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Some remarks on tensor invariants of $O(n)$, $U(n)$, $Sp(n)$.

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1. In a paper of Kuiper-Yano [1], tensor invariants of order $\leqq 4$ of special orthogonal groups $SO(n)$ are determined, and the results obtained are applied on the geometry of Riemannian spaces and Finsler spaces.

Using an analogous method on the tensor invariants of the real representations of unitary groups $U(n)$, T. Fukami [2] has obtained corresponding results for $U(n)$, and applied them on hermitian and Kaehlerian spaces.

Now, as we shall show in this note, these problems can be treated more conveniently by intrinsic method than in using tensor components. Thus the results of [1], [2] can be easily generalized for tensor invariants of higher orders, and the cases of groups $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ and $Sp(n)$ can be treated in parallel. In Kuiper-Yano [1], tensor invariants of the subgroup of $SO(n)$ consisting of proper orthogonal transformations which fix a given vector are also determined, however the corresponding problem is not treated in [2]. We shall also show that for the groups $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ and $Sp(n)$, the problem of the determination of tensor invariants of the subgroup consisting of transformations which fix a subspace elementwise is reduced to that of original groups.

2. Let G be a group and (ρ, U) a representation of G on a vector space U. Then

$$U^{\sharp}=U^{\sharp}(G)=\{x\in U;\ \rho(\sigma)x=x \quad \text{for every } \sigma \text{ in } G\}$$

is a subspace of U. An element of $U^{\sharp}$ is called invariant of G in the representation (ρ, U). Now, let G be a group of linear transformations of a vector space V, and U the space of all tensors of type (r, s) over V, i.e.

$$U=V_s^r=\underbrace{V\otimes\cdots\otimes V}_{r\text{-times}}\otimes\underbrace{V^*\otimes\cdots\otimes V^*}_{s\text{-times}},$$

where $\otimes$ means the tensor product and V^* means the dual vector space of V. Then, as is well-known, U becomes a representation space of G under the representation ρ defined as follows:

$$\rho(\sigma)(x_1\otimes\cdots\otimes x_r\otimes f_1\otimes\cdots\otimes f_s)=(\sigma x_1)\otimes\cdots\otimes(\sigma x_r)\otimes(\sigma^*f_1)\otimes\cdots\otimes(\sigma^*f_s)$$

where $\sigma\in G$, $x_i\in V$, $f_j\in V^*$ $(i=1,\cdots,r;\ j=1,\cdots,s)$ and σ^* is a linear transformation of V^* given by

$$(\sigma^* f)(x) = f(\sigma^{-1}x) \qquad (f \in V^*, x \in V)$$

i. e.

$$\sigma^* = {}^t\sigma^{-1} \qquad ({}^t\sigma\text{: the transposed of } \sigma)$$

In this case the invariant of G in the representation (ρ, U) is called *tensor invariant* of type (r, s) of G.

3. Now let V be an n-dimensional real vector space with positive definite, symmetric bilinear form (x, y). Then denoting by $GL(V)$ the group of all non-singular linear transformations, we consider the following subgroups of $GL(V)$:

The orthogonal group:
$O(n) = \{\sigma \in GL(V)\,;\ (\sigma x, \sigma y) = (x, y) \text{ for every } x, y \text{ in } V\}$,

The special orthogonal group:
$SO(n) = \{\sigma \in O(n)\,;\ \det(\sigma) = 1\}$.

We remark that, in these cases V and V^* give equivalent representations of $O(n)$, $SO(n)$. In fact, $a \in V$ defines an element $f_a \in V^*$ by $f_a(x) = (a, x)$. Then the linear mapping $a \to f_a$ gives an equivalence of V and V^*. Thus, we can identify V and V^* by identifying $a \in V$ and $f_a \in V^*$.[1)]

Then V_s^r, V_0^{r+s} and V_{r+s}^0 are also identified.[2)] Hence concerning $O(n)$, $SO(n)$, it is sufficient to consider only V_{r+s}^0 instead of V_s^r.

Now, a tensor f of type $(0, p)$, i. e. p-linear form f on V, is invariant of a subgroup G of $GL(V)$, if and only if

(1) $$f(\sigma x_1, \cdots, \sigma x_p) = f(x_1, \cdots, x_p)\,. \qquad (\sigma \in G, x_i \in V)$$

We denote by $V_p^0(G)^\#$ the subspace of V_p^0 consisting of p-linear forms f on V which satisfy (1), and put

$$\tilde{\nu}_n^{(p)} = \dim V_p^0(O(n))^\#$$
$$\nu_n^{(p)} = \dim V_p^0(SO(n))^\#\,.$$

Then we have immediately

(2) $$\tilde{\nu}_n^{(p)} = 0\,, \qquad \text{if } p \text{ is odd}$$

(3) $$\nu_n^{(p)} = 0\,, \qquad \text{if } p \text{ is odd and } n \text{ is even.}$$

1) In tensor components, the components of f_a are $\sum g_{ij}a^j$, where the a^j, g_{ij} are the components of a and the fundamental covariant tensor (x, y) respectively.

2) For example for a tensor $R^i{}_{jk}$ of type (1,3) there corresponds a tensor of type (0,4) $R_{ijkl} = \sum g_{i\alpha} R^\alpha{}_{jkl}$. More intrinsically, for a tensor of type (1,3) R, i. e. multilinear mapping R from $V \times V \times V$ into V, there corresponds a tensor of type (0,4) $\tilde{R}$, i. e. 4-linear forms on V, defined as follows:

$$(x_1, R(x_1, x_2, x_3)) = \tilde{R}(x_1, x_2, x_3, x_4)\,.$$

Then $\tilde{R}$ is $O(n)$-invariant if and only if R is $O(n)$-invariant. We identify R and $\tilde{R}$.

In fact, let $f(x_1,\cdots,x_p)\in V_p^0(O(n))^\#$. Then, as $(-1)I\in O(n)$,[3] we have

$$f(x_1,\cdots,x_p)=f(-x_1,\cdots,-x_p)=(-1)^p f(x_1,\cdots,x_p).$$

Hence, if p is odd, $f=-f$. Thus we have $f=0$ and (2) is proved. (3) is similarly proved.

Now let us consider $\mathfrak{V}_n^{(p)}$ for even p.

THEOREM 1. *If $p\leqq n$, we have*

$$\mathfrak{V}_n^{(2p)}=(2p-1)(2p-3)\cdots 3\cdot 1.$$

PROOF. We shall say that a mapping ρ from the set of $2p$ integers $\{1,2,\cdots,2p\}$ onto the set of p integers $\{1,2,\cdots,p\}$ is *admissible*, if for every integer i, $1\leqq i\leqq p$, $\rho^{-1}(i)$ consists of two integers. Let us identify two admissible mappings ρ,τ if $\{\rho^{-1}(1),\cdots,\rho^{-1}(p)\}$ and $\{\tau^{-1}(1),\cdots,\tau^{-1}(p)\}$ coincide up to their orders. Let us denote by A_p the set of all admissible mappings identified in this way and by N_p the number of elements in A_p. Then we have easily $N_1=1$, $N_p=N_{p-1}\cdot(2p-1)$ $(p=2,3,\cdots)$. Hence we obtain

$$N_p=(2p-1)(2p-3)\cdots 3\cdot 1.$$

Let us associate to $\rho\in A_p$ a $2p$-linear form F_ρ as follows:

$$F_\rho(x_1,x_2,\cdots,x_{2p})=(x_{k_1},x_{k_1'})(x_{k_2},x_{k_2'})\cdots(x_{k_p},x_{k_p'}) \tag{4}$$

where k_j, k_j' are determined by $\rho^{-1}(j)=\{k_j,k_j'\}$. Then F_ρ is obviously an invariant tensor of $O(n)$. We shall show that these F_ρ's form a base of $V_{2p}^0(O(n))^\#$. Let $e_1,\cdots,e_n$ be a ortho-normal base of V. Let us call an ordered set $\{e_{j_1},e_{j_2},\cdots,e_{j_{2p}}\}$ *of type* ρ $(\in A_p)$, if

$$\rho(j_1)=\rho(j_2),\ \rho(j_3)=\rho(j_4),\cdots,\rho(j_{2p-1})=\rho(j_{2p}). \tag{5}$$

