

Ordinal diagrams.

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In his paper [2] on the consistency-proof of the theory of natural numbers, G. Gentzen assigned to every proof-figure an ordinal number. In modifying his method, we may do this as follows:

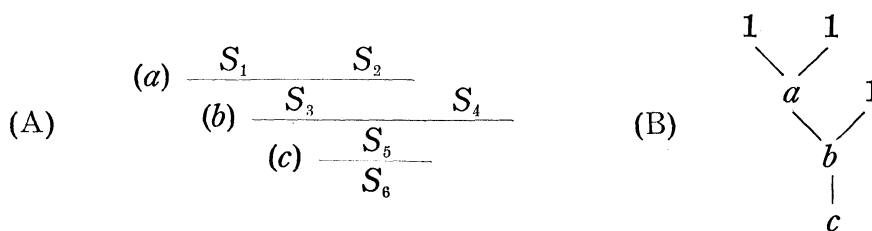


Fig. 1

Suppose, to fix our idea, a proof-figure (A) (in Fig. 1) is given. This is composed of beginning sequences S_1, S_2, S_4 and inferences $(a), (b), (c)$. To the inferences: weakening, contraction and exchange, we assign the value 0; to a cut of degree n , the value n ; to an induction of degree n , the value $n+1$; and the value 1 to all other inferences. We denote the values of inferences $(a), (b), (c)$ by a, b, c respectively. We replace the beginning sequences by 1, and draw the figure (B) according to the form of the proof-figure (A).

If we consider $\bigvee_a^{\alpha \beta}$ and $\big|_a^\alpha$ (α, β being ordinal numbers and a a non-negative integer) as operations defining ordinal numbers (to be defined properly, see below), then the figure like (B) represents itself an ordinal number. This may be called 'Gentzen's number' for the proof-figure (A). Although this is not the same ordinal number as assigned to (A) by Gentzen himself, we can accomplish the consistency-proof of the theory of natural numbers just as in [2], in proving that this 'Gentzen's number' is diminished by the reduction of the proof-figure.

The operations $\bigvee_a^{\alpha \beta}$ and $\big|_a^\alpha$ can be described by Ackermann's

construction in [1]. We shall write for simplicity (α, β) instead of Ackermann's $(1, \alpha, \beta)$, and use $\alpha + \beta$ in the meaning of natural sum in general, while Ackermann uses it only in case $\alpha \geq \beta$. Then $\bigvee_a^{\alpha \beta}$

and $\big|_a^\alpha$ mean $(a, \alpha + \beta)$ and (a, α) respectively. $((a, \alpha)$ is defined in [1] only for $a \geq 1$. We put $(0, \alpha) = \alpha$.)

The purpose of the present paper is to construct a system of ordinal numbers of the second "Zahlenkasse", represented by what we shall call "ordinal diagrams". Presumably our system contains the system constructed by Ackermann [1], but it is not proved. We have in view to apply our result to consistency-proof.

Ordinal diagrams are constructed in the following way. Consider 'trees' of the following form: e. g.

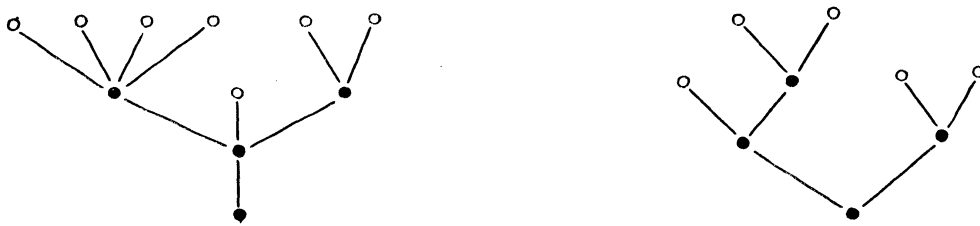
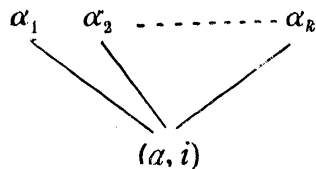


