Journal of the Mathematical Society of Japan Vol. 9, No. 1, January, 1957

Some theorems in dimension theory for non-separable spaces.

By Keiô NAGAMI

(Received Feb. 17, 1956)

This paper provides some theorems in dimension theory for non-separable spaces. Let R be a topological space, $\dim R$ the covering dimension of R, $\operatorname{ind} R$ the so-called "small" inductive dimension of R defined by means of boundaries of neighborhoods of points, and $\operatorname{Ind} R$ the so-called "large" inductive dimension of R defined by means of boundaries of neighborhoods of closed sets. (Cf. [14]. In the notations of [2], Appendix, p. 153, we have $\operatorname{ind} R=d_1(R)$, $\operatorname{Ind} R=d_2(R)$.) It is to be noted that when R is normal, $\dim R\leqq n$ is equivalent to the following condition: For any closed subset C of R and for any mapping (=continuous transformation) f from C into an n-sphere there exists a continuous extension g of f defined on the whole space R.

In § 1 we shall give the sum theorem of covering dimension for metric spaces which is a generalization of the known. In § 2 we shall study closed mappings which lower dimension and related problems. In § 3 we shall give a new definition of dimension-kernel and shall study some properties concerning it.

§ 1. Sum theorem of covering dimension.

Let R be a topological space and $\mathfrak{U}=\{U_\alpha; \alpha\in A\}$ be a collection of subsets of R. Then $\mathfrak{U}\cap S$, S being a subset of R, stands for $\{U_\alpha\cap S; \alpha\in A\}$.

LEMMA 1. *Let S be a closed subset of a normal space R and $\mathfrak{U}=\{U_\alpha; \alpha\in A\}$ be a finite open covering of S whose elements are F_σ. Then there exists a finite open collection $\mathfrak{V}=\{V_\alpha; \alpha\in A\}$ of R whose elements are F_σ such that the order of $\mathfrak{V}$ is not greater than that of $\mathfrak{U}$ and $V_\alpha\cap S=U_\alpha$ for every $\alpha\in A$.*

LEMMA 2. *An F_σ-subset of a normal space is also normal as a relative space.*

LEMMA 3. *Let R be a normal space and $F_i, i=1,2,\cdots$, be subsets of R. If $\bigcup_{i=1}^{j} F_i$ is closed for every $j<\infty$,* $\dim \bigcup_{i=1}^{\infty} F_i \leqq \sup_{i=1}^{\infty} \dim F_i$.

PROOF. It suffices only to prove the inequality for the case when $\sup_{i=1}^{\infty} \dim F_i = n < \infty$ and $\bigcup_{i=1}^{\infty} F_i = R$ by virtue of lemma 2. First we shall show by the induction on m that $\dim \bigcup_{i=1}^{m} F_i \leqq n$ for every $m<\infty$. Suppose that $\dim \bigcup_{i=1}^{m} F_i \leqq n$. Let $\mathfrak{G}=\{G_\alpha ; \alpha \in A\}$ be a finite open covering of $\bigcup_{i=1}^{m+1} F_i$. Then $\dim \bigcup_{i=1}^{m} F_i \leqq n$ asserts the existence of a finite open F_σ covering $\mathfrak{U}=\{U_\alpha ; \alpha \in A\}$ of $\bigcup_{i=1}^{m} F_i$ such that the order of $\mathfrak{U}$ is not greater than $n+1$ and $\mathfrak{U}$ refines $\mathfrak{G} \cap (\bigcup_{i=1}^{m} F_i)$. Then, by lemma 1, there exists a finite open F_σ collection $\mathfrak{V}=\{V_\alpha ; \alpha \in A\}$ of $\bigcup_{i=1}^{m+1} F_i$ such that i) the order of $\mathfrak{V}$ is not greater than $n+1$, ii) $V_\alpha \cap (\bigcup_{i=1}^{m} F_i) = U_\alpha$ for every $\alpha \in A$, iii) $\mathfrak{V}$ refines $\mathfrak{G}$. Let G be an open set of $\bigcup_{i=1}^{m+1} F_i$ such that $\bigcup_{i=1}^{m} F_i \subset G \subset \overline{G} \subset \bigcup_{\alpha \in A} V_\alpha$ and consider a finite open covering $\mathfrak{W}= (\mathfrak{V} \cap F_{m+1}) \cup (\mathfrak{G} \cap (F_{m+1} - \overline{G}))$ of F_{m+1}. Then $\dim F_{m+1} \leqq n$ asserts the existence of a finite open covering $\mathfrak{D}=\{D_\alpha, D'_\alpha ; \alpha \in A\}$ of F_{m+1} such that i) the order of $\mathfrak{D}$ is not greater than $n+1$, ii) $D_\alpha \subset V_\alpha \cap F_{m+1}$ for every $\alpha \in A$, iii) $D'_\alpha \subset G_\alpha \cap (F_{m+1} - \overline{G})$ for every $\alpha \in A$. It is almost evident that $E_\alpha = D_\alpha \cup (V_\alpha \cap G)$ is open in $\bigcup_{i=1}^{m+1} F_i$. Hence $\mathfrak{E}=\{E_\alpha, D'_\alpha ; \alpha \in A\}$ is a finite open covering of $\bigcup_{i=1}^{m+1} F_i$. Moreover we can easily see that $\mathfrak{E}$ refines $\mathfrak{G}$ and the order of $\mathfrak{E}$ is not greater than $n+1$, which completes the induction. Thus we have established the fact that $\dim \bigcup_{i=1}^{m} F_i \leqq n$ for every $m<\infty$. Hence $\bigcup_{i=1}^{\infty} F_i$ is a countable sum of closed sets whose dimension are at most n and we know, by the usual sum theorem, that $\dim \bigcup_{i=1}^{\infty} F_i \leqq n$.