Since $p\leqq n$, we can select for every $\rho\in A_p$ an ordered set

$$e_\rho=\{e_{j_1},e_{j_2},\cdots,e_{j_{2p}}\}$$

of type ρ. We denote $F(e_{j_1},e_{j_2},\cdots,e_{j_{2p}})$ by $F(e_\rho)$ for $F\in V_{2p}^0$. Then we have for $\rho,\rho'\in A_p$

$$F_{\rho'}(e_\rho)=\begin{cases}1 & \text{if } \rho=\rho'\\ 0 & \text{if } \rho\neq\rho'.\end{cases} \tag{6}$$

From (6) we can deduce that F_ρ's are linearly independent. In fact, if there is a linear relation

$$\sum_{\rho\in A_p} c_\rho F_\rho=0$$

with real coefficients c_ρ, then denoting the left hand side by F, we have $0=F(e_\rho)=c_\rho$ by (6).

3) I denotes the identity transformation of V.

Now it is sufficient to show that, for every $F\in V_{2p}^{0}(O(n))^{\#}$ there exist N_p real numbers c_ρ $(\rho\in A_p)$ such that

$$F=\sum_{\rho\in A_p} c_\rho F_\rho .$$

Define c_ρ by $c_\rho=F(e_\rho)$. Then, putting $F'=F-\sum_{\rho} c_\rho F_\rho$ we have $F'(e_\rho)=0$ for every $\rho\in A_p$. Moreover, F' is obviously an invariant tensor of $O(n)$. Let us show that $F'=0$, i.e. for every $e_{k_1},\cdots,e_{k_{2p}}$

$$F'(e_{k_1},\cdots,e_{k_{2p}})=0 . \tag{7}$$

We distinguish now several cases.

Case 1. $\{k_1,k_2,\cdots,k_{2p}\}$ contains a subset $\{k_\alpha,k_\beta,\cdots,k_\gamma\}$ consisting of odd number of elements such that for every $j\notin\{\alpha,\beta,\cdots,\gamma\}$ $(1\leqq j\leqq 2p)$ we have $k_j\notin\{k_\alpha,k_\beta,\cdots,k_\gamma\}$. In this case, there exists a $\sigma\in O(n)$ such that

$$\sigma e_{k_\alpha}=-e_{k_\alpha},\quad \sigma e_{k_\beta}=-e_{k_\beta},\cdots,\sigma e_{k_\gamma}=-e_{k_\gamma}$$

$$\sigma e_{k_j}=e_{k_j}\quad (j\notin\{\alpha,\beta,\cdots,\gamma\})$$

Then

$$F'(e_{k_1},\cdots,e_{k_{2p}})=F'(\sigma e_{k_1},\cdots,\sigma e_{k_{2p}})=-F'(e_{k_1},\cdots,e_{k_{2p}})$$

hence we have (7).

Thus, in the remaining case, for every j, $1\leqq j\leqq 2p$, the number μ_j of h such that $1\leqq h\leqq 2p$, $k_h=k_j$ is even. Denote by μ the maximum of $\mu_1,\mu_2,\cdots,\mu_{2p}$. We remark that in this case the number of different e_j's among $e_{k_1},\cdots,e_{k_{2p}}$ does not exceed p, and coincides with p if and only if $\mu=2$.

Case 2. $\mu\geqq 4$. There exists an integer i, $1\leqq i\leqq n$, such that $i\notin\{k_1,\cdots,k_{2p}\}$. Then we can assume without any loss of generality that $\mu=\mu_1$, $k_1=k_2=k_3=k_4=1$, $i=2$. Now since F' is $O(n)$-invariant, we have for every element S in the Lie algebra of $O(n)$, and for every $x_1,\cdots,x_{2p}\in V$,

$$F'(Sx_1,x_2,\cdots,x_{2p})+F'(x_1,Sx_2,\cdots,x_{2p})+\cdots+F'(x_1,x_2,\cdots,Sx_{2p})=0 .$$

(Remark that a linear endomorphism S of V is in the Lie algebra of $O(n)$ if and only if the matrix of S with respect to an ortho-normal base of V is skew-symmetric.)

Take as S the following linear endomorphism of V:

$$Se_1=-e_2,\ Se_2=e_1,\ Se_j=0\qquad (j=3,4,\cdots,n)$$

Then we have

$$\begin{aligned}&F'(e_{k_1},\cdots,e_{k_{2p}})=F'(Se_2,e_1,e_1,e_1,\cdots)\\&=-F'(e_2,Se_1,e_1,e_1,\cdots)-F'(e_2,e_1,Se_1,e_1,\cdots)-F'(e_2,e_1,e_1,Se_1,\cdots)-\cdots\\&=F'(e_2,e_2,e_1,e_1,\cdots)+F'(e_2,e_1,e_2,e_1,\cdots)+F'(e_2,e_1,e_1,e_2,\cdots)+\cdots .\end{aligned}$$

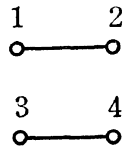
For every term on the right hand side, the number μ_1 is replaced by μ_1-2. Thus continuing this process, we reach finally the situation $\mu=2$.

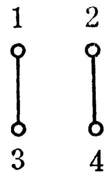
Case 3. $\mu=2$. In this case $\{e_{k_1}, e_{k_2}, \cdots, e_{k_{2p}}\}$ is of type ρ for some $\rho\in A_p$. Then denoting by $\{e_{j_1}, e_{j_2}, \cdots, e_{j_{2p}}\}$ the e_ρ which we have selected, we can find a $\sigma\in O(n)$ such that

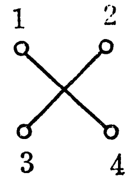
$$\sigma e_{j_s}=e_{k_s} \qquad (s=1,\cdots,2p)$$

Then, $F'(e_{k_1},\cdots,e_{k_{2p}})=F'(e_{j_1},\cdots,e_{j_{2p}})=F'(e_\rho)=0$. Thus the proof is accomplished.

