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A consistency-proof of a formal theory of Ackermann's ordinal numbers.

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In [1], W. Ackermann constructed a certain system of ordinal numbers of the "zweite Zahlenklasse" which will be called in this paper Ackermann's system of ordinal numbers.

In the following, we shall first formalize Ackermann's system of ordinal numbers in Gentzen's LK (see [2]). We shall denote by Γ_A the system thus formalized, and prove the consistency of Γ_A. The proof is performed as follows.

We shall consider the subsystem N^1 of G^1LC (see [8]) defined as follows: N^1 has the same beginning sequences and the same inferences as in G^1LC except the following restriction on the inference $\forall$ left on f-variables. Let $\mathfrak{P}$ be a proof-figure of N^1 containing an inference $\forall$ left on an f-variable of the form

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi), \Gamma \rightarrow \Delta}.$$

Then $F(\varphi)$ should contain no $\forall$ on an f-variable. We shall use only proof-figures in N^1 for our consistency-proof.

Let $\mathfrak{S}$ be a sequence of the form

$$A_1(a_1, \cdots, a_i),\ \cdots,\ A_n(a_1, \cdots, a_i) \\ \rightarrow B_1(a_1, \cdots, a_i),\ \cdots,\ B_m(a_1, \cdots, a_i).$$

Then the axiom

$$\forall x_1 \cdots \forall x_i (A_1(x_1, \cdots, x_i) \wedge \cdots \wedge A_n(x_1, \cdots, x_i) \\ \vdash B_1(x_1, \cdots, x_i) \vee \cdots \vee B_m(x_1, \cdots, x_i))$$

will be called the axiom obtained from $\mathfrak{S}$.

Let Γ_0 be a system of axioms obtained from some "mathematische Grundsequenzen" in the sense of [3]. Then the consistency in N^1 of the following system of axioms, which we shall call the system Γ_N:

$$\Gamma_0, \\ \forall \varphi \forall x \forall y (x = y \vdash (\varphi[x] \dashv\vdash \varphi[y])), \\ \forall \varphi \forall x (\varphi[0] \wedge \forall y (\varphi[y] \vdash \varphi[y+1]) \vdash \varphi[x])$$

follows from Theorem II, Chapter II of [6], as a special case. (In [6] ordinal diagrams of any order are used. It is to be noticed that for the proof of this special case, we have to use only ordinal diagrams of order 2.)

In § 1, we shall give a precise definition of Γ_A and in § 2 of Γ_0 and of Γ_N. Finally we shall construct in §§ 3–5 a model of Γ_A in Γ_N and thus complete our consistency-proof.

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§ 1. Formalization of Ackermann's system of ordinal numbers.

We define the system Γ_A by the following axioms in *LK*.

1.1. $\forall x(x=x)$.

1.2. $\forall x \forall y(x=y \vdash y=x)$.

1.3. $\forall x \forall y \forall z(x=y \wedge y=z \vdash x=z)$.

1.4. $\forall x_1 \forall x_2 \forall y_1 \forall y_2(x_1=y_1 \wedge x_2=y_2 \vdash x_1+x_2=y_1+y_2)$,

where '+' is a special function, and $a+b$ means the sum of the ordinal numbers a and b.

1.5. $\forall x_1 \forall x_2 \forall x_3 \forall y_1 \forall y_2 \forall y_3(x_1=y_1 \wedge x_2=y_2 \wedge x_3=y_3 \vdash \mathbf{ack}(x_1, x_2, x_3)=\mathbf{ack}(y_1, y_2, y_3))$,

where '**ack**' is a special function, which corresponds to the "Klammerzeichen (, ,)" in [1].

1.6. $\forall x \neg(x<1)$,

where '1' is a special variable and $a<b$ is read, "a is smaller than b."

1.7. $\forall x \forall y(x<x+y)$.

1.8. $\forall x \forall y \forall z(y<z \vdash x+y<x+z)$.

1.9. $\forall x \forall y(y<x \vdash \exists z(y+z=x))$.

1.10. $\forall x(x<\mathbf{ack}(1, 1, 1) \vdash x=1 \vee \exists y(x=y+1))$.

1.11. $\forall x \forall y \forall z \forall u \forall v(u<\mathbf{ack}(x, y, z) \wedge v<\mathbf{ack}(x, y, z) \vdash u+v<\mathbf{ack}(x, y, z))$.

1.12. $\forall x_1 \forall x_2 \forall x_3 \forall y_1 \forall y_2 \forall y_3(x_1=y_1 \wedge x_2=y_2 \wedge x_3<y_3 \vdash \mathbf{ack}(x_1, x_2, x_3)<\mathbf{ack}(y_1, y_2, y_3))$.

1.13. $\forall x_1 \forall x_2 \forall x_3 \forall y_1 \forall y_2 \forall y_3(x_1=y_1 \wedge x_2<y_2 \wedge x_3<\mathbf{ack}(y_1, y_2, y_3) \vdash \mathbf{ack}(x_1, x_2, x_3)<\mathbf{ack}(y_1, y_2, y_3))$.

1.14. $\forall x_1 \forall x_2 \forall x_3 \forall y_1 \forall y_2 \forall y_3(x_1<y_1 \wedge x_2<\mathbf{ack}(y_1, y_2, y_3) \wedge x_3<\mathbf{ack}(y_1, y_2, y_3) \vdash \mathbf{ack}(x_1, x_2, x_3)<\mathbf{ack}(y_1, y_2, y_3))$.

1.15. $\forall x \forall y \forall z \forall u(u \leqq x \vee u \leqq y \vee u \leqq z \vdash u<\mathbf{ack}(x, y, z))$,

where $a \leqq b$ is an abbreviation for $a<b \vee a=b$ as usual.

1.16. $\forall x \forall y(x<y \vee x=y \vee y<x)$.

1.17. $\forall x \forall y \neg (x = y \wedge x < y)$.

1.18. $\forall x \forall y \forall z (x < y \wedge y < z \vdash x < z)$.

1.19. $\forall A \forall x (\forall y (\forall z (z < y \vdash A(z)) \vdash A(y)) \vdash A(x))$.

$\forall A F(\{x\} A(x))$ means the system of all the axioms of the form

$$\forall z_1 \cdots \forall z_n (F(\{x\} A(x, z_1, \cdots, z_n)))$$

where $\{x\} A(x, z_1, \cdots, z_n)$ ranges over all the formulas with one argument-place (see [9] § 1 for this notion).

1.19 is called the axiom of transfinite induction, which corresponds to "Erreichbarkeit" of Ackermann's ordinal numbers.

§ 2. The axiom system Γ_N.

We shall now display the system Γ_N of the axioms of the theory of natural numbers in G^1LC, which is the basic system for our consistency-proof.

2.1. $\forall x (x = x)$.

2.2. $\forall \varphi \forall x \forall y (x = y \vdash (\varphi[x] \vdash \varphi[y]))$.

2.3. $\forall x \forall y (x < y \vee x = y \vee y < x)$.

2.4. $\forall x \forall y \neg (x = y \wedge x < y)$.

2.5. $\forall x \forall y \forall z (x < y \wedge y < z \vdash x < z)$.

2.6. $\forall x (0 \leqq x)$,

where '0' is a special variable and $a \leqq b$ means $a < b \vee a = b$.

2.7. $0 < 1$,

where '1' is a special variable.

2.8. $\forall x \forall y (x < y \vdash x + 1 \leqq y)$,

where '+' is a special function.

2.9. $\forall x (0 < x \vdash \exists y (x = y + 1))$.

2.10. $\forall x (x + 0 = x)$.

2.11. $\forall x \forall y (x + y = y + x)$.

2.12. $\forall x \forall y \forall z ((x + y) + z = x + (y + z))$.

2.13. $\forall x \forall y \forall z (x < y \vdash x + z < y + z)$.

2.14. $\forall x \forall y (y \leqq x \vdash y + (x - y) = x)$,

where '−' is a special function.

2.15. $\forall x \forall y (x < y \vdash x - y = 0)$.

2.16. $\forall x (x \cdot 1 = x)$,

where '·' is a special function.

2.17. $\forall x \forall y(x \cdot y = y \cdot x)$.
2.18. $\forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))$.
2.19. $\forall x \forall y \forall z((x + y) \cdot z = x \cdot z + y \cdot z)$.
2.20. $\forall x \forall y \forall z(0 < x \wedge y < z \vdash x \cdot y < x \cdot z)$.
2.21. $\forall x(x^1 = x)$,

where '$*^*$' is a special function.

2.22. $\forall x \forall y \forall z(x^y \cdot x^z = x^{y+z})$.
2.23. $\forall x \forall y \forall z((x^y)^z = x^{y \cdot z})$.
2.24. $\forall \varphi \forall x(\varphi[0] \wedge \forall y(\varphi[y] \vdash \varphi[y+1]) \vdash \varphi[x])$,

which is called the axiom of mathematical induction. Now we define $2=1+1$, $3=2+1$, $4=3+1$, $5=4+1$, $6=5+1$, $7=6+1$, $8=7+1$, $9=8+1$, $10=9+1$, $11=10+1$, etc.

2.25. $\forall x \forall y \forall z(\mathrm{ack}(x, y, z) = 2^x 3^y 5^z)$,

where 'ack' is a special function corresponding to '**ack**' in Γ_A.

2.26. $\forall x \forall y(x \dagger y = 7^x 11^y)$,

where '$\dagger$' is a special function.

2.27. $\mathrm{Ack}(1) = 0$,

where 'Ack' is a special function.