It is to be noted that when "dim = covering dimension" in the above lemma is replaced with "large inductive dimension", the proposition thus obtained also holds under some additional conditions: i) R is completely normal, ii) F_i are mutually disjoint. (Cf. [1].)

LEMMA 4. *Let R be a metric space and S be a subset of R. If every point of S has a neighborhood V (in R) such that $V \cap S$ is an F_σ in R, then S is an F_σ in R.*

This was proved in [5].

THEOREM 1. *Let R be a metric space and F_α subsets of R indexed by all ordinals α less than some fixed ordinal η. If $\bigcup_{\beta<\alpha} F_\beta = H_\alpha$ is closed for every $\alpha<\eta$, $\dim \bigcup_{\alpha<\eta} F_\alpha \leqq \sup_{\alpha<\eta} \dim F_\alpha$.*

PROOF. It suffices only to prove the inequality for the case when $\sup_{\alpha<\eta} \dim F_\alpha = n < \infty$ and $R = \bigcup_{\alpha<\eta} F_\alpha$. Moreover we can assume, by lemma 3, with no loss of generality that η is a limit ordinal. Let ρ be a metric on R which agrees with the preasigned topology of R. Setting $G_{\alpha i} = \{p\,; \rho(p, H_\alpha) < 1/i\}$, $i=1, 2, \cdots$, it can easily be seen that i) $H_\alpha = \bigcap_{i=1}^{\infty} G_{\alpha i}$, ii) $G_{\alpha i} \supset \overline{G}_{\alpha, i+1}$, iii) $G_{\alpha i} \subset G_{\beta i}$ for every $\alpha < \beta < \eta$. We set $G_{1i} = \phi$. Then $F_{\alpha i} = F_\alpha - G_{\alpha i}$ is a closed set with $\dim F_{\alpha i} \leqq n$. Let $D_{\alpha i} = G_{\alpha+1, i+1} - \overline{G}_{\alpha, i+1}$ and then $D_{\alpha i}$ is open and includes $F_{\alpha i}$. Moreover $\{D_{\alpha i}\,; \alpha < \eta\}$ is as can easily be seen a mutually disjoint collection. Therefore $\dim \bigcup_{\alpha<\eta} F_{\alpha i} \leqq n$. Setting $F_i = \bigcup_{\alpha<\eta} F_{\alpha i}$, F_i is, by lemma 4, an F_σ. Since $F_\alpha - H_\alpha = \bigcup_{i=1}^{\infty} F_{\alpha i}$ and hence $\bigcup_{\alpha<\eta} F_\alpha = \bigcup_{i=1}^{\infty} F_i$, we can see, by the usual sum theorem, that $\dim R \leqq n$, which completes the proof.

§ 2. Mappings which lower dimension.

It is well known that

$$\text{(A)} \qquad \dim R = \operatorname{ind} R = \operatorname{Ind} R$$

for every separable metric space. Recently M. Katětov [3] and K. Morita [9] have succeeded independently in proving the validity of

$$\text{(B)} \qquad \dim R = \operatorname{Ind} R$$

for every non-separable metric space. Moreover Morita [9] has proved the validity of (A) for every metric S_σ-space, where an S_σ-space is a space which is the sum of a countable number of closed subsets with the star-finite property.

Let φ be a mapping from a space R onto another space S. If

φ is a closed mapping, then the inequality

(C) $$\dim R \leqq \sup_{y \in S} \dim \varphi^{-1}(y) + \dim S$$

holds when R and S are separable metric [2]. Our concern is to establish the analogous inequalities to (C):

(D) $$\dim R \leqq \sup_{y \in S} \dim \varphi^{-1}(y) + \operatorname{Ind} S,$$

(E) $$\dim R \leqq \sup_{y \in S} \dim \varphi^{-1}(y) + \operatorname{ind} S,$$

for some class of non-separable spaces. First we shall show the validity of (D) for the case when R is paracompact Hausdorff and S is hereditarily paracompact Hausdorff (Theorem 2 below). By virtue of the validity of (B) for metric spaces, we can know that (C) is valid even if R and S are non-separable metric spaces (Corollary 2 below). Moreover the validity of (E) will be shown for the case when R is a normal S_σ-space (Theorem 3 below). This extends a theorem due to Morita [7] which asserts the validity of (D) for the case when R is compact Hausdorff. Two corollaries are deduced from theorem 3: One is Morita's theorem which asserts $\dim R \leqq \operatorname{ind} R$ for every normal S_σ-space and another is a generalization of N. Vedenisoff's theorem concerning a continuous decomposition of a compact space (Corollaries 3 and 4 below). Whether (E) holds for the case when R and S are metric is an open problem. The method used in this paper seems to be unable to apply to this case. If this question could be answered in the affirmative, we could determine $\dim R \leqq \operatorname{ind} R$ for every non-separable metric R; this is surely one of the most important unsolved problems in the dimension theory.