Fig. 2

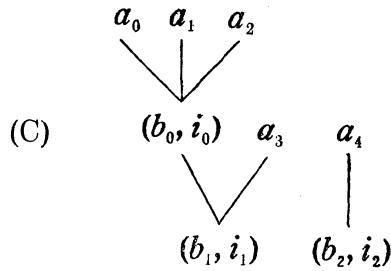
Such trees have two sorts of vertices, 'beginning vertices' marked with \circ and 'non-beginning vertices' marked with \bullet . We assign to each vertex a positive integer called 'value' of the vertex, and to each non-beginning vertex a positive integer called 'index' of the vertex, not exceeding an integer $n (> 0)$ fixed once for all, which we shall call the *order* of the system. If we consider



as 'operation' on diagrams and denote it by $(i; a, \alpha_1 + \dots + \alpha_k)$ (i is the index and a the value of the vertex (a, i)), then a diagram like

(C) can be described by

$$(i_1; b_1, (i_0; b_0, a_0 + a_1 + a_2) + a_3) + (i_2; b_2, a_4).$$



In the following lines, we shall give the formal definition of ordinal diagrams and the ordering between them, and prove that they are well-ordered.

In view of applications to consistency-proof, we should like to add here the following remark. If we denote the system of ordinal diagrams of order n with $O(n)$, it is clear that we have $O(1) \subset O(2) \subset \dots$ and it will be proved as was said above, that $O(n)$ is well-ordered. It will be also proved that $\bigcup_n O(n)$ is not well-ordered.

Let \tilde{N} be some theory including the theory N of natural numbers. A consistency-proof of such theory \tilde{N} may be carried out as follows. To each proof-figure P in \tilde{N} , we assign an ordinal diagram of a certain order n , and prove that the ordinal diagram is ‘diminished’ by a reduction of the proof-figure. This will not be in contradiction with Gödel’s result [3], that the consistency-proof of \tilde{N} is not formulable in \tilde{N} ,—just as Gentzen’s consistency-proof of N is not in contradiction with [3]—and this, even when \tilde{N} is a fairly ‘rich’ theory, in the following sense.

Denote the ordinal number

$$\left. \begin{matrix} \omega \\ \cdot \\ \cdot \\ \cdot \\ \omega \end{matrix} \right\} n$$

with ω_n and let $Q(n)$ mean the system of ordinal numbers less than ω_n . Then Gentzen has assigned to each proof-figure P in N an ordinal number of $Q(n)$ for a certain n , and proved that this ordinal

number is diminished by reduction. Although the transfinite induction in $Q(n)$ for a given n , and the system $\bigcup_n Q(n)$ itself are both formulable in N , the transfinite induction in $\bigcup_n Q(n)$ is not formulable in N , and thus Gentzen's consistency-proof is not in contradiction with Gödel's result. The same circumstances will arise when we replace $Q(n)$ by our $O(n)$.

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§ 1. Ordinal diagram of order n .

Hereafter let n be a fixed positive integer.

1. *Ordinal diagram of order n* is constructed by two operation $(i; ,)$ ($i=1, 2, \dots, n$) and $\#$, and is defined recursively as follows. (If no confusion is to be feared, we use 'ordinal diagram' or 'o. d.' in place of 'ordinal diagram of order n '. O. d.'s are denoted by $\alpha, \beta, \gamma, \dots$ (possibly with suffixes).

1.1. If a is a positive integer, then a is an o. d.

1.2. If a is a positive integer and α is an o. d., and i is an integer satisfying $0 < i \leq n$, then $(i; a, \alpha)$ is an o. d.

