi_i), \Pi_1, F(A_i) \rightarrow \Lambda_1$ in 6.6.4. Then $\alpha + \omega$ is the ordinal number of $\forall \varphi_i F(\varphi_i), \Pi_1 \rightarrow \Lambda_1$ in 6.6.2. Therefore if α_1 is the ordinal number of a sequence $\mathfrak{S}$ between $\forall \varphi_i F(\varphi_i), \Pi_1, F(A_i) \rightarrow \Lambda_1$ and $\Gamma_2, \Pi_2, F(A_i) \rightarrow \Delta_2, \Lambda_2$ and β_1 is the ordinal number of the sequence corresponding to $\mathfrak{S}$ in 6.6.2, then $\alpha_1 + \omega$ is not greater than β_1. Since the ordinal number of $\Gamma_0 \rightarrow \Delta_0, F(A_i)$ in 6.6.4 is less than ω, the ordinal number of $\Gamma_0, \Gamma_2, \Pi_2 \rightarrow \Delta_0, \Delta_2, \Lambda_2$ in 6.6.4 is less than the ordinal number of $\Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2$ in 6.6.2. Therefore the ordinal number of 6.6.4 is less than the one of 6.6.2.

Now we consider the normality and the conditions of potentials for 6.6.4. $\mathfrak{P}_2$ has no cut, and every implicit formulas in $\mathfrak{P}_2$ are ancestors of the left cut-formula $F(A_i)$ of $\mathfrak{F}_0$, so we consider only the conditions 3.1.1, 3.3.4 and 3.3.5 on $\mathfrak{F}_0$.

Let a fibre $\mathfrak{T}_1$ (or $\mathfrak{T}_2$) begin with a beginning formula and end with the right (or left) cut-formula of $\mathfrak{F}_0$ and $\mathfrak{T}_1'$ be the fibre corresponding to $\mathfrak{T}_1$ in 6.6.2. Since both cut-formulas of $\mathfrak{F}$ are not weakening, and $\mathfrak{T}_1'$ is affected by the inference $\forall$ left on f-variable, the beginning formula of $\mathfrak{T}_1'$ has no logical symbols. Therefore the beginning formula of $\mathfrak{T}_1$ has no logical symbols and $F(\alpha_1)$ has no $\forall$ on f-variable. On the other hand, as beginning sequences of $\mathfrak{P}_1$ have no logical symbol, the beginning formula of $\mathfrak{T}_2$ is an original formula of A_i in $F(A_i)$ of the left cut-formula $\mathfrak{F}_0$, so $\mathfrak{T}_2$ is not affected by $\forall$ on f-variable. Hence the condition 3.1.1 is satisfied.

From 3.2.5 and 3.3.4 we see $b+d+1\leq\sigma$, where b is the maximal number of the degrees of original formulas of A_i in $F(A_i)$ of $\mathfrak{F}_1$, and d is the rank of $F(\alpha_i)$. Let a or a' be the degree of left or right cut-formula of $\mathfrak{F}_0$, and c be the logical length of $\mathfrak{T}_2$. Then by 6.6.1 we have

$$a\leq 1+d, \quad a'\leq b+d \quad \text{and} \quad b+c\leq b+d.$$

Of course, b is the maximal number of the degrees of formulas related to the beginning formula of $\mathfrak{T}_2$, and $\mathfrak{P}_2$ has no cut, therefore 3.3.4 or 3.3.5 is obtained by these inequalities.

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