2.28. $\forall x \forall y \forall z(\mathrm{Ack}(\mathrm{ack}(x, y, z)) = 0 \dashv\vdash \mathrm{Ack}(x) = 0 \wedge \mathrm{Ack}(y) = 0 \wedge \mathrm{Ack}(z) = 0)$.
2.29. $\forall x(\mathrm{M}(x) = 0 \dashv\vdash x = 1 \vee \exists u \exists v \exists w(x = \mathrm{ack}(u, v, w) \wedge \mathrm{Ack}(x) = 0))$,

where 'M' is a special function and $\mathrm{M}(a) = 0$ means "a is monomial".

2.30. $\forall x \forall y(\mathrm{Ack}(x \dagger y) = 0 \vdash \mathrm{Ack}(x) = 0 \wedge \mathrm{M}(y) = 0)$.
2.31. $\forall x(\mathrm{M}(x) = 0 \dashv\vdash \mathrm{lh}(x) = 0)$,

where 'lh' is a special function and $\mathrm{lh}(a)$ means the length of a.

2.32. $\forall x \forall y(\mathrm{Ack}(x \dagger y) = 0 \vdash \mathrm{lh}(x \dagger y) = \mathrm{lh}(x) + 1)$.
2.33. $\forall x(\mathrm{M}(x) = 0 \vdash \mathrm{cp}(x, 0) = x)$,

where 'cp' is a special function, and $\mathrm{cp}(a, i)$ means the i-th component of a.

2.34. $\forall x \forall y(\mathrm{Ack}(x) = 0 \wedge y \leqq \mathrm{lh}(x) \vdash \mathrm{M}(\mathrm{cp}(x, y)) = 0)$.
2.35. $\forall x \forall y \forall z(\mathrm{Ack}(x) = 0 \wedge x = y \dagger z \vdash \forall u(u \leqq \mathrm{lh}(y) \vdash \mathrm{cp}(x, u) = \mathrm{cp}(y, u)) \wedge \mathrm{cp}(x, \mathrm{lh}(x)) = z)$.
2.36. $\forall x \forall y(\mathrm{lh}(x) < y \vdash \mathrm{cp}(x, y) = 0)$.
2.37. $\forall x(\mathrm{lc}(x) = \mathrm{cp}(x, \mathrm{lh}(x)))$,

where 'lc' is a special function, and $\mathrm{lc}(a)$ means the last component of a.

2.38. $\forall x \forall y(\mathrm{Ack}(x) = 0 \wedge y \leqq \mathrm{lh}(x) \vdash \mathrm{lh}(\mathrm{pt}(x, y)) = y)$,

where 'pt' is a special function, and $\mathrm{pt}(a, i)$ means the i-th part of a.

2.39. $\forall x\forall y\forall z(\mathrm{Ack}(x)=0\wedge z\leqq y\wedge y\leqq \mathrm{lh}(x)\vdash \mathrm{cp}(\mathrm{pt}(x, y), z)=\mathrm{cp}(x, z))$.

2.40. $\forall x(\mathrm{Ack}(x)=0\vdash \mathrm{M}(x)=0\vee x=\mathrm{pt}(x, \mathrm{lh}(x)-1)\dagger \mathrm{lc}(x))$.

2.41. $\forall x\neg(\mathrm{Ord}(x, 1)=0)$,

where 'Ord' is a special function, and $\mathrm{Ord}(a, b)=0$ corresponds to $a<b$ in Γ_A.

2.42. $\forall x(\mathrm{Ack}(x)=0\vdash x=1\vee \mathrm{Ord}(1, x)=0)$.

2.43. $\forall x\forall y(\mathrm{Ack}(x)=0\wedge \mathrm{Ack}(y)=0\wedge \mathrm{lh}(y)<\mathrm{lh}(x)$
$\wedge\forall z(z\leqq \mathrm{lh}(y)\vdash \mathrm{cp}(y, z)=\mathrm{cp}(x, z))\vdash \mathrm{Ord}(y, x)=0)$.

2.44. $\forall x\forall y(\mathrm{Ack}(x)=0\wedge \mathrm{Ack}(y)=0\wedge\exists z(\forall u(u<z\vdash \mathrm{cp}(y, u)=\mathrm{cp}(x, u))$
$\wedge \mathrm{Ord}(\mathrm{cp}(y, z), \mathrm{cp}(x, z))=0)\vdash \mathrm{Ord}(y, x)=0)$.

2.45. $\forall x\forall y\forall z\forall u\forall v\forall w(u=x\wedge v=y\wedge \mathrm{Ord}(w, z)=0\vdash \mathrm{Ord}(\mathrm{ack}(u, v, w), \mathrm{ack}(x, y, z))=0)$.

2.46. $\forall x\forall y\forall z\forall u\forall v\forall w(u=x\wedge \mathrm{Ord}(v, y)=0\wedge \mathrm{Ord}(w, \mathrm{ack}(x, y, z))=0$
$\vdash \mathrm{Ord}(\mathrm{ack}(u, v, w), \mathrm{ack}(x, y, z))=0)$.

2.47. $\forall x\forall y\forall z\forall u\forall v\forall w(\mathrm{Ord}(u, x)=0\wedge \mathrm{Ord}(v, \mathrm{ack}(x, y, z))=0\wedge \mathrm{Ord}(w, \mathrm{ack}(x, y, z))=0$
$\vdash \mathrm{Ord}(\mathrm{ack}(u, v, w), \mathrm{ack}(x, y, z))=0)$.

2.48. $\forall x\forall y\forall z\forall u(\mathrm{Ord}(u, x)=0\vee u=x\vee \mathrm{Ord}(u, y)=0\vee u=y\vee \mathrm{Ord}(u, z)=0\vee u=z$
$\vdash \mathrm{Ord}(u, \mathrm{ack}(x, y, z))=0)$.

2.49. $\forall x\forall y(\mathrm{Ord}(y, x)=0\vdash \mathrm{Ack}(x)=0\wedge \mathrm{Ack}(y)=0$
$\wedge((y=1\wedge 1<x)\vee(\mathrm{lh}(y)<\mathrm{lh}(x)\wedge\forall z(z\leqq \mathrm{lh}(y)\vdash \mathrm{cp}(y, z)=\mathrm{cp}(x, z)))$
$\vee\exists z(\forall u(u<z\vdash \mathrm{cp}(y, u)=\mathrm{cp}(x, u))\wedge \mathrm{Ord}(\mathrm{cp}(y, z), \mathrm{cp}(x, z))=0)$
$\vee\exists r\exists s\exists t\exists u\exists v\exists w(x=\mathrm{ack}(r, s, t)\wedge y=\mathrm{ack}(u, v, w)$
$\wedge((u=r\wedge v=s\wedge \mathrm{Ord}(w, t)=0)\vee(u=r\wedge \mathrm{Ord}(v, s)=0\wedge \mathrm{Ord}(w, x)=0)$
$\vee(\mathrm{Ord}(u, r)=0\wedge \mathrm{Ord}(v, x)=0\wedge \mathrm{Ord}(w, x)=0))$
$\vee\exists u\exists v\exists w(x=\mathrm{ack}(u, v, w)\wedge(\mathrm{Ord}(y, u)=0\vee y=u\vee \mathrm{Ord}(y, v)=0\vee y=v$
$\vee \mathrm{Ord}(y, w)=0\vee y=w))))$.

2.50. $\mathrm{O}(1)=0$,

where 'O' is a special function, and $\mathrm{O}(a)=0$ means that a corresponds to Ackermann's ordinal number.

2.51. $\forall x\forall y\forall z(\mathrm{O}(x)=0\wedge \mathrm{O}(y)=0\wedge \mathrm{O}(z)=0\vdash \mathrm{O}(\mathrm{ack}(x, y, z))=0)$.

2.52. $\forall x(\mathrm{Ack}(x)=0\wedge 0<\mathrm{lh}(x)\wedge\forall y(y\leqq \mathrm{lh}(x)\vdash \mathrm{O}(\mathrm{cp}(x, y))=0)$
$\wedge\forall z(z<\mathrm{lh}(x)\vdash \mathrm{Ord}(\mathrm{cp}(x, z+1), \mathrm{cp}(x, z))=0\vee \mathrm{cp}(x, z+1)=\mathrm{cp}(x, z))$
$\vdash \mathrm{O}(x)=0)$.

2.53. $\forall x(\mathrm{O}(x)=0\vdash \mathrm{Ack}(x)=0\wedge(x=1$
$\vee\exists u\exists v\exists w(x=\mathrm{ack}(u, v, w)\wedge \mathrm{O}(u)=0\wedge \mathrm{O}(v)=0\wedge \mathrm{O}(w)=0)$
$\vee(0<\mathrm{lh}(x)\wedge\forall y(y\leqq \mathrm{lh}(x)\vdash \mathrm{O}(\mathrm{cp}(x, y))=0)$
$\wedge\forall z(z<\mathrm{lh}(x)\vdash \mathrm{Ord}(\mathrm{cp}(x, z+1), \mathrm{cp}(x,z))=0\vee \mathrm{cp}(x,z+1)=\mathrm{cp}(x,z))))$.

2.54. $\forall x\forall y(\mathrm{O}(x)=0\wedge \mathrm{O}(y)=0\wedge(\mathrm{Ord}(\mathrm{cp}(y, 0), \mathrm{lc}(x))=0\vee \mathrm{cp}(y, 0)=\mathrm{lc}(x))$
$\vdash(0=\mathrm{lh}(y)\vdash x\# y=x\dagger y)$
$\wedge(0<\mathrm{lh}(y)\vdash x\# y=(x\#\mathrm{pt}(y, \mathrm{lh}(y)-1))\dagger \mathrm{lc}(y)))$,

where '#' is a special function corresponding to '+' in Γ_A.