Example 1. $p=2$. A_p consists of 3 elements ρ_1, ρ_2, ρ_3 ($N_2=3$), which is shown in the following diagrams:

ρ_1: 

ρ_2: 

ρ_3: 

where the existence of a segment connecting two integers means that their images under ρ_i coincide with each other. We have as a base of $V_4^0(O(n))^\#$, $n\geqq 2$, the following $F_{\rho_1}, F_{\rho_2}, F_{\rho_3}$:

$$F_{\rho_1}(x_1, x_2, x_3, x_4)=(x_1, x_2)\cdot(x_3, x_4)\,,$$

$$F_{\rho_2}(x_1, x_2, x_3, x_4)=(x_1, x_3)\cdot(x_2, x_4)\,,$$

$$F_{\rho_3}(x_1, x_2, x_3, x_4)=(x_1, x_4)\cdot(x_2, x_3)\,.$$

Theorem 2. (i) *If $p<n$, then $\nu_n^{(p)}=\tilde{\nu}_n^{(p)}$.* (ii) *If $p=n$, then $\nu_n^{(p)}=\tilde{\nu}_n^{(p)}+1$.*

Proof. (i) The proof of theorem 1 is easily seen to be valid in this case also. (ii) Let us orient V by a orthc-normal base $e_1,\cdots,e_n$. Then we denote by $[x_1,\cdots,x_n]$ the following n-form on V:

$$[x_1,\cdots,x_n]=\det(\xi_j{}^i), \text{ where } x_i=\sum_{j=1}^{n}\xi_i{}^j e_j \qquad (i=1,\cdots,n)$$

Then $[x_1,\cdots,x_n]$ is an invariant tensor of $SO(n)$ of type $(0,n)$.[4] It is seen that the base of $V_n^0(O(n))^\#$ which was given in the proof of theorem 1 and $[x_1,\cdots,x_n]$ form a base of $V_n^0(SO(n))^\#$ repeating an analogous discussion as in theorem 1.

Example 2. $p=4$, $n>4$. Every invariant tensor F of $SO(n)$ of type (0,4) is expressed as follows:

$$F(x_1, x_2, x_3, x_4)=c_1(x_1, x_2)(x_3, x_4)+c_2(x_1, x_3)(x_2, x_4)+c_3(x_1, x_4)(x_2, x_3)\,.$$

If moreover, F is skew-symmetric with respect to x_3, x_4, then

$$F(x_1, x_2, x_3, x_4)=c\{(x_1, x_3)(x_2, x_4)-(x_1, x_4)(x_2, x_3)\}\,.$$

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4) In [1], the components of this tensor are denoted by $e_{i_1 i_2\cdots i_n}$.

Remark 1. By means of analogous discussion, it is shown that theorems 1,2 are also valid for tensor invariants of Lorentz groups i. e. a linear group of n variables consisting of linear transformations leaving invariant a quadratic form of type

$$x_1^2+\cdots+x_r^2-x_{r+1}^2-\cdots-x_n^2 .$$

Also, for complex tensor invariants of complex orthogonal group $O(n, C)$, theorem 1,2 are valid. Of course the condition $p \leqq n$ is necessary for all these cases.

Remark 2. If we use results concerning 'vector invariants' of orthogonal groups stated in Chap. II of H. Weyl [3], then, the proof of theorem 1 will be extremely simplified. In fact, the F_ρ's ($\rho \in A_p$) span $V_{2p}^0(O(n))^\#$ for *every* p (not only for $p \leqq n$) by virtue of theorem (2.9.A) of [3]. Hence the linear independence of $\{F_\rho\}$ was sufficient for the proof of theorem 1. We remark also we have for any n, p.

$$\tilde{\nu}_n^{(2p)} \leqq (2p-1)(2p-3)\cdots 3\cdot 1 . \tag{8}$$

Also for $p > n$ every element F of $V_p^0(SO(n))^\#$ is of the form

$$F(x_1,\cdots, x_p)=F_1(x_1,\cdots, x_p)+\sum_{(i)} F_2^{(i)}(x_{i_1},\cdots, x_{i_{p-n}})\cdot[x_{i_{p-n+1}},\cdots, x_{i_p}]$$

where $i_1,\cdots, i_p$ is a permutation of $1,\cdots, p$, and $F_1 \in V_p^0(O(n))^\#$, $F_2^{(i)} \in V_{p-n}^0(O(n))^\#$. (cf. [3], loc. cit.)

4. Now let $\tilde{V}$ be an n-dimensional complex vector space with positive definite hermitian form (x, y). We consider the following subgroups of $GL(\tilde{V})$:

The unitary group

$$U(n)=\{\sigma \in GL(\tilde{V})\,;\,(\sigma x, \sigma y)=(x, y) \text{ for every } x, y \text{ in } \tilde{V}\},$$

The special unitary group

$$SU(n)=\{\sigma \in U(n)\,;\,\det(\sigma)=1\}.$$

Now $\tilde{V}$ can be regarded as a $2n$-dimensional real vector space. We denote this vector space by V. As a set, V coincides with $\tilde{V}$. Then every linear transformation σ of $\tilde{V}$ defines a linear transformation σ' of V, and the homomorphism $\sigma \to \sigma'$ is an into isomorphism from $GL(\tilde{V})$ into $GL(V)$. Thus we regard $GL(\tilde{V})$ as a subgroup of $GL(V)$. Then $U(n), SU(n)$ also become subgroups of $GL(V)$. These subgroups are called the *real representations* of $U(n), SU(n)$. In the following we consider only real representations of $U(n), SU(n)$, so we denote also by $U(n), SU(n)$ the real representations of $U(n), SU(n)$.

We denote by $\Omega_0(x, y)$, $\Omega_1(x, y)$ the real and imaginary parts of the hermitian form (x, y) respectively:

$$(x, y)=\Omega_0(x, y)+\Omega_1(x, y)i .$$

Then, as is easily seen, Ω_0 (resp. Ω_1) is a symmetric (resp. skew-symmetric) real valued bilinear form on the real vector space V. Moreover, $\Omega_0(x, y)$ is positive definite. So we can define the orthogonal group $O(2n)$ over V with respect to $\Omega_0(x, y)$. Then it is easily seen that

$$U(n) \subset SO(2n) .$$

Thus V and V^* are equivalent representation spaces of $U(n) \subset GL(V)$. Hence, concerning tensor invariant of $U(n)$, $SU(n)$, we may consider only V_p^0 as in the case of $O(n)$.

5. Before stating analogues of theorems 1,2 for $U(n)$ and $SU(n)$, we shall consider the real representation of symplectic group.

Let $\tilde{V}$ be an n-dimensinal vector space over the field of quaternions $Q=R+Ri+Rj+Rk$ ($i^2=j^2=k^2=-1$, $ij=-ji=k$, $jk=-kj=i$, $ki=-ik=j$). We shall denote by $GL(\tilde{V})$ the group of all linear transformations of $\tilde{V}$ over Q.

Let (x, y) be a mapping from $\tilde{V} \times \tilde{V}$ into Q which satisfy the following conditions:

$$(9) \qquad \begin{cases} (\alpha x+\beta y, z)=\alpha(x, z)+\beta(y, z) & (\alpha, \beta \in Q\,;\, x, y, z \in \tilde{V}) \\ (x, y)=\overline{(y, x)} & \\ (x, x)>0 \quad \text{for every } x \in \tilde{V},\ x \neq 0 & \end{cases}$$

where $\bar{q}$ means the conjugate of a quaternion $q=\alpha+\beta i+\gamma j+\delta k$, i.e.

$$\bar{q}=\alpha-\beta i-\gamma j-\delta k .$$

Then the symplectic group $Sp(n)$ with respect to the 'metric form' (x, y) is defined as a subgroup of $GL(\tilde{V})$ as follows:

$$Sp(n)=\{\sigma \in GL(\tilde{V})\,;\, (\sigma x, \sigma y)=(x, y) \quad \text{for every } x, y \text{ in } \tilde{V}\} .$$

Now, as in n°4, $\tilde{V}$ can be regarded as $4n$-dimensional real vector space. We shall denote this real vector space by V. As a set, V coincides with $\tilde{V}$. Then every element $\sigma \in GL(\tilde{V})$ defines a linear transformation $\sigma' \in GL(V)$, and $GL(\tilde{V})$ can be regarded as a subgroup of $GL(V)$ using the into isomorphism $\sigma \rightarrow \sigma'$. Then $Sp(n)$ can be regarded also as a subgroup of $GL(V)$. This subgroup is called the real representation of $Sp(n)$. In the following we consider only real representation of $Sp(n)$, which is also denoted by $Sp(n)$.

