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On an absolute constant in the theory of quasi-conformal mappings.[1)]

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1. A topological mapping $w=T(z)$ of a planer region D onto another such region Δ is called a quasi-conformal mapping with the parameter of quasi-conformality K, or, simply, a K-QC mapping, if

(i) it preserves the orientation of the plane; and

(ii) for any quadrilateral $\Omega(z_1, z_2, z_3, z_4)$ contained in D together with its boundary, the inequality

$$\operatorname{mod} T(\Omega(z_1, z_2, z_3, z_4)) \leqq K \operatorname{mod} \Omega(z_1, z_2, z_3, z_4),$$

holds, where K is a constant $\geqq 1$. (See Mori [3], [4] and also Ahlfors [1].)

Let $w=T(z)$ be a K-QC mapping of $|z|<1$ onto $|w|<1$ such that $T(0)=0$. Then, as is already known, this mapping can be regarded as a topological mapping of $|z|\leqq 1$ onto $|w|\leqq 1$. (See Ahlfors [1], Mori [3], [4].) And, if z_1, z_2 are arbitrary two points on $|z|\leqq 1$, we have

$$|T(z_1)-T(z_2)| \leqq C|z_1-z_2|^{\frac{1}{K}}, \tag{1}$$

where C is a numerical constant.

To the author's knowledge this was first proved by Yûjôbô [6], though under a narrower definition than that given above, the author proved it with $C=48$. (Mori [3], [4]). Though with C depending on K, ($C=12^{K^2}$), Ahlfors proved (1) under the same definition. Further, Lavrentieff is reported to have proved (1) in a paper to which the author has not access (Lavrentieff [2]), so the author does not know with what C and under what definition Lavrentieff proved it.

1) The author of this paper, A. Mori suddenly died on July 5, 1955 at the age of 30. This paper was edited by Z. Yûjôbô after a manuscript of A. Mori (written in Japanese) found after his death.

The purpose of this paper is to show that 16 is the best possible value of C (as a constant not depending on K); i.e., to prove the following

THEOREM. *Let $w=T(z)$ be an arbitrary K-QC mapping of $|z|<1$ onto $|w|<1$, such that $T(0)=0$. Then*

$$\sup_{K,T,z_1\neq z_2} \frac{|T(z_1)-T(z_2)|}{|z_1-z_2|^{\frac{1}{K}}}=16 \qquad (|z_1|\leqq 1, \quad |z_2|\leqq 1). \tag{2}$$

(However, there is no mapping T which attains this value 16.)

2. *Preliminaries.* Let A be an annulus[2]. We always suppose that neither of the two complementary continua of the annulus is not reduced to one point (including the point at infinity). Then we can map A conformally onto a circular annulus $q<|\zeta|<1$, $(0<q<1)$. We call $\log(1/q)$ the "modulus" of A and denote it by mod A. Then, it is easily proved, that for any K-QC mapping $w=T(z)$ of a planer region D onto another such region and for any annulus $A\subset D$, we have

$$\frac{1}{K}\operatorname{mod} A\leqq \operatorname{mod} T(A)\leqq K\operatorname{mod} A. \tag{3}$$

(See Mori [3], [4].)

Next, we enumerate some known facts concerning the moduli of annuli. (For proofs, see Teichmüller [5].)

(I) (Grötzsch) For any real number P such that $1<P<+\infty$, we denote by G_P the annulus whose two complementary continua are respectively $\{z; |z|\leqq 1\}$ and $\{z; P\leqq \Re z\leqq +\infty, \Im z=0\}$. ($G_P$ is called Grötzsch's extremal region.) Then, if one of the complementary continua of an annulus A contains $\{z; |z|\leqq 1\}$, and if the other contains $z=\infty$ and also a point on $|z|=P$, we have

$$\operatorname{mod} A\leqq \operatorname{mod} G_P, \tag{4}$$

and moreover the equality holds if and only if A is an annulus obtained by a revolution of G_P around the point $z=0$.