We start from some preliminary lemmas.

LEMMA 5. *Let R be a non-empty hereditarily paracompact Hausdorff space, i.e. a Hausdorff space any of whose subspace is paracompact. Then* $\operatorname{Ind} R \leqq n (< \infty)$ *if and only if for every open covering* $\mathfrak{U}$ *there exists a collection* $\mathfrak{V} = \{V\}$ *of mutually disjoint open sets such that* i) $\overline{\mathfrak{V}} = \{\overline{V}\}$ *refines* $\mathfrak{U}$, ii) $\mathfrak{V}$ *is locally finite,* iii) $R - \cup V = \cup (\overline{V} - V)$, iv) $\operatorname{Ind}(R - \cup V) \leqq n - 1$.

AN OUTLINE OF THE PROOF. Since if-part is evident, we shall prove only-if-part. Since every paracompact Hausdorff space is strongly screenable [13], [10], there exists an open covering $\mathfrak{W} = \{W_\alpha;\ \alpha \in \bigcup_{i=1}^{\infty} A_i\}$ and a closed covering $\mathfrak{F} = \{F_\alpha;\ \alpha \in \bigcup_{i=1}^{\infty} A_i\}$ such that i) $\overline{\mathfrak{W}}$ is locally finite and refines $\mathfrak{U}$, ii) for every i, $\{W_\alpha ; \alpha \in A_i\}$ is mutually disjoint, iii) for every $\alpha \in \bigcup_{i=1}^{\infty} A_i$, $W_\alpha \supset F_\alpha$. Setting $F_i = \bigcup_{\alpha \in A_i} F_\alpha$ and $W_i = \bigcup_{\alpha \in A_i} W_\alpha$, there exists an open set G_1 with $F_1 \subset G_1 \subset \overline{G}_1 \subset W_1$ and $\operatorname{Ind}(\overline{G}_1 - G_1) \leqq n-1$. Since $\operatorname{Ind}(R-G_1) \leqq n$, there exists a relatively open set G_2 of $R-G_1$ with $F_2 - G_1 \subset G_2 \subset \overline{G}_2 \subset W_2 - G_1$ and $\operatorname{Ind}(\overline{G}_2 - G_2) \leqq n-1$. Proceeding this procedure successively, we get for every i a relatively open set G_i of $R - \bigcup_{j<i} G_j$ with $F_i - \bigcup_{j<i} G_j \subset G_i \subset \overline{G}_i \subset W_i - \bigcup_{j<i} G_j$ and $\operatorname{Ind}(\overline{G}_i - G_i) \leqq n-1$. Setting for $\alpha \in A_1$, $V_\alpha = W_\alpha \cap G_1$ and for $\alpha \in A_i (i>1)$ $V_\alpha = W_\alpha \cap G_i - \bigcup_{j<i} \overline{G}_j$, it is not so hard to see that $\{V_\alpha ; \alpha \in \bigcup_{i=1}^{\infty} A_i\}$ satisfies the conditions i), ii), iii) in the lemma. Since $\bigcup_{\alpha \in \cup A_i} (\overline{V}_\alpha - V_\alpha) = \bigcup_{i=1}^{\infty} (\overline{G}_i - G_i)$, $\operatorname{Ind}(R - \bigcup_{\alpha \in \cup A_i} V_\alpha) \leqq n-1$ by the sum theorem of large inductive dimension [1], which shows $\mathfrak{B}$ satisfies condition iv) and the lemma is proved.

As a corollary of this lemma we get the decomposition theorem for non-separable metric spaces which has recently proved by Katětov [3] and Morita [9] independently.

COROLLARY 1. *Let R be a metric space with* $\operatorname{Ind} R \leqq n < \infty$. *Then R can be decomposed into the sum of $n+1$ subsets R_i with* $\operatorname{Ind} R_i \leqq 0$ *for* $i = 1, \cdots, n+1$.

PROOF. When $n=0$, the assertion is trivial. Now let $n=m \geqq 1$ and put the induction assumption that the assertion is valid for $n = m-1$. Let $\mathfrak{U}_i$ be a collection of mutually disjoint open sets such that $\operatorname{Ind}(R - \bigcup_{U \in \mathfrak{U}_i} U) \leqq m-1$ and a diameter of each set of $\mathfrak{U}_i$ is less than $1/i$. This $\mathfrak{U}_i$ exists by lemma 5, since every metric space is paracompact. Set $R_1 = \bigcap_{i=1}^{\infty} \bigcup_{U \in \mathfrak{U}_i} U$ and we can show $\operatorname{Ind} R_1 \leqq 0$ as follows. Let F and G be respectively a closed and an open subsets of R_1 with $F \subset G$. Let V_i be the sum of sets $U \in \mathfrak{U}_i$ such that $U \cap R_1 \subset G$. Then $H = R_1 \cap (\bigcup_{i=1}^{\infty} V_i)$ is evidently open in R_1 and $F \subset H \subset G$. Moreover H is closed: Let p be a point of R_1 with $p \notin H$. Let j be a positive integer such that $S_{2/j}(p) \cap F = \phi$ where $S_{2/j}(p)$ denotes an open sphere of radius $2/j$ with centre p. Then $(\bigcup_{i \geqq j} V_i) \cap S_{1/j}(p) = \phi$. Since $V_i \cap R_1$

