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# Locally convex lattices.

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Since normed vector lattice was considered first by L. Kantorovitch [1] (the numbers in brackets refer to the list of References at the end of this paper), the theory of Banach lattices has undergone a considerable development, but very little effort seems to have been made to extend this theory to more general topological lattices (see Nakano [3]. On a similar situation in the theory of topological algebra, see Michael [1]). On the other hand, the theory of Banach spaces was considerably extended in its range of application by introducing the notion of locally convex spaces (Mackey [1], [2]; Dieudonné et Schwartz [1]; Bourbaki [1], [2], [3]).

The purpose of this paper is to generalize the theory of normed vector lattices in an analogous fashion, by introducing the concept of locally convex lattices.

A locally convex lattice (for Definition, see § 1) is a locally convex Hausdorff space over the reals as well as a vector lattice such that whenever a net (a directed system) $\{x_\lambda\}$ converges to 0 and $|x_\lambda| \geqq |y_\lambda|$ for each $\lambda$, then $\{y_\lambda\}$ converges to 0. If a locally convex lattice is an $\mathfrak{L}\mathfrak{F}$-space, it will be called an $\mathfrak{L}\mathfrak{F}$-lattice. An $\mathfrak{F}$-lattice is a metrizable and topologically complete locally convex lattice.

This paper is divided into six sections, the first of which is concerned with the definition and fundamental properties of locally convex lattices. Every locally convex lattice $E$ is generated, in a certain sense, by normed vector lattices $\{E_\alpha\}$ (Theorem 1.3, 1.4) and then most of problems about locally convex lattices can be reduced to similar ones about normed vector lattices. Thus, in § 3, we shall study the relations between $E$ and $\{E_\alpha\}$.

§ 2 is concerned with the conjugate spaces and duals of locally convex lattices.

The completion $\hat{E}$ (as a locally convex space) of any locally convex lattice $E$ is also a locally convex lattice. § 4 is devoted to the study of the relation between $E$ and $\hat{E}$.

In § 5, we give miscellaneous theorems about locally convex lattices. Most of propositions in § 2, § 5 are generalization of known results in the case where $E$ is normed.

§ 6 is devoted to discussions about $\mathfrak{LF}$-lattices. We give various results about $\mathfrak{LF}$-lattices closely related to the general theory of locally convex lattices here and there in § 1—§ 5, but in this section, it is intended to establish the propositions whose range of application seems not to go beyond $\mathfrak{LF}$- (and $\mathfrak{F}$-) lattices. We shall see first that an $\mathfrak{LF}$-lattice $E$ can be defined by an ascending sequence $\{E_n\}$ of $\mathfrak{F}$-spaces which are ideals of $E$ (Theorem 6.1). This fact is the corner stone of our whole theory of $\mathfrak{LF}$-lattices and it will enable us to reduce most of problems about $\mathfrak{LF}$-lattices to similar ones about $\mathfrak{F}$-lattices. Next, we shall give an example of the $\mathfrak{LF}$-lattice which is a lattice of continuous functions and about which we know much. We shall consider in detail this function lattice and conclude this section with its characterization.

Let $E$ be a locally convex lattice with the topology $T$ (in cases where precision is necessary, we write $E(T)$ instead of $E$), then the semi-norm associated with the convex, circled neighborhood $V$ of the origin is the function $p(x) = \inf\{|\lambda| : (1/\lambda)x \text{ in } V\}$ and the topology $T$ can be described by specifying its semi-norms.

Topological terms used in this paper such as bounded, closed, continuous, complete, convergent with respect to the topology $T$ are qualified as $T$-bounded, $T$-closed and so on. If a net $\{a_\lambda\}$ converges to $a$ in the topology $T$, we write $a_\lambda \to a(T)$.

Topological terms in the weak (weak-star) topology on $E$ (on the dual $E'$ of $E$) are qualified as weakly (weak-star) bounded, weakly (weak-star) convergent and so on.

Unqualified terms such as complete, $\sigma$-complete and terms such as $o$-convergent, $o$-continuous refer to the order topology in $E$ (in $E'$). If a net $\{a_\lambda\}$ converges to $a$ in the order-topology, we write $a_\lambda \to a(o)$.

The author will conclude this introduction by offering his thanks to Prof. T. Ogasawara and Dr. I. Amemiya for many valuable suggestions.

## § 1. Definition and fundamental properties of locally convex lattices.

DEFINITION. A vector lattice $E$ is called *locally convex* if it is a locally convex Hausdorff space over the reals (with the topology $T$). and it satisfies the following condition ($A$): Whenever a net (a directed system, Kelley [1] p. 277) $x_\lambda \to 0$ ($T$) and $|x_\lambda| \geqq |y_\lambda|$ for each $\lambda$, then $y_\lambda \to 0$ ($T$).

If a locally convex lattice is an $\mathfrak{L}\mathfrak{F}$-space, it will be called an $\mathfrak{L}\mathfrak{F}$-lattice.

The next assertions are immediate from the definition.

(1) $E$ is archimedean.

(2) $x+y$, $x \cup y$, $x \cap y$ are $T$-continuous functions of $x$ and $y$.

(3) For nets $\{x_\lambda\}$, $\{y_\lambda\}$ such that $x_\lambda \to x$ ($T$), $y_\lambda \to y$ ($T$) and $x_\lambda \geqq y_\lambda$ for each $\lambda$, we have $x \geqq y$.

The significance of condition ($A$) can be illustrated by the corresponding properties of semi-norms as follows:

THEOREM 1.1. *Let $E$ be a locally convex lattice with the topology $T$. Then there exists a family $\{p_\alpha\}$ of semi-norms on $E$ describing the topology $T$ and satisfying the condition: $|y| \leqq |x|$ implies $p_\alpha(y) \leqq p_\alpha(x)$ for each $\alpha$.* In the sequel, such a family $\{p_\alpha\}$ of semi-norms as given in our present theorem will be said briefly to satisfy the condition ($A$).

PROOF. Let $\{p\}$ be a family of semi-norms describing the topology $T$ of $E$. Put $\bar{p}(x) = \sup_{0 \leqq y \leqq |x|} p(y)$ for every $x \in E$, then $\bar{p}$ is a semi-norm on $E$ such that $2\bar{p}(x) \geqq p(x)$ for each $x \in E$ and $|x| \geqq |y|$ implies $\bar{p}(x) \geqq \bar{p}(y)$. Now, we shall show: $\bar{p}$ is $T$-continuous.

Suppose $x_\lambda \to 0$ ($T$), but $\bar{p}(x_\lambda) \geqq \eta$ for some real $\eta > 0$. Take $y_\lambda$ with $|x_\lambda| \geqq y_\lambda \geqq 0$, $p(y_\lambda) \geqq \eta/2$. By the condition ($A$), we have $y_\lambda \to 0$ ($T$), which implies $p(y_\lambda) \to 0$, but this is impossible.

Now $\{\bar{p}\}$ have the required properties.

COROLLARY 1.1.1. *Let $E$ be a bornographic* (bornologique in French, Bourbaki [3] p. 13) *space over the reals as well as a vector lattice and suppose whenever a sequence $x_n \to 0$ ($T$) and $|x_n| \geqq |y_n|$, then $y_n \to 0$ ($T$). Then $E$ is a locally convex lattice* (which will be called a *bornographic lattice*).

PROOF. Let $\bar{p}$ have the meaning indicated in the proof of Theorem 1.1. We can see easily $\bar{p}$ is bounded on each $T$-bounded set and thus $\bar{p}$ is $T$-continuous.

The family $\{p\}$ of all $T$-continuous semi-norms satisfying the condition ($A$) is directed by agreeing that $p \geqq q$ if and only if $p(x) \geqq q(x)$ for each $x \in E$ and whence $\Omega = \{p\}$ becomes a net. Throughout this paper, $\Omega$ will be used always in this sense.

A linear subspace $N$ of $E$ will be called an *ideal* of $E$ if $N$ contains with $y$ all $x$ satisfying $|x| \leqq |y|$ (normal subspace in the terminology of Birkhoff [1]). Given a subset $M$ of $E$, the totality of all $x \in E$ orthogonal to $M$ is denoted by $M^{\perp}$. A subset $M$ of $E$ such that $M = M^{\perp\perp}$ will be called a *normal subspace* and in addition, if $E = M + M^{\perp}$ (the direct sum decomposition), $M$ will be called a *complemented normal subspace. Interval* means closed interval (Birkhoff [1] p. 1) throughout this paper.

A locally convex spaces $E$ is said to be *boundedly closed* if every $T$-bounded linear functional on $E$ is $T$-continuous (Donoghue and Smith [1] p. 325). Every bornographic space is boundedly closed.

It is known to Grothendieck (Bourbaki [1] p. 11) that the closed subspace of the bornographic space may fail to be bornographic. Concerning an ideal of the bornographic lattice, we have:

THEOREM 1.2. *An ideal $H$ of any boundedly closed locally convex lattice $E$ is boundedly closed in the relative topology.*

*An ideal $H$ of any bornographic lattice $E$ is bornographic in the relative topology.*

PROOF. We shall see the first part. Let $f$ be a linear functional defined on $H$ which is bounded on every $T$-bounded set of $H$, then $f \in \tilde{H}$ (the totality of all $o$-bounded linear functionals on $H$; see Birkhoff [1] p. 244) and we shall assume $f \geqq 0$. Define $\bar{f}$ on $E$ by $\bar{f}(a) = \sup\{f(x) : a \geqq x \geqq 0, x \in H\}$ for every $0 \leqq a \in E$. Then we can see if $a, b \geqq 0$ and real $\lambda \geqq 0$, then $\bar{f}(a+b) = \bar{f}(a) + \bar{f}(b)$, $\bar{f}(\lambda a) = \lambda \bar{f}(a)$. Now putting $\bar{f}(a) = \bar{f}(a^{+}) - \bar{f}(a^{-})$ for any $a \in E$, then $\bar{f}$ becomes a linear functional on $E$.

Let $B$ be any $T$-bounded set of $E$. Without loss of generality, we may assume whenever $|x| \geqq |y|$, $x \in B$, then $y \in B$. Put $C = B \cap H$, then $C$ is a $T$-bounded set of $H$ and since $f$ is bounded on $C$, $\bar{f}$ is bounded on $B$ clearly. By hypothesis, we have $\bar{f} \in E'$ (the dual of $E$) which implies $f \in H'$ (the dual of $H$).

To prove the second part, let us remark that a locally convex space is bornographic if and only if every semi-norm which is bounded on every bounded set is continuous (Bourbaki [3] p. 13).

Let $p$ be a semi-norm defined on $H$ which is bounded on every $T$-bounded set of $H$. Define $\bar{p}$ on $E$ by $\bar{p}(a)=\sup\{p(x): |a|\geqq x\geqq 0, x\in H\}$, then $\bar{p}$ is a semi-norm on $E$ and $2\bar{p}(x)\geqq p(x)$ for each $x\in H$. Since $\bar{p}$ is bounded on every $T$-bounded set of $E$, $\bar{p}$ is $T$-continuous and then $p$ is $T$-continuous.

REMARK. A locally convex space $E$ is called "infratonnelé" if it satisfies the following condition: if $N$ is a closed, convex, circled set such that, for any $T$-bounded set $B$ of $E$, there exists a real $\lambda>0$ with $\mu B\subset N$ $(|\mu|\leqq\lambda)$, then $N$ is a neighborhood of 0 (Bourbaki [3] p. 13). Following the same line as the proof of Theorem 1.2, we can see an ideal of the "infratonnelé" lattice is "infratonnelé" too.

The topology $T$ of $E$ is said to be *o-continuous* if $a_n\downarrow 0$ (i. e., $a_1\geqq a_2\geqq a_3\geqq\cdots$ and $\cap a_n=0$) implies $a_n\to 0$ $(T)$ and *o-continuous in the sense of Moore-Smith* (Abbreviation: M. S. *o-continuous*) if a net $a_\lambda\downarrow 0$ (i. e., $\lambda\geqq\mu$ implies $a_\lambda\leqq a_\mu$ and $\cap a_\lambda=0$) implies $a_\lambda\to 0$ $(T)$. The topology $T$ of $E$ is said to be (M. S.) *semi-o-continuous* if $0\leqq a_n\uparrow a$ $(0\leqq a_\lambda\uparrow a)$ implies $p(a_n)\uparrow p(a)$ $(p(a_\lambda)\uparrow p(a))$ for every $p$ contained in some family $\{p\}$ of semi-norms describing $T$ and satisfying the condition $(A)$.

In the sequel, if we are considering (M. S.) semi-$o$-continuity, a family of semi-norms describing $T$ shall be chosen to be such a family $\{p\}$ as above.

When no confusion seems possible, we shall say briefly $E$ to be (M. S.) (semi-) $o$-continuous.

The vector lattice of all $o$-bounded linear functionals on $E$ is called the *conjugate space* $\tilde{E}$ of $E$. (This terminology is due to Birkhoff [1] p. 246 and different from Nakano [1] p. 63.)

$E^*$ denotes all M. S. $o$-continuous linear functionals $\{f\}$ (i. e., $a_\lambda\to 0\,(o)$ implies $f(a_\lambda)\to 0$) on $E$ (Nakano [1] p. 81). The totality of all $o$-continuous linear functionals on $E$ will be denoted by $\dot{E}$.

$\tilde{E}$ is a complete vector lattice (Nakano [1] p. 64) and we have $\tilde{E}\supset\dot{E}\supset E^*$.

We would now give some examples of locally convex lattices and methods for constructing new ones out of old.