1.3. If α and β are o. d.'s, then $\alpha \# \beta$ is an o. d.

2. Let α, β be o. d.'s, and i an integer satisfying $0 < i \leq n$. We define recursively the relation $\beta \subset_i \alpha$ (to read: β is an i -section of α) as follows:

2.1. If α is an integer, then $\beta \subset_i \alpha$ never holds. (α has no i -section.)

2.2. Let α be of the form $(j; a, \alpha_0)$.

2.2.1. If $j < i$, then $\beta \subset_i \alpha$ if and only if $\beta \subset_i \alpha_0$.

2.2.2. If $j = i$, then $\beta \subset_i \alpha$ if and only if β is α_0 .

2.2.3. If $j > i$, then $\beta \subset_i \alpha$ never holds.

2.3. Let α be of the form $\alpha_1 \# \alpha_2$. Then $\beta \subset_i \alpha$ if and only if either $\beta \subset_i \alpha_1$ or $\beta \subset_i \alpha_2$ holds.

3. An o. d. α is called a c. o. d. (*connected ordinal diagram*), if and only if the operation used in the final step of construction of α is not $\#$.

Let α be an o. d. We define *components* of α recursively as follows:

3.1. If α is a c. o. d., then α has only one component which is α itself.

3.2. If α is an o. d. of the form $\alpha_1 \# \alpha_2$ and components of α_1 and α_2 are β_1, \dots, β_k and r_1, \dots, r_l respectively, then components of $\alpha_1 \# \alpha_2$ are $\beta_1, \dots, \beta_k, r_1, \dots, r_l$.

4. Let α and β be o. d. 's. We define $\alpha = \beta$ recursively as follows:

4.1. Let α be an integer. Then $\alpha = \beta$, if and only if β is the same integer as α .

4.2. Let α be an o. d. of the form $(i; a, \alpha_0)$. Then $\alpha = \beta$, if and only if β is of the form $(i; a, \beta_0)$ and $\alpha_0 = \beta_0$.

4.3. Let α be a non-connected o. d. with k components $\alpha_1, \dots, \alpha_k$. Then $\alpha = \beta$, if and only if β has the same number of components, and β_1, \dots, β_k being these components, there exists a permutation (l_1, \dots, l_k) of $(1, \dots, k)$ such that $\alpha_m = \beta_{l_m}$, $m = 1, \dots, k$.

4.4. $\beta = \alpha$ holds, if and only if $\alpha = \beta$.

5. Let α and β be two o. d. 's. We define the relations $\alpha <_0 \beta$, $\alpha <_1 \beta, \dots, \alpha <_n \beta$ recursively as follows. Sometimes $\alpha <_0 \beta$ is denoted by $\alpha \ll \beta$ and $\alpha <_n \beta$ by $\alpha < \beta$.

5.1. Let α and β be two integers. Then $\alpha <_0 \beta, \dots, \alpha <_n \beta$ all mean $\alpha < \beta$ in the sense of integer.

5.2. Let the components of α and β be $\alpha_1, \dots, \alpha_k$ and β_1, \dots, β_h respectively. $\alpha <_i \beta$ ($i = 0, 1, \dots, n$) holds, if and only if one of the following conditions is fulfilled.

5.2.1. There exists β_m ($1 \leq m \leq h$) such that for every l ($1 \leq l \leq k$) $\alpha_l <_i \beta_m$ holds.

5.2.2. $k = 1$, $h > 1$ and $\alpha_1 = \beta_m$ for suitable m ($1 \leq m \leq h$).

5.2.3. $k > 1$, $h > 1$ and there exist α_l ($1 \leq l \leq k$) and β_m ($1 \leq m \leq l$) such that $\alpha_l = \beta_m$ and

$$\alpha_1 \# \dots \# \alpha_{l-1} \# \alpha_{l+1} \# \dots \# \alpha_k <_i \beta_1 \# \dots \# \beta_{m-1} \# \beta_{m+1} \# \dots \# \beta_h.$$

5.3. Let α and β be c. o. d. 's. Then $\alpha <_i \beta$ ($i = 1, 2, \dots, n$), if and only if one of the following conditions is fulfilled.

5.3.1. There exists an i -section β_0 of β such that $\alpha \leq_i \beta_0$.

5.3.2. $\alpha_0 <_i \beta$ for every i -section α_0 of α , and $\alpha <_{i-1} \beta$.