is closed in R_1, $(\bigcup_{i<j} V_i)\cap R_1$ is closed in R_1. Setting $V(p)=S_{1/j}(p)\cap(\bigcup_{i<j} V_i)\cap R_1$, $V(p)\cap H=\phi$. Hence H is closed in R_1. Thus H is open and closed in R_1, which proves $\operatorname{Ind} R_1\leqq 0$. Since the sum theorem of large inductive dimension holds in metric spaces [1], and $R-R_1$ is the sum of a countable number of closed subsets with large inductive dimension $\leqq m-1$, $\operatorname{Ind}(R-R_1)\leqq m-1$. Hence $R-R_1$ is the sum of an m number of subsets R_i, $i=2,\cdots,m+1$ with $\operatorname{Ind} R_i\leqq 0$. Thus R is the sum of $m+1$ subsets R_i, $i=1,\cdots,m+1$, and hence the induction is completed.

LEMMA 6. *Let H and F be closed subsets of a normal space R and let f and g be mappings defined respectively on H and on F with values in an n-sphere S_n. Let a subset $A=\{x\,;f(x)\neq g(x)\}\subset H\cap F$. If* $\dim A\leqq n-1$, *then f can be extended continuously to the whole space R.*

This will be verified by a quite analogous method used in [2] in view of [8, Theorem 6.3] and hence its proof is omitted.

LEMMA 7. *Let C be a closed subset of a normal space R and f be a mapping defined on C with values in S_n. If for any open set G with $G\supset C$ there exists a closed subset F with $C\subset F\subset G$ such that* $\dim(R-F)\leqq n$, *f can be extended continuonsly to the whole space R.*

PROOF. Since S_n is a neighborhood-extensor for normal spaces, there is a continuous extension g defined on some open set G with $C\subset G$. Then from the hypothesis-condition there exists a closed subset F with $C\subset F\subset G$ and $\dim(R-F)\leqq n$. By [8], there is a continuous extension h of $g|F$, defined on the whole space, which is the desired extension.

THEOREM 2. *Let φ be a closed mapping from a non-empty paracompact Hausdorff space R onto a non-empty hereditarily paracompact Hausdorff space S. Then*

$$\dim R\leqq \sup_{y\in S}\dim\varphi^{-1}(y)+\operatorname{Ind} S.$$

PROOF. When either $\sup_{y\in S}\dim\varphi^{-1}(y)=\infty$ or $\operatorname{Ind} S=\infty$, the inequality trivially holds. We treat the case when $\sup_{y\in S}\dim\varphi^{-1}(y)=m<\infty$ and $\operatorname{Ind} S=n<\infty$ and prove the theorem by the induction on n. Let C be an arbitrary closed subset of R and f a mapping from C into S_{m+n}. For any point $y\in S$ there is, by lemma 7, a continuous extension f_y of f, defined on $C\cup\varphi^{-1}(y)$. Since S_{m+n} is a neighborhood-extensor for normal spaces, there is an open subset $G_y\supset C\cup\varphi^{-1}(y)$

and a mapping $g_y: G_y \to S_{m+n}$ with $g_y | C \cup \varphi^{-1}(y) = f_y$. Since φ is a closed mapping, $\varphi(R-G_y)$ is closed in S and does not contain y. Hence $V(y) = S - \varphi(R-G_y)$ is an open neighborhood of y. Then by lemma 5 there exists a collection $\mathfrak{V} = \{V_\alpha ; \alpha \in A\}$ of mutually disjoint open sets of S such that i) $\overline{\mathfrak{V}}$ refines $\{V(y) ; y \in S\}$, ii) $\{\overline{V}_\alpha ; \alpha \in A\}$ is locally finite, iii) $S - \bigcup_{\alpha \in A} \overline{V}_\alpha = \bigcup_{\alpha \in A} (\overline{V}_\alpha - V_\alpha)$, iv) $\operatorname{Ind}(S - \bigcup_{\alpha \in A} V_\alpha) \leqq n-1$. Set $T = \bigcup_{\alpha \in A} (\overline{V}_\alpha - V_\alpha)$.

When $n=0$, we have $T = \phi$. Hence $\{\varphi^{-1}(V_\alpha) ; \alpha \in A\}$ is a mutually disjoint open covering of R which refines $\{G_y ; y \in S\}$. For every $\alpha \in A$, there exists $G_{y(\alpha)}$ with $\varphi^{-1}(V_\alpha) \subset G_{y(\alpha)}$. Let $g: R \to S_m$ be a mapping defined as follows: $g | \varphi^{-1}(V_\alpha) = g_{y(\alpha)} | \varphi^{-1}(V_\alpha)$. Then g is uniquely defined and continuous and $g | C = f$, which proves $\dim R \leqq m$. Thus the theorem is valid for $n=0$.