2.55. $\forall x \forall y(\mathrm{O}(x)=0 \wedge \mathrm{O}(y)=0 \wedge \mathrm{Ord}(\mathrm{lc}(x), \mathrm{cp}(y, 0))=0$
$\vdash(0=\mathrm{lh}(y) \vdash x \# y=y)$
$\wedge(0<\mathrm{lh}(y) \vdash x \# y=\mathrm{pt}(x, \mathrm{lh}(x)-1) \# y))$.

2.56. $\forall x \forall y(\mathrm{O}(x)=0 \wedge \mathrm{O}(y)=0 \vdash \mathrm{O}(x \# y)=0)$.

2.57. $\forall x \forall y(\mathrm{O}(x)=0 \wedge \mathrm{O}(y)=0 \vdash \mathrm{Ord}(x, x \# y)=0)$.

2.58. $\forall x \forall y \forall z(\mathrm{O}(x)=0 \wedge \mathrm{O}(y)=0 \wedge \mathrm{O}(z)=0 \wedge \mathrm{Ord}(y, z)=0 \vdash \mathrm{Ord}(x \# y, x \# z)=0)$.

2.59. $\forall x \forall y \forall z(\mathrm{O}(x)=0 \wedge \mathrm{O}(y)=0 \wedge \mathrm{O}(z)=0 \wedge \mathrm{Ord}(z, x \# y)=0 \vdash \mathrm{Ord}(z, x)=0 \vee z=x$
$\vee \exists u(\mathrm{O}(u)=0 \wedge \mathrm{Ord}(u, y)=0 \wedge(\mathrm{Ord}(z, x \# u)=0 \vee z=x \# u))))$.

2.60. $\forall x \forall y \forall z \forall u \forall v(\mathrm{O}(x)=0 \wedge \mathrm{O}(y)=0 \wedge \mathrm{O}(z)=0 \wedge \mathrm{O}(u)=0 \wedge \mathrm{O}(v)=0$
$\wedge \mathrm{Ord}(u, \mathrm{ack}(x, y, z))=0 \wedge \mathrm{Ord}(v, \mathrm{ack}(x, y, z))=0$
$\vdash \mathrm{Ord}(u \# v, \mathrm{ack}(x, y, z))=0)$.

2.61. $\forall x \forall y(\mathrm{O}(x)=0 \wedge \mathrm{O}(y)=0 \wedge \mathrm{Ord}(y, x)=0 \vdash x=y \#(x \dot{-} y))$,

where '$\dot{-}$' is a special function.

2.62. $\forall x \forall y(\mathrm{O}(x)=0 \wedge \mathrm{O}(y)=0 \vdash \mathrm{O}(x \dot{-} y)=0)$.

2.63. $\forall x \forall y(\mathrm{O}(x)=0 \wedge \mathrm{O}(y)=0 \vdash \mathrm{Ord}(x, y)=0 \vee x=y \vee \mathrm{Ord}(y, x)=0)$.

2.64. $\forall x \forall y(\mathrm{O}(x)=0 \wedge \mathrm{O}(y)=0 \vdash \neg(x=y \wedge \mathrm{Ord}(x, y)=0))$.

2.65. $\forall x \forall y \forall z(\mathrm{O}(x)=0 \wedge \mathrm{O}(y)=0 \wedge \mathrm{O}(z)=0$
$\wedge \mathrm{Ord}(x, y)=0 \wedge \mathrm{Ord}(y, z)=0 \vdash \mathrm{Ord}(x, z)=0)$.

2.66. $\mathrm{rb}(1)=0$,

where 'rb' is a special function, and $\mathrm{rb}(a)$ means the number of ack's contained in a.

2.67. $\forall x \forall y \forall z(\mathrm{O}(\mathrm{ack}(x, y, z))=0 \vdash \mathrm{rb}(\mathrm{ack}(x, y, z))=\mathrm{rb}(x)+\mathrm{rb}(y)+\mathrm{rb}(z)+1)$.

2.68. $\forall x(\mathrm{O}(x)=0 \wedge 0<\mathrm{lh}(x) \vdash \mathrm{rb}(x)=\mathrm{rb}(\mathrm{pt}(x, \mathrm{lh}(x)-1))+\mathrm{rb}(\mathrm{lc}(x)))$.

2.69. $\mathrm{pl}(1)=0$,

where 'pl' is a special function, and $\mathrm{pl}(a)$ means the number of †'s contained in a.

2.70. $\forall x \forall y \forall z(\mathrm{O}(\mathrm{ack}(x, y, z))=0 \vdash \mathrm{pl}(\mathrm{ack}(x, y, z))=\mathrm{pl}(x)+\mathrm{pl}(y)+\mathrm{pl}(z))$.

2.71. $\forall x(\mathrm{O}(x)=0 \wedge 0<\mathrm{lh}(x) \vdash \mathrm{pl}(x)=\mathrm{pl}(\mathrm{pt}(x, \mathrm{lh}(x)-1))+\mathrm{pl}(\mathrm{lc}(x))+1)$.

All the special functions contained in Γ_N are "entscheidbar" in Gentzen's sense (cf. [4]). That is why Γ_N is consistent in N^1 by [6], Chapter II.

§ 3. The consistency of Γ_A.

We are to prove the consistency of Γ_A by the restriction theory. (See [5], § 7 for the notions and notations on restriction.) Consider the system of restriction which contains only the formula $\mathrm{O}(a)=0$. Let $\Gamma_A^{\mathrm{O}(\,)}$ denote the restriction of Γ_A by this system. According to 7.8 of [5], we have only to prove for our purpose that the following axioms are provable under Γ_N in N^1:

$\Gamma_A^{O(\ )}$,

$O(1)=0$,

$\exists x(O(x)=0)$,

$\forall x\forall y(O(x)=0\wedge O(y)=0\vdash O(x+y)=0)$,

$\forall x\forall y\forall z(O(x)=0\wedge O(y)=0\wedge O(z)=0\vdash O(\mathbf{ack}(x,y,z))=0)$.

By replacing $\{x,y\}(x<y)$, $\{x,y\}(x+y)$ and $\{x,y,z\}\mathbf{ack}(x,y,z)$ by $\{x,y\}(\mathrm{Ord}(x,y)=0)$, $\{x,y\}(x\#y)$ and $\{x,y,z\}\mathrm{ack}(x,y,z)$ respectively in the above axioms, we obtain the following axioms.

3.1. $\forall x(O(x)=0\vdash x=x)$.

3.2. $\forall x\forall y(O(x)=0\wedge O(y)=0\wedge x=y\vdash y=x)$.

3.3. $\forall x\forall y\forall z(O(x)=0\wedge O(y)=0\wedge O(z)=0\wedge x=y\wedge y=z\vdash x=z)$.

3.4. $\forall x\forall y\forall z\forall u\forall v\forall w(O(x)=0\wedge O(y)=0\wedge O(z)=0\wedge O(u)=0\wedge O(v)=0\wedge O(w)=0 \wedge x=u\wedge y=v\wedge z=w\vdash \mathrm{ack}(x,y,z)=\mathrm{ack}(u,v,w))$.

3.5. $\forall x\forall y\forall u\forall v(O(x)=0\wedge O(y)=0\wedge O(u)=0\wedge O(v)=0\wedge x=u\wedge y=v\vdash x\#y=u\#v)$

3.6. $\forall x(O(x)=0\vdash \neg\mathrm{Ord}(x,1)=0)$.

3.7. $\forall x\forall y(O(x)=0\wedge O(y)=0\vdash \mathrm{Ord}(x,x\#y)=0)$.

3.8. $\forall x\forall y\forall z(O(x)=0\wedge O(y)=0\wedge O(z)=0\wedge \mathrm{Ord}(y,z)=0\vdash \mathrm{Ord}(x\#y,x\#z)=0)$.

3.9. $\forall x\forall y(O(x)=0\wedge O(y)=0\wedge \mathrm{Ord}(y,x)=0\vdash \exists z(O(z)=0\wedge y\#z=x))$.

3.10. $\forall x(O(x)=0\wedge \mathrm{Ord}(x,\mathrm{ack}(1,1,1))=0\vdash x=1\vee\exists y(O(y)=0\wedge x=y\#1))$.

3.11. $\forall x\forall y\forall z\forall u\forall v(O(x)=0\wedge O(y)=0\wedge O(z)=0\wedge O(u)=0\wedge O(v)=0 \wedge \mathrm{Ord}(u,\mathrm{ack}(x,y,z))=0\wedge \mathrm{Ord}(v,\mathrm{ack}(x,y,z))=0 \vdash \mathrm{Ord}(u\#v,\mathrm{ack}(x,y,z))=0)$.

3.12. $\forall x\forall y\forall z\forall u\forall v\forall w(O(x)=0\wedge O(y)=0\wedge O(z)=0\wedge O(u)=0\wedge O(v)=0\wedge O(w)=0 \wedge u=x\wedge v=y\wedge \mathrm{Ord}(w,z)=0\vdash \mathrm{Ord}(\mathrm{ack}(u,v,w),\mathrm{ack}(x,y,z))=0)$.

3.13. $\forall x\forall y\forall z\forall u\forall v\forall w(O(x)=0\wedge O(y)=0\wedge O(z)=0\wedge O(u)=0\wedge O(v)=0\wedge O(w)=0 \wedge u=x\wedge \mathrm{Ord}(v,y)=0\wedge \mathrm{Ord}(w,\mathrm{ack}(x,y,z))=0 \vdash \mathrm{Ord}(\mathrm{ack}(u,v,w),\mathrm{ack}(x,y,z))=0)$.

