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On the family of connected subsets and the topology of spaces.

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Let A be a set of points. If we introduce a topology into A, then the family $\{C\}$ of all connected subsets of A will be determined. But it is easily known by a simple example that the topology introduced into A can not be determined in general by $\{C\}$. But under some conditions it happens that the topology of the space can be determined by $\{C\}$, as will be shown in the following.

In this paper we shall use the notions and terminologies in G. T. Whyburn's "Analytic topology" [1]. We shall assume that all spaces to be considered are separable metric spaces. We begin with the definition of a biconnected transformation which plays an important role in this paper.

DEFINITION. *Let A and B be spaces. A transformation f of A onto B will be called a biconnected transformation if the following conditions are satisfied:*

(1) *f is one-to-one.*

(2) *Connected subsets of A are transformed to connected subsets of B under f, and conversely.*

Then we consider the properties of A and B under which the biconnected transformation f becomes a continuous or topological transformation.

THEOREM 1. *Let f be a biconnected transformation of A onto B, where A is a space and B is a semi-locally-connected space*[1]. *Then f is a continuous transformation.*

PROOF. Suppose, on the contrary, that f is not continuous at a point p of A. Let $\{p_n\}$ be a sequence of points of A converging to p such that $\{f(p_n)\}$ does not converge to $f(p)$. Since B is semi-locally-connected, there exists a neighborhood U of $f(p)$ such that $B-U$ contains an infinite number of points of $\{f(p_n)\}$ and consists of a finite number of components $C_1, \dots C_m$. Now without loss of generality we may suppose that every point of $\{f(p_n)\}$ is in C_1. Since C_1 is a component of $B-U$, $C+f(p)$ is not connected. On the other hand $f^{-1}(C_1)+p$ is connected, for $f^{-1}(C_1)$ is connected and p is a limit point of $\{p_n\}$, where p_n belongs to $f^{-1}(C_1)$. This contradicts the hypothesis for f. Thus f is a continuous transformation.

COROLLARY. *Let f be a biconnected transformation of A onto B, where A is a space and B is a locally connected generalized continuum. Then f is a cntinuous transformation.*

PROOF. A locally connected generalized continuum is semi-locally-connected[2]. Thus the Corollary is an immediate consequence of Theorem 1.

If A and B are semi-locally-connected in Theorem 1, then f is topological. Accordingly it is known that the topology of a space can be determined by the family of connected subsets under the condition that the space is semi-locally-connected.

In order to consider the case where the space is not always semi-locally-connected, we first state some properties of biconnected transformations.

LEMMA 1. *Let f be a biconnected transformation of A onto B, where A and B are spaces, and let C be a connected subset of A. Then we have $f(\overline{C})\subset\overline{f(C)}$.*

PROOF. Let p be a point of $\overline{C}$. Then p belongs either to C or to $\overline{C}-C$. In the first case, it is obvious that $f(p)$ belongs to $f(C)$. In the second case, p is a limit point of C. Therefore $C+p$ is connected, and hence $f(C)+f(p)$ is also connected. Therefore $f(p)$ is a limit point of $f(C)$, and hence $f(p)$ belongs to $\overline{f(C)}$. Thus Lemma 1 is proved.

LEMMA 2. *Let f be a biconnected transformation of A onto B, where A is a space and B is a compact space. Then we have the following relation:*

(2.1) *A subcontinuum of A is transformed to a subcontinuum of B under f.*

(2.2) *A locally connected subcontinuum of A is transformed topologically to a locally connected subcontinuum of B under f.*

Proof of (2.1). Let C be a subcontinuum of A. Since we have

$C=\bar{C}=\overline{f^{-1}(f(C))}\supset f^{-1}(\overline{f(C)})$ by Lemma 1, $f(C)=\overline{f(C)}$ holds. Moreover, since B is compact and $f(C)$ is connected, $f(C)$ is a subcontinuum of B.