(II) (Teichmüller) For any real number P such that $1<P<+\infty$, we denote by H_P the annulus whose two complementary continua are

2) We call "annulus" any doubly connected planar region.

respectively $\{z;\ -1\leqq\Re z\leqq 0,\ \Im z=0\}$ and $\{z;P\leqq\Re z\leqq+\infty,\ \Im z=0\}$. ($H_P$ is called Teichmüller's extremal region.) Then, if one of the complementary continua of an annulus A contains both $z=0$ and $z=-1$, and if the other contains $z=\infty$ and also a point on $|z|=P$, we have

$$\text{mod}\, A\leqq \text{mod}\, H_P\,, \tag{5}$$

and moreover the equality holds if and only if A is H_P.

(III) We write

$$\text{mod}\, G_P=\log \Phi(P)\,,$$

$$\text{mod}\, H_P=\log \Psi(P)\,.$$

Then the following facts hold.

$$\Psi(P)=[\Phi(\sqrt{1+P}\,)]^2\,, \tag{6}$$

$$P<\Phi(P)<4P\,,\qquad \Phi(P)/P\uparrow 4\quad\text{as}\quad P\to+\infty\,,^{3)} \tag{7}$$

$$P<\Psi(P)<16P+8\,.\qquad \Psi(P)/P\to 16\quad\text{as}\quad P\to+\infty\,. \tag{8}$$

3. For any real number λ such that $0<\lambda\leqq 2$, we denote by A_λ the annulus whose two complementary continua are respectively $\left\{z;|z|=1,\ |\arg z|\leqq\sin^{-1}\dfrac{\lambda}{2}\right\}$ and $\{z;\ -\infty\leqq\Re z\leqq 0,\ \Im z=0\}$, and write

$$\text{mod}\, A_\lambda=\log X(\lambda)\,.$$

Then we have

LEMMA 1. *Let A be an annulus on the z-plane, and Γ,Γ' be respectively the two complementary continua of A. Then, if*

$$\text{diam.}\,(\Gamma\cap\{|z|\leqq 1\})\geqq\lambda>0\,,$$

$$\Gamma'\ni z=0,\ z=\infty\,,$$

we have

$$\text{mod}\, A\leqq\text{mod}\, A_\lambda\,, \tag{9}$$

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3) We mean, by this, that $\Phi(P)/P$ is an increasing function of P, ($P>0$) and moreover $\lim\limits_{P\to+\infty}(\Phi(P)/P)=4$.

and moreover the equality holds if and only if A is an annulus obtained by a revolution of A_λ, around the point $z=0$.

LEMMA 2. *We have*

$$X(\lambda)=\Phi\left(\frac{2}{\lambda}\sqrt{2+\sqrt{4-\lambda^2}}\right) \tag{10}$$
$$=\Phi\left(\frac{2}{\sqrt{2-\sqrt{4-\lambda^2}}}\right),$$

$$\lambda X(\lambda)\uparrow 16 \qquad \text{as} \quad \lambda\to+0\,.\text{[4]} \tag{11}$$

PROOF OF LEMMA 1 AND LEMMA 2. We may assume, without loss of generality, that Γ contains $z=1$ and also a point z_0 such that $|z_0|\leqq 1$, $|z_0-1|\geqq\lambda$; because, if not so, we can transform A into such one by a suitable transformation of the form $Z=\alpha z$, $(|\alpha|\geqq 1)$ without varying mod A.