5.4. Let α and β be c. o. d. of the form $(i; a, \alpha_0)$ and $(j; b, \beta_0)$ respectively. $\alpha \ll \beta$, if and only if one of the following conditions is fulfilled.

5.4.1. $a < b$.

5.4.2. $a = b$ and $j < i$.

5.4.3. $a = b$, $i = j$ and $\alpha_0 <_i \beta_0$.

5.5. Let a be a positive integer and β be a c. o. d. of the form $(j; b, \beta_0)$. Then $a \ll \beta$, if and only if $a \leq b$. And $\beta \ll a$, if and only if $b < a$.

Under these definitions the following propositions are easily proved.

PROPOSITION 1. $=$ is an equivalence relation between o. d.'s, i. e. $\alpha = \alpha$ and $\alpha = \beta, \beta = \gamma$ imply $\alpha = \gamma$.

PROPOSITION 2. $\alpha_1 = \alpha_2, \beta_1 = \beta_2$ imply $\alpha_1 \# \beta_1 = \alpha_2 \# \beta_2, (i; \alpha, \alpha_1) = (i; \alpha, \alpha_2)$.

PROPOSITION 3. $\alpha_1 = \alpha_2, \beta_1 = \beta_2, \alpha_1 <_i \beta_1$, imply $\alpha_2 <_i \beta_2$.

PROPOSITION 4. Everyone of the relations $<_i$ ($i = 0, 1, \dots, n$) defines a linear order between o. d.'s, i. e. $\alpha <_i \beta, \beta <_i \gamma$ imply $\alpha <_i \gamma$; and one and only one relation $\alpha <_i \beta, \alpha = \beta, \beta <_i \alpha$ holds for every pair of o. d.'s α, β .

§ 2. Transfinite induction.

1. Let \mathfrak{S} be a system with a linear order. An element s of \mathfrak{S} is called 'accessible in this system (or accessible for this order)', if the subsystem of \mathfrak{S} consisting of elements, which are not 'greater' than s , is well-ordered. The following propositions are easily proved.

PROPOSITION 1. Let α be an o. d. If every o. d. less than α in the sense of $<_i$ is accessible for $<_i$, then α is accessible for $<_i$.

PROPOSITION 2. Let α be an o. d. If α is accessible for $<_i$, then every o. d. less than α in the sense of $<_i$ is accessible for $<_i$.

PROPOSITION 3. Let $\alpha_1, \dots, \alpha_k$ be o. d.'s. If $\alpha_1, \dots, \alpha_k$ are accessible for $<_i$, then $\alpha_1 \# \dots \# \alpha_k$ is accessible for $<_i$.

2. Let α be an o. d. and i an integer satisfying $0 \leq i \leq n$. We define recursively ' α is an i -fan' and ' α is i -accessible' as follows:

2.1. Every o. d. is an n -fan.

2.2. α is i -accessible, if and only if α is an i -fan and α is accessible for $<_i$ in the system of i -fans.

2.3. α is an i -fan ($0 \leq i \leq n$), if and only if α is an $(i+1)$ -fan and every $(i+1)$ -section of α is $(i+1)$ -accessible.

Every 0-fan is also called a *fan*. A fan α is said to be 'accessible in the sense of fan', if α is 0-accessible. We see clearly that propositions 1, 2, and 3 remain correct, if we replace

'o. d.' with '*i*-fan' and 'accessible for $<_i$ ' with '*i*-accessible'.

We obtain easily the following propositions.

PROPOSITION 4. *The following two conditions on an o. d. α are equivalent:*

2.4. α is accessible for $<$.

2.5. α is *n*-accessible.

PROPOSITION 5. *If α is an *i*-fan, then α is an $(i+1)$ -fan.*

PROPOSITION 6. *If every positive integer is *i*-accessible, then every *i*-fan is *i*-accessible.*

PROOF. Let α be an *i*-fan and a be the maximal number of integers, of which α is composed. Then clearly $\alpha <_i (a+1)$, whence the proposition 6 follows directly.