3.14. $\forall x\forall y\forall z\forall u\forall v\forall w(O(x)=0\wedge O(y)=0\wedge O(z)=0\wedge O(u)=0\wedge O(v)=0\wedge O(w)=0 \wedge \mathrm{Ord}(u,x)=0\wedge \mathrm{Ord}(v,\mathrm{ack}(x,y,z))=0\wedge \mathrm{Ord}(w,\mathrm{ack}(x,y,z))=0 \vdash \mathrm{Ord}(\mathrm{ack}(u,v,w),\mathrm{ack}(x.y,z))=0)$.

3.15. $\forall x\forall y\forall z\forall u(O(x)=0\wedge O(y)=0\wedge O(z)=0\wedge O(u)=0\wedge(\mathrm{Ord}(u,x)=0\vee u=x \vee \mathrm{Ord}(u,y)=0\vee u=y\vee \mathrm{Ord}(u,z)=0\vee u=z)\vdash \mathrm{Ord}(u,\mathrm{ack}(x,y,z))=0)$.

3.16. $\forall x\forall y(O(x)=0\wedge O(y)=0\vdash \mathrm{Ord}(x,y)=0\vee x=y\vee \mathrm{Ord}(y,x)=0)$.

3.17. $\forall x\forall y(O(x)=0\wedge O(y)=0\vdash \neg(x=y\wedge \mathrm{Ord}(x,y)=0))$.

3.18. $\forall x\forall y\forall z(O(x)=0\wedge O(y)=0\wedge O(z)=0 \wedge \mathrm{Ord}(x,y)=0\wedge \mathrm{Ord}(y,z)=0\vdash \mathrm{Ord}(x,z)=0)$.

3.19. $\forall\varphi\forall x(O(x)=0\wedge\forall y(O(y)=0\wedge\forall z(O(z)=0 \wedge \mathrm{Ord}(z,y)=0\vdash\varphi[z])\vdash\varphi[y])\vdash\varphi[x])$.

3.20. $O(1)=0$.

3.21. $\exists x(O(x)=0)$.

3.22. $\forall x\forall y(O(x)=0\wedge O(y)=0\vdash O(x\# y)=0)$.

3.23. $\forall x\forall y\forall z(O(x)=0\wedge O(y)=0\wedge O(z)=0\vdash O(\mathrm{ack}(x, y, z))=0)$.

Each of 3.1–3.18, 3.20–3.23 follows directly or easily from the axioms of Γ_N. We have only to prove that 3.19 is provable under Γ_N in N^1. Let $Acc(a)$ denote

$$O(a)=0\wedge\forall\varphi(\forall x(O(x)=0\wedge\forall y(O(y)=0\wedge \mathrm{Ord}(y, x)=0\vdash\varphi[y])\vdash\varphi[x])\vdash\varphi[a]),$$

which means that a is accessible. Since 3.19 follows from $\forall x(O(x)=0\vdash Acc(x))$, our proof will be finished by constructing proof-figures to the following sequences T1—T10 under Γ_N in N^1.

T1. $\rightarrow Acc(1)$.

T2. $Acc(a), O(b)=0, \mathrm{Ord}(b, a)=0\rightarrow Acc(b)$.

T3. $O(a)=0, \forall x(O(x)=0\wedge \mathrm{Ord}(x, a)=0\vdash Acc(x))\rightarrow Acc(a)$.

T4. $Acc(a), O(b)=0, \mathrm{Ord}(b, \mathrm{lc}(a))=0\vee b=\mathrm{lc}(a)\rightarrow Acc(a\# b)$.

T5. $O(a)=0, Acc(\mathrm{cp}(a, 0))\rightarrow Acc(a)$.

T6. $Acc(a), Acc(b), Acc(c)$,
$\forall x\forall y\forall z(O(x)=0\wedge \mathrm{Ord}(x, a)=0\wedge Acc(y)\wedge Acc(z)\vdash Acc(\mathrm{ack}(x, y, z)))$,
$\forall x\forall y(O(x)=0\wedge \mathrm{Ord}(x, b)=0\wedge Acc(y)\vdash Acc(\mathrm{ack}(a, x, y)))$,
$\forall x(O(x)=0\wedge \mathrm{Ord}(x, c)=0\vdash Acc(\mathrm{ack}(a, b, x)))\rightarrow Acc(\mathrm{ack}(a, b, c))$.

T7. $Acc(a), Acc(b), Acc(c)$,
$\forall x\forall y\forall z(O(x)=0\wedge \mathrm{Ord}(x, a)=0\wedge Acc(y)\wedge Acc(z)\vdash Acc(\mathrm{ack}(x, y, z)))$,
$\forall x\forall y(O(x)=0\wedge \mathrm{Ord}(x, b)=0\wedge Acc(y)\vdash Acc(\mathrm{ack}(a, x, y)))\rightarrow Acc(\mathrm{ack}(a, b, c))$.

T8. $Acc(a), Acc(b), Acc(c)$,
$\forall x\forall y\forall z(O(x)=0\wedge \mathrm{Ord}(x, a)=0\wedge Acc(y)\wedge Acc(z)\vdash Acc(\mathrm{ack}(x, y, z)))$
$\rightarrow Acc(\mathrm{ack}(a, b, c))$.

T9. $Acc(a), Acc(b), Acc(c)\rightarrow Acc(\mathrm{ack}(a, b, c))$.

T10. $O(a)=0\rightarrow Acc(a)$.

Now we shall see that every chief-formula of the inference $\forall$ left on an f-variable appearing in the following proof-figures is always either of the form

$$\forall\varphi(\forall x(O(x)=0\wedge\forall y(O(y)=0\wedge \mathrm{Ord}(y, x)=0\vdash\varphi[y])\vdash\varphi[x])\vdash\varphi[a])$$

(i. e. a predecessor of a formula of the form $Acc(a)$), or of the form

$$\forall\varphi\forall x(\varphi[0]\wedge\forall y(\varphi[y]\vdash\varphi[y+1])\vdash\varphi[x]),$$

or of the form

$$\forall\varphi\forall x(\forall y(\forall z(z<y\vdash\varphi[z])\vdash\varphi[y])\vdash\varphi[x]),$$

so that all these proof-figures are contained in N^1. The last formula cited above is sometimes called the axiom of course-of-values induction, and we see easily that it is provable under Γ_N in N^1. Proof-figures to T1, T8—9 are easy to construct, so that we may omit the construction. Since it takes too much space to show the proof-figures themselves, we shall only show how to construct those to T2—T7 and T10 in the following paragraphs.

§ 4. A preparation.

Hereafter we say briefly that $\Gamma \to \Delta$ is provable or that we have $\Gamma \to \Delta$, when $\Gamma \to \Delta$ is provable under Γ_N in N^1 (i. e. $\Gamma_N, \Gamma \to \Delta$ is provable in N^1). In this section we shall mention some provable sequences useful for the next sections.

LEMMA. *The following sequences are provable.*

4.1. $Acc(a) \to \mathrm{O}(a)=0$.

4.2. $\mathrm{O}(a)=0, \mathrm{O}(b)=0, \mathrm{Ord}(b, \mathrm{lc}(a))=0 \vee b=\mathrm{lc}(a) \to \mathrm{Ord}(b, a)=0, b=a$.

4.3. $\mathrm{O}(a)=0 \to \mathrm{cp}(\mathrm{pt}(a, k), 0)=\mathrm{cp}(a, 0)$.

4.4. $\mathrm{O}(a)=0, \mathrm{lh}(a)=k+1 \to \mathrm{lh}(\mathrm{pt}(a, k))=k$.

4.5. $\mathrm{O}(a)=0, \mathrm{lh}(a)=k+1 \to \mathrm{O}(\mathrm{pt}(a, k))=0$.

4.6. $\mathrm{O}(a)=0 \to \mathrm{O}(\mathrm{lc}(a))=0$.

4.7. $\mathrm{O}(a)=0, \mathrm{lh}(a)=k+1 \to \mathrm{Ord}(\mathrm{lc}(a), \mathrm{lc}(\mathrm{pt}(a, k)))=0, \mathrm{lc}(a)=\mathrm{lc}(\mathrm{pt}(a, k))$.

4.8. $\mathrm{lh}(a)=0 \to \mathrm{cp}(a, 0)=a$.

4.9. $\mathrm{O}(\mathrm{ack}(d_1, d_2, d_3))=0, \mathrm{rb}(\mathrm{ack}(d_1, d_2, d_3))=k \to (\mathrm{rb}(\mathrm{cp}(d_3, 0))<k)$.

4.10. $\mathrm{Ord}(\mathrm{ack}(d_1, d_2, d_3), \mathrm{ack}(a, b, c))=0 \to d_1=a \wedge d_2=b \wedge \mathrm{Ord}(d_3, c)=0$,
$d_1=a \wedge \mathrm{Ord}(d_2, b)=0 \wedge \mathrm{Ord}(d_3, \mathrm{ack}(a, b, c))=0$,
$\mathrm{Ord}(d_1, a)=0 \wedge \mathrm{Ord}(d_2, \mathrm{ack}(a, b, c))=0 \wedge \mathrm{Ord}(d_3, \mathrm{ack}(a, b, c))=0$,
$\mathrm{Ord}(\mathrm{ack}(d_1, d_2, d_3), a)=0, \mathrm{ack}(d_1, d_2, d_3)=a$,
$\mathrm{Ord}(\mathrm{ack}(d_1, d_2, d_3), b)=0, \mathrm{ack}(d_1, d_2, d_3)=b$,
$\mathrm{Ord}(\mathrm{ack}(d_1, d_2, d_3), c)=0, \mathrm{ack}(d_1, d_2, d_3)=c$.