We construct the Riemann surface F of the analytic function $\zeta=\sqrt{z}$ above the z-plane. Denote by B the annulus which is obtained by excluding the two replicas of Γ from F. Then, since B contains the two replicas of A and since, moreover, each of them separates the boundary continua of B, we have

$$\operatorname{mod} A\leqq\frac{1}{2}\operatorname{mod} B. \tag{12}$$

(See Teichmüller [5])

Now, we shall try to maximize mod B. We map F onto the whole w-plane by the composition of the two transformations $\zeta^2=z$, $w=i\dfrac{1-\zeta}{1+\zeta}$. Then, the whole part of F lying above $|z|<1$, is mapped onto the upper half-plane of the w-plane, the two points lying above $z=1$ are transformed respectively to $w=0$, ∞; $z=0$ is transformed to $w=i$, and $z=\infty$ is transformed to $w=-i$. Let the images of z_0 be $w_0=\rho_0e^{i\varphi_0}$, $(0\leqq\varphi_0\leqq\pi)$ and $w_0'=\rho_0'e^{i\varphi_0'}$, $(0\leqq\varphi_0'\leqq\pi)$. We then have

$$0<\lambda\leqq|z_0-1|=\left|\left(\frac{i-\rho_0e^{i\varphi_0}}{i+\rho_0e^{i\varphi_0}}\right)^2-1\right|=\frac{4\rho_0}{1+2\rho_0\sin\varphi_0+\rho_0^2}\,.$$

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4) We mean, by this, that $\lambda X(\lambda)$ is a decreasing function of λ, $(0<\lambda\leqq 2)$ and moreover $\lim\limits_{\lambda\to+0}\lambda X(\lambda)=16$.

Consequently, since $\sin\varphi \geqq 0$ for $0 \leqq \varphi \leqq \pi$, we have

$$0 < \lambda \leqq \frac{4\rho_0}{1+\rho_0^2},$$

whence follows

$$\rho_1 \leqq \rho_0 \leqq \rho_2, \tag{13}$$

where

$$\rho_1 = \frac{2-\sqrt{4-\lambda^2}}{\lambda}, \qquad \rho_2 = \frac{2+\sqrt{4-\lambda^2}}{\lambda}.$$

Similarly, we can prove

$$\rho_1 \leqq \rho_0' \leqq \rho_2. \tag{14}$$

It follows easily from (13) and (14) that one of the images of the two replicas of Γ lying on $\boldsymbol{F}$ on the w-plane contains $w=0$ and a point on $|w|=\rho_1$, and that the other contains $w=\infty$ and a point on $|w|=\rho_2$. Hence we have, by (II) in **2**,

$$\operatorname{mod} \boldsymbol{B} \leqq \log \Psi\left(\frac{\rho_2}{\rho_1}\right) = \log \Psi\left(\frac{(2+\sqrt{4-\lambda^2})^2}{\lambda^2}\right); \tag{15}$$

where the equality holds if and only if the two images of Γ are respectively

$$-\rho_1 \leqq \Re w \leqq 0, \qquad \Im w = 0$$

and

$$\rho_2 \leqq \Re w \leqq +\infty, \qquad \Im w = 0,$$

or

$$-\infty \leqq \Re w \leqq -\rho_2, \qquad \Im w = 0$$

and

$$0 < \Re w \leqq \rho_1, \qquad \Im w = 0.$$

The condition for equality in (15) may be also stated as follows. It holds, if and only if Γ is a minor arc $\widehat{z_0 z_1}$ of the unit circle, with $z_0 = 1$, $|z_0 - z_1| = \lambda$. There exist two such minor arcs: $\widehat{z_0 z_1}$. We denote by $\Gamma_\lambda^{(1)}$ one of them, which lies in the upper half-plane and by $\Gamma_\lambda^{(2)}$ the other.