PROPOSITION 7. *Every fan is accessible in the sense of fan.*

3. Now we shall prove the following proposition.

PROPOSITION 8. *If every $(i-1)$ -fan is $(i-1)$ -accessible, then every *i*-fan is *i*-accessible ($i=1, 2, \dots, n$).*

PROOF. Let α be an arbitrary $(i-1)$ -fan. By the proposition 6 we have only to prove that α is *i*-accessible. Without loss of generality, we may assume the following condition 3.1 on α :

3.1. β is *i*-accessible, if β is an $(i-1)$ -fan and $\beta <_{i-1} \alpha$.

Now, let γ be an arbitrary connected *i*-fan and suppose $\gamma <_i \alpha$. We have only to prove that γ is *i*-accessible. We prove this by induction on the number of operations in the construction of γ . If γ has no *i*-section, then γ is an $(i-1)$ -fan and one of the following conditions follows from $\gamma <_i \alpha$:

3.2. $\gamma <_{i-1} \alpha$.

3.3. There exists an *i*-section δ of α such that $\gamma \leq_i \delta$.

In case 3.2, the proposition 8 follows from 3.1. In case 3.3, the proposition 8 follows from the condition that α is an $(i-1)$ -fan.

Now, suppose γ has an *i*-section. Since every *i*-section of γ is less than α for $<_i$ and is an *i*-fan, it follows from the hypothesis of the induction, that every *i*-section of γ is *i*-accessible. Hence γ is an $(i-1)$ -fan. Therefore, from $\gamma <_i \alpha$ one of the following conditions follows:

3.4. $\gamma <_{i-1} \alpha$.

3.5. There exists *i*-section δ_0 of α such that $\gamma \leq_i \delta_0$.

In case 3.4, the proposition 8 follows from 3.1. In case 3.5, the proposition 8 follows from the condition that α is an $(i-1)$ -fan.

From propositions 7 and 8 follows:

THEOREM. *The system of all the o. d.'s is well-ordered for $<$.*

§ 3. **Some properties of o. d.'s.**

The following propositions on o. d.'s follow easily from the above.

PROPOSITION 1. *Let α and β be c. o. d.'s and i be an integer satisfying $0 < i \leq n$. If $\alpha_0 <_j \beta$ holds for every j satisfying $j \leq i$ and for every j -section α_0 of α and $\alpha \ll \beta$, then $\alpha <_i \beta$.*

PROPOSITION 2. *Let α be a c. o. d. and β be an i -section of α . Then $\beta <_i \alpha$.*

PROPOSITION 3. *Let α and β be c. o. d.'s and i, k integers satisfying $0 < i \leq n$, and $0 < k \leq i$ respectively. If α_0 is a k -section of α and the following conditions 1.1~1.3 are fulfilled, then $\alpha <_i \beta$.*

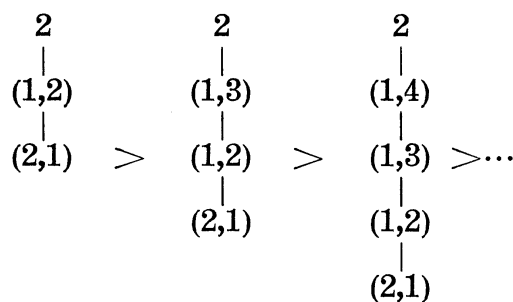
1.1. Let j be any integer satisfying $0 < j \leq i$ and α_1 a j -section of α other than α_0 . Then there exists a j -section β_1 of β such that $\alpha_1 \leq_j \beta_1$.

1.2. $\alpha_0 <_k \beta$.

1.3. $\alpha \ll \beta$.

PROPOSITION 4. *In the notation of the introduction $\bigcup_n O(n)$ is not well-ordered.*

PROOF. This is easily seen by the following example.



References

- [1] W. Ackermann: Konstruktiver Aufbau eines Abschnitts der zweiten Cantorschen Zahlenklasse; *Math. Z.* **53** (1951), 403–413.
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 - [3] K. Gödel: Über formal unentscheidbare Sätze der Principia Mathematica und verwandter System I; *Monatsh. f. Math. Phys.* **38** (1931), 173–198.
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