(1): Every normed vector lattice is a locally convex lattice.

(2): A vector sublattice of a locally convex lattice is a locally convex lattice in the relative topology.

(3): The cartesian product of locally convex lattices is a locally convex lattice in the product topology.

(4): Let $E$ be a locally convex lattice and if $N$ is a $T$-closed ideal, then $E/N$ is a locally convex lattice in the quotient topology.

(5): Let $E$ be a locally convex lattice, then the $T$-completion $\hat{E}$ of $E$ is a locally convex lattice.

This assertion follows from Theorem 1.4 which will be given later on.

(6): Let $E$ be a vector lattice and suppose whenever $f(x)=0$ for each $f\in E^*$, then $x=0$. Define the semi-norm $p_f$ by $p_f(x)=f(|x|)$ for each $0<f\in E^*$, then the collection of all such $p_f$ defines the locally convex lattice topology on $E$ and $E$ becomes M. S. $o$-continuous.

It is known to Amemiya and Mori [1] that all M. S. $o$-continuous locally convex lattice topologies on the vector lattice $E$ are equivalent on any interval of $E$ (see Remark after Theorem 5.3). Now, if we are considering the M. S. $o$-continuous lattice topology, the properties of the interval of $E$ owe exclusively to $E$ considered as the vector lattice merely. Thus, the important significance of (6) can be seen.

(7): If $E$ be a lattice of continuous real-valued functions on a topological space $S$ in the topology of uniform convergence on some collection of compact subsets $\{K_\alpha\}$ of $S$ whose union is $S$, then $E$ is locally convex.

We now proceed to show that every locally convex lattice is generated in a certain sense, by normed vector lattices. This fact will enable us to reduce most of problems about locally convex lattices to similar ones about normed vector lattices.

Let $\Omega=\{p_\alpha\}$, then, for each $\alpha$, the null-space $N_\alpha=p_\alpha^{-1}(0)$ of $p_\alpha$ is a $T$-closed ideal of $E$. If we define $\dot{p}_\alpha$ on $E/N_\alpha$ by $\dot{p}_\alpha(x+N_\alpha)=p_\alpha(x)$, then $\dot{p}_\alpha$ is a norm on $E/N_\alpha$, which makes $E/N_\alpha$ into a normed vector lattice. Denote by $E_\alpha$, the normed vector lattice which we get by equipping $E/N_\alpha$ with the norm $\dot{p}_\alpha$ above and let $\hat{E}_\alpha$ be the $\dot{p}_\alpha$-completion of $E_\alpha$. Denoting the natural homomorphism $E$ onto $E_\alpha$ by $\pi_\alpha$, $\pi_\alpha$ is evidently continuous. Let $[x]_\alpha$ denote $\pi_\alpha(x)$.

THEOREM 1.3. *A topological vector lattice $E$ is locally convex if and only if it is isomorphic to a vector sublattice of a cartesian product of normed vector lattices* (*Banach lattices*).

In this paper, "Isomorphism" and "isomorphic" imply that topological structure is also preserved.

PROOF. If $E$ be locally convex, then $E$ is naturally isomorphic to a vector sublattice of the cartesian product of $\{E_\alpha\}$ ($\{\hat{E}_\alpha\}$) (see Bourbaki [2] p. 99). Q. E. D.

Next, if $p_\alpha \geqq p_\beta$ and if $[x]_\alpha = [y]_\alpha$, then clearly $[x]_\beta = [y]_\beta$; we may therefore define $\pi_{\beta\alpha}$: $E_\alpha \rightarrow E_\beta$ by $\pi_{\beta\alpha}([x]_\alpha) = [x]_\beta$ and $\pi_{\beta\alpha}$ is evidently a continuous homomorphism onto $E_\beta$. Consequently, $\pi_{\beta\alpha}$ can be uniquely extended to a continuous homomorphism $\hat{\pi}_{\beta\alpha}$ from $\hat{E}_\alpha$ into $\hat{E}_\beta$. Consider now the cartesian product $\prod\otimes \hat{E}_\alpha$ and let $\hat{\pi}_\alpha$ be the projection from $\prod\otimes \hat{E}_\alpha$ onto $\hat{E}_\alpha$. Following to Weil, consider the set $\mathrm{LP}(E) = \{x \in \prod\otimes \hat{E}_\alpha : \hat{\pi}_{\beta\alpha}(\hat{\pi}_\alpha(x)) = \hat{\pi}_\beta(x)$ whenever $p_\alpha \geqq p_\beta\}$, then $\mathrm{LP}(E)$ is a vector sublattice of $\prod\otimes \hat{E}_\alpha$. (We shall call this set the *projective limit* of $\{\hat{E}_\alpha\}$ with respect to $\{\hat{\pi}_{\beta\alpha}\}$). Now we have

THEOREM 1.4. *A locally convex lattice $E$ is isomorphic to a vector sublattice of the projective limit* LP($E$) *of the Banach lattices and $E$ is dense in* LP($E$). *Further,* LP($E$) *is the T-completion of $E$ and if $E$ be T-complete, then $E$ coincides with* LP($E$).

(The similar assertions are found in Takenouchi [1] p. 62, 65 for locally convex spaces and in Michael [1] p. 17 for algebras.)

$E$ is said to satisfy the *o-countability condition* if for any system $a_\lambda \geqq 0$ ($\lambda \in \Lambda$) such that $\cap a_\lambda$ exists, there exist countable $a_{\lambda(n)}$ ($\lambda(n) \in \Lambda$, $n = 1, 2, 3, \cdots$) for which $\cap a_\lambda = \cap a_{\lambda(n)}$. In addition, if $E$ be complete, $E$ is said to be *supercomplete* (superuniversally continuous in Nakano [1] p. 41).

We assume $E$ be M. S. semi-$o$-continuous throughout the rest of this section and denote by $\bar{E}$ the cut extension of $E$ (Nakano [2] p. 140).

Then there exists a mapping of $E$ into $\bar{E}$ which assigns to every element $a \in E$ an element $\bar{a} \in \bar{E}$ such that

(1): $\overline{\lambda a + \mu b} = \lambda \bar{a} + \mu \bar{b}$ for every $a, b \in E$ and real $\lambda, \mu$.

(2): $\bar{a} \geqq 0$ if and only if $a \geqq 0$.

(3): $\bar{E}$ is complete.

(4): $\cap a_\lambda = 0$ ($a_\lambda \in E$) implies $\cap \bar{a}_\lambda = 0$ in $\bar{E}$.

(5): To every element $x \in \bar{E}$, there exists a system $\{a_\lambda\}$ of elements of $E$ such that $x = \cap \bar{a}_\lambda$.

An archimedean vector lattice has a cut extension uniquely determined up to an isomorphism. Clearly, $a=\bigcup a_\lambda$ or $a=\bigcap a_\lambda$ in $E$ implies $\bar{a}=\bigcup \bar{a}_\lambda$ or $\bar{a}=\bigcap \bar{a}_\lambda$ respectively and to every element $x\in\bar{E}$, there exists two systems $\{a_\lambda\}$, $\{b_\mu\}$ of elements of $E$ such that $x=\bigcup \bar{a}_\lambda = \bigcap \bar{b}_\mu$.

To every $x\in\bar{E}$, take $\bar{a}_\lambda \uparrow |x|$ $(0\leqq a_\lambda\in E)$ and put $\bar{p}_\alpha(x)=\sup_\lambda p_\alpha(a_\lambda)$. Then $\bar{p}_\alpha(x)$ can be defined uniquely and we can verify $\{\bar{p}_\alpha\}$ becomes a family of semi-norms on $\bar{E}$ and $\bar{p}_\alpha(\bar{a})=p_\alpha(a)$ for each $a\in E$. Further, if $|x|\geqq|y|$ in $\bar{E}$, then $\bar{p}_\alpha(x)\geqq\bar{p}_\alpha(y)$ and if $0\leqq x_\lambda\uparrow x$ in $\bar{E}$, then $\bar{p}_\alpha(x_\lambda)\uparrow\bar{p}_\alpha(x)$.

Now we have the following theorem:

THEOREM 1.5. *Let $E$ be a locally convex lattice with the* M. S. *semi-o-continuous topology. Then the cut extension $\bar{E}$ of $E$ is a complete locally convex lattice with the* M. S. *semi-o-continuous topology which induces on $E$ the original topology of $E$. Further, $(\bar{E})_\alpha$ is isomorphic to a cut extension of $E_\alpha$ for each $\alpha$.*

This theorem will enable us to reduce some problems about M. S. semi-$o$-continuous locally convex lattices to similar ones about complete, M. S. semi-$o$-continuous locally convex lattices. Moreover, if we consider the $T$-completion $\hat{\bar{E}}$ of $\bar{E}$, $\hat{\bar{E}}$ is $T$-complete, complete, M. S. semi-$o$-continuous and $\bar{E}$ is an ideal of $\hat{\bar{E}}$ by Theorem 4.2 which will be given later on.

To put to use later on, we shall go to work on $\bar{E}$ moreover.

If $E$ be M. S. $o$-continuous, then $E$ is M. S. $o$-continuous too.

If $E$ satisfies the $o$-countability condition, then $\bar{E}$ is supercomplete and in addition, if a sequence $a_n\in E\rightarrow 0\,(o)$ in $\bar{E}$, then $a_n\rightarrow 0\,(o)$ in $E$.

Let $A(\bar{A})$ be a normal subspace of $E\,(\bar{E})$, then we have $A^{\perp\perp}\cap E=A$ ($\perp$-operation shall be considered in $\bar{E}$), $(A^\perp\cap E)^\perp=A^{\perp\perp}$ and $\bar{A}\cap E$ is a normal subspace of $E$. Now, we may set up a one to one correspondence between the totality of all normal subspaces of $E$ and those of $\bar{E}$.

Let $0\leqq f\in E^*$ and if we put $\bar{f}(x)=\sup\{f(a): 0\leqq a\leqq x, a\in E\}$ for each $0\leqq x\in\bar{E}$ and $\bar{f}(x)=\bar{f}(x^+)-\bar{f}(x^-)$ for each $x\in E$, then $\bar{f}\in(\bar{E})^*$ and $\bar{f}$ coincides with $f$ on $E$. Now, if we put $\bar{f}=\bar{f}^+-\bar{f}^-$ for each $f\in E^*$, we can identify $E^*$ with $(\bar{E})^*$ as a vector lattice by the correspondence $f\leftrightarrows\bar{f}$.

Assume $E$ be M. S. $o$-continuous throughout the rest of this section. Then, $E'((\bar{E})')$ is contained in $E^*((\bar{E})^*)$ and since $\bar{f}\in(\bar{E})'$ for each $f\in E'$, we can identify $E'$ with $(\bar{E})'$ as a vector lattice by the correspondence $f\leftrightarrows\bar{f}$.

Let $A(\mathfrak{A})$ be a normal subspace of $E$ $(E')$ and put $\bar{A}=A^{\perp\perp}$ ($\perp$-operation shall be considered in $\bar{E}$), $\bar{\mathfrak{A}}=\{\bar{f}: f\in\mathfrak{A}\}$, then we have seen $\bar{A}\cap E=A$, $(\bar{A}\cap E)^{\perp}=(\bar{A})^{\perp}$. Further, denoting by $A^\circ$, $(\bar{A})^\circ$, $\mathfrak{A}^\circ$, $(\bar{\mathfrak{A}})^\circ$, the polar set of $A, \bar{A}, \mathfrak{A}, \bar{\mathfrak{A}}$ into $E'$ $(\bar{E})'$, $E, \bar{E}$ respectively (Dieudonné et Schwartz [1] p. 63; Bourbaki [3] p. 51), then we have $(\bar{A})^\circ=\bar{A}^\circ$, $(\bar{\mathfrak{A}})^\circ=\bar{\mathfrak{A}}^\circ$.

## § 2. The conjugate spaces and duals.

In this section, we shall consider the conjugate space $\tilde{E}$ and the dual $E'$ of a locally convex lattice $E$ with the topology $T$.

Let $E^*$, $\dot{E}$ have the significances indicated in § 1, then we have $\tilde{E}\supset\dot{E}\supset E^*$ and $\tilde{E}\supset E'$.

$\tilde{E}$ is a complete vector lattice and for any $f, g, f_\alpha\in\tilde{E}$, we have

$$f^+(x)=\sup\{f(y): 0\leqq y\leqq x\} \quad \text{for any} \quad x\geqq 0$$
$$|f|(x)=\sup\{f(y): |y|\leqq x\} \quad \text{for any} \quad x\geqq 0$$
$$f\cup g(x)=\sup\{f(y)+g(z): x=y+z \text{ and } y, z\geqq 0\} \text{ for any } x\geqq 0$$
$$f\cap g(x)=\inf\{f(y)+g(z): x=y+z \text{ and } y, z\geqq 0\} \text{ for any } x\geqq 0,$$

and if $0\leqq f_\alpha\leqq g$ $(\alpha\in\Gamma)$,

$$\bigcup_\alpha f_\alpha(x)=\sup\left\{\sum_{i=1}^n f_{\alpha(i)}(x_i): x=\sum_{i=1}^n x_i, x_i\geqq 0, \alpha(i)\in\Gamma\right\}$$

for any $x\geqq 0$.

It is well known that $E^*$ is a complemented normal subspace of $\tilde{E}$ (Nakano [1] p. 81).