4.11. $\mathrm{O}(\mathrm{ack}(a, b, c))=0, \mathrm{O}(d)=0, \mathrm{Ord}(d, \mathrm{ack}(a, b, c))=0 \to \mathrm{cp}(d, 0)=1$,
$\exists x \exists y \exists z(\mathrm{cp}(d, 0)=\mathrm{ack}(x, y, z) \wedge \mathrm{O}(\mathrm{ack}(x, y, z))=0$
$\wedge \mathrm{Ord}(\mathrm{ack}(x, y, z), \mathrm{ack}(a, b, c))=0)$.

4.12. $\mathrm{O}(a)=0, \mathrm{pl}(a)+\mathrm{rb}(a)=k, a=\mathrm{ack}(b, c, d) \to \mathrm{pl}(b)+\mathrm{rb}(b)<k$.

4.13. $\mathrm{O}(a)=0, \mathrm{pl}(a)+\mathrm{rb}(a)=k, a=\mathrm{ack}(b, c, d) \to \mathrm{pl}(c)+\mathrm{rb}(c)<k$.

4.14. $\mathrm{O}(a)=0, \mathrm{pl}(a)+\mathrm{rb}(a)=k, a=\mathrm{ack}(b, c, d) \to \mathrm{pl}(d)+\mathrm{rb}(d)<k$.

4.15. $\mathrm{O}(a)=0, \mathrm{pl}(a)+\mathrm{rb}(a)=k, 0<\mathrm{lh}(a) \to \mathrm{pl}(\mathrm{cp}(a, 0))+\mathrm{rb}(\mathrm{cp}(a, 0))<k$.

4.16. $\mathrm{O}(a)=0 \to a=1, \exists x \exists y \exists z(a=\mathrm{ack}(x, y, z)), 0<\mathrm{lh}(a)$.

PROOFS. 4.1. It is obvious, because $Acc(a)$ denotes

$$\mathrm{O}(a)=0 \wedge \forall \varphi(\forall x(\forall y(\mathrm{Ord}(y, x)=0 \vdash \varphi[y]) \vdash \varphi[x]) \vdash \varphi[a]).$$

4.2. By 2.2, 2.21, 2.31, 2.34, 2.37, 2.43, 2.53 and 2.65.

4.3. By 2.6, 2.39 and 2.53.

4.4. By 2.2, 2.7, 2.10, 2.11 and 2.38.

4.5. By 2.2, 2.14, 2.52 and 2.53.
4.6. By 2.1, 2.2, 2.37 and 2.53.
4.7. By 2.2, 2.7, 2.37, 2.38, 2.39 and 2.53.
4.8. By 2.31 and 2.33.
4.9. By 2.2, 2.7, 2.13, 2.14, 2.24, 2.29, 2.31, 2.33, 2.53 and 2.62.
4.10. By 2.1 and 2.49.
4.11. By 2.2, 2.29, 2.34, 2.43, 2.53 and 2.65.
4.12. By 2.1, 2.2, 2.6, 2.7, 2.10, 2.67 and 2.70.
4.13 and 4.14. By the same way as in 4.12.
4.15. By 2.2, 2.6, 2.7, 2.10, 2.11, 2.24, 2.33, 2.38, 2,39, 2.40, 2.53 and 2.71.
4.16. By 2.53.

§ 5. Proofs of T2—T7 and T10.

We shall now outline the proof-figures to T2—T7 and T10. To save space, we shall often use the following abbreviations:

$\{x, y\}D(x, y)$ for $\{x, y\}(\mathrm{O}(x)=0\wedge \mathrm{Ord}(x, y)=0)$,
$\{x, \varphi\}E(x, \varphi)$ for $\{x, \varphi\}(\mathrm{O}(x)=0\wedge \forall y(D(y, x)\vdash\!\!- \varphi[y])\vdash\!\!- \varphi[x])$,
$\{x, y\}L(x, y)$ for $\{x, y\}(\mathrm{O}(x)=0\wedge \mathrm{lh}(x)=y\wedge Acc(\mathrm{cp}(x, 0))\vdash\!\!- Acc(x))$,
$\{x\}A(x)$ for $\{x\}(\forall u\forall v\forall w(D(u, x)\wedge Acc(v)\wedge Acc(w)\vdash\!\!- Acc(\mathrm{ack}(u, v, w))))$,
$\{x, y\}B(x, y)$ for $\{x, y\}(\forall u\forall v(D(u, y)\wedge Acc(v)\vdash\!\!- Acc(\mathrm{ack}(x, u, v))))$,
$\{x, y, z\}C(x, y, z)$ for $\{x, y, z\}(\forall u(D(u, z)\vdash\!\!- Acc(\mathrm{ack}(x, y, u))))$,
$\{x, y\}R(x, y)$ for $\{x, y\}(\mathrm{rb}(\mathrm{cp}(x, 0))=y\wedge D(x, \mathrm{ack}(a, b, c))\vdash\!\!- Acc(x))$,
$\{x, \varphi\}F(x, \varphi)$ for $\{x, \varphi\}(\forall y(y<x\vdash\!\!- \varphi[y])\vdash\!\!- \varphi[x])$,
$\{x, y, z\}K(x, y, z)$ for $\{x, y, z\}(Acc(z)\vdash\!\!- Acc(\mathrm{ack}(x, y, z)))$,
$\{x\}pr(x)$ for $\{x\}(\mathrm{pl}(x)+\mathrm{rb}(x))$,
$\{x, y\}P(x, y)$ for $\{x, y\}(\mathrm{O}(x)=0\wedge pr(x)=y\vdash\!\!- Acc(x))$.

Moreover, the left side of Ti will sometimes be denoted by Γ_i ($2\leqq i\leqq 10$).

5.1. Proof of T2.

We have the following sequences separated by semicolons successively:

$$\forall z(D(z, d)\vdash\!\!- \alpha[z])\rightarrow \forall y(D(y, d)\vdash\!\!- \alpha[y]);$$

adding $D(d, c)\rightarrow D(d, c)$, we obtain

$$D(d, c)\vdash\!\!- \forall z(D(z, c)\vdash\!\!- \alpha[z]), D(d, c)\rightarrow \forall y(D(y, d)\vdash\!\!- \alpha[y]);$$

$$\forall y(D(y, c)\vdash\!\!- \forall z(D(z, y)\vdash\!\!- \alpha[z])), D(d, c)\rightarrow \forall y(D(y, d)\vdash\!\!- \alpha[y]);$$

adding $\mathrm{O}(d)=0\rightarrow \mathrm{O}(d)=0$ and remembering the meaning of $D(d, c)$,

$$\forall y(D(y, c)\vdash\!\!- \forall z(D(z, y)\vdash\!\!- \alpha[z])), D(d, c)\rightarrow \mathrm{O}(d)=0\wedge \forall y(D(y, d)\vdash\!\!- \alpha[y]);$$

adding $\alpha[d]\rightarrow\alpha[d]$,

$$E(d, \alpha), \forall y(D(y, c)\vdash\!\!- \forall z(D(z, y)\vdash\!\!- \alpha[z])), D(d, c)\rightarrow \alpha[d];$$

$$\forall xE(x, \alpha), \forall y(D(y, c)\vdash\!\!- \forall z(D(z, y)\vdash\!\!- \alpha[z])), D(d, c)\rightarrow \alpha[d];$$

$\forall xE(x, \alpha), \forall y(D(y, c) \vdash \forall z(D(z, y) \vdash \alpha[z])) \rightarrow D(d, c) \vdash \alpha[d]$;

$\forall xE(x, \alpha), \forall y(D(y, c) \vdash \forall z(D(z, y) \vdash \alpha[z])) \rightarrow \forall z(D(z, c) \vdash \alpha[z])$;

$O(c)=0 \wedge \forall y(D(y, c) \vdash \forall z(D(z, y) \vdash \alpha[z]), \forall xE(x, \alpha) \rightarrow \forall z(D(z, c) \vdash \alpha[z])$;

$\forall xE(x, \alpha) \rightarrow E(c, \{y\}(\forall z(D(z, y) \vdash \alpha[z])))$;

$\forall xE(x, \alpha) \rightarrow \forall xE(x, \{y\}(\forall z(D(z, y) \vdash \alpha[z])))$;

using $\forall z(D(z, a) \vdash \alpha[z]), O(b)=0, \mathrm{Ord}(b, a)=0 \rightarrow \alpha[b]$,

$\forall xE(x, \{y\}(\forall z(D(z, y) \vdash \alpha[z]))) \vdash \forall z(D(z, a) \vdash \alpha[z]), \forall xE(x, \alpha)$,
$O(b)=0, \mathrm{Ord}(b, a)=0 \rightarrow \alpha[b]$;

by $\forall$ left on an f-variable,

$\forall \varphi(\forall x(E(x, \varphi) \vdash \varphi[a]), \forall xE(x, \alpha), O(b)=0, \mathrm{Ord}(b, a)=0 \rightarrow \alpha[b]$;

$\forall xE(x, \alpha), \Gamma_2 \rightarrow \alpha[b]$;

$\Gamma_2 \rightarrow \forall xE(x, \alpha) \vdash \alpha[b]$;

$\Gamma_2 \rightarrow \forall \varphi(\forall xE(x, \varphi) \vdash \varphi[b])$;

adding $O(b)=0 \rightarrow O(b)=0$,

$\Gamma_2 \rightarrow Acc(b)$, q. e. d.