Now, we consider the case where Γ coincides either with $\Gamma_\lambda^{(1)}$ or with $\Gamma_\lambda^{(2)}$. We rotate the z-plane around $z=0$, until the middle point of Γ coincides with $z=1$. Then, in formula (12): $\operatorname{mod} A \leqq \frac{1}{2} \operatorname{mod} B$, the equality holds if and only if Γ' is mapped onto $|Z|=\sqrt{q}$ by the conformal mapping which maps B onto a circular annulus $q<|Z|<1$. (See Teichmüller [5].) As is easily seen, this happens if and only if Γ' is the negative real axis. Consequently, since we have in such a case

$$\operatorname{mod} A = \frac{1}{2} \operatorname{mod} B = \frac{1}{2} \log \Psi \left(\frac{(2+\sqrt{4-\lambda^2}\,)^2}{\lambda^2} \right),$$

and also $A=A_\lambda$, we have

$$\log X(\lambda) = \operatorname{mod} A_\lambda = \frac{1}{2} \log \Psi \left(\frac{(2+\sqrt{4-\lambda^2}\,)^2}{\lambda^2} \right). \tag{16}$$

Now, (9) follows immediately from (12), (15) and (16). Furthermore, it is easily observed from the facts obtained until now, that the equality in the formula (9) holds if and only if A is the annulus obtained by a revolution of A_λ around the point $z=0$. Next, (10) follows from (16) and (6). (10) being proved, we have

$$\lambda X(\lambda) = 2\sqrt{2+\sqrt{4-\lambda^2}} \cdot \frac{\Phi\left(\frac{2}{\lambda}\sqrt{2+\sqrt{4-\lambda^2}} \right)}{\frac{2}{\lambda}\sqrt{2+\sqrt{4-\lambda^2}}}. \tag{17}$$

Now, since $\frac{2}{\lambda}\sqrt{2+\sqrt{4-\lambda^2}}$ is a decreasing function of λ, the second fraction in the right-hand side is a decreasing function of λ by (7). As $2\sqrt{2+\sqrt{4-\lambda^2}}$ decreases also when λ increases, $\lambda X(\lambda)$ is a decreasing function of λ. Therefore we have, by (6) and (17),

$$\lambda X(\lambda) \uparrow 16 \qquad \text{as} \quad \lambda \to +0,$$

which proves (11).

4. *Proof of the theorem.* First, we prove that

$$\sup_{K,T,z_1\neq z_2} \frac{|T(z_1)-T(z_2)|}{|z_1-z_2|^{\frac{1}{K}}} \leqq 16\,. \tag{18}$$

For this purpose, it suffices to show

$$|T(z_1)-T(z_2)|<16|z_1-z_2|^{\frac{1}{K}}\,, \tag{19}$$

for an arbitrary K-QC mapping $w=T(z)$ of $|z|<1$ onto $|w|<1$ such that $T(0)=0$ and for arbitrary two points z_1, z_2 such that $|z_1|\leqq 1$, $|z_2|\leqq 1$, $z_1\neq z_2$. We set $T(z_1)=w_1$, $T(z_2)=w_2$.

In case $|z_1-z_2|\geqq 1$, (19) is trivial. So we may assume that $0<|z_1-z_2|<1$. Now, $w=T(z)$ can be extended to a K-QC mapping from $|z|<+\infty$ to $|w|<+\infty$. (See Ahlfors [1], Mori [3], [4].) We denote by A the annulus $\left\{z\,;\ \frac{1}{2}|z_1-z_2|<\left|z-\frac{z_1+z_2}{2}\right|<\frac{1}{2}\right\}$. Then we have

$$\operatorname{mod} A=\log\frac{1}{|z_1-z_2|}$$

and consequently, by the fact stated at the beginning of **2**,

$$\log\frac{1}{|z_1-z_2|^{\frac{1}{K}}}\leqq \operatorname{mod} T(A)\,. \tag{20}$$

Now, we shall estimate the right-hand side of this formula. Suppose first that

$$1^\circ \qquad \left|\frac{z_1+z_2}{2}\right|\leqq\frac{1}{2}\,.$$

Then, A is contained in $|z|<1$, and so $T(A)$ is contained in $|w|<1$, and *a fortiori*, in $|w-w_1|<2$. Consequently, one of the complementary continua of $T(A)$ contains both w_1 and w_2, and the other contains $\{w\,;|w-w_1|\geqq 2\}$. Therefore, we have by (I) in **2**

$$\operatorname{mod} T(A)\leqq \log\Phi\left(\frac{1}{|w_1-w_2|}\right).$$

Consequently we have, by (20) and (7),

$$\frac{1}{|z_1-z_2|^{\frac{1}{K}}} \leqq \Phi\left(\frac{2}{|w_1-w_2|}\right) < \frac{8}{|w_1-w_2|},$$

whence (18) follows.