Proof of (2.2). Let C be a locally connected subcontinuum of A. It is obvious from the definition of the biconnected transformation that the inverse f^{-1} of a biconnected transformation f is also such a transformation. Moreover, since a connected subset of a subspace is also a connected subset of a space, any respection of a biconnected transformation is also a biconnected transformation. Accordingly, the transformation f^{-1} of $f(C)$ onto C is continuous by the corollary of Theorem 1. Moreover, since $f(C)$ is compact by (2.1), the transformation f^{-1} of $f(C)$ onto C is topological. Thus (2.2) is proved.

LEMMA 3. *Let f be a biconnected transformation of A onto B, where A and B are compact spaces, and let $\{M_n\}$ be a sequence of subcontinua M_n of A containing a point o. Then we have $f(\overline{\lim}\, M_n)=\overline{\lim}\, f(M_n)$.*

PROOF. Since every point of $\overline{\lim}\, M_n$ is a limit point of $\sum M_n$ and each M_n is a continuum containing a point o, $\overline{\lim}\, M_n+\sum M_n$ is connected. Moreover, since $\overline{\lim}\, M_n$ and M_n are closed, $\overline{\lim}\, M_u+\sum M_n$ is closed. Therefore $\overline{\lim}\, M_n+\sum M_n$ is a continuum. Hence $f(\overline{\lim}\, M_n)+\sum f(M)$ is also a continuum by Lemma 2. Accordingly, to complete the proof it is sufficient to prove the following two facts:

(3.1) $f(\overline{\lim}\, M_n)$ is contained in $\overline{\lim}\, f(M)$.

(3.2) $f(M_m)-f(\overline{\lim}\, M_n)$ and $\overline{\lim}\, f(M_n)$ are disjoint.

Proof of (3.1). Let $f(p)$ be any point of $f(\overline{\lim}\, M_n)$. Then there exists either an infinite number of M_{n_i} containing p or a sequence of points $\{p_{n_i}\}$ such that $p_{n_i}\in M_{n_i}$, $p\notin M_{n_i}$ and $\lim p_{n_i}=p$. In the first case it is obvious that $f(p)\in\overline{\lim}\, f(M_n)$. In the second case $p+\sum M_{n_i}$ is connected and hence $f(p)+\sum f(M_{n_i})$ is connected. Moreover, since $f(M_{n_i})$ is a continuum not containing $f(p)$, we have $f(p)\in\overline{\lim}\, f(M_{n_i})$. Thus (3.1) is proved.

Proof of (3.2). Let $f(p)$ be any point of $f(M_m)-f(\overline{\lim}\, M_n)$. Suppose, on the contrary, that $f(p)\in\overline{\lim}\, f(M_n)$. Then by the same way as in (3.1) it results that $p \in \lim M_n$. This contradicts the assumption that $f(p) \notin f(\lim M_n)$.

Thus Lemma 3 is completely proved.

THEOREM 2. *Let f be a biconnected transformation of A onto B, where A and B are compact spaces satisfying the following condition:*

(*) *For any point p and any sequence of points $\{p_n\}$ converging to p, there exist a poiut o, a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ and a sequence of arcs $\{op_{n_i}\}$ such that $\{op_{n_i}\}$ converges to an arc op.*

Then f is a topological transformation.