5.2. PROOF OF T3.

We have the following sequences separated by semicolons successively:

$\forall xE(x, \alpha) \vdash \alpha[b], \forall xE(x, \alpha) \rightarrow \alpha[b]$;

by $\forall$ left on an f-variable,

$\forall \varphi(\forall xE(x, \varphi) \vdash \varphi[b]), \forall xE(x, \alpha) \rightarrow \alpha[b]$;

$Acc(b), \forall xE(x, \alpha) \rightarrow \alpha[b]$;

adding $D(b, a) \rightarrow D(b, a)$,

$D(b, a) \vdash Acc(b), D(b, a), \forall xE(x, \alpha) \rightarrow \alpha[b]$;

$\forall x(D(x, a) \vdash Acc(x)), D(b, a), \forall xE(x, \alpha) \rightarrow \alpha[b]$;

$\forall x(D(x, a) \vdash Acc(x)), \forall xE(x, \alpha) \rightarrow D(b, a) \vdash \alpha[b]$;

$\forall x(D(x, a) \vdash Acc(x)), \forall xE(x, \alpha) \rightarrow \forall y(D(y, a) \vdash \alpha[y])$;

adding $O(a)=0 \rightarrow O(a)=0$,

$\forall xE(x, \alpha), \Gamma_3 \rightarrow 0(a)=0 \wedge \forall y(D(y, a) \vdash \alpha[y])$;

adding $\alpha[a] \rightarrow \alpha[a]$,

$E(a, \alpha), \forall xE(x, \alpha), \Gamma_3 \rightarrow \alpha[a]$;

$\forall xE(x, \alpha), \Gamma_3 \rightarrow \alpha[a]$;

$\Gamma_3 \rightarrow \forall xE(x, \alpha) \vdash \alpha[a]$;

$\Gamma_3 \rightarrow \forall \varphi xE(x, \varphi) \vdash \varphi[a])$;

adding $O(a)=0 \rightarrow O(a)=0$ again,

$\Gamma_3 \rightarrow Acc(a)$, q. e. d.

5.3. PROOF OF T4.

We have the following sequences separated by semicolons successively:

$\forall y(D(y, c) \vdash Acc(a \# y)), D(e, c) \rightarrow Acc(a \# e)$;

using $d=a\# e, Acc(a\# e)\rightarrow Acc(d)$ and $\mathrm{O}(d)=0, \mathrm{Ord}(d, a\# e)=0, Acc(a\# e)\rightarrow Acc(d)$,

$$D(e, c)\wedge(\mathrm{Ord}(d, a\# e)=0\vee d=a\# e), \forall y(D(y, c)\vdash Acc(a\# y)), \mathrm{O}(d)=0\rightarrow Acc(d);$$

$$\exists z(D(z, c)\wedge(\mathrm{Ord}(d, a\# z)=0\vee d=a\# z)),$$
$$\forall y(D(y, c)\vdash Acc(a\# y)), \mathrm{O}(d)=0\rightarrow Acc(d);$$

using $d=a, Acc(a)\rightarrow Acc(d)$, and $\mathrm{O}(d)=0, \mathrm{Ord}(d, a)=0, Acc(a)\rightarrow Acc(d)$,

$$\mathrm{Ord}(d, a)=0\vee d=a\vee\exists z(D(z, c)\wedge(\mathrm{Ord}(d, a\# z)=0\vee d=a\# z)),$$
$$\forall y(D(y, c)\vdash Acc(a\# y)), \mathrm{O}(d)=0, Acc(a)\rightarrow Acc(d);$$

by the help of 2.59 and 4.1,

$$\mathrm{O}(c)=0, \mathrm{O}(d)=0, \mathrm{Ord}(d, a\# c)=0, \forall y(D(y, c)\vdash Acc(a\# y)), Acc(a)\rightarrow Acc(d);$$
$$\mathrm{O}(c)=0, D(d, a\# c), \forall y(D(y, c)\vdash Acc(a\# y)), Acc(a)\rightarrow Acc(d);$$
$$\mathrm{O}(c)=0, \forall y(D(y, c)\vdash Acc(a\# y)), Acc(a)\rightarrow D(d, a\# c)\vdash Acc(d);$$
$$\mathrm{O}(c)=0, \forall y(D(y, c)\vdash Acc(a\# y)), Acc(a)\rightarrow \forall y(D(y, a\# c)\vdash Acc(y));$$

by the help of 2.56 and 4.1,

$$\mathrm{O}(c)=0, \forall y(D(y, c)\vdash Acc(a\# y)), Acc(a)\rightarrow 0(a\# c)=0\wedge\forall y(D(y, a\# c)\vdash Acc(y));$$

using $\mathrm{O}(a\# c)=0\wedge\forall y(D(y, a\# c)\vdash Acc(y))\rightarrow Acc(a\# c)$ (cf. T3),

$$\mathrm{O}(c)=0, \forall y(D(y, c)\vdash Acc(a\# y)), Acc(a)\rightarrow Acc(a\# c);$$
$$\mathrm{O}(c)=0\wedge\forall y(D(y, c)\vdash Acc(a\# y)), Acc(a)\rightarrow Acc(a\# c);$$
$$Acc(a)\rightarrow E(c, \{y\}Acc(a\# y));$$
$$Acc(a)\rightarrow \forall xE(x, \{y\}Acc(a\# y));$$

adding $Acc(a\# b)\rightarrow Acc(a\# b)$,

$$\forall xE(x, \{y\}Acc(a\# y))\vdash Acc(a\# b), Acc(a)\rightarrow Acc(a\# b);$$

by $\forall$ left on an f-variable,

$$\forall\varphi(\forall xE(x, \varphi)\vdash\varphi[b]), Acc(a)\rightarrow Acc(a\# b);$$

(1) $Acc(b), Acc(a)\rightarrow Acc(a\# b)$.

On the other hand, from 4.2 and $b=a, Acc(a)\rightarrow Acc(b)$, we have successively,

$$\Gamma_4\rightarrow Acc(b), \mathrm{Ord}(b, a)=0;$$

using T2,

(2) $\Gamma_4\rightarrow Acc(b)$.

From (1) and (2) we have T4, q. e. d.

5.4. Proof of T5.

We have the following sequences successively: from 4.3 and

$\mathrm{cp}(\mathrm{pt}(a, k), 0)=\mathrm{cp}(a, 0), Acc(\mathrm{cp}(a, 0))\rightarrow Acc(\mathrm{cp}(\mathrm{pt}(a, k), 0))$,

$$\mathrm{O}(a)=0, Acc(\mathrm{cp}(a, 0))\rightarrow Acc(\mathrm{cp}(\mathrm{pt}(a, k), 0));$$

using 4.4,

$$\mathrm{O}(a)=0, \mathrm{lh}(a)=k+1, Acc(\mathrm{cp}(a, 0))\rightarrow \mathrm{lh}(\mathrm{pt}(a, k))=k\wedge Acc(\mathrm{cp}(\mathrm{pt}(a, k), 0));$$

using 4.5,

$$\mathrm{O}(a)=0, \mathrm{lh}(a)=k+1, Acc(\mathrm{cp}(a, 0))$$
$$\rightarrow \mathrm{O}(\mathrm{pt}(a, k))=0\wedge \mathrm{lh}(\mathrm{cp}(\mathrm{pt}(a, k), 0))=k\wedge Acc(\mathrm{cp}(\mathrm{pt}(a, k), 0));$$

adding $Acc(\mathrm{pt}(a, k))\rightarrow Acc(\mathrm{pt}(a, k))$,

$L(\mathrm{pt}(a, k), k), \mathrm{O}(a)=0, \mathrm{lh}(a)=k+1, Acc(\mathrm{cp}(a, 0)) \rightarrow Acc(\mathrm{pt}(a, k))$;

(1) $\forall z L(z, k), \mathrm{O}(a)=0, \mathrm{lh}(a)=k+1, Acc(\mathrm{cp}(a, 0)) \rightarrow Acc(\mathrm{pt}(a, k))$.

By T4 (and by 2.2, 2.40, 2.53 and 2.54) we have,

$Acc(\mathrm{pt}(a, k)), \mathrm{O}(\mathrm{lc}(a))=0,$

$\mathrm{Ord}(\mathrm{lc}(a), \mathrm{lc}(\mathrm{pt}(a, k)))=0 \vee \mathrm{lc}(a)=\mathrm{lc}(\mathrm{pt}(a, k)) \rightarrow Acc(a)$,

then by 4.6 and 4.7,

(2) $Acc(\mathrm{pt}(a, k)), \mathrm{O}(a)=0, \mathrm{lh}(a)=k+1 \rightarrow Acc(a)$.

From (1) and (2), we have successively,

$\mathrm{O}(a)=0, \mathrm{lh}(a)=k+1, Acc(\mathrm{cp}(a, 0)), \forall z L(z, k) \rightarrow Acc(a)$;

$\mathrm{O}(a)=0 \wedge \mathrm{lh}(a)=k+1 \wedge Acc(\mathrm{cp}(a, 0)), \forall z L(z, k) \rightarrow Acc(a)$;

$\forall z L(z, k) \rightarrow L(a, k+1)$;

$\forall z L(z, k) \rightarrow \forall z L(z, k+1)$;

$\rightarrow \forall z L(z, k) \vdash \forall z L(z, k+1)$;

(3) $\rightarrow \forall y(\forall z L(z, y) \vdash \forall z L(z, y+1))$;

On the other hand, from 4.8 and $\mathrm{cp}(a, 0)=a, Acc(\mathrm{cp}(a, 0)) \rightarrow Acc(a)$, we have easily

(4) $\rightarrow \forall z L(z, 0)$.