THEOREM 2.1. *Let $E$ be a locally convex lattice. Then*

(1) *$E'$ is an ideal of $\tilde{E}$.*

(2) *If $E$ be sequentially $T$-complete and boundedly closed, we have $E'=\tilde{E}$.*

PROOF. To prove (1), suppose $|f|\geqq|g|$ where $f\in E'$, $g\in\tilde{E}$ and let a net $x_\lambda\to 0$ $(T)$. Then we can find $\{y_\lambda\}$ such that $|y_\lambda|\leqq|x_\lambda|$ and $|f|(|x_\lambda|)\leqq f(y_\lambda)+\varepsilon$ for given $\varepsilon>0$. Since $y_\lambda\to 0$ $(T)$ and $f\in E'$, we have $|f|(|x_\lambda|)\to 0$. Now, by $|f|(|x_\lambda|)\geqq g^\pm(x_\lambda^\pm)\geqq 0$, we have $g^\pm(x_\lambda^\pm)\to 0$ which implies $g^+, g^-\in E'$.

To see (2), if the contrary were true, we could find $f \in \tilde{E}$ such that $f \notin E'$. Then we could find a $T$-bounded set $B$ and a sequence $\{x_n\}$ of elements of $B$ such that $n^3 \leqq |f(x_n)|$. Now, $\sum_{n=1}^{\infty}(1/n^2)|x_n|$ is $T$-convergent to some $a \in E$ and we can see easily $|f|(a)=\infty$ which is absurd. Q. E. D.

Now, since $E'$ is an ideal of a complete vector lattice $\tilde{E}$, $E'$ is itself complete. For any $T$-bounded set $B$ of $E$, put $I(B)=\{x: |x| \leqq |a|, a \in B\}$. Then $I(B)$ is $T$-bounded and the strong topology $T'$ of $E'$ (Bourbaki [3] p. 85) can be described by polars of all $I(B)$. Define $\pi_x$ by $\pi_x(f)=|f|(x)$ for each $0<x \in E$ and denote by $T'_0$ the topology on $E'$ defined by all $\{\pi_x: 0<x \in E\}$ throughout the rest of this section.

Now we can obtain easily the following theorem:

THEOREM 2.2. *$E'$ with the strong topology $T'$ (the topology $T'_0$) is a complete locally convex lattice and* M. S. *semi-o-continuous* (M. S. *o-continuous*).

A locally convex lattice with the topology $T$ is called to be (M. S.) *monotone complete* if it satisfies the following condition: whenever a sequence $0 \leqq a_n \uparrow$ (a net $0 \leqq a_\lambda \uparrow$) is $T$-bounded, then $\cup a_n$ ($\cup a_\lambda$) exists (Nakano [1] p. 129).

Now we can see if $E$ be a boundedly closed (tonnelé) locally convex lattice, then $E'$ with $T'$ is (both $E'$ with $T'$ and $E'$ with $T'_0$ are) M. S. monotone complete and it follows in view of Nakano [3] Theorem 4.2 (p. 96) that if $E$ be tonnelé, then $E'$ is $T'_0$- and $T'$-complete. (Compare with Bourbaki [3] Proposition 3 of p. 87).

Next, we shall be concerned with the polar sets in $E$, $E'$.

THEOREM 2.3. *Let $E$ be a locally convex lattice. Then:*

(1): *For each ideal $A$ of $E$, the polar set $A^\circ$ of $A$ is a normal subspace of $E'$ and every $f \in E'$ can be represented as a sum $f=g+h$ where $g \in A^\circ$, $h \in (A^\perp)^\circ$ (not necessarily unique).*

*Furthermore, we have $A^{\perp\perp} \supset A^{\circ\circ} \supset A$.*

(2): *Suppose $E$ be* M. S. *o-continuous. Then for any ideal $\mathfrak{A}$ of $E'$, the polar set $\mathfrak{A}^\circ$ of $\mathfrak{A}$ is a normal subspace of $E$.*

*For every ideal $A$, $\mathfrak{A}$ of $E$, $E'$, we have:*

$$A^{\circ\circ}=A^{\perp\perp},\ \mathfrak{A}^{\circ\circ}=\mathfrak{A}^{\perp\perp},\ (A^\circ)^\perp=(A^\perp)^\circ,\ (\mathfrak{A}^\circ)^\perp=(\mathfrak{A}^\perp)^\circ.$$

*Now we may set up a one to one correspondence between the totality of all normal subspaces of $E$ and those of $E'$ by $A \to (A^\circ)^\perp$, $\mathfrak{A} \to (\mathfrak{A}^\circ)^\perp$.*

*Further, we have a direct sum decomposition:* $E'=A^\circ+(A^\perp)^\circ$ *for any ideal* $A$ *of* $E$.

(3): *Suppose* $E$ *be complete and let* $A$ *be a normal subspace of* $E$. *Then we have a direct sum decomposition:* $E'=A^\circ+(A^\perp)^\circ$ *and* $(A^\circ)^\perp=(A^\perp)^\circ$.

To prove our theorem, we need the following Lemma which will be used frequently later on.

LEMMA. 2.3.1. *Let* $E$ *be an archimedean vector lattice and* $A$ *be an ideal of* $E$. *Then, for each positive element* $x\in E$, *we have:*

$$x=\bigcup\{y+z:\ 0\leqq y\leqq x,\ y\in A,\ 0\leqq z\leqq x,\ z\in A^\perp\}.$$

*Similarly, let* $A$ *be a subset of* $E$ *such that if* $x\in A$, *then* $nx\in A$ $(n=1,2,3,\cdots)$ *and if* $x\in A$, $|y|\leqq|x|$, *then* $y\in A$. *Then, for every* $0\leqq x\in A^{\perp\perp}$, *we have* $x=\bigcup\{y:\ 0\leqq y\leqq x,\ y\in A\}$.

PROOF. We can see easily that $S=\{y+z:\ y\in A,\ z\in A^\perp\}$ is an ideal of $E$. Suppose $0<x\notin S$. Putting $M=\{u:\ 0\leqq u\leqq x,\ u\in S\}$, we shall prove $x=\bigcup\{u:\ u\in M\}$. If this were untrue, we could find $y$ such that $x>y\geqq u$ for every $u\in M$. Since $S^\perp=\{0\}$, there exists an element $a\in S$ such that $b=(x-y)\cap a>0$ and we have $b\in S$, $x\geqq y+b$. For any $u\in M$, it is clear that $b+u\in M$, from which $nb+u\leqq x$ for every integer $n$. Since $E$ is archimedean, we have $b=0$, which is absurd.

PROOF OF THEOREM 2.3. First, we shall see (1).

$A^\circ$ coincides with the set $\{f\in E':|f|(a)=0$ for each $0\leqq a\in A\}$ and then we can see easily $A^\circ$ is a normal subspace of $E'$. Thus we have a direct sum decomposition: $E'=A^\circ+(A^\circ)^\perp$. To see $(A^\perp)^\circ\supset(A^\circ)^\perp$, suppose $0<f\in(A^\circ)^\perp$ and $f(a)>0$ for some $0<a\in A^\perp$. Define $g$ on $S=\{x+y:\ x\in A, y\in A^\perp\}$ by $g(z)=f(y)$ if $z=x+y$ $(x\in A, y\in A^\perp)$, then $g\in S'$. Extending $g$ to $h\in E'$ (Hahn-Banach extension theorem), we have $h\in A^\circ$ and $h$ coincides with $f$ on $A^\perp$. Now, $f\cap|h|(a)=f(a)>0$ which contradicts to $f\in(A^\circ)^\perp$.

Next we shall see $A^{\circ\circ}=A$ in the case where $A$ is a normal subspace of $E$. By Bourbaki [3] Prop. 4 (p. 53), it is enough to see $A$ is weakly closed. If the contrary were true, we could take $a\in A^\perp$ which is the cluster point of $A$ in the weak topology. Indeed, take $0<x\in A^{\circ\circ}-A$ and then take $a$ with $0<a\leqq x, a\in A^\perp$. Take $f\in E'$ with $f(a)\neq0$ and in the same way as above, take $h\in E'$ such that $h$ coincides with $f$ on $A^\perp$ and $h=0$ on $A$. Since $a$ is a cluster point of $A$ in the weak topology, we have $h(a)=0$ which is absurd

To see (2), let us remark $A^\circ=(A^{\perp\perp})^\circ$, $\mathfrak{A}^\circ=(\mathfrak{A}^{\perp\perp})^\circ$. Now we may assume $A(\mathfrak{A})$ to be a normal subspace of $E(E')$. $A^{\circ\circ}=A$ has been seen in (1).

Let $0\leqq f\in A^\circ$, $0\leqq g\in(A^\perp)^\circ$, then $f\cap g=0$ on $S=\{x+y: x\in A, y\in A^\perp\}$. Since $S$ is $T$-dense in $E$ by Lemma 2.3.1 and M. S. $o$-continuity of $E$, we have $f\cap g=0$. Therefore we have $(A^\perp)^\circ\subset(A^\circ)^\perp$ from which it follows $(A^\perp)^\circ=(A^\circ)^\perp$ by (1).

The other properties: $\mathfrak{A}^{\circ\circ}=\mathfrak{A}$ and $(\mathfrak{A}^\circ)^\perp=(\mathfrak{A}^\perp)^\circ$ can be verified directly, but we shall give the proof by use of the cut extension $\bar{E}$ of $E$. By the discussion about $\bar{E}$ in the last part of § 1, we may assume $E$ to be complete and M. S. $o$-continuous without loss of generality. Every interval of $E$ is weakly compact by Theorem 5.1 which will be given later on and whence the dual of $E'(T_0')$ is $E$ itself (Bourbaki [3] Theorem 2 (p. 68)). Now, applying to $\mathfrak{A}$ the propositions established for $A$ just above, we have $\mathfrak{A}^{\circ\circ}=\mathfrak{A}$, $(\mathfrak{A}^\circ)^\perp=(\mathfrak{A}^\perp)^\circ$.

THEOREM 2.4. *Let $E$ be a locally convex lattice with the topology $T$. Then $E'\subset\dot{E}$ if and only if $T$ is o-continuous and $E'\subset E^*$ if and only if $T$ is* M. S. *o-continuous.*

PROOF. By the following lemma, we can easily complete the proof.

LEMMA 2.4.1. *Let $E$ be a locally convex lattice. If a net $a_\lambda\to a_0$ weakly and $a_\lambda\downarrow$ or $a_\lambda\uparrow$, then $a_\lambda\to a_0$ $(T)$.*

PROOF. Let $\{V\}$ be a family of convex, circled, $T$-closed neighborhoods of 0 in $E$ satisfying the condition $(A)$. Then $f\in V^\circ$ implies $f^+, f^-\in V^\circ$. Now put $\mathrm{H}(V)=\{f: f\geqq 0\}\cap V^\circ$, then $\mathrm{H}(V)$ is weak-star compact as well-known.

Suppose $a_\lambda\downarrow$ and $a_\lambda\to a_0$ weakly, then $a_\lambda\geqq a_0$ for every $\lambda$ and $(a_\lambda-a_0)(f)\downarrow 0$ for every $f\in\mathrm{H}(V)$. Now $a_\lambda-a_0$ can be regarded as a continuous function defined on the compact $\mathrm{H}(V)$ and therefore, $(a_\lambda-a_0)(f)\downarrow 0$ uniformly on $\mathrm{H}(V)$ which implies $f(a_\lambda-a_0)\to 0$ uniformly for all $f\in V^\circ$. Hence for sufficiently large $\lambda$, we have $a_\lambda-a_0\in V^{\circ\circ}=V$.

REMARK. In view of the above lemma, we can see the following assertion: Let $E$ be a locally convex lattice and semi-reflexive. Then the strong topology $T'$ of $E'$ is M. S. $o$-continuous.

COROLLARY 2.4.2. *Let $E$ be a sequentially $T$-complete and boundedly closed locally convex lattice. Then $\tilde{E}=E'=\dot{E}=E^*$ if and only if $T$ is* M. S. *o-continuous and $\tilde{E}=E'=\dot{E}$ if and only if $T$ is o-continuous.*

A locally convex lattice is an $\mathfrak{LF}$-lattice if it is an $\mathfrak{LF}$-space as said before. An $\mathfrak{LF}$-space $E$ is a locally convex space whose topology is defined by an ascending sequence of $\mathfrak{F}$-spaces $E_n$. It is supposed that $\{E_n\}$ have compatible topologies and their set union is $E$. A neighborhood of the origin in $E$ is any convex, circled set whose intersection with each $E_n$ is a neighborhood of the origin in $E_n$. In the sequel, such an ascending sequence $\{E_n\}$ will be called *an ascending sequence of subspaces which defines* $E$ (Dieudonné et Schwarts [1]; Bourbaki [2] p. 64). We shall see later (Theorem 6.1) every $E_n$ can be taken to be an ideal of $E$.

COROLLARY 2.4.3. *Let $E$ be an $\mathfrak{LF}$- or an $\mathfrak{F}$-lattice and suppose it is $\sigma$-complete. Then $\tilde{E}=E'=\dot{E}=E^*$ if and only if $T$ is o-continuous.*

PROOF. Evidently it is enough to see that if $E$ is a $\sigma$-complete $\mathfrak{LF}$- or $\mathfrak{F}$-lattice with the $o$-continuous topology, then we can see the supercompleteness of $E$. In view of Theorem 6.1 noted above, our problem will be reduced to the case where $E$ be an $\mathfrak{F}$-lattice and this is known to Ogasawara [1] p. 50.