We make the induction assumption that the theorem is valid for $n < i$, $i > 0$. Now let $n = i$ and assume that A consists of all ordinals less than some fixed ordinal η. Let $f_2 : \overline{\varphi^{-1}(V_1)} \cup C \to S_{m+i}$ be $g_{y(1)} | \overline{\varphi^{-1}(V_1)} \cup C$. Take an arbitrary ordinal $\alpha < \eta$ and put the transfinite induction assumption that there is a mapping $f_\beta : \bigcup_{\delta < \beta} \overline{\varphi^{-1}(V_\delta)} \cup C \to S_{m+i}$ with $f_\gamma = f_\beta | \bigcup_{\delta < \gamma} \overline{\varphi^{-1}(V_\delta)} \cup C$ for any $\gamma < \beta < \alpha$. We construct $f_\alpha : \bigcup_{\delta < \alpha} \overline{\varphi^{-1}(V_\delta)} \cup C \to S_{m+i}$ as $f_\alpha | \bigcup_{\delta < \beta} \overline{\varphi^{-1}(V_\delta)} \cup C = f_\beta$, $\beta < \alpha$. This f_α is continuous, since $\{\overline{\varphi^{-1}(V_\delta)} ; \delta < \eta\}$ is locally finite. Let $g'_{y(\alpha)} = g_{y(\alpha)} | \overline{\varphi^{-1}(V_\alpha)}$ and then $B = \{x ; g'_{y(\alpha)}(x) \neq f_\alpha(x)\}$ is an F_σ-set with $B \subset \varphi^{-1}(T)$. $\varphi | \varphi^{-1}(T)$ is a closed mapping from $\varphi^{-1}(T)$ onto T. Since $\operatorname{Ind} T \leqq i-1$, $\dim \varphi^{-1}(T) \leqq m+i-1$ by the induction assumption. Hence $\dim B \leqq m+i-1$. Therefore there exists a mapping $f_{\alpha+1} : \bigcup_{\delta < \alpha+1} \overline{\varphi^{-1}(V_\delta)} \cup C \to S_{m+i}$ with $f_{\alpha+1} | \bigcup_{\delta < \alpha} \overline{\varphi^{-1}(V_\delta)} \cup C = f_\alpha$. Thus we can construct a mapping $f_\alpha : \bigcup_{\delta < \alpha} \overline{\varphi^{-1}(V_\delta)} \cup C \to S_{m+i}$ for any $\alpha < \eta$ such that $f_\alpha | \bigcup_{\delta < \beta} \overline{\varphi^{-1}(V_\delta)} \cup C = f_\beta$ for any $\beta < \alpha$. Define $h : \bigcup_{\alpha < \eta} \overline{\varphi^{-1}(V_\alpha)} \cup C \to S_{m+i}$ as follows: $h | \bigcup_{\delta < \alpha} \overline{\varphi^{-1}(V_\delta)} \cup C = f_\alpha$. Then h is continuous, since $\{\overline{\varphi^{-1}(V_\alpha)} ; \alpha < \eta\}$ is locally finite. Since $\dim \varphi^{-1}(T) \leqq m+i-1$ and $\bigcup_{\alpha < \eta} \overline{\varphi^{-1}(V_\alpha)} \cup C$ is closed, there exists by lemma 7 a mapping $g : R \to S_{m+i}$ with $g | \bigcup_{\alpha < \eta} \overline{\varphi^{-1}(V_\alpha)} \cup C = h$. It is evident that

$g|C=f$, which proves $\dim R \leqq m+i$. Thus the theorem is completely proved.

Since $\operatorname{Ind} R = \dim R$ for metric spaces, we get at once the following

COROLLARY 2. *Let R and S be non-empty metric spaces and φ be a closed mapping from R onto S. Then*

$$\dim R \leqq \sup_{y\in S} \dim \varphi^{-1}(y) + \dim S.$$

THEOREM 3. *Let φ be a closed mapping from a non-empty normal S_σ-space R onto a non-empty space S. Then*

$$\dim R \leqq \sup_{y\in S} \dim \varphi^{-1}(y) + \operatorname{ind} S.$$

PROOF. First we remark that if we assume that the theorem is valid for the case when R is a normal space with the star-finite property, our theorem can be easily deduced from the assumption: Let $R=\bigcup_{i=1}^{\infty} R_i$ where R_i are closed subspaces with the star-finite property. Let $\varphi_i=\varphi\,|\,R_i$ and φ_i is a closed mapping from R_i onto $\varphi(R_i)$. Since $\varphi_i^{-1}(y)$, $y\in\varphi(R_i)$, is a closed subset of $\varphi^{-1}(y)$, $\dim\varphi_i^{-1}(y) \leqq \dim\varphi^{-1}(y)$. Hence by the assumption and by the monotonicity of small inductive dimension $\dim R_i \leqq \sup_{y\in\varphi(R_i)} \dim\varphi_i^{-1}(y) + \operatorname{ind}\varphi(R_i) \leqq \sup_{y\in S} \dim\varphi^{-1}(y) + \operatorname{ind} S$. Since every R_i is closed, $\dim R \leqq \sup_{i=1}^{\infty} \dim R_i$ and hence $\dim R \leqq \sup_{y\in S} \dim\varphi^{-1}(y) + \operatorname{ind} S$.