By (3) and (4),

$\rightarrow \forall z L(z, 0) \wedge \forall y(\forall z L(z, y) \vdash \forall z L(z, y+1))$;

using $\forall z L(z, \mathrm{lh}(a)), \mathrm{O}(a)=0, Acc(\mathrm{cp}(a, 0)) \rightarrow Acc(a)$ (by 2.1),

$\forall z L(z, 0) \wedge \forall y(\forall z L(z, y) \vdash \forall z L(z, y+1)) \vdash \forall z L(z, \mathrm{lh}(a)), \Gamma_5 \rightarrow Acc(a)$;

$\forall x(\forall z L(z, 0) \wedge \forall y(\forall z L(z, y) \vdash \forall z L(z, y+1)) \vdash \forall z L(z, x)), \Gamma_5 \rightarrow Acc(a)$;

by $\forall$ left on an f-variable,

$\forall \varphi \forall x(\varphi[0] \wedge \forall y(\varphi[y] \vdash \varphi[y+1]) \vdash \varphi[x]), \Gamma_5 \rightarrow Acc(a)$;

by 2.24,

$\Gamma_5 \rightarrow Acc(a)$, q. e. d.

5.5. Proof of T6.

From $d_1=a, Acc(\mathrm{ack}(a, b, d_3)) \rightarrow Acc(\mathrm{ack}(d_1, b, d_3))$ and $d_2=b, Acc(\mathrm{ack}(d_1, b, d_3)) \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3))$, we have the following sequences successively:

$d_1=a, d_2=b, Acc(\mathrm{ack}(a, b, d_3)) \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3))$;

adding $\mathrm{O}(d_3)=0, \mathrm{Ord}(d_3, c)=0 \rightarrow D(d_3, c)$,

$D(d_3, c) \vdash Acc(\mathrm{ack}(a, b, d_3)), d_1=a, d_2=b, \mathrm{O}(d_3)=0, \mathrm{Ord}(d_3, c)=0$

$\rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3))$;

$C(a, b, c), d_1=a, d_2=b, \mathrm{O}(d_3)=0, \mathrm{Ord}(d_3, c)=0 \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3))$;

(1) $C(a, b, c), d_1=a \wedge d_2=b \wedge \mathrm{Ord}(d_3, c)=0, \mathrm{O}(d_3)=0 \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3))$.

We have also the following sequences successively:

$R(d_3, \mathrm{rb}(\mathrm{cp}(d_3, 0))), \mathrm{O}(d_3)=0, \mathrm{Ord}(d_3, \mathrm{ack}(a, b, c))=0 \rightarrow Acc(d_3)$;

$\forall u R(u, \mathrm{rb}(\mathrm{cp}(d_3, 0))), \mathrm{O}(d_3)=0, \mathrm{Ord}(d_3, \mathrm{ack}(a, b, c))=0 \rightarrow Acc(d_3)$;

adding $\mathrm{rb}(\mathrm{cp}(d_3, 0))<k \rightarrow \mathrm{rb}(\mathrm{cp}(d_3, 0))<k$,

$\mathrm{rb}(\mathrm{cp}(d_3, 0) < k \vdash \forall u R(u, \mathrm{rb}(\mathrm{cp}(d_3, 0))), \mathrm{rb}(\mathrm{cp}(d_3, 0)) < k, \mathrm{O}(d_3) = 0,$
$\quad \mathrm{Ord}(d_3, \mathrm{ack}(a, b, c)) = 0 \rightarrow Acc(d_3);$

$\forall z(z < k \vdash \forall u R(u, z)), \mathrm{rb}(\mathrm{cp}(d_3, 0)) < k,$
$\quad \mathrm{O}(d_3) = 0, \mathrm{Ord}(d_3, \mathrm{ack}(a, b, c)) = 0 \rightarrow Acc(d_3);$

(2) $\mathrm{rb}(\mathrm{cp}(d_3, 0)) < k, \forall z(z < k \vdash \forall u R(u, z)),$
$\quad \mathrm{O}(d_3) = 0, \mathrm{Ord}(d_3, \mathrm{ack}(a, b, c)) = 0 \rightarrow Acc(d_3);$

using 4.9,

$\mathrm{O}(d_3) = 0, \mathrm{Ord}(d_3, \mathrm{ack}(a, b, c)) = 0, \mathrm{rb}(\mathrm{ack}(d_1, d_2, d_3)) = k,$
$\quad \forall z(z < k \vdash \forall u R(u, z)) \rightarrow Acc(d_3);$

using $\mathrm{O}(d_2) = 0, \mathrm{Ord}(d_2, b) = 0 \rightarrow D(d_2, b),$

$\mathrm{Ord}(d_2, b) = 0, \mathrm{Ord}(d_3, \mathrm{ack}(a, b, c)) = 0, \mathrm{rb}(\mathrm{ack}(d_1, d_2, d_3)) = k,$
$\quad \forall z(z < k \vdash \forall u R(u, z)), \mathrm{O}(d_2) = 0, \mathrm{O}(d_3) = 0 \rightarrow D(d_2, b) \wedge Acc(d_3);$

using $d_1 = a, Acc(\mathrm{ack}(a, d_2, d_3)) \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3)),$

$D(d_2, b) \wedge Acc(d_3) \vdash Acc(\mathrm{ack}(a, d_2, d_3)), d_1 = a, \mathrm{Ord}(d_2, b) = 0,$
$\quad \mathrm{Ord}(d_3, \mathrm{ack}(a, b, c)) = 0, \mathrm{rb}(\mathrm{ack}(d_1, d_2, d_3)) = k, \forall z(z < k$
$\quad \vdash \forall u R(u, z)), \mathrm{O}(d_2) = 0, \mathrm{O}(d_3) = 0 \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3));$

$B(a, b), d_1 = a, \mathrm{Ord}(d_2, b) = 0, \mathrm{Ord}(d_3, \mathrm{ack}(a, b, c)) = 0, \mathrm{rb}(\mathrm{ack}(d_1, d_2, d_3)) = k,$
$\quad \forall z(z < k \vdash \forall u R(u, z)), \mathrm{O}(d_2) = 0, \mathrm{O}(d_3) = 0 \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3));$

(3) $d_1 = a \wedge \mathrm{Ord}(d_2, b) = 0 \wedge \mathrm{Ord}(d_3, \mathrm{ack}(a, b, c)) = 0, \mathrm{rb}(\mathrm{ack}(d_1, d_2, d_3)) = k,$
$\quad \forall z(z < k \vdash \forall u R(u, z)), \mathrm{O}(d_2) = 0, \mathrm{O}(d_3) = 0, B(a, b) \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3)).$

In the similar method as we obtained (3), we have

(4) $\mathrm{Ord}(d_1, a) = 0 \wedge \mathrm{Ord}(d_2, \mathrm{ack}(a, b, c)) = 0 \wedge \mathrm{Ord}(d_3, \mathrm{ack}(a, b, c)) = 0,$
$\quad \mathrm{rb}(\mathrm{ack}(d_1, d_2, d_3)) = k, \forall z(z < k \vdash \forall u R(u, z)), \mathrm{O}(d_1) = 0,$
$\quad \mathrm{O}(d_2) = 0, \mathrm{O}(d_3) = 0, A(a) \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3)).$

Moreover, we have the following sequences by T2 or by 2.2:

(5) $\mathrm{Ord}(\mathrm{ack}(d_1, d_2, d_3), a) = 0, \mathrm{O}(\mathrm{ack}(d_1, d_2, d_3)) = 0, Acc(a) \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3))$

(6) $\mathrm{ack}(d_1, d_2, d_3) = a, Acc(a) \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3))$

(7) $\mathrm{Ord}(\mathrm{ack}(d_1, d_2, d_3), b) = 0, \mathrm{O}(\mathrm{ack}(d_1, d_2, d_3)) = 0, Acc(b) \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3))$

(8) $\mathrm{ack}(d_1, d_2, d_3) = b, Acc(b) \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3))$

(9) $\mathrm{Ord}(\mathrm{ack}(d_1, d_2, d_3), c) = 0, \mathrm{O}(\mathrm{ack}(d_1, d_2, d_3)) = 0, Acc(c) \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3))$

(10) $\mathrm{ack}(d_1, d_2, d_3) = c, Acc(c) \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3)).$

Then using 4.10, (1), (3)—(10) and 2.51 successively, we have

$D(\mathrm{ack}(d_1, d_2, d_3), \mathrm{ack}(a, b, c)), \mathrm{rb}(\mathrm{ack}(d_1, d_2, d_3)) = k,$
$\quad \forall z(z < k \vdash \forall u R(u, z)), \Gamma_6 \rightarrow Acc(\mathrm{ack}(d_1, d_2, d_3));$

using $\mathrm{cp}(d, 0) = \mathrm{ack}(d_1, d_2, d_3), Acc(\mathrm{ack}(d_1, d_2, d_3)) \rightarrow Acc(\mathrm{cp}(d, 0)),$