We denote by $\Omega_0(x, y)$, $\Omega_1(x, y)$, $\Omega_2(x, y)$, and $\Omega_3(x, y)$ the coefficients of $1, i, j, k$ in the 'metric form' (x, y) respestively:

$$(x, y)=\Omega_0(x, y)+\Omega_1(x, y)i+\Omega_2(x, y)j+\Omega_3(x, y)k .$$

Then, as is easily seen, Ω_0(resp. $\Omega_1, \Omega_2, \Omega_3$) is a symmetric (resp. skew-symmetric), real valued bilinear form on the real vector space V. Moreover, $\Omega_0(x, y)$ is positive definite. So we can define the orthogonal group $O(4n)$

over V with respect to $\Omega_0(x, y)$. Then it is easily seen that

$$Sp(n) \subset SO(4n).$$

Thus V and V^* are equivalent representation spaces of $Sp(n) \subset GL(V)$. Hence, concerning the tensor invariants of $Sp(n)$, we may consider only V_p^0 as in the case of $O(n)$.

REMARK. The vector space $\tilde{V}$ can be also regarded as a $2n$-dimensional complex vectorspace $\tilde{V}_c$. Then $GL(\tilde{V})$ and $Sp(n)$ can be regarded as subgroups of $GL(\tilde{V}_c)$. Now 'metric form' (x, y) in $\tilde{V}$ can be written as follows:

$$(x, y) = \Phi_0(x, y) + \Phi_1(x, y)j \tag{10}$$

where $\Phi_0(x, y) = \Omega_0(x, y) + \Omega_1(x, y)i$ and $\Phi_1(x, y) = \Omega_2(x, y) + \Omega_3(x, y)i$ are complex numbers. Then Φ_0 is a positive definite hermitian form on $\tilde{V}_c$. Then, $Sp(n)$ is contained as a subgroup of $GL(\tilde{V}_c)$ in the special unitary group $SU(2n)$ over $\tilde{V}_c$ with respect to $\Phi_0(x, y)$. Thus the following relation holds for real representations of $Sp(n)$, $U(2n)$.

$$Sp(n) \subset SU(2n) \subset U(2n) \subset SO(4n).$$

Now we consider $V_p^0(Sp(n))^\sharp$. Put

$$\kappa_n^{(p)} = \dim V_p^0(Sp(n))^\sharp. \tag{11}$$

As in (2), we have

$$\kappa_n^{(2p-1)} = 0. \tag{12}$$

THEOREM 3. *If $p \leqq n$, we have*

$$\kappa_n^{(2p)} = 4^p \cdot (2p-1)(2p-3) \cdots 3 \cdot 1. \tag{13}$$

PROOF. For every $\rho \in A_p$ (cf. the proof of theorem 1 as to the definition of A_p), and for every $\alpha_1, \alpha_2, \cdots, \alpha_p$, $(0 \leqq \alpha_i \leqq 3)$, we denote by $F_\rho^{\alpha_1 \cdots \alpha_p}$ the following $2p$-form on V:

$$F_\rho^{\alpha_1 \cdots \alpha_p}(x_1, \cdots, x_{2p}) = \Omega_{\alpha_1}(x_{k_1}, x_{k_1'})\Omega_{\alpha_2}(x_{k_2}, x_{k_2'}) \cdots \Omega_{\alpha_p}(x_{k_p}, x_{k_p'})$$

where $\{k_j, k_j'\} = \rho^{-1}(j)$, $k_j < k_j'$ $(j = 1, \cdots, p)$. The $Sp(n)$-invariance of (x, y) implies that of Ω_α $(0 \leqq \alpha \leqq 3)$, so $F_\rho^{\alpha_1 \cdots \alpha_p}$ is also $Sp(n)$-invariant. Now, to prove (13), it is sufficient to show that these $F_\rho^{\alpha_1 \cdots \alpha_p}$'s form a base of $V_p^0(S_p(n))^\sharp$. As is seen easily by induction on n, there exists a base $e_1, \cdots, e_n$ of $\tilde{V}$ such that $(e_i, e_j) = \delta_{ij}$ $(1 \leqq i, j \leqq n)$. Then $\{\varepsilon_i e_j\}$ $(0 \leqq i \leqq 3, 1 \leqq j \leqq n)$ is a base of V, where $\varepsilon_0 = 1$, $\varepsilon_1 = i$, $\varepsilon_0 = j$, $\varepsilon_3 = k$. Now we note that the following relations hold:

$$\begin{cases} \Omega_s(\varepsilon_s e_q, e_r) = \delta_{qr} & (0 \leqq s \leqq 3, 1 \leqq q, r \leqq n) \\ \Omega_s(\varepsilon_t e_q, e_r) = 0 & (0 \leqq s \neq t \leqq 3, 1 \leqq q, r \leqq n). \end{cases} \tag{14}$$

Now let us select for every $\rho \in A_p$ an ordered system $e_\rho = \{e_{j_1}, \cdots, e_{j_{2p}}\}$ of type ρ in the sence of n°3. This is possible by the assumption $p \leqq n$. Then we denote by $e^{\alpha_1\cdots\alpha_p}_{\rho}$ the following ordered system consisting of $\{\varepsilon_t e_s\}$

$$e^{\alpha_1\cdots\alpha_p}_{\rho}=\{\varepsilon_{\alpha_1}e_{k_1}, e_{k_1'}, \varepsilon_{\alpha_2}e_{k_2}, e_{k_2'},\cdots, \varepsilon_{\alpha_p}e_{k_p}, e_{k_p'}\}$$

where $\{k_j, k_j'\}=\rho^{-1}(j)$, $k_j<k_j'$ $(1\leqq j\leqq p)$. Then we have by (14)

$$F^{\alpha_1\cdots\alpha_p}_{\rho}(e^{\beta_1\cdots\beta_p}_{\rho'})=\delta_{\rho\rho'}\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\cdots\delta_{\alpha_p\beta_p}, \tag{15}$$

for $\rho, \rho'\in A_p$ and $0\leqq\alpha_i, \beta_i\leqq 3$. From (15) the linear independence of $\{F^{\alpha_1\cdots\alpha_p}_{\rho}\}$ follows as in the proof of theorem 1.

Next we must show that any $F\in V^0_{2p}(Sp(n))^\#$ is a linear combination of the $F^{\alpha_1\cdots\alpha_p}_{\rho}$'s. To show this, it is sufficient to prove the following lemma as in the proof of theorem 1.

LEMMA. *Let $F\in V^0_{2p}(Sp(n))^\#$ and*

$$F(e^{\alpha_1\cdots\alpha_p}_{\rho})=0$$

for every $e^{\alpha_1\cdots\alpha_p}_{\rho}$ $(0\leqq\alpha_i\leqq 3, \rho\in A_p)$. Then $F=0$, i.e.

$$F(\varepsilon_{h_1}e_{l_1}, \varepsilon_{h_2}e_{l_2},\cdots, \varepsilon_{h_{2p}}e_{l_{2p}})=0 \tag{16}$$

for every $0\leqq h_i\leqq 3$, $1\leqq l_j\leqq n$.