Thus it suffices to prove the theorem for the case when R is a normal space with the star-finite property and we shall treat R having this property. It is to be noted that a regular space with the star-finite property is always paracompact. When either $\sup_{y\in S} \dim\varphi^{-1}(y) = \infty$ or $\operatorname{ind} S=\infty$, the theorem is evidently true. Hence we shall prove the theorem for the case when $\sup_{y\in S} \dim\varphi^{-1}(y) = m < \infty$ and $\operatorname{ind} S = n < \infty$ and shall prove it by the induction on n.

Let C be an arbitrary closed subset of R and f be an arbitrary mapping from C into S_{m+n}. For every $y\in S$, there exists a mapping $f_y : \varphi^{-1}(y)\cup C \to S_{m+n}$ with $f_y\,|\,C=f$. Then there exist an open set $G_y \supset \varphi^{-1}(y)\cup C$ and a mapping $g_y : G_y \to S_{m+n}$ with $g_y\,|\,\varphi^{-1}(y)\cup C = f_y$. Since φ is a closed mapping, $U(y)=S-\varphi(R-G_y)$ is an open neighborhood of y. From the condition $\operatorname{ind} S=n$, there exists an open neighborhood $V(y)$ of y with $\overline{V(y)}\subset U(y)$ and $\operatorname{ind}(\overline{V(y)}-V(y))\leqq n-1$.

When $n=0$, $V(y)$ is open and closed, and hence $\varphi^{-1}(V(y))$ is open and closed and is contained in G_y. Since $\mathfrak{V}=\{\varphi^{-1}(V(y))\,;\,y\in S\}$ is an open covering of R, there exists a star-finite open covering $\mathfrak{W}=\{W_\alpha\,;\,\alpha\in\bigcup_{\lambda\in\Lambda}A_\lambda\}$ of R which refines $\mathfrak{V}$ such that $|A_\lambda|\leqq\aleph_0$ for any $\lambda\in\Lambda$ and $W_\alpha\cap W_\beta=\phi$ for any $\alpha\in A_\lambda$ and $\beta\in A_\mu$ where $\lambda\neq\mu$. For every $\alpha\in A_\lambda$, take $\varphi^{-1}(V(y_\alpha))$ such that $\varphi^{-1}(V(y_\alpha))\supset W_\alpha$. Since A_λ consists of at most a countable number of indices, we can construct, by the successive process, a mapping $f_\lambda:\bigcup_{\alpha\in A_\lambda}\varphi^{-1}(V(y_\alpha))\to S_m$ such that the values of f_λ are identical with those of f on $(\bigcup_{\alpha\in A_\lambda}\varphi^{-1}(V(y_\alpha)))\cap C$. Define $g:R\to S_m$ as follows: $g|\bigcup_{\alpha\in A_\lambda}W_\alpha=f_\lambda|\bigcup_{\alpha\in A_\lambda}W_\alpha$. Then g is uniquely defined and continuous on R. Moreover it is evident that $g|C=f$. Thus the theorem is valid for $n=0$.

Next we treat the case when $n=i>0$ and assume that the theorem is valid for $n\leqq i-1$. In this case $\varphi_y=\varphi\,|\,\varphi^{-1}(\overline{V(y)})-\varphi^{-1}(V(y))$ is a closed mapping and hence by the induction assumption $\dim(\varphi^{-1}(\overline{V(y)})-\varphi^{-1}(V(y)))\leqq m+i-1$. Then lemmas 6 and 7 can be applied and by the analogous arguments used in the above and in the proof of theorem 2 we can find a continuous extension g of f to the whole spaec R. Thus the induction is completed and the theorem is proved.

When we consider φ as the identity mapping from R onto itself, we get at once the following

COROLLARY 3. *When R is a normal* S_σ*-space*, $\dim R\leqq\operatorname{ind}R$.

COROLLARY 4. *Let φ be a mapping from a locally compact paracompact Hausdorff space onto a space S. Then*

$$\dim R\leqq\sup_{y\in S}\dim\varphi^{-1}(y)+\operatorname{ind}S.$$

PROOF. Let x be an arbitrary point of R and $F(x)$ be a compact neighborhood of x. Then $\varphi_x=\varphi\,|\,F(x)$ is a closed mapping from $F(x)$ onto $\varphi(F(x))$. For every point $y\in\varphi(F(x))$, $\varphi_x^{-1}(y)$ is a closed subset of $\varphi^{-1}(y)$ and hence $\dim\varphi_x^{-1}(y)\leqq\dim\varphi^{-1}(y)$. Therefore $\dim F(x)\leqq\sup_{y\in\varphi(F(x))}\dim\varphi_x^{-1}(y)+\operatorname{ind}\varphi(F(x))\leqq\sup_{y\in S}\dim\varphi^{-1}(y)+\operatorname{ind}S$. Since $\dim R\leqq\sup_{x\in R}\dim F(x)$ [11], we get at once the desired inequality.