Next, suppose that

$$2^\circ \qquad \left|\frac{z_1+z_2}{2}\right| > \frac{1}{2}.$$

Then A does not contain $z=0$, and so $T(A)$ does not contain $w=0$. So that one of the complementary continua of $T(A)$ contains both $w=0$ and $w=\infty$, and the other contains both w_1 and w_2. Therefore, we have by Lemma 1

$$\mathrm{mod}\, T(A) \leqq \log X(|w_1-w_2|).$$

Consequently, by (20) and (11), we have

$$\frac{1}{|z_1-z_2|^{\frac{1}{K}}} \leqq X(|w_1-w_2|) < \frac{16}{|w_1-w_2|},$$

whence (19) follows immediately.

Thus we have proved (18).

Next, let us prove

$$\sup_{K,T,z_1 \neq z_2} \frac{|T(z_1)-T(z_2)|}{|z_1-z_2|^{\frac{1}{K}}} \geqq 16. \tag{21}$$

For any small positive number s, we denote by $A_s^{(z)}$ the annulus $\{z;|z|<+\infty\}-\{z;-\infty<\Re z\leqq 0, \Im z=0\}-\{z;|z|=1, |\arg z|\leqq s/2\}$. We map the planar region $\{z;|z|<+\infty\}-\{z;-\infty<\Re z\leqq 0, \Im z=0\}$ conformally onto the planar region $\{Z;|Z|<1\}$ on the Z-plane by a regular function $Z=f(z)$ in such a manner that $z=0$ and $z=-\infty$ correspond to $Z=-1$ and $Z=1$ respectively. Then the image of $|z|=1$ is the part of the imaginary axis of the Z-plane contained in $|Z|<1$, and consequently the image of the arc $\{z;|z|=1, |\arg z|\leqq s/2\}$ is a segment on the imaginary axis, whose middle point is $Z=0$. We denote the length of this segment by l. Then we can easily obtain

$$\lim_{s\to 0}\frac{l}{s}=\frac{1}{4}. \tag{22}$$

The image of $A_s^{(z)}$ is the annulus obtained by excluding this segment from $|Z|<1$. We denote this image by $A_s^{(Z)}$.

Next, we map $A_s^{(Z)}$ conformally onto a circular annulus $A_s^{(\zeta)}$: $\gamma<|\zeta|<1$ on the ζ-plane by a regular function $\zeta=\varphi(Z)$. We can easily obtain

$$\lim_{l\to 0}\frac{\gamma}{l}=\frac{1}{4}. \tag{23}$$

Then we map $A_s^{(\zeta)}$ onto a circular annulus $A_s^{(\omega)}: \gamma^{\frac{1}{K}}<|\omega|<1$ on the ω-plane by the K-QC mapping $\omega=\tau(\zeta)=|\zeta|^{\frac{1}{K}}e^{i\arg\zeta}$.

Further, we map $A_s^{(\omega)}$ conformally onto $A_s^{(W)}$ which is the annulus obtained by excluding from $|W|<1$ a segment lying on the imaginary axis of the W-plane whose middle point is $W=0$. We denote by $W=\psi(\omega)$ the mapping function and by l^* the length of this segment. We can easily obtain