PROOF OF LEMMA. We distinguish several cases.

Case 1. $\{l_1, l_2,\cdots, l_{2p}\}$ contains a subset $\{k_\alpha, k_\beta,\cdots, k_\gamma\}$ consisting of odd number of elements such that for every $j\notin\{\alpha, \beta,\cdots,\gamma\}$ $(1\leqq j\leqq 2p)$ $k_j\notin\{k_\alpha, k_\beta,\cdots, k_\gamma\}$. Then we see as in the case 1 of the proof of theorem 1 that the left hand side of (16) is equal to zero.

Now define $\mu_j(j=1, 2,\cdots, p)$ and μ just as in the proof of theorem 1. Then as in theorem 1, the proof reduces to the following

Case 2. $\mu=2$. Then $\{e_{l_1},\cdots, e_{l_{2p}}\}$ is of type ρ for some $\rho\in A_p$. We may limit ourselves only to the typical case $\rho^{-1}(1)=\{1, 2\}$, $\rho^{-1}(2)=\{3, 4\},\cdots, \rho^{-1}(p)=\{2p-1, 2p\}$. Then we have to show

$$F(\varepsilon_{h_1}e_1, \varepsilon_{h_2}e_1, \varepsilon_{h_3}e_2, \varepsilon_{h_4}e_2,\cdots, \varepsilon_{h_{2p-1}}e_p, \varepsilon_{h_{2p}}e_p)=0, \quad (0\leqq h_i\leqq 3). \tag{17}$$

There is a $\sigma\in Sp(n)$ such that

$$\sigma e_1=\varepsilon^{-1}_{h_2}e_1,\ \sigma e_2=\varepsilon^{-1}_{h_4}e_1,\cdots, \sigma e_p=\varepsilon^{-1}_{h_{2p}}e_p.$$

Then the left hand side of (17) is equal to

$$F(\varepsilon_{h_1}\varepsilon^{-1}_{h_2}e_1, e_1, \varepsilon_{h_3}\varepsilon^{-1}_{h_4}e_2, e_2,\cdots, \varepsilon_{h_{2p-1}}\varepsilon^{-1}_{h_{2p}}e_p, e_p)$$

which in turn is equal to $\pm F(e^{\alpha_1\cdots\alpha_p}_{\rho})$ for some $\alpha_1,\cdots,\alpha_p$. So it is equal to zero by our assumption. Thus the proof of the lemma is completed, and theorem 3 is also proved.

EXAMPLE 3. $p=2, n\geqq 2$. $V^0_4(Sp(n))^\#$ has the following base consisting of 48 elements

$$(18)\qquad \begin{cases} F_{\rho_1}^{\alpha\beta}(x_1, x_2, x_3, x_4)=\Omega_\alpha(x_1, x_2)\Omega_\beta(x_3, x_4) \\ F_{\rho_2}^{\alpha\beta}(x_1, x_2, x_3, x_4)=\Omega_\alpha(x_1, x_3)\Omega_\beta(x_2, x_4) \qquad (0\leqq\alpha, \beta\leqq 3) \\ F_{\rho_3}^{\alpha\beta}(x_1, x_2, x_3, x_4)=\Omega_\alpha(x_1, x_4)\Omega_\beta(x_2, x_3). \end{cases}$$

To determine a base of the subspace $V_4^0(Sp(n))^{\#}_{[3,4]}$ of $V_4^0(Sp(n))^{\#}$ consisting of $F\in V_4^0(Sp(n))^{\#}$ such that

$$F(x_1, x_2, x_3, x_4)=-F(x_1, x_2, x_4, x_3),$$

we apply the alternation operator A_{34} on every $F_{\rho_i}^{\alpha\beta}$, where

$$(19)\qquad (A_{34}F)(x_1, x_2, x_3, x_4)=F(x_1, x_2, x_3, x_4)-F(x_1, x_2, x_4, x_3).$$

Then

$$(20)\qquad \begin{cases} A_{34}F_{\rho_1}^{\alpha 0}=0,\ A_{34}F_{\rho_1}^{\alpha\beta}=2F_{\rho_1}^{\alpha\beta} \quad (0\leqq\alpha\leqq 3,\ 1\leqq\beta\leqq 3) \\ A_{34}F_{\rho_2}^{\alpha\beta}=-A_{34}F_{\rho_3}^{\alpha\beta}=F_{\rho_2}^{\alpha\beta}-F_{\rho_3}^{\alpha\beta}. \end{cases}$$

Thus $4\times 3+16=28$ elements $F_{\rho_1}^{\alpha\beta}$ $(0\leqq\alpha\leqq 3,\ 1\leqq\beta\leqq 3)$, $F_{\rho_2}^{\alpha\beta}-F_{\rho_3}^{\alpha\beta}$ $(0\leqq\alpha, \beta\leqq 3)$ form a base of the subspace $V_4^0(Sp(n))^{\#}_{[3,4]}$ of $V_4^0(Sp(n))^{\#}$.

6. Analogous theorems hold for the real representations of $U(n)$, $SU(n)$ (cf. n°4). Put

$$(21)\qquad \tilde{\mu}_n^{(p)}=\dim V_p^0(U(n))^{\#}$$

and

$$(22)\qquad \mu_n^{(p)}=\dim V_p^0(SU(n))^{\#}.$$

Then we have like (2), (3)

$$(23)\qquad \tilde{\mu}_n^{(2p-1)}=0$$

and

$$(24)\qquad \mu_{2n}^{(2p-1)}=0.$$

THEOREM 4. *If $p\leqq n$, then we have*

$$\tilde{\mu}_n^{(2p)}=2^p\cdot(2p-1)(2p-3)\cdots 3\cdot 1.$$

THEOREM 5. (i) *If $p<n$, then $\mu_n^{(p)}=\tilde{\mu}_n^{(p)}$.* (ii) *If $p=n$, then $\mu_n^{(p)}=\tilde{\mu}_n^{(p)}+2$.*

These theorems are analogously proved as theorems 1, 2, 3, namely, we can define $F_\rho^{\alpha_1\cdots\alpha_p}$ $(\rho\in A_p,\ \alpha_i=0, 1)$ just as in theorem 3. Then these $2^p(2p-1)\cdot(2p-3)\cdots 3\cdot 1$ forms are linearly independent and span $V_{2p}^0(U(n))^{\#}$. We only remark about (ii) of theorem 5. Let us fix an ortho-normal base $e_1, e_2, \cdots, e_n$ of $\tilde{V}$ (n-dimensional complex vector space with positive definite hermitian form (x, y)). Then for $x_i\in\tilde{V}$ $(i=1, \cdots, n)$ we define $[x_1, x_2, \cdots, x_n]_R$ and $[x_1, \cdots, x_n]_I$ as follows:

$$(25)\qquad [x_1, \cdots, x_n]_R=\text{real part of } \det(\xi_i^{\,j}),$$

$$(26)\qquad [x_1, \cdots, x_n]_I=\text{imaginary part of } \det(\xi_i^{\,j})$$

where $x_i=\sum_{i=1}^{n}\xi_i{}^j e_j$ $(i=1,\cdots,n)$. Then $[x_1,\cdots,x_n]_R$ and $[x_1,\cdots,x_n]_I$ are in $V_n^0(SU(n))^\sharp$. It is easily seen as before that the base of $V_n^0(U(n))^\sharp$, $[x_1,\cdots,x_n]_R$ and $[x_1,\cdots,x_n]_I$ form a base of $V_n^0(SU(n))^\sharp$.