Finally let us consider the case when $\dim S=0$.

THEOREM 4. *Let φ be a closed mapping from a normal space R*

onto a paracompact Hausdorff space S with $\dim S=0$. *Then*

$$\dim R \leqq \sup_{y\in S} \dim \varphi^{-1}(y) .$$

PROOF. When $\sup \dim \varphi^{-1}(y)=\infty$, the theorem is evidently true. Let us consider the case when $\sup \dim \varphi^{-1}(y)=m<\infty$. Let $\mathfrak{U}=\{U\}$ be an arbitrary finite open covering of R. Since $\dim \varphi^{-1}(y)\leqq m$, there exists a finite open covering $\mathfrak{U}_y$ of $\varphi^{-1}(y)$ whose order is at most $m+1$. Then by lemma 1 there exists a collection $\mathfrak{V}_y$ of open sets of R whose order is at most $m+1$ such that i) $\mathfrak{V}_y$ covers $\varphi^{-1}(y)$, ii) $\mathfrak{V}_y$ refines $\mathfrak{U}$, iii) $\{V\cap\varphi^{-1}(y)\,;\,V\in\mathfrak{V}_y\}$ refines $\mathfrak{U}_y$. Let $G_y=\bigcup_{V\in\mathfrak{V}_y} V$ and $F_y=R-G_y$. Then $W(y)=S-\varphi(F_y)$ is an open neighborhood of y. Since $\dim S=0$, there exists an open covering $\mathfrak{W}=\{W_\alpha\,;\,\alpha\in A\}$ of S whose order is 1 such that $\mathfrak{W}$ refines $\{W(y)\,;\,y\in S\}$. Since $\{\varphi^{-1}(W_\alpha)\,;\,\alpha\in A\}$ is an open covering of R and refines $\{G_y\,;\,y\in S\}$, we can choose, for every $W_\alpha\in\mathfrak{W}$, $G_{y(\alpha)}$ with $G_{y(\alpha)}\supset\varphi^{-1}(W_\alpha)$. Then it can easily be seen that $\{V\cap\varphi^{-1}(W_\alpha)\,;\,V\in\mathfrak{V}_{y(\alpha)},\ \alpha\in A\}$ is an open covering of R whose order is at most $m+1$ which refines $\mathfrak{U}$. Thus the theorem is proved.

§ 3. A new definition of dimension-kernel.

It is well-known that the dimension-kernel of an n-dimensional separable metric space is of dimension $\geqq n-1$ and that the equality sign of this cannot be redundant [4, § 22]. It is to be noted that the classical dimension-kernel is defined by means of boundaries of neighborhoods of points. On the other hand our new dimension-kernel given in this paper is defined by means of neighborhoods of points. Then our dimension-kernel of an n-dimensional space becomes also an n-dimensional closed subset for some class of spaces and is homogeneous in the sense of dimension. Our definition seems to be more natural than the classical, because so-called small-inductive method cannot be so useful for non-separable spaces and the covering dimension of a paracompact Hausdorff space or the so-called large inductive dimension of a hereditarily paracompact Hausdorff space is respectively completely determined by the local covering dimension or by the local large inductive dimension [11][1].

1) In [11] we were concerned with only the covering dimension but the analogous statements about the large inductive dimension of a hereditarily paracompact Hausdorff space can easily be obtained by use of the sum theorem of the large inductive dimension for hereditarily paracompact Hausdorff spaces [1].

DEFINITIONS AND NOTATIONS. Let R be a non-empty space with $\dim R=n$ or with $\operatorname{Ind} R=n, n<\infty$. Let kR or KR be respectively the aggregate of points which have no closed neighborhoods U with $\dim U<n$ or $\operatorname{Ind} U<n$. kR or KR is called respectively the cov-dimension-kernel or the Ind-dimension-kernel of R. When $R=kR$ or $=KR$, R is called respectively cov-homogeneous or Ind-homogeneous.

Let us study some properties about the dimension-kernel just defined. In the following propositions R is always a non-empty paracompact Hausdorff space with $\dim R=n<\infty$.

A) *kR is a non-empty closed subset of R.*

PROOF. kR is evidently closed. If kR is empty, R is covered by open sets U with $\dim U<n$ and hence by [11] $\dim R<n$, which is a contradiction.

B) $\dim kR=n$.

PROOF. Let $\mathfrak{U}=\{U_i\}$ be an arbitrary finite open covering of R. If $\dim kR<n$, there would exist a same-indexed open covering $\mathfrak{V}=\{V_i\}$ of kR such that $V_i\subset U_i\cap kR$ for every i and that the order of $\mathfrak{V}$ is at most n. Since kR is closed, there would exist a same-indexed open collection $\mathfrak{W}=\{W_i\}$ of R with $W_i\subset U_i$ and $W_i\cap kR\subset V_i$ such that the order of $\mathfrak{W}$ is at most n. Let G be an open set with $kR\subset G\subset \bar{G}\subset \bigcup_{W\in\mathfrak{W}} W$. Then $R-G$ is closed and hence paracompact. Therefore $\dim (R-G)<n$ by [11]. Then by [7, Theorem 2.2] $\mathfrak{U}$ can be refined by an open covering of R whose order is at most n, which proves $\dim R<n$. This is a contradiction.