$\mathrm{cp}(d, 0) = \mathrm{ack}(d_1, d_2, d_3), \mathrm{rb}(\mathrm{ack}(d_1, d_2, d_3)) = k, D(\mathrm{ack}(d_1, d_2, d_3), \mathrm{ack}(a, b, c)),$
$\quad \forall z(z < k \vdash \forall u R(u, z)), \Gamma_6 \rightarrow Acc(\mathrm{cp}(d, 0));$

using $\mathrm{cp}(d, 0) = \mathrm{ack}(d_1, d_2, d_3), \mathrm{rb}(\mathrm{cp}(d, 0)) = k \rightarrow \mathrm{rb}(\mathrm{ack}(d_1, d_2, d_3)) = k,$

$\mathrm{cp}(d, 0) = \mathrm{ack}(d_1, d_2, d_3), \mathrm{rb}(\mathrm{cp}(d, 0)) = k, D(\mathrm{ack}(d_1, d_2, d_3), \mathrm{ack}(a, b, c)),$
$\quad \forall z(z < k \vdash \forall u R(u, z)), \Gamma_6 \rightarrow Acc(\mathrm{cp}(d, 0));$

$$\mathrm{cp}(d, 0)=\mathrm{ack}(d_1, d_2, d_3)\wedge D(\mathrm{ack}(d_1, d_2, d_3), \mathrm{ack}(a, b, c)), \mathrm{rb}(\mathrm{cp}(d, 0))=k,$$
$$\forall z(z<k\vdash \forall uR(u, z)), \Gamma_6\rightarrow Acc(\mathrm{cp}(d, 0));$$
$$\exists x\exists y\exists z(\mathrm{cp}(d, 0)=\mathrm{ack}(x, y, z)\wedge D(\mathrm{ack}(x, y, z), \mathrm{ack}(a, b, c))),$$
$$\mathrm{rb}(\mathrm{cp}(d, 0))=k, \forall z(z<k\vdash \forall uR(u, z)), \Gamma_6\rightarrow Acc(\mathrm{cp}(d, 0));$$

using 4.11, 4.1 and 2.51,

$$\mathrm{rb}(\mathrm{cp}(d, 0))=k, D(d, \mathrm{ack}(a, b, c)), \forall z(z<k\vdash \forall uR(u, z)),$$
$$\Gamma_6\rightarrow Acc(\mathrm{cp}(d,0)), \mathrm{cp}(d, 0)=1;$$

using $\mathrm{cp}(d, 0)=1\rightarrow Acc(\mathrm{cp}(d, 0))$ (following from T1 and 2.2),

$$\mathrm{rb}(\mathrm{cp}(d, 0))=k, D(d, \mathrm{ack}(a, b, c)), \forall z(z<k\vdash \forall uR(u, z)), \Gamma_6\rightarrow Acc(\mathrm{cp}(d, 0));$$

then by the help of T5,

$$\mathrm{rb}(\mathrm{cp}(d, 0))=k, D(d, \mathrm{ack}(a, b, c)), \forall z(z<k\vdash \forall uR(u, z)), \Gamma_6\rightarrow Acc(d);$$
$$\mathrm{rb}(\mathrm{cp}(d, 0))=k\wedge D(d, \mathrm{ack}(a, b, c)), \forall z(z<k\vdash \forall uR(u, z)), \Gamma_6\rightarrow Acc(d);$$
$$\forall z(z<k\vdash \forall uR(u, z)), \Gamma_6\rightarrow R(d, k);$$
$$\forall z(z<k\vdash \forall uR(u, z)), \Gamma_6\rightarrow \forall uR(u, k);$$
$$\Gamma_6\rightarrow F(k, \{z\}(\forall uR(u, z)));$$
$$\Gamma_6\rightarrow \forall yF(y, \{z\}(\forall uR(u, z)));$$

using $\forall uR(u, \mathrm{rb}(\mathrm{cp}(d, 0))), D(d, \mathrm{ack}(a, b, c))\rightarrow Acc(d)$,

$$\forall yF(y, \{z\}(\forall uR(u, z)))\vdash \forall uR(u, \mathrm{rb}(\mathrm{cp}(d, 0))), D(d, \mathrm{ack}(a, b, c)),$$
$$\Gamma_6\rightarrow Acc(d);$$
$$\forall x(\forall yF(y, \{z\}(\forall uR(u, z)))\vdash \forall uR(u, x)), D(d, \mathrm{ack}(a, b, c)), \Gamma_6\rightarrow Acc(d);$$

by $\forall$ left on an f-variable,

$$\forall\varphi\forall x(\forall yF(y, \varphi)\vdash \varphi[x]), D(d, \mathrm{ack}(a, b, c)), \Gamma_6\rightarrow Acc(d);$$

i. e. (cf. the end of § 3),

$$D(d, \mathrm{ack}(a, b, c)), \Gamma_6\rightarrow Acc(d);$$
$$\Gamma_6\rightarrow D(d, \mathrm{ack}(a, b, c))\vdash Acc(d);$$
$$\Gamma_6\rightarrow \forall x(D(x, \mathrm{ack}(a, b, c))\vdash Acc(x));$$

by the help of T3,

$$\Gamma_6\rightarrow Acc(\mathrm{ack}(a, b, c)), \text{ q. e. d.}$$

5.6. Proof of T7.

We have the following sequences successively on one hand: by the help of T2,

$$D(e, d)\vdash K(a, b, e), D(e, d), Acc(d)\rightarrow Acc(\mathrm{ack}(a, b, e));$$
$$\forall y(D(y, d)\vdash K(a, b, y)), D(e, d), Acc(d)\rightarrow Acc(\mathrm{ack}(a, b, e));$$
$$\forall y(D(y, d)\vdash K(a, b, y)), Acc(d)\rightarrow D(e, d)\vdash Acc(\mathrm{ack}(a, b, e));$$

(1) $\forall y(D(y, d)\vdash K(a, b, y)), Acc(d)\rightarrow C(a, b, d)$.

On the other hand, by T6 we have

(2) $Acc(a), Acc(b), Acc(d), A(a), B(a, b), C(a, b, d)\rightarrow Acc(\mathrm{ack}(a, b, d))$.

From (1) and (2) we have successively,

$$\forall y(D(y, d)\vdash K(a, b, y)), Acc(a), Acc(b), Acc(d), A(a), B(a, b)\rightarrow Acc(\mathrm{ack}(a, b, d));$$

$\forall y(D(y, d) \vdash K(a, b, y)), Acc(a), Acc(b), A(a), B(a, b) \to K(a, b, d)$;

$Acc(a), Acc(b), A(a), B(a, b) \to E(d, \{y\}K(a, b, y))$;

$Acc(a), Acc(b), A(a), B(a, b) \to \forall x E(x, \{y\}K(a, b, y))$;

adding $K(a, b, c), Acc(c) \to Acc(\mathrm{ack}(a, b, c))$,

$\forall x E(x, \{y\}K(a, b, y)) \vdash K(a, b, c), \Gamma_7 \to Acc(\mathrm{ack}(a, b, c))$;

by $\forall$ left on an f-variable,

$\forall \varphi(\forall x E(x, \varphi) \vdash \varphi[c]), \Gamma_7 \to Acc(\mathrm{ack}(a, b, c))$;

$\Gamma_7 \to Acc(\mathrm{ack}(a, b, c))$, q. e. d.

T8 and T9 are proved quite similarly.

5.7. PROOF OF T10.

In the same way as we obtained (2) in 5.5, we have

$pr(b) < k, \forall z(z < k \vdash \forall u P(u, z)), \mathrm{O}(b) = 0 \to Acc(b)$,

then using 4.12 and 2.53,

(1) $pr(a) = k, a = \mathrm{ack}(b, c, d), \mathrm{O}(a) = 0, \forall z(z < k \vdash \forall u P(u, z)) \to Acc(b)$.

By 4.13 and 4.14 in place of 4.12,

(2) $pr(a) = k, a = \mathrm{ack}(b, c, d), \mathrm{O}(a) = 0, \forall z(z < k \vdash \forall u P(u, z)) \to Acc(c)$

and

(3) $pr(a) = k, a = \mathrm{ack}(b, c, d), \mathrm{O}(a) = 0, \forall z(z < k \vdash \forall u P(u, z)) \to Acc(d)$

respectively. Using (1), (2) and (3) successively, by the help of T9 and 2.2, we have

(4) $pr(a) = k, a = \mathrm{ack}(b, c, d), \mathrm{O}(a) = 0, \forall z(z < k \vdash \forall u P(u, z)) \to Acc(a)$.

From 4.15 and $\forall x P(x, pr(\mathrm{cp}(a, 0))), \mathrm{O}(\mathrm{cp}(a, 0)) = 0 \to Acc(\mathrm{cp}(a, 0))$ (by 2.1), we have

$0 < \mathrm{lh}(a), pr(a) = k, \forall z(z < k \vdash \forall u P(u, z)) \to Acc(\mathrm{cp}(a, 0))$,

then by the help of T5,

(5) $0 < \mathrm{lh}(a), \mathrm{O}(a) = 0, pr(a) = k, \forall z(z < k \vdash \forall u P(u, z)) \to Acc(a)$.

By T1 and 2.2,

(6) $a = 1 \to Acc(a)$.

Then, using (4), (5) and (6) successively by the help of 4.16,

$\mathrm{O}(a) = 0 \wedge pr(a) = k, \forall z(z < k \vdash \forall u P(u, z)) \to Acc(a)$;

$\forall z(z < k \vdash \forall u P(u, z)) \to P(a, k)$;

$\forall z(z < k \vdash \forall u P(u, z)) \to \forall u P(u, k)$;

$\to F(k, \{y\}(\forall u P(u, y)))$;

$\to \forall x F(x, \{y\}(\forall u P(u, y)))$.

Now we can conclude the proof in the same way as in 5.5, q. e. d.

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