EXAMPLE 4. $p=2$, $n\geqq 2$. $V_4^0(U(n))^\sharp$ has the following base consisting of 12 elements:

$$(27)\qquad \begin{cases} F_{\rho_1}^{\alpha\beta}(x_1,x_2,x_3,x_4)=\Omega_\alpha(x_1,x_2)\Omega_\beta(x_3,x_4) \\ F_{\rho_2}^{\alpha\beta}(x_1,x_2,x_3,x_4)=\Omega_\alpha(x_1,x_3)\Omega_\beta(x_2,x_4) \qquad (\alpha,\beta=0,1) \\ F_{\rho_3}^{\alpha\beta}(x_1,x_2,x_3,x_4)=\Omega_\alpha(x_1,x_4)\Omega_\beta(x_2,x_3)\,. \end{cases}$$

The subspace $V_4^0(U(n))^\sharp_{[3,4]}$ of $V_4^0(U(n))^\sharp$ consisting of $F\in V_4^0(U(n))^\sharp$ which is skew-symmetric with respect to x_3, x_4 is also determined as in example 3. Namely, $2+4=6$ elements $F_{\rho_1}^{\alpha 1}$ $(\alpha=0,1)$, $F_{\rho_2}^{\alpha\beta}-F_{\rho_3}^{\alpha\beta}$ $(\alpha,\beta=0,1)$ form a base of the subspace $V_4^0(U(n))^\sharp_{[3,4]}$ (cf. [2]).

7. Let $\tilde{V}$ be an n-dimensional complex vector space with positive definite hermitian form (x,y), and $\tilde{U}$ an r-dimensional complex subspace of $\tilde{V}$. Then we denote by $U(n)_r$ the subgroup of $U(n)$ given by

$$(28)\qquad U(n)_r=\{\sigma\in U(n)\,;\ \sigma x=x \text{ for every } x \text{ in } \tilde{U}\}\,.$$

Put

$$(29)\qquad \tilde{\mu}_{n,r}^{(p)}=\dim V_p^0(U(n)_r)^\sharp$$

i.e. $\tilde{\mu}_{n,r}^{(p)}$ is the dimension of the space of invariant tensors of type $(0,p)$ under (the real representation of) the group $U(n)_r$. In the following we shall show that the determination of $\mu_{n,r}^{(p)}$ is reduced to that of $\mu_{n-r}^{(1)}, \mu_{n-r}^{(2)},\cdots, \tilde{\mu}_{n-r}^{(p)}$.

Let $\tilde{W}$ be the orthogonal complement of $\tilde{U}$:

$$\tilde{W}=\{x\in\tilde{V}\,;\ (x,y)=0 \text{ for every } y \text{ in } \tilde{U}\}\,.$$

Then $\tilde{W}$ is an $(n-r)$-dimensional subspace of $\tilde{V}$ and

$$\tilde{V}=\tilde{U}+\tilde{W} \qquad \text{(direct sum)}\,.$$

Every $\sigma\in U(n)_r$ induces a unitary transformation σ' on $\tilde{W}$, and $\sigma\to\sigma'$ is an onto isomorphism from $U(n)_r$ onto $U(n-r)$ (the unitary group of $\tilde{W}$). Let V, U and W be real vector spaces associated to $\tilde{V}$, $\tilde{U}$ and $\tilde{W}$ respectively. Then we have obviously $V=U+W$ (direct sum). Taking into consideration that we have identified V, U and W with their dual spaces respectively, we obtain by a well-known rule of tensor product the following decomposition of V_p^0 into direct sum of subspaces:

$$(30)\qquad V_p^0=X_0+X_1+\cdots+X_p$$

where

$$(31)\quad \begin{cases} X_0 = U_p^0 = U\otimes\cdots\otimes U \\ X_1 = W\otimes U\otimes\cdots\otimes U + U\otimes W\otimes U\otimes\cdots\otimes U + \cdots + U\otimes\cdots\otimes U\otimes W \\ X_2 = W\otimes W\otimes U\otimes\cdots\otimes U + W\otimes U\otimes W\otimes\cdots\otimes U + \cdots + U\otimes\cdots\otimes U\otimes W\otimes W \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ X_p = W_p^0 = W\otimes\cdots\otimes W \end{cases}$$

where every tensor product consits of p factors. Then $V_p^0(U(n)_r)^\# = \sum_{q=0}^{p} X_q(U(n)_r)^\#$. Now since $U(n)_r$ leaves invariant every element of U, we have easily

$$(32)\qquad (W_s{}^0\otimes U_t{}^0)(U(n)_r)^\# = W_s{}^0(U(n)_r)^\#\otimes U_t{}^0 .$$

Then the dimension of the left hand side of (32) is equal to $\tilde{\mu}_{n-r}^{(s)}(2r)^t$. Since X_q is a direct sum of $\binom{p}{q}$ $U(n)_r$-invariant subspaces each of which is equivalent as a representation space of $U(n)_r$ to $W_q{}^0\otimes U_{p-q}^0$, we have

$$(33)\qquad \dim X_q(U(n)_r)^\# = \binom{p}{q}\tilde{\mu}_{n-r}^{(q)}\cdot(2r)^{p-q} .$$

Thus we arrive finally at the following

THEOREM 6.

$$(34)\qquad \tilde{\mu}_{n,r}^{(p)} = \sum_{q=0}^{p}\binom{p}{q}\tilde{\mu}_{n-r}^{(q)}\cdot(2r)^{p-q}, \quad \tilde{\mu}_{n-r}^{(0)} = 1 .$$

We remark that above consideration contains not only the determination of $\tilde{\mu}_{n,r}^{(p)}$, but also that of a base of $V_p^0(U(n)_r)^\#$, as is explained in the following.

From (32) we see that, if $(f_\lambda), (g_\mu)$ form bases for $W_s{}^0(U(n)_r)^\#$, $U_t{}^0$ respectively, then $(f_\lambda\otimes g_\mu)$ form a base for $(W_s{}^0\otimes U_t{}^0)(U(n)_r)^\#$. For example, taking $p=4, s=t=2$, we exhibit bases for $U(n)_r$-invariants of $W\otimes W\otimes U\otimes U$ and of $W\otimes U\otimes W\otimes U$. For simplicity we assume $n-r\geqq 1$. Let $(u_1,\cdots,u_{2r})$ be a base of U over the field of real numbers. Then $(u_i\otimes u_j)_{1\leqq i,j\leqq 2r}$ form a base for $U\otimes U$. On the other hand, $W_2{}^0(U(n)_r)^\#$ has the following base (since $n-r\geqq 1$) by theorem 4:

$$\Omega_0(x_1, x_2), \quad \Omega_1(x_1, x_2) .$$

Then the following $2\times(2r)^2$ elements

$$(35)\qquad F_{ij}^{\alpha}(x_1, x_2, x_3, x_4) = \Omega_\alpha(x_1, x_2)\Omega_0(u_i, x_3)\Omega_0(u_j, x_4)$$
$$(\alpha = 0, 1;\ i, j = 1, 2, \cdots, 2r)$$

form a base of $(W_2{}^0\otimes U_2{}^0)(U(n)_r)^\#$.

REMARK 1. Let $(v_1,\cdots. v_r)$ be a base of U over the field of complex numbers. Then $(v_1,\cdots, v_r, iv_1,\cdots, iv_r)$ is a base of U over the field of real numbers.