COROLLARY 2.4.4. *Let $E$ be a $\sigma$-complete* M. S. *semi-o-continuous locally convex lattice, then $E'\subset E^*$ if and only if $T$ is o-continuous.*

PROOF. It is enough to see that if $E$ be $\sigma$-complete and M. S. semi-$o$-continuous, then $o$-continuity of $T$ implies M. S. $o$-continuity. By Theorem 3.2, 3.3 in the following section, we can complete the proof easily.

We now conclude this section with the lemma which will be used frequently in § 5.

LEMMA 2.5. *Suppose whenever $f(x)=0$ for each $f\in E'\cap E^*$, then $x=0$. If an o-bounded net $a_\lambda\to b$ weakly, then $f(a_\lambda)\to f(b)$ for each $f\in E^*$.*

PROOF. First we shall see $E^*=(E'\cap E^*)^{\perp\perp}$ in $E^*$.

Suppose $0<f\in E^*$ and $f\cap g=0$ for each $0\leqq g\in E'\cap E^*$. Let $f(a)>0$ $(0<a\in E)$ and without loss of generality, we may assume that $f(x)=0$ $(0\leqq x\leqq a)$ implies $x=0$. From $f\cap g(a)=0$ $(0\leqq g\in E'\cap E^*)$ we can see there exists a sequence $\{x_n\}$ with $0\leqq x_n\leqq a$, $f(x_n)\leqq 1/2^n$, $g(a-x_n)\leqq 1/2^n$. Evidently we have $\bigcap\{x_n: n\geqq N\}=0$ for each integer $N$. Then it follows:

$$g(a)=g\left(\bigcup_{n=N}^{\infty}(a-x_n)\right)\leqq\sum_{n=N}^{\infty}g(a-x_n)\leqq 1/2^{N-1}.$$

Now we have $g(a)=0$ for every $g\in E'\cap E^*$ and thus $a=0$, which is absurd.

Next, we shall assume $0 \leqq a_\lambda, b \leqq c$ without loss of generality. For every $0 < f \in E^*$, we can take a net $\{g_\alpha\}$ such that $0 \leqq g_\alpha \uparrow f$, $g_\alpha \in E' \cap E^*$ ($\{\alpha\} \in \Gamma$) by Lemma 2.3.1. For each $\varepsilon > 0$, take $\beta \in \Gamma$ with $(f - g_\beta)(c) < \varepsilon$, then we have easily $|f(a_\lambda - b)| < 2\varepsilon + |g_\beta(a_\lambda - b)|$ which implies $f(a_\lambda) \rightarrow f(b)$.

## § 3. Miscellaneous relations between $E$ and $\{E_\alpha\}$.

Let $E$, $N_\alpha$, $E_\alpha$, $\hat{E}_\alpha$, $[\ ]_\alpha$, and the net $\Omega = \{p_\alpha\}$ have the significances indicated in § 1.

The following statements are immediate:

(1): If $E$ be M. S. semi-$o$-continuous, then $N_\alpha$ is a normal subspace of $E$ and a net $a_\lambda \rightarrow a\,(o)$ implies $[a_\lambda]_\alpha \rightarrow [a]_\alpha\,(o)$.

(2): If $E$ be semi-$o$-continuous, then a sequence $a_n \rightarrow a\,(o)$ implies $[a_n]_\alpha \rightarrow [a]_\alpha\,(o)$.

Theorem 3.1.

(1): *If $E$ be* M. S. *semi-o-continuous, then $E_\alpha$ is* M. S. *semi-o-continuous.*

(2): *If $E$ be* M. S. *o-continuous, then $E_\alpha$ is* M. S. *o-continuous.*

Proof. To see (1), suppose $0 \leqq [a_\lambda] \uparrow [a]$ (for brevity, we shall omit $\alpha$ of $p_\alpha$, $N_\alpha$ and $[\ ]_\alpha$). Without loss of generality, we may assume $0 \leqq a_\lambda \leqq a$ for each $\lambda$. In view of Lemma 2.3.1, we can take a net $b_\beta + c_\beta \uparrow a$ $(0 \leqq b_\beta \in N^\perp, 0 \leqq c_\beta \in N)$. Clearly we have $p(b_\beta) \uparrow p(a)$. For each $\beta$, we have $[b_\beta \cap a_\lambda] \uparrow [b_\beta]$ and since $b_\beta, b_\beta \cap a_\lambda \in N^\perp$, we have $b_\beta \cap a_\lambda \uparrow b_\beta$. Therefore, $p(b_\beta \cap a_\lambda) \uparrow p(b_\beta)$ which implies $\underline{\lim}\, p(a_\lambda) \geqq p(b_\beta)$ for each $\beta$. Now we can complete easily the proof.

Theorem 3.2.

(1): *If $E$ be complete and* M. S. *semi-o-continuous, then both $E_\alpha$ and $\hat{E}_\alpha$ are complete,* M. S. *semi-o-continuous and $E_\alpha$ is an ideal of $\hat{E}_\alpha$.*

(2): *If $E$ be $\sigma$-complete and o-continuous (semi-o-continuous), then both $E_\alpha$ and $\hat{E}_\alpha$ are supercomplete ($\sigma$-complete),* M. S. *o-continuous (semi-o-continuous) and $E_\alpha$ is an ideal of $\hat{E}_\alpha$.*

(3): *If $E$ be supercomplete and semi-o-continuous, then $E_\alpha$ is supercomplete.*

Proof. The only part which requires the proof is the relations between $E_\alpha$ and $\hat{E}_\alpha$. These are known to Ogasawara [1] p. 50, Kawai [1] Theorem 5, 11.

Theorem 3.3.

(1): *If $E$ be* M. S. *semi-o-continuous and every $E_\alpha$ is* M. S. *o-continuous,* *then $E$ is* M. S. *o-continuous.*

(2): *If $E$ be semi-o-continuous and every $E_\alpha$ is o-continuous, then $E$ is o-continuous.*

(3): *Let $E$ be $T$-complete and* M. S. *semi-o-continuous.*
*If every $E_\alpha$ is $[\sigma$-$]$ complete, then $E$ is $[\sigma$-$]$ complete.*
*If every $E_\alpha$ is* M. S. *o-continuous, then $E$ is complete and* M. S. *o-continuous.*

(4): *Let $E$ be $T$-complete.*
*If every $E_\alpha$ is $\sigma$-complete and o-continuous, then $E$ is complete and* M. S. *o-continuous.*

PROOF. To see the first part of (3), suppose $a_\lambda \uparrow$ and $0 \leqq a_\lambda \leqq a$. Then we have $[a_\lambda]_\alpha \uparrow [b_\alpha]_\alpha$ for each $\alpha$. Now we can see if $p_\beta \geqq p_\alpha$, then $[a_\lambda]_\alpha \uparrow [b_\beta]_\alpha$ which implies $p_\alpha(b_\beta - b_\alpha) = 0$. Let $b_\alpha \to b_0\ (T)$, then $p_\alpha(b_\alpha - b_0) = 0$ and we have $[a_\lambda]_\alpha \uparrow [b_0]_\alpha$ for each $\alpha$. Thus $a_\lambda \uparrow b_0$.

To see the last part of (3), let us remark $T$-completeness and M. S. $o$-continuity implies completeness.

To see (4), observe that every $E_\alpha$ is complete and M. S. $o$-continuous. Suppose $a_\lambda \uparrow$, $0 \leqq a_\lambda \leqq a$. Then we have $[a_\lambda]_\alpha \uparrow [b_\alpha]_\alpha$ for each $\alpha$ and $p_\alpha(b_\alpha - a_\lambda) \to 0$. Therefore we can see easily $p_\alpha(b_\beta - b_\alpha) = 0$ if $p_\beta \geqq p_\alpha$. In the same way as the proof of (3), we can complete the proof. Q. E. D.

Let $E$ be M. S. semi-$o$-continuous. Then we can see easily $e$ is a weak unit of $E$ if and only if $[e]_\alpha$ is a weak unit of $E_\alpha$ for each $\alpha$ and $\{e_\lambda\}$ is a complete system of orthogonal elements of $E$ if and only if $\{[e_\lambda]_\alpha\}$ is a complete system of orthogonal elements of $E_\alpha$ for each $\alpha$.

Next, we shall consider the dual $E'$. If $f \in E'$, then we can find $p_\alpha$ and real $k > 0$ such that $|f(x)| \leqq k p_\alpha(x)$ for every $x \in E$ (Bourbaki [3] p. 100) and it follows $f$ defines $\varphi \in E_\alpha'$. Conversely, to each $\varphi \in E_\alpha'$, put $f(x) = \varphi([x]_\alpha)$, then $f \in E'$.

By this observation, we have:

THEOREM 3.4. *A net $a_\lambda \to a$ weakly in $E$ if and only if $[a_\lambda]_\alpha \to [a]_\alpha$ weakly in $E_\alpha$ for each $\alpha$.*

Let us turn to the study of maximal ideals.

An ideal $M$ of a vector lattice $E$ is maximal if and only if for any $a \notin M$, every element $x$ of $E$ can be expressed as a sum $x = \lambda a + m$ where $\lambda$ is real and $m \in M$. Now, taking $a$ with $a > 0$ and setting $h_M(x) = \lambda \alpha_M$ (choose any real $\alpha_M > 0$), then $M$ defines a homomorphism

$h_M$ of $E$ onto the vector lattice $R$ of all real numbers. If $x \geqq 0$, then $h_M(x) \geqq 0$. Hence $h_M \in \tilde{E}$. Further, if $M$ is $T$-closed, then we can see $h_M \in E'$.

Let $M$ be $T$-closed. Since we can take $p_\alpha$ with $|h_M(x)| \leqq k p_\alpha(x)$ for each $x \in E$, it follows that $M \supset N_\alpha$ and $[M]_\alpha$ becomes a $p_\alpha$-closed maximal ideal of $E_\alpha$. Conversely, if $\mathfrak{M}$ is a maximal ideal of $E_\alpha$, then $M = \pi_\alpha^{-1}(\mathfrak{M})$ is a maximal ideal of $E$ and further, if $\mathfrak{M}$ is $p_\alpha$-closed, then $M$ is $T$-closed.

We shall say a locally convex lattice $E$ to be *semi-simple* if the intersection of its $T$-closed maximal ideals is 0 and *semi-simple in the wide sense* if the intersection of its maximal ideals is 0.

Then, the following assertions are immediate:

(1): Let $E$ be M. S. semi-$o$-continuous. If $E$ be semi-simple in the wide sense, then every $E_\alpha$ is semi-simple in the wide sense.

(2): If the set $\{p_\beta\}$ such that $E_\beta$ is semi-simple (in the wide sense) is cofinal in the net $\Omega = \{p_\alpha\}$, then $E$ is semi-simple (in the wide sense).

Every maximal ideal (if any exists) of $\hat{E}_\alpha$ is $p_\alpha$-closed. More generally, in view of Theorem 2.1, we have:

THEOREM 3.5. *If $E$ be sequentially $T$-complete and boundedly closed, then every maximal ideal* (*if any exists*) *of $E$ is $T$-closed.*

One to one correspondence between the family of all $p_\alpha$-closed maximal ideals of $E_\alpha$ and those of $\hat{E}_\alpha$ can be established. Then if $\hat{E}_\alpha$ is semi-simple (in the wide sense), $E_\alpha$ is also semi-simple.

## § 4. *T*-Completion.

Let $E$ be a locally convex lattice with the topology $T$, then the $T$-completion $\hat{E}$ of $E$ is a locally convex lattice too as seen in Theorem 1.4 and the topology of $\hat{E}$ is described by $\{\hat{p}\}$ where $\hat{p}$ is an extension of $p \in \Omega$ to $\hat{E}$ (Bourbaki [2] p. 98).

It is known that $a_n \to 0\,(o)$ in $E$ may fail to be $a_n \to 0\,(o)$ in $\hat{E}$. Concerning this fact, we have:

THEOREM 4.1. *Let $E$ be a locally convex lattice. In order that whenever a net $a_\lambda \to 0\,(o)$ in $E$, then $a_\lambda \to 0\,(o)$ in $\hat{E}$, it is necessary and sufficient that for each Cauchy net $\{b_\alpha\}$ such that $0 \leqq b_\alpha \in E$ and $b_\alpha \to 0\,(o)$ in $E$, we have $b_\alpha \to 0(T)$.*

PROOF. To establish the sufficiency, suppose $a_\lambda \downarrow 0$ in $E$ and $a_\lambda \geqq x \geqq 0$ $(x \in \hat{E})$. For each $p \in \Omega$, take $b_{\lambda,p} \in E$ such that $0 \leqq b_{\lambda,p} \leqq a_\lambda$, $p(a_\lambda - x - b_{\lambda,p}) \leqq 1$. We order the cartesian product $\{\lambda\} \times \Omega$ by agreeing that $\{\lambda, q\} \geqq \{\mu, p\}$ if and only if $\lambda \geqq \mu$, $q \geqq p$, then $\{\lambda\} \times \Omega$ is directed by $\geqq$. Now $\{a_\lambda - b_{\lambda,p}\}$ is a Cauchy net and $a_\lambda - b_{\lambda,p} \to 0\,(o)$. By hypothesis, $a_\lambda - b_{\lambda,p} \to 0\,(T)$ which implies $x=0$.

To prove the necessity, suppose $\{a_\lambda\}$ be a Cauchy net such that $0 \leqq a_\lambda \in E$, $a_\lambda \to 0\,(o)$ in $E$. There exists a net $\{b_\lambda\}$ such that $a_\lambda \leqq b_\lambda \in E$ and $b_\lambda \downarrow 0$ in $E$. By hypothesis, we have $b_\lambda \downarrow 0$ in $\hat{E}$. Let $a_\lambda \to x\,(T)$, then $0 \leqq x \leqq b_\lambda$ for each $\lambda$. Now we have $x=0$. Q. E. D.