C) *$kkR=kR$ i. e. kR is cov-homogeneous.*

PROOF. If not, there would exist a point $p\in kR-kkR$. Then there would exist a closed neighborhood F (in R) of p with $F\cap kkR=\phi$ and $\dim (F\cap kR)$. Since F is closed in R by A) and hence paracompact, the analogous arguments used above can be applied and we can conclude $\dim F<n$, which contradicts $p\in kR$.

D) *$R=kR$ implies $\beta R=k\beta R$, where βR is the Čech-Stone-compactification of R.*

PROOF. It is to be noted that $\dim R=\dim \beta R$ [6, Theorem 8]. If $\beta R\neq k\beta R$, there would exist a point $p\in\beta R-k\beta R$ and its closed neighborhood F (in βR) with $F\cap k\beta R=\phi$ and $\dim F<n$. Then $F\cap R$ is a non-empty closed neighborhood (in R) of some point $q\in R$. Since $\overline{F\cap R}$ is contained in F and is essentially the same with $\beta(F\cap R)$,

$\dim(F \cap R) = \dim \overline{F \cap R} \leqq \dim F < n$. Hence $q \in R - kR$, which is a contradiction.

E) *βkR is essentially the same as $k\beta R$.*

PROOF. By the analogous arguments used above it holds that $\overline{kR} \supset k\beta R \supset k\beta kR$. On the other hand $\beta kR = k\beta kR$ by C) and D). Since $\overline{kR}$ is essentially the same as βkR, βkR is essentially the same as $k\beta R$.

It is to be noted that when R is hereditarily paracompact, $\dim(R - kR) < n$, but when R is not so, this is not the case as the following example shows.

Let A be an ordered space consists of all ordinals $\leqq \omega$ (=the first ordinal of the second class). Let B be an ordered space consists of all ordinals $\leqq \Omega$ (=the first ordinal of the third class). Let C be a space consists of all limit ordinals $< \Omega$. Let D be a space $B \cup \{\alpha + 2 + x; \alpha \in C, 0 \leqq x \leqq 1\}$ with the usual order topology. Then $R = A \times D$ is a paracompact Hausdorff space and not hereditarily paracompact, with $\dim R = 1$. On the other hand $R - kR$ is homeomorphic to $A \times B - (\omega, \Omega)$ and $\dim(A \times B - (\omega, \Omega)) > 0$ [2, p. 155].

When "dim", "paracompact Hausdorff" and "kR" in the preceding propositions are replaced respectively with "Ind", "hereditarily paracompact Hausdorff" and "KR", all propositions thus obtained also hold.

Addendum. After the manuscript of this paper was completed, I have learnt that there appeared two papers on the same subject as ours: C. H. Dowker: Local dimension of normal spaces, Quart. J. math., 6 (1955), 101–120 and K. Morita: On closed mappings and dimension, Proc. Japan Acad., 32 (1956), 161–165. In paticular, the paper of Prof. Morita covers the content of § 2 of our present paper (with different proofs). Prof. Morita was kind enough to inform me that he also obtained independently these results.

Ehime University.

Bibliography

[1] C. H. Dowker: Inductive dimension of completely normal spaces, Quart. J. Math., Oxford Ser. (2), 4 (1953), 267–281.

[2] W. Hurewicz-H. Wallman: Dimension theory, 1941.

[3] M. Katětov: On the dimension of non-separable space I, Čehoslovack Mat. Ž., 2 (77) (1953), 333–368.

[4] C. Kuratowski: Topologie I, 1934.

[5] D. Montgomery: Non-separable metric spaces, Fund. Math., 25 (1935), 527-533.
[6] K. Morita: On uniform spaces and the dimension of compact spaces, Proc. Phys.-Math. Soc. Japan, 22 (1940), 969-977.
[7] ———: On the dimension of normal spaces I, Japanese J. Math., 20 (1950), 5-36.
[8] ———: On the dimension of normal spaces II, J. Math. Soc. Japan, 2 (1950), 16-33.
[9] ———: Normal families and dimension theory for metric spaces, Math. Ann., 128 (1954), 350-362.
[10] K. Nagami: Paracompactness and strong screenability, Nagoya Math. J., 8 (1955), 83-88.
[11] ———: On the dimension of paracompact Hausdorff spaces, Nagoya Math. J., 8 (1955), 69-70.
[12] N. Vedenisoff: Généralisation de quelques théorèmes sur la dimension, Comp. Math., 7 (1939), 194-200.
[13] A. H. Stone: Paracompactness and product spaces, Bull. Amer. Math. Soc., 54 (1948), 977-982.
[14] Yu. M. Smirnov: Some relations in the theory of dimensions, Mat. Sbornik N. S., 29 (71) (1951), 157-172.