Then, using the relation

$$\Omega_0(iv, x) = -\Omega_1(v, x)$$

we have, instead of (35), following base.

$$F'^{\alpha\beta\gamma}_{st}(x_1, x_2, x_3, x_4) = \Omega_\alpha(x_1, x_2)\Omega_\beta(v_s, x_3)\Omega_\gamma(v_t, x_4) \tag{36}$$

$$(\alpha, \beta, \gamma = 0, 1;\ s, t = 1, 2, \cdots, r).$$

REMARK 2. Theorem 6 has analogues for $O(n), SO(n), SU(n), Sp(n)$. Defining similarly $\tilde{\nu}^{(p)}_{n,r}$, $\nu^{(p)}_{n,r}$, $\mu^{(p)}_{n,r}$, $\kappa^{(p)}_{n,r}$ we have in fact

$$\tilde{\nu}^{(p)}_{n,r} = \sum_{q=0}^{p} \binom{p}{q} \tilde{\nu}^{(q)}_{n-r} \cdot r^{p-q}, \tag{37}$$

$$\nu^{(p)}_{n,r} = \sum_{q=0}^{p} \binom{p}{q} \nu^{(q)}_{n-r} \cdot r^{p-q}, \tag{38}$$

$$\mu^{(p)}_{n,r} = \sum_{q=0}^{p} \binom{p}{q} \mu^{(q)}_{n-r} \cdot (2r)^{p-q}, \tag{39}$$

$$\kappa^{(p)}_{n,r} = \sum_{q=0}^{p} \binom{p}{q} \kappa^{(q)}_{n-r} \cdot (4r)^{p-q}. \tag{40}$$

The determination of bases is also done analogously.

EXAMPLE 5. $p=4$, $r=1$, $n \geqq 3$, for $U(n)_r$.

$$\tilde{\mu}^{(4)}_{n,1} = 2^4 + \binom{4}{1}\tilde{\mu}^{(1)}_{n-1}\cdot 2^3 + \binom{4}{2}\tilde{\mu}^{(2)}_{n-1}\cdot 2^2 + \binom{4}{3}\tilde{\mu}^{(3)}_{n-1}\cdot 2 + \tilde{\mu}^{(4)}_{n-1}$$

$$= 2^4 + 24\tilde{\mu}^{(2)}_{n-1} + \tilde{\mu}^{(4)}_{n-1} = 76.$$

Base of $V_4{}^0(U(n)_1)^\sharp$: Let u be a (complex) base of $\tilde{U}$, then

base of $U_4{}^0$: $F_{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) = \Omega_\alpha(u, x_1)\Omega_\beta(u, x_2)\Omega_\gamma(u, x_3)\Omega_\delta(u, x_4)$

$(\alpha, \beta, \gamma, \delta = 0, 1)$

base of $(U\otimes U\otimes W\otimes W)^\sharp$: $F^{(12)}_{\alpha\beta\gamma}(x_1, x_2, x_3, x_4) = \Omega_\alpha(u, x_1)\Omega_\beta(u, x_2)\Omega_\gamma(x_3, x_4)$

base of $(U\otimes W\otimes U\otimes W)^\sharp$: $F^{(13)}_{\alpha\beta\gamma}(x_1, x_2, x_3, x_4) = \Omega_\alpha(u, x_1)\Omega_\beta(u, x_3)\Omega_\gamma(x_2, x_4)$

base of $(U\otimes W\otimes W\otimes U)^\sharp$: $F^{(14)}_{\alpha\beta\gamma}(x_1, x_2, x_3, x_4) = \Omega_\alpha(u, x_1)\Omega_\beta(u, x_4)\Omega_\gamma(x_2, x_3)$

base of $(W\otimes U\otimes U\otimes W)^\sharp$: $F^{(23)}_{\alpha\beta\gamma}(x_1, x_2, x_3, x_4) = \Omega_\alpha(u, x_2)\Omega_\beta(u, x_3)\Omega_\gamma(x_1, x_4)$

base of $(W\otimes U\otimes W\otimes U)^\sharp$: $F^{(24)}_{\alpha\beta\gamma}(x_1, x_2, x_3, x_4) = \Omega_\alpha(u, x_2)\Omega_\beta(u, x_4)\Omega_\gamma(x_1, x_3)$

base of $(W\otimes W\otimes U\otimes U)^\sharp$: $F^{(34)}_{\alpha\beta\gamma}(x_1, x_2, x_3, x_4) = \Omega_\alpha(u, x_3)\Omega_\beta(u, x_4)\Omega_\gamma(x_1, x_2)$

$(\alpha, \beta, \gamma = 0, 1)$

base of $(W_4{}^0)^\sharp$:

$$\begin{cases} F^{\alpha\beta}_{\rho_1}(x_1, x_2, x_3, x_4) = \Omega_\alpha(x_1, x_2)\Omega_\beta(x_3, x_4) \\ F^{\alpha\beta}_{\rho_2}(x_1, x_2, x_3, x_4) = \Omega_\alpha(x_1, x_3)\Omega_\beta(x_2, x_4) \qquad (\alpha, \beta = 0, 1) \\ F^{\alpha\beta}_{\rho_3}(x_1, x_2, x_3, x_4) = \Omega_\alpha(x_1, x_4)\Omega_\beta(x_2, x_3). \end{cases}$$

To find a base of the subspace $V_4^0(U(n)_1)^{\#}_{[3,4]}$ of $V_4^0(U(n)_1)^{\#}$ consisting of skew-symmetric forms with respect to x_3, x_4, we only need to apply the alternation A_{34} with respect to x_3, x_4 on these bases. Then we have

$$\dim V_4^0(U(n)_1)^{\#}_{[3,4]}=32 \qquad (n\geqq 3).$$

EXAMPLE 6. $p=3, r=1, n\geqq 4$, for $SO(n)_r$.

$$\nu_{n,1}^{(3)}=1+\binom{3}{1}\nu_{n-1}^{(1)}+\binom{3}{2}\nu_{n-1}^{(2)}+\nu_{n-1}^{(3)}.$$

Thus,

$$\nu_{n,1}^{(3)}=\begin{cases} 4 & (n\geqq 5) \\ 5 & (n=4). \end{cases}$$

For $n\geqq 5$, $V_3^0(SO(n)_1)^{\#}$ has following base (u being a base of U)

$$\begin{cases} F_0(x_1, x_2, x_3)=(u, x_1)\cdot(u, x_2)\cdot(u, x_3) \\ F_1(x_1, x_2, x_3)=(u, x_1)\cdot(x_2, x_3) \\ F_2(x_1, x_2, x_3)=(u, x_2)\cdot(x_1, x_3) \\ F_3(x_1, x_2, x_3)=(u, x_3)\cdot(x_1, x_2). \end{cases} \tag{41}$$

For $n=4$, $V_3^0(SO(4)_1)^{\#}$ has as base besides F_0, F_1, F_2, F_3 in (41) the following

$$F_4(x_1, x_2, x_3)=[u, x_1, x_2, x_3]. \tag{42}$$

8. Now we shall give formulas of $\nu_n^{(p)}$, $\tilde{\mu}_n^{(p)}$, $\mu_n^{(p)}$, $\kappa_n^{(p)}$ *for any* n, p.

Let G be a compact group and (ρ, U) a representation of G. Then as is well-known, the dimension r of the space of invariants of G in the representation (ρ, U) is given by

$$r=\int_G \chi_\rho(\sigma)d\sigma$$

where $\chi_\rho(\sigma)$ is the character of the representation (ρ, U) and the Haar measure $d\sigma$ of G is normalized by $\int_G d\sigma=1$.