PROOF. Suppose, on the contrary, that f is not continuous at a point p of A. Then there exist a sequence of points $\{p_{1n}\}$ and a point q, different from p, such that $\lim p_{1n}=p$ and $\lim f(p_{1n})=f(q)$. For $\lim p_{1n}=p$ we consider a sequence of arcs $\{op_{2n}\}$[3] and an arc op satisfying the condition (*). Then by Lemma (2.2), $f(op_{2n})$ and $f(op)$ are arcs, and by Lemma 3 $f(q)$ belongs to $f(op)$. Let S be the sphere with center $f(q)$ and radius $\frac{1}{2}\cdot d[f(q), f(p)]$, where $d[f(q), f(p)]$ is the distance between $f(p)$ and $f(q)$. We may suppose that every point of $\{f(p_{2n})\}$ is in the interior of S. Let $f(a_{2n})$ be the first common point of S and $f(p_{2n}o)$ ordered from $f(p_{2n})$ to $f(o)$, $f(b)$ the first common point of S and $f(qo)$, $f(c)$ the first common point of S and $f(qp)$ and $f(d)$ the last common point of S and $f(qp)$. Then the sequence of arcs $\{f(p_{2n}a_{2n})\}$, where $f(p_{2n}a_{2n})$ is a subarc of $f(p_{2n}o)$, contains a convergent subsequence $\{f(p_{3n}a_{3n})\}$ whose limit is a subarc of $f(op)$ containing $f(qb)$ or $f(qc)$, say $f(qc)$, and also the sequence of arcs $\{p_{3n}a_{3n}\}$ contains a convergent subsequence $\{p_{4n}a_{4n}\}$ whose limit is a subarc L of op having p as an end point.[4] Accordingly $\{f(p_{4n}a_{4n})\}$ converges to a subarc of $f(op)$ containing $f(qc)$, and $\{p_{4n}a_{4n}\}$ converges to a subarc L of op having p as an end point. In the following we consider two possible cases.

(1) The case where L is a subset of the arc dp. Let $f(r)$ be an inner point of $f(qc)$ and let $\{f(r_{4n})\}$ be a sequence of points converging to $f(r)$, where $f(r_{4n}) \in f(p_{4n}a_{4n})$. And for $\lim f(r_{4n})=f(r)$ we consider a sequence of arcs $\{f(o'r_{5n})\}$ and an arc $f(o'r)$ satisfying the condition (*). Now let $f(s)$ be a point of $f(qc)$ such that $f(s) \notin f(o'r)$ Then as $f(o'r)$ is the limit of $\{f(o'r_{5n})\}$, we may suppose that

$f(s)$ does not belong to any $f(o'r_{5n})$. Hence $f(s)$ is separated from $\sum f(o'r_{5n})$. Therefore s is separated from $\sum o'r_{5n}$. And also since s does not belong to L, similarly we may consider that s is separated from $\sum p_{5n}a_{5n}$. Hence $s+\sum o'r_{5n}+\sum p_{5n}a_{5n}$ is not connected, while $f(s)+\sum f(o'r_{5n})+\sum f(p_{5n}a_{5n})$ is connected. This contradicts the hypothesis for f.

(2) The case where L is not a subset of the arc dp. Then, of course, dp is a subset of L. Let r be an inner point of dp and let $\{r_{4n}\}$ be a sequence of points converging to r, where $r_{4n}\in a_{4n}p_{4n}$. And for $\lim r_{4n}=r$ we consider a sequence of arcs $o'r_{5n}$ and an arc $o'r$ satisfying the condition (*).

Now let s be a point of dp such that $s\notin o'r$. In the same way as in case 1, it results that $s+\sum o'r_{5n}+\sum p_{5n}a_{5n}$ is connected, while $f(s)+\sum f(o'r_{5n})+\sum f(p_{5n}a_{5n})$ is not. This contradicts the hypothesis for f.

Accordingly f is continuous. Similarly f^{-1} is continuous. Thus f is a topological transformation.

From Theorem 2 it is known that the topology of a space can be determined by the family of connected subsets under the condition that the space is compact and satisfies the condition (*) in Theorem 2.

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Notes

1) A connected space A is semi-locally-connected if for any positive number ε every point of A has a neighborhood U of diameter less than ε such that $A-U$ consists of a finite number of components.

2) [1], pp. 19-20.

3) $\{p_{2n}\}$ is a subsequence of $\{p_{1n}\}$

4) [1], pp. 14-15.

Reference

[1] G. T. Whyburn, Analytic topology, Amer. Math. Soc. Colloquium Publication, vol. 28, New York, 1942.