Moreover, $a_n \to 0\,(o)$ $(a_n \in E)$ in $\hat{E}$ may fail to be $a_n \to 0\,(o)$ in $E$. However, we can see if a net $a_\lambda \to b\,(o)$ in $E$ and $a_\lambda \to c\,(o)$ in $\hat{E}$ where $c \in E$, then $b=c$.

Next, we shall give the following theorem which has been refered to before.

THEOREM 4.2. *Let $E$ be a locally convex lattice.*

(1): *$\hat{E}$ is complete,* M. S. *semi-o-continuous and $E$ becomes an ideal of $\hat{E}$, if and only if $E$ is complete and* M. S. *semi-o-continuous.*

(2): *$\hat{E}$ is (complete and)* M. S. *o-continuous and $E$ becomes an ideal of $\hat{E}$, if and only if $E$ is complete and* M. S. *o-continuous.*

(3): *$\hat{E}$ is supercomplete, o-continuous and $E$ becomes an ideal of $\hat{E}$, if and only if $E$ is supercomplete, o-continuous and every monotone increasing Cauchy net $\{a_\lambda\}$ of positive elements of $E$ includes a Cauchy subsequence equivalent to $\{a_\lambda\}$.*

PROOF. We shall prove the sufficiency of (1). By Theorem 3.3 of Nakano [3] (p. 93), we can see $E$ is an ideal of $\hat{E}$.

Let $x_\lambda, y \in \hat{E}$, $0 \leqq x_\lambda \uparrow$, $x_\lambda \leqq y$. We can see easily there exists a net $a_\alpha$ $(\alpha \in \Gamma)$ of elements of $E$ such that $0 \leqq a_\alpha \leqq y$, $a_\alpha \uparrow$ and $a_\alpha \to y$ $(T)$. Put $a_\alpha \cap x_\lambda = b_{\alpha,\lambda} \in E$ and $b_\alpha = \bigcup_\lambda b_{\alpha,\lambda}$, then $\{b_\alpha\}$ is a Cauchy net. Indeed, if $\beta \geqq \alpha$ $(\alpha, \beta \in \Gamma)$, we have:

$$0 \leqq b_\beta - b_\alpha = o\text{-}\lim_\lambda (b_{\beta,\lambda} - b_{\alpha,\lambda}) \leqq a_\beta - a_\alpha\,.$$

Let $b_\alpha \to z\,(T)$. Since $b_\alpha \uparrow$, we have $b_\alpha \uparrow z$. Now, we can verify easily $z = \bigcup x_\lambda$.

Next, for each $p \in \Omega$ and any $\varepsilon > 0$, take $\alpha$ with $p(z - b_\alpha) \leqq \varepsilon$ and then take $\lambda$ with $0 \leqq p(b_\alpha) - p(b_{\alpha,\lambda}) \leqq \varepsilon$. Then $0 \leqq p(z) - p(b_{\alpha,\lambda}) \leqq 2\varepsilon$ and since $b_{\alpha,\lambda} \leqq x_\lambda \leqq z$, we have $0 \leqq p(z) - p(x_\lambda) \leqq 2\varepsilon$.

Now we have seen $\hat{E}$ is M. S. semi-$o$-continuous.

COROLLARY 4.2.1. *Let $E$ be a supercomplete locally convex lattice with the semi-o-continuous topology. Then, if $E$ has a complete orthogonal sequence, every orthogonal system of $\hat{E}$ consists of at most countable elements.*

PROOF. Let $\{e_n\}$ be a complete orthogonal sequence of $E$ and $\{h_\lambda\}$ $(\lambda \in \Gamma)$ be any orthogonal system of $\hat{E}$. Put $e_{\lambda,n} = h_\lambda \cap e_n$, then $e_{\lambda,n} \in E$ by our Theorem 4.2 and $e_{\lambda,n} \cap e_{\mu,m} = 0$ if $\lambda \neq \mu$ or $n \neq m$. For every $n$, in view of supercompleteness of $E$, we can take a sequence $\lambda(i;n)$ $(i=1,2,3,\cdots)$ with $\bigcup_\lambda e_{\lambda,n} = \bigcup_{i=1}^{\infty} e_{\lambda(i;n)}$ $(\lambda(i;n) \in \Gamma)$ which implies $e_{\lambda,n} = 0$ if $\lambda \neq \lambda(i;n)$ $(i=1,2,3,\cdots)$. Now we have seen $e_{\lambda,n}$ are 0 for all but at most countable pairs $\{\lambda, n\}$. Since $\{e_n\}$ becomes to be a complete orthogonal sequence of $\hat{E}$, $\{h_\lambda\}$ with $h_\lambda \neq 0$ consists of at most countable elements. Q. E. D.

From this Corollary, we can see if $E$ be supercomplete, semi-$o$-continuous and has a complete orthogonal sequence, then $\hat{E}$ is supercomplete (Nakano [1] p. 42).

THEOREM 4.3. *Let $E$ be a locally convex lattice. Then $\hat{E}$ is $\sigma$-complete and o-continuous if and only if every monotone decreasing sequence $\{a_n\}$ of positive elements of $E$ is a Cauchy sequence.*

PROOF. To establish the sufficiency, we shall show if $0 \leqq x_n \in \hat{E}$ and $x_n \downarrow$, then $\{x_n\}$ is a Cauchy sequence. For $p \in \Omega$, take $0 \leqq a_n \in E$ with $p(x_n - a_n) < 1/2^{n+2}$ and put $b_n = \bigcap \{a_k : k=1,2,3,\cdots,n\}$. Then $b_n \downarrow$ and we can see easily $p(x_n - b_n) < 1/4$. By hypothesis, $\{b_n\}$ is a Cauchy sequence and it follows:

$$p(x_n - x_m) \leqq p(x_n - b_n) + p(x_m - b_m) + p(b_n - b_m) \leqq 1$$

for sufficiently large $n, m$. Now we can see $\{x_n\}$ is a Cauchy sequence and the proof can be completed immediately. Q. E. D.

Finally we shall be concerned with the normal subspaces. Let $E$ be M. S. semi-$o$-continuous. We shall establish a one to one correspondence between the totality of all normal subspaces of $E$ and those of $\hat{E}$.

Since the cut extension $\bar{E}$ of $E$ is complete and M. S. semi-$o$-continuous, $\bar{E}$ is an ideal of $\hat{\bar{E}}$ by Theorem 4.2 and it follows that whenever $0 < x \in \hat{\bar{E}}$, then $x = \sup \{a : 0 \leqq a \leqq x, a \in E\}$. Now, every

$0<x\in\hat{E}$ can be represented as $x=\sup\{a: 0\leqq a\leqq x, a\in E\}$.

Let $A(\mathfrak{A})$ be a normal subspace of $E(\hat{E})$, then we can see easily $\mathfrak{A}^{\perp}=(\mathfrak{A}\cap E)^{\perp}$ and it follows $A^{\perp\perp}\cap E=A$ ($\perp$-operation shall be considered in $\hat{E}$) and $\mathfrak{A}\cap E$ is a normal subspace of $E$. Thus, we may set up a one to one correspondence between the totality of all normal subspaces of $E$ and those of $\hat{E}$.

## § 5. Miscellaneous theorems.

In this section, we give miscellaneous theorems, most of which are generalization of known results in the case where $E$ is normed.

Let $\Omega=\{p_\alpha\}$, $N_\alpha$, $E_\alpha$, $\hat{E}_\alpha$, $[\ ]_\alpha$ have the significances indicated in § 1 and § 3. If we are considering different topologies for the same $E$, we shall write $E$ equipping with the topology $T$ by $E(T)$ precisely as said in Introduction.

THEOREM 5.1. *Let $E$ be a locally convex lattice with the topology $T$. Then the following assertions are equivalent:*

(1): *$E$ is complete and* M. S. *o-continuous.*

(2): *$E$ is complete,* M. S. *semi-o-continuous and o-continuous.*

(3): *$E$ is complete and $E'\subset E^*$.*

(4): *$E$ is an ideal of $E''$.*

(4′): *$E$ is an ideal of $(E')^*$. In this case, no element $\neq 0$ of $(E')^*$ can be orthogonal to all elements of $E$.*

(5): *Every interval of $E$ is weakly compact.*

*If $E$ be an $\mathfrak{L}\mathfrak{F}$- or an $\mathfrak{F}$-lattice, then* (1)—(5) *and the following assertions are equivalent:*

(6): *$E$ is $\sigma$-complete and o-continuous.*

(7): *$E$ is supercomplete and* M. S. *o-continuous.*

(8): *Every interval of $E$ is sequentially weakly compact.*

PROOF. In view of Theorem 2.4 and the proof of Corollary 2.4.4, we can see easily (1), (2), (3) are equivalent.

To see (3)→(4), suppose $0\leqq x\leqq a$ ($a\in E$, $x\in E''$). Then $x\in(E')^*$. Since $E'$ is an ideal of $E^*$ by our hypothesis and Theorem 2.1, we can take $X\in E^{**}$ such that $0\leqq X\leqq a$ and $X(f)=x(f)$ for all $f\in E'$ (Nakano [1] Theorem 23.5). Since $E$ is an ideal of $E^{**}$ (Nakano [1] Theorem 24.3), we can see easily $x\in E$.

The proof of (3)→(4′) proceeds just as above.

To see (4) ((4′))→(5), we must prove the interval $I=\{x: a\leqq x\leqq b\}$

is weakly closed but this is an immediate consequence of (4) ((4')). Now by the well-known argument (Loomis [1] 9.B. p. 22), we can complete the proof.

(5)→(1) is immediate and (6)→(7) has been noted in Corollary 2.4.3.

(7)→(8) follows from the following Lemma 5.1.1.

LEMMA 5.1.1. *Let $E$ be a supercomplete, o-continuous locally convex lattice. Then every interval of $E$ is sequentially weakly compact.*

PROOF. By Lemma 2.5 and Nakano [1] Theorem 27.6, we can complete the proof immediately, but here we shall inform an another proof to put to use later on.

We shall show the interval $I=\{x: 0\leqq x\leqq e\}$ be sequentially weakly compact. By Grothendieck [1] Prop. 6, it is enough to see there exist countable $\{p_{\alpha(n)}\}$ $(p_{\alpha(n)}\in\Omega)$ such that $p_{\alpha(n)}(x)=0$ $(n=1, 2, 3,\cdots)$, $0\leqq x\leqq e$ implies $x=0$.

Corresponding to the direct sum decomposition: $E=N_\alpha+N_\alpha^\perp$, we have $e=b_\alpha+c_\alpha$ $(0\leqq b_\alpha\in N_\alpha,\ 0\leqq c_\alpha\in N_\alpha^\perp)$. By $\bigcap b_\alpha=0$ and supercompleteness of $E$, we can take a sequence $\{\alpha(n)\}$ with $\bigcap b_{\alpha(n)}=0$. Now $\{p_{\alpha(n)}\}$ is the required system.

REMARK 1. If we would give a condition for the prescribed interval $I=\{x: a\leqq x\leqq b\}$ to be (sequentially) weakly compact, apply our Theorem (Lemma) to $F=\{x: x\in E, |x|\leqq\lambda(b-a)$ for some real $\lambda\}$. This remark is also available for Theorem 5.4.

REMARK 2. Since every interval of $E'$ is equi-continuous, we have the following assertion:

Let $E$ be a locally convex lattice. Then every interval of $E'$ is weak-star compact.

THEOREM 5.2. *Let $E$ be an* M. S. *semi-o-continuous locally convex lattice satisfying the o-countability condition. Then, for any o-bounded sequence $\{a_n\}$ such that $a_n\to c\,(T)$, we can find a subsequence $\{a_{k(n)}\}$ such that $a_{k(n)}\to c\,(o)$.*

PROOF. By considering the cut extension of $E$, we may assume $E$ is supercomplete, M. S. semi-$o$-continuous and $0\leqq a_n\leqq e$, $c=0$ without loss of generality (by the discussion after Theorem 1.5).

Put $F=\{x: |x|\leqq\lambda e$ for some real $\lambda,\ x\in E\}$. Let $\{p_{\alpha(n)}\}$ have the same significance as in the proof of Lemma 5.1.1 and put

$$q(x)=\sum_{n=1}^{\infty}p_{\alpha(n)}(x)/2^n p_{\alpha(n)}(e)$$

for every $x \in F$. Now $F$ equipping with the norm $q$ is a complete, M. S. semi-$o$-continuous normed vector lattice and by Theorem 4.2, $F$ is an ideal of the $q$-completion $\hat{F}$. In any Banach lattice, metric convergence is equivalent to relative uniform star-$o$-convergence*) and thus we can take $\{a_{k(n)}\}$ and $\{x_n\}$ such that $0 \leqq a_{k(n)} \leqq x_n$, $x_n \in \hat{F}$, $x_n \downarrow 0$. Now we have $a_{k(n)} \leqq x_n \cap e \in F$ which completes the proof.

COROLLARY 5.2.1. *Let $E$ be a locally convex lattice satisfying the $o$-countability condition and suppose whenever $f(x)=0$ for each $f \in E' \cap E^*$, then $x=0$.*

*If $\{a_n\}$ be an $o$-bounded sequence such that $|a_n| \to 0$ weakly, then we can find a subsequence $\{a_{k(n)}\}$ such that $a_{k(n)} \to 0$ $(o)$.*

PROOF. For each $0 < f \in E' \cap E^*$, put $p_f(x) = f(|x|)$. $E$ equipping with the topology defined by $\{p_f : 0 < f \in E' \cap E^*\}$ satisfies the condition in the statements of Theorem 5.2.