When G is one of the classical compact Lie groups $SO(n), SU(n), U(n), Sp(n)$ and $f(\sigma)$ is a class function on G i.e. $f(\tau\sigma\tau^{-1})=f(\sigma)$ for every σ, τ in G, then the value of the integral $\int_G f(\sigma)d\sigma$ is calculated by the following formula (cf. Weyl [3], Chap. VII or [4], 'exposé' n°21)

$$\int_G f(\sigma)d\sigma=\frac{1}{w}\int_H f(h)\Phi(h)dh$$

where H is a maximal torus of G and dh is a Haar measure of H such that

$$\int_H dh=1,$$

and the function $\Phi(h)$ is given by

$$\Phi(h)=\prod_{\alpha}(e^{\alpha(h)}-1),$$

the product being extended for all non-zero root forms α of the complex form $\mathfrak{g}^c$ of the Lie algebra $\mathfrak{g}$ of G,[5] and w is the order of the Weyl group of $\mathfrak{g}^c$. $\alpha(h)$ is defined by

$$ad(h)E_\alpha=e^{\alpha(h)}E_\alpha$$

where E_α is a root vector for the root α.

Let us consider in particular $SO(2n+1)$. Then the root forms are $\pm\lambda_i$ $(1\leqq i\leqq n)$; $\pm(\lambda_i-\lambda_k)$, $\pm(\lambda_i+\lambda_k)$ $(1\leqq i<k\leqq n)$. A maximal torus H consists of matrices of the form:

$$h=1\dot{+}\begin{pmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{pmatrix}\dot{+}\cdots\dot{+}\begin{pmatrix}\cos\theta_n & -\sin\theta_n\\ \sin\theta_n & \cos\theta_n\end{pmatrix}$$

and we have

$$\Phi(h)=2^{n^2}\prod_{i=1}^{n}(1-\cos\theta_i)\prod_{i<k}(1-\cos(\theta_i-\theta_k))(1-\cos(\theta_i+\theta_k)).$$

Now the character $\chi(h)$ of the representation of $SO(2n+1)$ on $(0,p)$-tensors is given by

$$\chi(h)=(1+2(\cos\theta_1+\cdots+\cos\theta_n))^p$$

and the order w of the Weyl group is given by

$$w=2^n n!.$$

Thus, putting

$$F(\theta)=F(\theta_1,\cdots,\theta_n)=\cos\theta_1+\cdots+\cos\theta_n,$$

$$G(\theta)=G(\theta_1,\cdots,\theta_n)=\prod_{i<k}(1-\cos(\theta_i-\theta_k)),$$

$$H(\theta)=H(\theta_1,\cdots,\theta_n)=\prod_{i<k}(1-\cos(\theta_i+\theta_k))$$

we have the following formula:

$$(42)\qquad \nu_{2n+1}^{(p)}=\frac{2^{n^2}}{(2\pi)^n2^n n!}\int_0^{2\pi}\cdots\int_0^{2\pi}(1+2F(\theta))^pG(\theta)H(\theta)\prod_{i=1}^{n}(1-\cos\theta_i)d\theta_1\cdots d\theta_n.$$

Analogously we obtain the following formulas:

$$(43)\qquad \nu_{2n}^{(p)}=\frac{2^{n(n-1)}}{(2\pi)^n2^{n-1}n!}\int_0^{2\pi}\cdots\int_0^{2\pi}2^pF(\theta)^pG(\theta)H(\theta)d\theta_1\cdots d\theta_n,$$

$$(44)\qquad \tilde{\mu}_n^{(p)}=\frac{2^{\frac{n(n-1)}{2}}}{(2\pi)^n n!}\int_0^{2\pi}\cdots\int_0^{2\pi}2^pF(\theta)^pG(\theta)d\theta_1\cdots d\theta_n,$$

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5) The root forms are considered with respect to the Cartan subalgebra of $\mathfrak{g}^c$ associated to H.

(45) $$\mu_n^{(p)}=\frac{2^{\frac{n(n-1)}{2}}}{(2\pi)^{n-1}n!}\int_0^{2\pi}\cdots\int_0^{2\pi}2^pF^*(\theta_1,\cdots,\theta_{n-1})^pG^*(\theta_1,\cdots,\theta_{n-1})d\theta_1\cdots d\theta_{n-1},$$

where $F^*(\theta_1,\cdots,\theta_{n-1})$ and $G^*(\theta_1,\cdots,\theta_{n-1})$ are given by

$$\begin{cases} F^*(\theta_1,\cdots,\theta_{n-1})=F(\theta_1,\cdots,\theta_{n-1},\ -\theta_1-\cdots-\theta_{n-1}) \\ G^*(\theta_1,\cdots,\theta_{n-1})=G(\theta_1,\cdots,\theta_{n-1},\ -\theta_1-\cdots-\theta_{n-1}), \end{cases}$$

(46) $$\kappa_n^{(p)}=\frac{2^{n^2}}{(2\pi)^n n!}\int_0^{2\pi}\cdots\int_0^{2\pi}4^pF(\theta)^pG(\theta)H(\theta)\prod_{i=1}^n(1-\cos^2\theta_i)d\theta_1\cdots d\theta_n.$$

Example 7. Let us calculate (42)~(46) for small values of n:

$$\nu_3^{(p)}=\frac{1}{2\pi}\int_0^{2\pi}(1+2\cos\theta)^p(1-\cos\theta)d\theta$$

$$=\sum_{j=0}^{[\frac{p}{2}]}\binom{p}{2j}\binom{2j}{j}-\frac{1}{2}\sum_{j=1}^{[\frac{p+1}{2}]}\binom{p}{2j-1}\binom{2j}{j},$$

$$\nu_2^{(2p)}=\frac{1}{2\pi}\int_0^{2\pi}2^{2p}\cos^{2p}\theta d\theta=\binom{2p}{p},\qquad (\nu_2^{(2p-1)}=0),$$

$$\nu_4^{(2p)}=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}2^{2p}(\cos\theta_1+\cos\theta_2)^{2p}(1-\cos(\theta_1-\theta_2))(1-\cos(\theta_1+\theta_2))d\theta_1 d\theta_2$$

$$=\frac{1}{2}\left\{\sum_{j=0}^{p}\binom{2p}{2j}\binom{2j+2}{j+1}\binom{2p-2j}{p-j}-\sum_{j=1}^{p}\binom{2p}{2j-1}\binom{2j}{j}\binom{2p-2j+2}{p-j+1}\right\},$$

$$\tilde{\mu}_1^{(2p)}=\nu_2^{(2p)},$$

$$\tilde{\mu}_2^{(2p)}=\sum_{j=0}^{p}\binom{2p}{2j}\binom{2j}{j}\binom{2p-2j}{p-j}-\frac{1}{4}\sum_{j=1}^{p}\binom{2p}{2j-1}\binom{2j}{j}\binom{2p-2j+2}{p-j+1},$$

$$\mu_2^{(2p)}=2^{2p-1}\left(2^2\binom{2p}{p}-\binom{2p+2}{p+1}\right),$$

$\kappa_1^{(2p)}=\mu_2^{(2p)}$ (Note that the real representations of $Sp(1)$ and $SU(2)$ coincide.)

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