$$\lim_{\gamma\to 0}\frac{l^*}{\gamma^{\frac{1}{K}}}=4. \tag{24}$$

Now, as is easily ascertained, two boundary points of $A_s^{(Z)}$ lying on its slit which are in the same position in the Z-plane are transformed by the composite mapping $W=\psi(\tau(\varphi(Z)))$ to two boundary points of $A_s^{(W)}$ lying on its slit in the same position in the W-plane. Consequently $W=\psi(\tau(\varphi(Z)))$ can be regarded as a continuous function in $|Z|<1$, and hence this mapping can be regarded as a K-QC mapping of $|Z|<1$ onto $|W|<1$. (See Ahlfors [1], Mori [3], [4].) By this extended mapping $W=\psi(\tau(\varphi(Z)))$, the part of the imaginary axis contained in $|Z|<1$ is transformed into the part of the imaginary axis contained in $|W|<1$.

Next we map $|W|<1$ conformally onto the planar region $\{w; |w|<+\infty\}-\{w; -\infty<\Re w\leqq 0, \Im w=0\}$ by a regular function $w=g(W)$ in such a manner that $W=-1$ and $W=1$ correspond respectively to $w=0$ and $w=-\infty$. Then, the part of the imaginary axis contained in $|W|<1$ corresponds to the unit circle on the w-plane, and consequently, $A_s^{(W)}$ is transformed into a planar annulus $A_s^{(w)}$ obtained by excluding from $\{w; |w|<+\infty\}-\{w; -\infty<\Re w\leqq 0, \Im w=0\}$ an arc lying on $|w|=1$ which is symmetric with respect to the real axis. We denote by s^* the length of this arc. Then we can easily obtain

$$\lim_{l^*\to 0}\frac{s^*}{l^*}=4. \tag{25}$$

We denote by $w=T_s(z)$ the mapping from $A_s^{(z)}$ onto $A_s^{(w)}$ which is obtained by combining the above-mentioned five mappings one after another. Then $w=T_s(z)$ is a K-QC mapping in $A_s^{(z)}$. As is easily ascertained, two boundary points of $A_s^{(z)}$ lying on the negative real axis in the same position in the z-plane are transformed by $w=T_s(z)$ to two boundary points of $A_s^{(w)}$ lying on the negative real axis in the same position in the w-plane. On the other hand, since $f(z)$ and $g(W)$ are regular in $\{z; |z|<+\infty\}-\{z; -\infty<\Re z\leqq 0, \Im z=0\}$ and in $\{W; |W|<1\}$ respectively, and since, as was shown above, $\psi(\tau(\varphi(Z)))$ can be regarded as a K-QC mapping in $|Z|<1$, $w=T_s(z)$ can be regarded as a K-QC mapping of $\{z; |z|<+\infty\}-\{z; -\infty<\Re z\leqq 0, \Im z=0\}$ onto $\{w; |w|<+\infty\}-\{w; -\infty<\Re w\leqq 0, \Im w=0\}$. Therefore $w=T_s(z)$ can be regarded as a K-QC mapping of $|z|<+\infty$ onto $|w|<+\infty$.

Next, as was remarked above, the unit circle on the z-plane is transformed into the part of the imaginary axis contained in $|Z|<1$ by $Z=f(z)$. The interval on the imaginary axis contained in $|Z|<1$ is transformed into $\{W; |\Im W|<1, \Re W=0\}$ by $W=\psi(\tau(\varphi(Z)))$, and the part of the imaginary axis contained in $|W|<1$ is transformed into the unit circle on the w-plane by $w=g(W)$. Consequently $|z|<1$ is mapped onto $|w|<1$ by $w=T_s(z)$.

Now, we have obviously

$$\sup_{K, T, z_1\neq z_2}\frac{|T(z_1)-T(z_2)|}{|z_1-z_2|^{\frac{1}{K}}}\geqq\frac{s^*}{s^{\frac{1}{K}}}.$$

On the other hand, we have, by (22), (23), (24) and (25),

$$\lim_{s\to 0}\frac{s^*}{s^{\frac{1}{K}}}=16^{1-\frac{1}{K}}.$$

(21) follows immediately from these two formulas.

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