REMARK. In view of Lemma 2.5, this corollary is an immediate consequence of Nakano [1] Theorem 27.10.

THEOREM 5.3.

(1): *Let $E$ be a $\sigma$-complete, $o$-continuous locally convex lattice with the topology $T$. Then, if $\mathfrak{T}$ be any locally convex lattice topology on $E$ such that the dual of $E(\mathfrak{T})$ coincides with the dual of $E(T)$, then $T$- and $\mathfrak{T}$-convergence of an $o$-bounded sequence are equivalent.*

(2): *Let $E$ be a locally convex lattice with the topology $T$ such that every monotone decreasing sequence of positive elements of $E$ is a Cauchy sequence. Then for an $o$-bounded sequence $\{a_n\}$, $a_n \to 0\,(T)$ if and only if $|a_n| \to 0$ weakly.*

PROOF. To see (1), we shall show if $0 \leqq a_n \leqq e$ and $a_n \to 0$ weakly, then $a_n \to 0\,(T)$. Theorem 3.2, 3.4 enable us to reduce our problem to similar one about $E_\alpha$ (for each $\alpha$) and this is evident from Corollary 5.2.1. Now, we can complete the proof immediately.

To see (2), consider the $T$-completion of $E$ and apply Theorem 4.3.

REMARK. We shall here refer to the Theorem due to I. Amemiya and T. Mori.

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*) A sequence $a_n$ of elements of a vector lattice $E$ is said to be relatively uniformly $o$-convergent to $a \in E$, if there exist $0 \leqq b \in E$ and a sequence of real numbers $\varepsilon_n \to 0$ such that $|a_n - a| \leqq \varepsilon_n b$ for $n=1, 2, \cdots$. A sequence $a_n$ is said to be relatively uniformly star-$o$-convergent, if we can take a relatively uniformly $o$-convergent subsequence from every subsequence of $a_n$.

Let $E$ be M. S. $o$-continuous, then, in the similar way to the above (by considering the cut extension of $E$, $E$ may be assumed to be complete and M. S. $o$-continuous), we can prove the following assertion:

Let $\{a_\lambda\}$ be an $o$-bounded net, then $a_\lambda \to 0\,(T)$ if and only if $|a_\lambda| \to 0$ weakly.

Now, by Lemma 2.5, $T$ is equivalent on every interval of $E$ to the topology (which will be denoted by $T_0$ in the rest of this section) defined by all semi-norms $\{p_f\}$ where $0 < f \in E^*$ and $p_f(x) = f(|x|)$ for each $x \in E$.

Thus we have seen all M. S. $o$-continuous locally convex lattice topologies are equivalent on every interval of $E$. This significant assertion is due to I. Amemiya and T. Mori [1]. Further we can say as follows: Let $E$ be a locally convex lattice with the topology $T$ and suppose whenever $f(x)=0$ for each $f \in E' \cap E^*$, then $x=0$. Then on every interval of $E$, $T$ is finer than any M. S. $o$-continuous locally convex lattice topology on $E$.

Consequently, we can say, if we are considering the M. S. $o$-continuous lattice topology, the properties of the interval of $E$ owe exclusively to $E$ considered as the vector lattice merely.

An element $a \in E$ is said to be *discrete* if for every element $x \in E$ such that $|x| \leqq |a|$, there exists a real $\lambda$ for which $x = \lambda a$. $E$ is said to be *discrete* if $E$ is complete and has a complete orthogonal system consisting of discrete elements (Halperin and Nakano [1]).

We shall denote by $T_0'$ the topology of $E'$ defined by all semi-norms $\{\pi_a : 0 < a \in E\}$ where $\pi_a(f) = |f|(a)$ for each $f \in E'$.

The following theorem was first dealt with by Nakamura [2] for the separable Banach lattices.

THEOREM 5.4. *Let $E$ be a locally convex lattice with the topology $T$. Then the following assertions are equivalent:*

(1): *Every interval of $E$ is $T$-compact.*

(2): *Every interval of $E$ is $T_0$-compact and $T$ is* M. S. *$o$-continuous.*

(3): *$E$ is complete,* M. S. *semi-$o$-continuous and for each $\alpha$, any interval of $E_\alpha$ is $p_\alpha$-(sequentially) compact.*

(4): *$E$ is complete,* M. S. *$o$-continuous and $T$-topology and weak topology ($T$-convergence and weak convergence of a sequence) are equivalent on every interval of $E$.*

(5): *$E$ is complete,* M. S. *$o$-continuous and for each $o$-bounded net $\{a_\lambda\}$*

*($o$-bounded sequence $\{a_n\}$) of elements of $E$, $a_\lambda \to 0$ ($a_n \to 0$) weakly implies $|a_\lambda| \to 0$ ($|a_n| \to 0$) weakly.*

(6): *$E$ is complete,* M. S. *$o$-continuous and for each $o$-bounded net $\{f_\lambda\}$ ($o$-bounded sequence $\{f_n\}$) of elements of $E'$, $f_\lambda \to 0$ ($f_n \to 0$) in weak-star topology implies $|f_\lambda| \to 0$ ($|f_n| \to 0$) in weak-star topology.*

(7): *$E$ is complete,* M. S. *$o$-continuous and every interval of $E'$ is $T_0'$-compact.*

(8): *$E$ is discrete and $T$ is* M. S. *$o$-continuous.*

PROOF. (1)$\leftrightarrows$(2) is immediate by the remark after Theorem 5.3 and (1)$\to$(3) is obvious.

To see (3)$\to$(1), let $\{a_\lambda\}$ be a universal net in $E$ (Kelley [1] p. 281) such that $0 \leqq a_\lambda \leqq e$. By hypothesis, we can see easily $\{a_\lambda\}$ is a Cauchy net and thus, $a_\lambda$ converges to an element of $E$ by Theorem 4.2. (1).

(1)$\to$(4) is immediate from Theorem 5.1.

(4)$\to$(5) is obvious in the case of nets.

Next, we shall see (5) in case of sequences implies (3). Suppose $0 \leqq [a_n]_\alpha \leqq [e]_\alpha$. Without loss of generality, we may assume $0 \leqq a_n \leqq e$, $e \in N_\alpha^\perp$ where $N_\alpha = p_\alpha^{-1}(0)$. By Theorem 3.2, $E_\alpha$ is supercomplete and hence $N_\alpha^\perp$ is supercomplete. Now, by Lemma 5.1.1, we can take a subsequence $\{a_{k(n)}\}$ such that $a_{k(n)} \to c$ weakly in $E$ and, by hypothesis and Theorem 5.3, we have $a_{k(n)} \to c\ (T)$. Thus, every interval of $E_\alpha$ is sequentially $p_\alpha$-compact.

Now, we have seen (1), (2), (3), (4), (5) are equivalent.

To see (6)$\leftrightarrows$(7), observe $E$ to be the dual of $E'(T_0')$ (see the proof of Theorem 2.3) and apply (1)$\leftrightarrows$(5) to $E'$, then in view of Theorem 2.2, we can complete the proof.

(1)$\to$(6) for nets follows from Prop. 5 of Bourbaki [3] p. 23, since the interval $\{g: -f \leqq g \leqq f\}$ of $E'$ is an equi-continuous subset of $E'$.

Similarly, the proof of (6)$\to$(5) now proceeds just as above, since the interval $\{x: -a \leqq x \leqq a\}$ is an equi-continuous subset of $E$ (considering as the continuous functionals on $E'(T_0')$).

(5)$\leftrightarrows$(8) follows immediately from Lemmas 1, 2 of Halperin-Nakano [1] (p. 406, 407) and our Lemma 2.5.

REMARK. If $E$ be a $T$-complete bornographic lattice, then we can omit the word "$o$-bounded" in the statements of (5), (6) for the case of sequences.

We have seen if $E$ be M. S. $o$-continuous, we may set up a one to one correspondence between the totality of all normal subspaces of $E$ and those of $E'$ by Theorem 2.3.

The least normal subspace containing the prescribed element of the vector lattice will be called a *principal normal subspace.*

THEOREM 5.5. *Let $E$ be an* M. S. *o-continuous locally convex lattice satisfying the o-countability condition. Then, by the correspondence between the totality of all normal subspaces of $E$ and those of $E'$ stated in Theorem* 2.3, *the principal normal subspace of $E$ corresponds to the principal normal subspace of $E'$.*

*In particular, if $E$ has a weak unit, then $E'$ also has a weak unit.*

PROOF. Considering the cut extension of $E$, we may assume $E$ to be supercomplete and M. S. $o$-continuous (refer to the discussion in the last part of § 1).

We shall prove that for any $0<e\in E$, $(e^{\perp\perp})^{\circ\perp}=(e^{\perp})^{\circ}$ is a principal normal subspace of $E'$.

Because of the direct sum decomposition: $E'=(e^{\perp})^{\circ}+(e^{\perp\perp})^{\circ}$ and by considering $e^{\perp\perp}$ instead of $E$, it is enough to see if $E$ has a weak unit $e$, then $E'$ has a weak unit too.

By the supercompleteness of $E$, we can take a sequence $\{p_{\alpha(n)}\}$ of semi-norms $\in\Omega$ such that $0\leqq x\leqq e$, $p_{\alpha(n)}(x)=0$ $(n=1,2,3,\cdots)$ implies $x=0$ as shown in the proof of Lemma 5.1.1. Put $F=\{x: |x|\leqq\lambda e$ for some real $\lambda\}$ and $q(x)=\sum_{n=1}^{\infty}p_{\alpha(n)}(x)/2^n p_{\alpha(n)}(e)$ for every $x\in F$. Since $F$ equipping with the norm $q$ is complete, $o$-continuous, the $q$-completion $\hat{F}$ of $F$ is complete and $o$-continuous. Hence the dual of $F$ has a weak unit $g>0$ (Ogasawara [1] p. 81, Theorem 3).

Put $f(x)=\sup\{g(a): 0\leqq a\leqq x, a\in F\}$ for each $0\leqq x\in E$ and $f(x)=f(x^+)-f(x^-)$ for each $x\in E$. Then we can see easily $f\in E'$. $f$ coincides with $g$ on $F$ and it follows that whenever $f(x)=0$ and $0\leqq x\leqq e$, then $x=0$.

Suppose $f\cap h=0$ $(0\leqq h\in E')$, then $f\cap h(e)=0$ and we can see $h(e)=0$ in the same way as in the proof of Lemma 2.5. For each $0<x\in E$, we can take a net $\{a_\lambda\}$ such that $0\leqq a_\lambda\uparrow x$, $a_\lambda\in F$ by Lemma 2.3.1. Since $h(a_\lambda)=0$, we have $h(x)=0$. Therefore $h=0$ and $f$ is a weak unit of $E'$.

COROLLARY 5.5.1. *Let $E$ be a T-separable $\mathfrak{LF}$- ($\mathfrak{F}$-) lattice and suppose it is o-continuous or it is σ-complete, semi-o-continuous. Then $E'$ has a complete orthogonal sequence.*

PROOF. This follows from Theorem 6.4 and Corollary 6.4.1 given later on.

We have given a necessary and sufficient condition for $E$ to be an ideal of $E''$. Now we shall conclude this section with giving the condition for $E$ to be a normal subspace of $E''$.

THEOREM 5.6. *Let $E$ be a locally convex lattice, then the following assertions are equivalent:*

(1): *$E$ is a normal subspace of $E''$.*

(2): $E=(E')^*$

(3): *Every monotone increasing $T$-bounded net of positive elements of $E$ is $T$-convergent to some element of $E$.*

(4): *$E$ is* M. S. *o-continuous and* M. S. *monotone complete.*

*Let $E$ be an $\mathfrak{L}\mathfrak{F}$- or $\mathfrak{F}$-lattice, then* (1)—(4) *and the following assertions are equivalent:*

(5): *$E$ is o-continuous and for any sequence $\{a_n\}$ of elements of $E$ such that $\lim f(a_n)$ exists for every $f\in E'$, we can find $a\in E$ with $a_n\to a$ weakly.*

(6): *$E$ is o-continuous and monotone complete.*

(7): *$E$ is $\sigma$-complete, o-continuous and satisfies the following condition: whenever $\{a_n\}$ be a monotone increasing sequence of positive elements of $E$ such that $\lambda_n a_n\to 0\,(o)$ always for any sequence $\lambda_n\downarrow 0$ of real numbers, then $\{a_n\}$ is o-bounded.*

PROOF. To see (1)→(3), suppose $0\leqq a_\lambda\uparrow$ and $\{a_\lambda\}$ be $T$-bounded. Putting $x(f)=\lim f(a_\lambda)$ for each $f\in E'$, then we can see easily $x\in E''$. Now $a_\lambda\uparrow x$ in $E''$ and by (1) we have $x\in E$. Thus, by Lemma 2.4.1, we can complete the proof.

To see (3)→(1), suppose $0\leqq a_\lambda\in E$, $a_\lambda\uparrow x$ in $E''$. Now $\{a_\lambda\}$ is weakly bounded and thus $T$-bounded (Corollary of Bourbaki [3] p. 70). Then, it follows $a_\lambda\to a\,(T)$ for some $a\in E$ which implies $x=a$.

To see (3)→(2), follow the same line as (3)→(1) and see the remark given in the statement (4′) of Theorem 5.1.

