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On the radial order of a univalent function.

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1. In a recent note in this Journal Gehring [**5**] has given a new proof of the following theorem of Denjoy [**1**] and Seidel and Walsh [**6**].*

THEOREM 1. *Suppose that $f(z)$ is regular and univalent in $|z|<1$. Then for almost all θ*

$$f'(z)=o((1-|z|)^{-\frac{1}{2}}) \tag{1.1}$$

uniformly as $z\rightarrow e^{i\theta}$ in each Stolz domain.

Gehring's proof, though short, is far from elementary, since it depends on a difficult maximal theorem of Hardy and Littlewood. In this note we give an alternative proof of Theorem 1 which is considerably more elementary.

2. We require a simple identity concerning Cesàro means. Let $f(z)=\sum c_n z^n$, and let $\tau_n^{\alpha}(\theta)$ denote the n-th Cesàro mean of order α of the sequence $nc_n e^{ni\theta}$. Then it is well known that for any α and for $|z|<1$

$$\frac{zf'(ze^{i\theta})}{(1-z)^{\alpha}}=\sum_{1}^{\infty} E_n^{\alpha}\tau_n^{\alpha}(\theta)z^n\,, \tag{2.1}$$

where (as usual)

$$E_n^{\alpha}=\binom{\alpha+n}{n}=\frac{(\alpha+1)(\cdots)(\alpha+n)}{n\,!} \qquad (n>0)\,.$$

3. Consider now the proof of the theorem. A familiar argument [**6**] allows us to assume that the image of $|z|<1$ under $\zeta=f(z)$ has finite area, or that

$$\int_0^1\int_{-\pi}^{\pi}|f'(\rho e^{i\theta})|^2\rho d\theta d\rho<\infty\,. \tag{3.1}$$

We show first that if f satisfies (3.1), and if $\alpha>1/2$, then the series $\sum|\tau_n^{\alpha}(\theta)|^2$ is convergent p. p. This is a particular case of a more general result (Flett [**4**]), but we give the proof for the sake of completeness.

Applying Parseval's theorem to the function (2.1) we obtain

$$\sum_{1}^{\infty}(E_n^{\alpha})^2|\tau_n^{\alpha}(\theta)|^2\rho^{2n}\leqq\frac{\rho}{2\pi}\int_{-\pi}^{\pi}\frac{|f'(\rho e^{i\theta+it})|^2}{|1-\rho e^{it}|^{2\alpha}}dt\,. \tag{3.2}$$

* Various generalizations of the theorem are known (see, for example, Ferrand [**3**]), but we do not consider these here.

Multiplying both sides of (3.2) by $(1-\rho)^{2\alpha-1}$, integrating with respect to ρ from 0 to 1, and observing that*

$$\int_0^1 (1-\rho)^{2\alpha-1}\rho^{2n}d\rho \geqq A(\alpha)n^{-2\alpha},$$

we obtain

$$\sum_1^\infty |\tau_n^\alpha(\theta)|^2 \leqq A(\alpha)\int_0^1 (1-\rho)^{2\alpha-1}\rho d\rho \int_{-\pi}^{\pi} \frac{|f'(\rho e^{i\theta+it})|^2}{|1-\rho e^{it}|^{2\alpha}}dt. \tag{3.3}$$

Now integrate both sides of (3.3) with respect to θ and interchange the order of integration on the right: we get

$$\int_{-\pi}^{\pi}(\sum_1^\infty |\tau_n^\alpha(\theta)|^2)d\theta \leqq A(\alpha)\int_0^1 (1-\rho)^{2\alpha-1}\rho d\rho \int_{-\pi}^{\pi}\frac{dt}{|1-\rho e^{it}|^{2\alpha}}\int_{-\pi}^{\pi}|f'(\rho e^{i\theta+it})|^2 d\theta .$$

Here the innermost integral on the right is actually independent of t and is equal to

$$\int_{-\pi}^{\pi}|f'(\rho e^{i\theta})|^2 d\theta .$$

Moreover,

$$\int_{-\pi}^{\pi}\frac{dt}{|1-\rho e^{it}|^{2\alpha}} \leqq \frac{A(\alpha)}{(1-\rho)^{2\alpha-1}}$$

(since $2\alpha > 1$), so that

$$\int_{-\pi}^{\pi}(\sum_1^\infty |\tau_n^\alpha(\theta)|^2)d\theta \leqq A(\alpha)\int_0^1 \rho d\rho \int_{-\pi}^{\pi}|f'(\rho e^{i\theta})|^2 d\theta = A(\alpha)\int_0^1\int_{-\pi}^{\pi}|f'(\rho e^{i\theta})|^2 \rho d\theta d\rho < \infty .$$

Thus the sum-function of the series $\sum|\tau_n^\alpha(\theta)|^2$ belongs to $L^2(-\pi,\pi)$, and so is finite p. p.

It remains now to show that (1.1) holds at any point θ for which $\sum|\tau_n^\alpha(\theta)|^2$ converges (where α is any fixed number greater than 1/2). Let θ be such a point. Then given $\varepsilon > 0$ we can find an integer N such that

$$\sum_N^\infty |\tau_n^\alpha(\theta)|^2 < \varepsilon^2 ,$$

and then also

$$|\sum_N^\infty E_n^\alpha \tau_n^\alpha(\theta)z^n| \leqq \{\sum_N^\infty (E_n^\alpha)^2|z|^{2n}\}^{\frac{1}{2}}\{\sum_N^\infty |\tau_n^\alpha(\theta)|^2\}^{\frac{1}{2}}$$
$$\leqq A(\alpha)\varepsilon(1-|z|)^{-\alpha-\frac{1}{2}} .$$

Hence, by (2.1),

$$\frac{|zf'(ze^{i\theta})|}{|1-z|^\alpha} \leqq \sum_1^{N-1} E_n^\alpha|\tau_n^\alpha(\theta)| + A(\alpha)\varepsilon(1-|z|)^{-\alpha-\frac{1}{2}} .$$

Since $|1-z|/(1-|z|)$ lies between two positive constants when z belongs to

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* We use $A(\alpha)$ to denote a positive constant depending only on α, not necessarily the same on any two occurrences.

any given Stolz domain with vertex at $z=1$, Theorem 1 follows.

4. The argument of §3 generalizes without difficulty to prove the following result of Dufresnoy [**2**].

THEOREM 2. *Suppose that* $f(z)$ *is regular in* $|z|<1$ *and that*

$$\int_0^1\int_{-\pi}^{\pi}(1-\rho)^{k-kr-1}|f'(\rho e^{i\theta})|^k d\rho d\theta<\infty\,,$$

where $k\geqq 1$ *and* $0<r\leqq 1$. *Then for almost all* θ

$$f'(z)=o((1-|z|)^{r-1}) \tag{4.1}$$

uniformly as $z\rightarrow e^{i\theta}$ *in each Stolz domain.*

We have now that for almost all θ

$$\sum n^{kr-1}|\tau_n{}^{\alpha}(\theta)|^k<\infty$$

provided that $\alpha>$ sup $(1/k, 1-1/k)$ [**4**, Theorem 11], and this implies (4.1).

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References

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