(2)→(1) is immediate from $(E')^*$ being a normal subspace of $(\tilde{E}')$. Next, we shall consider $E$ to be an $\mathfrak{L}\mathfrak{F}$- ($\mathfrak{F}$-) lattice.

(2)→(5) follows from Theorem 8 of Ogasawara [1] p. 44 or Theorem 2 of Nakamura [1] (by some modification).

(5)→(6)→(7) are immediate.

To see (7)→(6), suppose $0\leqq a_n\uparrow$ and $\{a_n\}$ be $T$-bounded. We shall show $\lambda_n a_n\to 0\,(o)$ for every $\lambda_n\downarrow 0$. Take $\{\lambda_{n(k)}\}$ such that $n(k)\uparrow\infty$

and $\lambda_{n(k)} \leqq 1/k^3$, then, by $T$-completeness of $E$, we can define the element: $u=\sum_{k=1}^{\infty}(\lambda_{n(k)})^{1/2}a_{n(k+1)}$. Now, if $n(k)\leqq e<n(k+1)$, we have $\lambda_e a_e \leqq \lambda_{n(k)}a_{n(k+1)} \leqq \lambda_{n(k)}^{1/2}u$ which implies $\lambda_e a_e \to 0\ (o)$.

To see (6)$\to$(4), suppose $0\leqq a_\lambda \uparrow$ and $\{a_\lambda\}$ be $T$-bounded. We shall show $\bigcup a_\lambda$ exists. Let $\{E_n\}$ be an ascending sequence of subspaces which defines $E$ and we may assume every $E_n$ is an ideal of $E$ by Theorem 6.1 which will be given later on.

Since the set $\{a_\lambda\}$ is contained in some $E_n$, our problem can be reduced to the case where $E$ be an $\mathfrak{F}$-lattice and the proof now proceeds just as the proof of Nakano [1] Theorem 30.20.

## § 6. $\mathfrak{L}\mathfrak{F}$-Lattices.

We have given already various theorems about $\mathfrak{L}\mathfrak{F}$-lattices which are immediate consequences of general theory. In this section, it is intended to supplement the discussions about $\mathfrak{L}\mathfrak{F}$-lattices until now.

The following theorem is the corner stone of our whole theory of $\mathfrak{L}\mathfrak{F}$-lattices, for it will enable us to reduce most of our problems about $\mathfrak{L}\mathfrak{F}$-lattices to similar ones about $\mathfrak{F}$-lattices.

THEOREM 6.1. *Let $E$ be an $\mathfrak{L}\mathfrak{F}$-lattice. Then, we can find an ascending sequence $\{E_n\}$ of subspaces which defines $E$ such that every $E_n$ is an ideal of $E$.*

In the sequel, unless otherwise stated, we shall take always $\{E_n\}$ in such a way that each $E_n$ is an ideal of $E$.

PROOF. Let $F_n$ be the set of all elements $x\in E_n$ such that $|x|\geqq|y|$ implies $y\in E_n$. Then, $F_n$ is a $T$-closed ideal of $E$ such that for each $E_n$, we can find some $F_{d(n)}$ with $F_{d(n)}\supset E_n$. In fact, suppose this is false. Then, we could find a sequence $\{a_k\}$ of elements of $E_n$ such that $a_k \notin F_k$. Take $x_k$ such that $|a_k|\geqq|x_k|$ and $x_k\notin E_k$.

Let $\{p_e\}$ be a monotone increasing sequence of semi-norms describing the topology of $E_n$ and set $\lambda_k=1/kp_k(a_k)$ (if $p_k(a_k)=0$, then $\lambda_k$ may be taken arbitarily). Then $\lambda_k a_k\to 0\ (T)$ and whence $\lambda_k x_k \to 0\ (T)$. Now $\{\lambda_k x_k\}$ is $T$-bounded and it follows $\{\lambda_k x_k\}\subset E_m$ for some $m$, but this is impossible. Therefore, we have seen $\{F_n\}$ is a required ascending sequence of subspaces which defines $E$.

COROLLARY 6.1.1. *Let $E$ be an $\mathfrak{L}\mathfrak{F}$-lattice and $\{E_n\}$ be an ascending sequence of subspaces (not necessarily to be an ideal of $E$) which defines*

*E. Then an ideal H of E is T-closed if and only if $H\cap E_n$ is a T-closed subset of $E_n$ for each n.*

PROOF. To prove the sufficiency, let $\{F_n\}$ have the significance indicated in the proof of Theorem 6.1. Then we can see easily $H\cap F_n$ is $T$-closed for each $n$. Suppose a net $x_\lambda\in H\to x\,(T)$, then we have $|x_\lambda|\cap|x|\to|x|\,(T)$. If $x\in F_k$, then $|x_\lambda|\cap|x|\in H\cap F_k$ and it follows $|x|\in H\cap F_k$. Now we have $x\in H$. Q. E. D.

Corollary 6.1.1 and the following assertion will give answers under certain additional assumptions to the problems (1), (2) of Dieudonné et Schwartz [1] (p. 97).

*Let E be an $\mathfrak{L}\mathfrak{F}$-lattice with the topology T and H be a T-closed ideal of E. Then H is an $\mathfrak{F}$-lattice or $\mathfrak{L}\mathfrak{F}$-lattice in the relative topology.*

PROOF. Let $H\not\subset E_n$ for all $n$. Put $H_n=E_n\cap H$, then $H_n$ is a $T$-closed subspace of $H$. Denote by $T_n$, $T$ the topology of $H_n$, $H$ induced by $T$ respectively. We shall prove $T$ is an inductive limit of $\{T_n\}$ (Bourbaki [2] p. 63, 65).

Let $V$ be a convex, circled subset of $H$ such that $V\cap H_n$ is a $T_n$-neighborhood of 0 in $H_n$. To see $V$ is a $T$-neighborhood of 0 in $H$, it is enough to see $V$ can swallow any $T$-bounded set of $H$ by Theorem 1.2 and this is immediate.

Now, we can see $\{H_n\}$ is an ascending sequence of subspaces which defines $H$.

THEOREM 6.2. *Let E be an* M. S. *semi-o-continuous $\mathfrak{L}\mathfrak{F}$-lattice or an $\mathfrak{L}\mathfrak{F}$-lattice such that T-bounded implies o-bounded. Then, we can find an ascending sequence $\{E_n\}$ of subspaces which defines E such that every $E_n$ is a normal subspace of E.*

PROOF. We need only prove $E_n^{\perp\perp}\subset E_m$ for some $m$. If this is false, we could find a sequence $\{a_k\}$ of positive elements of $E_n^{\perp\perp}$ such that $a_k\notin E_k$. By Lemma 2.3.1, we can find a net $\{x_\lambda^{(k)}\}$ such that $x_\lambda^{(k)}\uparrow a_k$ and $0\leqq x_\lambda^{(k)}\in E_n$ $(\lambda\in\Gamma_k)$. Since $\{x_\lambda^{(k)}\}$ is $T$-bounded in $E_n$ for each $k$, we can take $\alpha_k>0$ such that $B=\{\alpha_k x_\lambda^{(k)}:\lambda\in\Gamma_k,\ k=1,2,3,\cdots\}$ is $T$-bounded. Now, from our assumptions we can see that $\{\alpha_k a_k\}$ is $T$-bounded which implies $\{\alpha_k a_k\}$ is contained in some $E_m$. This is impossible.

REMARK. If there exists a sequence $\{e_n\}$ of positive elements of $E$ such that for each $x\in E$, we can find $e_n$ and real $\lambda>0$ with $|x|\leqq\lambda e_n$, then $T$-bounded implies $o$-bounded. In fact, for each $E_n$, we can take $e_m$ such that whenever $x\in E_n$, then $|x|\leqq\lambda e_m$ for some real $\lambda>0$. Put $\|x\|=\inf\{\lambda : |x|\leqq\lambda e_m\}$ for each $x\in E_n$, then in view of the following Theorem 6.3, we can see $\|\ \|$-topology is equivalent to $T$ on $E_n$. Q. E. D.

Evidently, the $\mathfrak{L}\mathfrak{F}$-lattice $E$ has no archimedean unit.

If $E$ satisfies the conditions in the statements of Theorem 6.2, then no weak unit can exist in $E$.

We now turn to the study of order topology in an $\mathfrak{L}\mathfrak{F}$-lattice $E$. If we are considering an $o$-convergent net $\{a_\lambda\}$, then $\{a_\lambda\}$ is contained in some $E_n$ and hence, problems about $o$-convergence in $\mathfrak{L}\mathfrak{F}$-lattices are reduced to ones in $\mathfrak{F}$-lattices.

Now, the following theorem and some of its corollaries are well known in substance (Birkhoff [1] p. 247, Nakano [1] Theorem 33.4, 33.5, Ogasawara [1] p. 48, 50).

THEOREM 6.3. *Let $E$ be an $\mathfrak{L}\mathfrak{F}$-lattice [$\mathfrak{F}$-lattice] with the topology $T$. Then, $T$-convergence of a sequence is equivalent to relative uniform star o-convergence.*

*A $T$-bounded net [A net] $a_\lambda\to a\ (T)$ $(\lambda\in\Gamma)$ if and only if there exists a monotone increasing sequence $\{\gamma_n\}$ $(\gamma_n\in\Gamma)$ such that for every $\beta(n)$ with $\beta(n)\geqq\gamma_n$, $\{a_{\beta(n)}\}$ is relatively uniformly o-convergent to $a$.*

COROLLARY 6.3.1. *Let $E$ be an $\mathfrak{L}\mathfrak{F}$-lattice with the topology $T$, then $T$ is finest in the collection of all locally convex lattice topologies for $E$.*

PROOF. Let $\mathfrak{T}$ be any locally convex lattice topology for $E$ and $p$ be any $\mathfrak{T}$-continuous semi-norm on $E$ satisfying the condition $(A)$.

To see $p$ is $T$-continuous, it is enough to see $p$ is $T$-continuous on each $E_n$ and this follows immediately from our Theorem 6.3.

COROLLARY 6.3.2. *Let $E$ be an $\mathfrak{L}\mathfrak{F}$- or $\mathfrak{F}$-lattice with the o-continuous topology. Then,*

(1): *o-convergence of a sequence is equivalent to relative uniform o-convergence.*

(2): *for each sequence $\{a_n\}$ such that $a_n\to 0\ (o)$, we can find a sequence $\{\lambda_n\}$ of reals such that $0<\lambda_n\uparrow\infty$ and $\lambda_n a_n\to 0\ (o)$.*

By Theorem 5.3 and Corollary 6.3.2, we have

COROLLARY 6.3.3. *Let $E$ be an $\mathfrak{L}\mathfrak{F}$- or $\mathfrak{F}$-lattice and suppose it is $\sigma$-complete and o-continuous. Then, the following assertions are equivalent:*

(1): *A sequence $a_n\to a\ (T)$.*

(2): *A sequence $\{a_n\}$ is star-o-convergent to $a$.*

(3): *A sequence $\{a_n\}$ is relatively uniformly star-o-convergent to $a$.*

(4): *$|a_n-a|\to 0$ weakly and for every subsequence of $\{a_n\}$, we can find* *an o-bounded subsubsequence.*

Next, we shall consider a $T$-separable $\mathfrak{L}\mathfrak{F}$-lattice $E$. Let $\{E_n\}$ be an ascending sequence of subspaces which defines $E$, then $E$ is $T$-separable if and only if every $E_n$ is $T$-separable.

THEOREM 6.4. *Let $E$ be an $\mathfrak{L}\mathfrak{F}$- or $\mathfrak{F}$-lattice and suppose it is $T$-separable. Then:*

(1): *No (strictly) monotone increasing well-ordered set of positive elements defined for all ordinals of the first and the second classes can exist in $E$.*

(2): *Every orthogonal system of $E$ consists of at most countable elements.*

PROOF. First, we shall show (1). Our assertion is known to Ogasawara [1] p. 51 for the case where $E$ is an $\mathfrak{F}$-lattice. Therefore, our proof is completed if we show that an ascending set defined for all ordinals of the first and the second classes: $0<x_1<x_2<\cdots\cdots<x_\omega<\cdots<x_\alpha<\cdots$ is $T$-bounded.

Suppose this is false, then we can take a semi-norm $p\in\Omega$ and a subsequence $\{x_{\alpha(n)}\}$ such that $p(x_{\alpha(n)})>n$. Let $\beta$ be the ordinal of the second class such that $\beta\geqq\alpha(n)$ $(n=1, 2, 3,\cdots)$, then $p(x_\beta)=\infty$ which is absurd.

To see (2), suppose there exists a non-countable orthogonal system. Then, there exists $E_n$ which contains a non-countable orthogonal system, but since $E_n$ is $T$-separable we can verify this is impossible.

COROLLARY 6.4.1. *Let $E$ be a $T$-separable $\mathfrak{L}\mathfrak{F}$- or $\mathfrak{F}$-lattice and suppose* (1)*: it is o-continuous or* (2)*: it is $\sigma$-complete and semi-o-continuous, then $E$ is supercomplete and* M. S. *o-continuous.*

PROOF. We shall prove our assertion in the case (1).

By Theorem 6.4, $E$ satisfies the $o$-countability condition and thus $o$-continuity implies M. S. $o$-continuity. By the $T$-completeness of $E$, M. S. $o$-continuity implies completeness.

Next, to prove the case (2), we shall show the following assertion: Let $E$ be a $\sigma$-complete, $T$-separable locally convex lattice with the M. S. semi-$o$-continuous topology. Then $E$ is M. S. $o$-continuous.

By Theorem 3.2, 3.3 and every $E_\alpha$ being $T$-separable, our problem can be reduced to similar one about $E_\alpha$ which is known to Nakano ([1] Theorem 30.27).

Finally, we shall give an example of the $\mathfrak{L}\mathfrak{F}$-lattice and discuss about it.

EXAMPLE. Let $G$ be a locally compact, $\sigma$-compact Hausdorff space (espaces localement compacts dénombrables à l'infini) (Bourbaki [4] chap. 1, § 10, n°11). Then there exists a sequence $\{G_n\}$ of open sets such that every $G_n$ is relatively compact and $\hat{G}_n$ (the closure of $G_n$) is included in $G_{n+1}$ and further, $G=\bigcup G_n$. Denote by $\mathfrak{K}(G)$ the totality of all real-valued continuous functions with compact carriers defined on $G$. For each continuous function $h(t)$ on $G$ such that $h(t)>0$ always, denote by $V(h)$ the totality of all $f(t)\in\mathfrak{K}(G)$ such that $|f(t)|\leqq h(t)$ for all $t\in G$. If we take all $V(h)$ as neighborhoods of the origin, $\mathfrak{K}(G)$ becomes an $\mathfrak{L}\mathfrak{F}$-lattice (Bourbaki [5] chap. 3, p. 64; Dieudonné et Schwartz [1]).

Let $X$ be a locally compact Hausdorff space and denote by $\mathfrak{C}_k(X)$ the totality of all real-valued continuous functions on $X$ with compact carriers. A function lattice $E$ defined on $X$ will be called *completely separating* if to every (ordered) pair $(s, t)$ of points of $X$, we can find an $f\in E$ such that $f(s)=1$, $f(t)=0$. If $E$ be a completely separating vector sublattice of $\mathfrak{C}_k(X)$, then every maximal ideal of $E$ coincides with the set $N_s=\{f\in E: f(s)=0\}$ for some $s\in X$.

In fact, if this is false, we could find a completely separating maximal ideal $M$ of $E$ and we can see $M$ is uniformly dense in $\mathfrak{C}_k(X)$ (Loomis [1] p. 8, Lemma 4.C). Thus, for every compact subset $K$ of $X$, we can find $f\in M$ such that $f(t)\geqq 1$ on $K$ and it follows every $g\in E$ is contained in $M$, which is absurd.

Now, a one to one correspondence between the totality of all maximal ideals of $E$ and the totality of all points of $X$ is established and we can identify $s\in X$ with the maximal ideal $N_s$.

The *kernel* of a set of maximal ideals is the ideal which is their intersection. The *hull* of an ideal $I$ is the set of all maximal ideals which include $I$.

Then, for each subset $S$ of $X$, the closure $\hat{S}$ coincides with the hull (kernel $(S)$).

Indeed, let $I_S$ be the kernel of $S$, that is $I_S=\{f\in E: f=0 \text{ on } S\}$, then $I_S=I_{\hat{S}}$.

If $p$ is not in $\hat{S}$, take $f\in E$ with carrier $K$ such that $f(p)=1$ and $f>0$. For every $q\in K\cap\hat{S}$ (if not void), we can find $g_q\in E$ such that $g_q(p)=1$, $g_q(q)=0$ and $g_q>0$. Choose a neighborhood $V(q)$ of $q$ such that $g_q\leqq 1/4$ on $V(q)$. Since $K\cap\hat{S}$ is compact, there exists a finite set of $V(q)$ such that $K\cap\hat{S}$ is included in $\bigcup_{i=1}^{m} V(q(i))$. Set

$h=\bigcap_{i=1}^{m} g_{q(i)} \cap f$, then $0<h \in E$, $h(p)=1$, $h \leqq 1/4$ on $K \cap \hat{S}$ and $h=0$ on the complement of $K$. Similarly, we can take $0<h' \in E$ such that $h'(p)=0$ and $h' \geqq 1/2$ on $K \cap \hat{S}$.

Now, $(h-h')^+ \in E$ vanishes on $\hat{S}$ but not at $p$. Thus, $I_{\hat{S}} \subset I_p$ if and only if $p \in \hat{S}$ and we get hull $(I_{\hat{S}})=\hat{S}$.

By this observation, $E$ determines $X$ up to a homeomorphism.

In particular, $\mathfrak{K}(G)$ *as a vector lattice determines* $G$ *up to a homeomorphism.*

Next we shall be concerned with the approximation theorem.

THEOREM 6.5. *Let* $A$ *be a linear subspace of* $\mathfrak{K}(G)$ *which is closed under the lattice operations:* $f \cup g$ *and* $f \cap g$. *Then* $A$ *is dense in* $\mathfrak{K}(G)$ *if and only if* $A$ *is completely separating.*

When this is the case, for given $h \in \mathfrak{K}(G)$, take any open relatively compact set $U$ which contains the carrier of $h$. Then, for any $\varepsilon>0$, we can take $g_\varepsilon \in A$ such that $|h(t)-g_\varepsilon(t)|<\varepsilon$ uniformly on $G$ and the carrier of $g_\varepsilon$ is contained in $\hat{U}$ (the closure of $U$).

PROOF. We shall see the sufficiency.

Let $h$, $U$, $\hat{U}$ have the significances indicated in the statements of our present theorem. Let $A'$ be the totality of all elements of $A$ vanishing on the complement of $U$.

We shall show $A'$ is completely separating on $U$. Let $s, t \in U$. Now, by following the same line as the argument in the proof which will be found just before the present theorem, we can see there exists $0<f \in A$ such that $f(s)=1$ and $f(u)=0$ if $u \notin U$. Take $0<g \in A$ with $g(s)=1$, $g(t)=0$, then $f \cap g \in A'$ and $f \cap g(s)=1$, $f \cap g(t)=0$. Now, we have seen $A'$ is completely separating on $U$.

Considering on $\hat{U}$, we can see easily $h$ can be approximated uniformly by functions of $A'$ (Loomis [1] p. 8, Lemma 4.C). Q. E. D.

We would conclude this section with the characterization of $\mathfrak{K}(G)$.

THEOREM 6.6. *An* $\mathfrak{LF}$*-lattice* $E$ *is isomorphic to some* $\mathfrak{K}(G)$ *if and only if there exists a sequence* $\{e_n\}$ *of positive elements of* $E$ *such that whenever* $x \in E$, *then we can find an* $e_n$ *and real* $\lambda>0$ *with* $|x| \leqq \lambda e_n$.

PROOF. We shall show the sufficiency.

Let $\{E_n\}$ be an ascending sequence of subspaces which defines $E$.

(1): In view of Theorem 6.2 and Remark after this theorem, we may assume

($\alpha$): every $E_n$ is a normal subspace of $E$,

($\beta$): whenever $x\in E_n$, then we can find real $\lambda>0$ with $|x|\leqq\lambda e_n$,

($\gamma$): $0\leqq e_n\uparrow$, $e_n\in E_{n+1}$ and $\notin E_n$.

Further we shall show we can assume the following ($\delta$) without loss of generality:

($\delta$): $e_{n+1}-e_n\in E_n^\perp$.

We define a sequence $\{\bar{e}_n\}$ by induction in such a way that $\{\bar{e}_n\}$ have all required properties ($\beta$), ($\gamma$), ($\delta$).

Put $\bar{e}_1=e_1$ and suppose $\bar{e}_1, \bar{e}_2, \bar{e}_3, \cdots, \bar{e}_n$ have been defined.

Set $K=\{x\in E_n: 0\leqq x\leqq\bar{e}_n\cup e_{n+1}\}$. Define $||x||$ and $|||x|||$ for each $x\in E_n$ by $||x||=\inf\{\lambda: |x|\leqq\lambda\bar{e}_n\}$ and $|||x|||=\inf\{\lambda: |x|\leqq\lambda(\bar{e}_n\cup e_{n+1})\}$. In view of Theorem 6.3, we can see the topology $T$ of $E$ coincides with $||\ ||$ ($|||\ |||$) on $E_n$ and thus $||\ ||$ and $|||\ |||$ are equivalent on $E_n$. Now $\{||x||: x\in K\}$ is bounded and then we put $\lambda_0=\sup\{||x||: x\in K\}$. Clearly $x\in K$ implies $x\leqq\lambda_0\bar{e}_n$ and $\lambda_0\geqq 1$.

Set $e_{n+1}\cup\bar{e}_n=(e_{n+1}\cup\bar{e}_n)\cap\lambda_0\bar{e}_n+h$, then $h>0$. We shall show $h\in E_n^\perp\cap E_{n+2}$. If the contrary were true, we could take $y$ with $0<y\leqq h$, $y\in E_n$. For every $x\in K$, we have $0\leqq x\leqq(\bar{e}_n\cup e_{n+1})\cap\lambda_0\bar{e}_n$ and thus $e_{n+1}\cup\bar{e}_n\geqq x+y>0$, $x+y\in E_n$ which implies $x+y\in K$. Now we can see $x+ky\in K$ for each positive integer $k$, which implies $y=0$, but this is absurd.

Next, put $\bar{e}_{n+1}=\bar{e}_n+h$, then by $e_{n+1}\leqq\lambda_0\bar{e}_{n+1}$, we can see easily $\bar{e}_{n+1}$ is the desired one.

(2): We shall consider the quotient vector lattice $E/E_n^\perp$. Denote by $[a]_n$ the coset containing $a$. Then $[e_n]_n=[e_{n+1}]_n=\cdots=[e_{n+k}]_n=\cdots$ and $[e_n]_n$ is an archimedean unit of $E/E_n^\perp$.

(3): Since $E/E_n^\perp$ has an archimedean unit, then $E/E_n^\perp$ is semi-simple in the wide sense. From this, we can see $E$ is semi-simple (in the wide sense).

(4): Let $G=\{M\}$ be the totality of all maximal ideals of $E$ and put $\Omega_n=\{M: M\supset E_n^\perp\}$, then $\Omega_n\uparrow G$. For each $M\in G$, take any $n$ with $M\in\Omega_n$, then we have $e_n\notin M$. Let $x\in E$ be expressed as a sum $x=\lambda e_n+m$ where $\lambda$ is real and $m\in M$, then put $x(M)=\lambda$. The function $x(\cdot)$ is now defined uniquely on $G$ and we have $e_n(M)=1$ if $M\in\Omega_n$, $e_n(M)=0$ if $M\notin\Omega_{n+1}$. $E$ is thus isomorphic (as a vector lattice) to a completely separating lattice $E(G)$ of functions defined on $G$ and $G$ is locally compact in the weak topology defined by the functions of $E(G)$.

In fact, suppose a net $x(M_\lambda)$ is convergent for each $x\in E$. Put $\lim x(M_\lambda)=f(x)$, then $f\in\tilde{E}$ such that $f(x \overset{\cup}{\cap} y)=f(x)\overset{\cup}{\cap}f(y)$ and if $f\neq 0$, there exists $e_n$ such that $e_n(M_\lambda)>0$ for $\lambda\geqq\lambda_0$. $e_n\notin M_\lambda$ implies $E_{n+1}^\perp\subset e_n^\perp\subset M_\lambda$. Hence $e_{n+1}(M_\lambda)=1$ and it follows $f(e_{n+1})=1$. From this, we can see easily there exists an $M\in G$ such that $f(x)=x(M)$ for each $x\in E$. Now, by the well-known reasoning (Loomis [1] p. 53) and by the fact that if we take $M_n\in\Omega_{n+1}-\Omega_n$ ($\Omega_{n+1}\neq\Omega_n$ will be seen by the following (5)), $x(M_n)\to 0$ for each $x\in E$, we can see $G$ is locally compact and non-compact.

Compactness of $\Omega_n$ can be proved in the same way as above. Then $G$ is $\sigma$-compact.

(5): $E_n$ coincides with the totality of all elements of $E$ having carriers contained in $\Omega_n$.

We shall show $x\in E_n$ if and only if $x(M)=0$ for each $M\notin\Omega_n$. Let $M\notin\Omega_n$ and $x\in E_n$. If $x\notin M$, then we have $M\supset x^\perp\supset E_n^\perp$ which is absurd. Hence we have $x(M)=0$.

Conversely let $x(M)=0$ for each $M\notin\Omega_n$ and $x\geqq 0$. If $0\leqq a\leqq x$, $a\in E_n^\perp$, then $a(M)=0$ for each $M\in\Omega_n$ and thus $a(M)=0$ for each $M\in G$. In view of Lemma 2.3.1, we have $x=\bigcup\{y: 0\leqq y\leqq x, y\in E_n\}$ and since $E_n$ is a normal subspace, we have $x\in E_n$.

(6): The topology on $E_n$ is equivalent to the uniform convergence on $\Omega_n$. Indeed, if we put $\|x\|_n=\inf\{\lambda: |x|\leqq\lambda e_n\}$ for every $x\in E_n$, the topology on $E_n$ is equivalent to $\|\ \|_n$ -topology.

(7): We shall consider $\mathfrak{K}(G)$. Let $f\in\mathfrak{K}(G)$ and $K$ be its carrier. Since $e_n=1$ on $\Omega_n$ and $e_n=0$ on the complement of $\Omega_{n+1}$, we can see the interior $\Omega_{n+1}^0$ of $\Omega_{n+1}$ contains $\Omega_n$. Hence we can take $n$ with $K\subset\Omega_n^0$.

In view of Theorem 6.5, the uniform closure of $E_n$ contains $f$ and by (6) and $T$-completeness of $E_n$, $E_n$ contains $f$. Now we have $E=\mathfrak{K}(G)$.

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