Journal of the Mathematical Society of Japan Vol. 11, No. 4, October, 1959

Characteristic classes of M-spaces I.

By Itiro TAMURA

(Received April 24, 1959)
(Revised July 14, 1959)

Introduction.

In the present paper, we shall introduce the concept of M-spaces, define characteristic classes of these spaces and study their properties.

The definition of M-spaces will be given in Section 1; we shall give here a rough explanation about what they are. Let $\tilde{M}$ be a connected manifold, say a C^{∞}-manifold, and G a finite group of transformations (C^{∞}-transformations, when we are dealing with C^{∞}-manifold) of $\tilde{M}$ onto itself. The quotient space $\tilde{M}/G$ is a manifold if each element of G has no fixed point; otherwise $\tilde{M}/G$ is a "manifold with singularities" (which is not a manifold in proper sense). Our concept of M-spaces is a generalization of the notion of "manifolds with singularities" $\tilde{M}/G$. Let W be a submanifold of $\tilde{M}$ and G a finite group of transformations of W onto itself (with or without fixed points). The space $M=(\tilde{M}-W)\cup(W/G)$, obtained as the set-sum of $\tilde{M}-W$ and W/G, the topology being so defined that the natural mapping φ of $\tilde{M}$ onto M becomes continuous and open, is an M-space, which we shall denote by $\{M, \tilde{M}, W, G, \varphi\}$.

The concept of characteristic classes of M-spaces is a generalization of the concept of characteristic classes of manifolds. This has an interest also on the theory of manifolds in the following sense. Pontrjagin classes are diffeomorphy invariants but not homotopy type invariants, and so there is a possibility to obtain finer classifications of manifolds than those according to homotopy types by means of Pontrjagin classes. However, in case where Pontrjagin classes vanish (for instance, when the manifolds are 3-dimensional), there is no such possibility. But in certain cases (for example, when the manifolds are lens spaces), we can imbed n-dimensional manifolds in $(n+1)$-dimensional M-spaces, and obtain a "finer classification" by means of characteristic classes of these M-spaces (Tamura [**6**]).

Moreover, by virtue of considerations of characteristic classes of M-spaces, especially those of "manifolds with singularities" $\tilde{M}/G$, we shall be able to discuss the differentiable or complex analytic structure ("with singularities") of such spaces in like manner as differentiable or complex analytic structures of manifolds. It seems that such a consideration is also useful to make clear various properties of characteristic classes of manifolds.

The present paper is divided into three parts.

We shall give definitions of M-spaces and describe their properties in the first half of Part I. In the second half of Part I, D-bundles over M-spaces will be defined. In particular case where the base space is a quotient space, D-bundles become V-bundles (Satake [**4**], Baily [**1**]) over a quotient space. We shall study especially D-bundles having vector spaces as fibres which we shall call vector D-bundles. A typical example of vector D-bundle is tangent D-bundle of an M-space. Associated D-bundles of vector D-bundles and generalized associated D-bundles of tangent D-bundles will be introduced and studied for later use.

Part II contains the obstruction theory of D-bundles over M-spaces. In case M-spaces have singularities, we have to impose several conditions upon the neighbourhoods of singular points.

Part III is devoted to the definitions of characteristic classes of D-bundles and of M-spaces. The obstruction theory of Part II is used here. We shall generalize Stiefel-Whitney classes and Pontrjagin classes of $O(m)$-bundles and of C^∞-manifolds to our cases, and shall call the corresponding classes SW-classes and P-classes of $O(m)$-D-bundles and of C^∞-M-spaces respectively. We shall also generalize Chern classes of $U(m)$-bundles and of almost complex manifolds to our cases, and shall call the corresponding classes C-classes of $U(m)$-D-bundles and of almost complex M-spaces respectively (Section 7).

Furthermore the Euler-Poincaré characteristic of quotient space will be defined. This can be done under a looser assumption than the conditions of Section 7. Our Euler-Poincaré characteristic can be written by Betti numbers as usual.

In a subsequent paper [**7**], characteristic classes of 2-fold symmetric products of spheres will be computed.

The author wishes to express his thanks to Prof. N. Shimada for his valuable remarks.

Part I. *M*-spaces and *D*-bundles.

1. *M*-spaces.

Let M be a paracompact connected Hausdorff space. We shall define a C^∞-M-space as follows.

Definition 1.1. By a *C^∞-M-space* $\boldsymbol{M}^m=\{M, \tilde{M}, W, G, \varphi\}$ we mean a collection of the following objects:

(i) $\tilde{M}$ is a connected m-dimensional C^∞-manifold (with or without boundary) and is called the *covering manifold* of $\boldsymbol{M}^m$.

(ii) $W=\sum_{i=1}^{l}{}^iW$ is a finite union of (not necessarily connected) submanifolds (with or without boundaries) ${}^iW\,(i=1,\cdots,l)$ of $\tilde{M}$ whose underlying topological spaces are closed subsets of $\tilde{M}$.

(iii) G is a finite group of C^∞-automorphisms of W which transform each ${}^iW$ onto itself. G is called the *group* of $\boldsymbol{M}^m$.

(iv) φ is a continuous and open map from $\tilde{M}$ onto M which is satisfying $\varphi(x)=\varphi(x')$ $(x, x'\in\tilde{M}, x\neq x')$ if and only if $x, x'\in W$ and there exists a $g\in G$ such that $g(x)=x'$.

(v) $\varphi({}^iW)$ is connected for $i=1,\cdots,l$.

M is called the *underlying topological space* of $\boldsymbol{M}^m$. We shall call m the *dimension* of $\boldsymbol{M}^m$.

(In future we shall sometimes write $\boldsymbol{M}$ instead of $\boldsymbol{M}^m$, if the dimension is of no importance.)

It is clear that $\varphi(g(x))=\varphi(x)$ for all $x\in W, g\in G$, and that φ is a homeomorphic map on $\tilde{M}-W$.

We denote by ${}_iG$ the (normal) subgroup of G consisting of all elements operating trivially on ${}^iW$ and by ${}^iG$ the factor group $G/{}_iG$ $(i=1,\cdots,l)$.

A C^∞-M-space becomes a usual C^∞-manifold, if each ${}^iG$ is the unit group or each ${}^iW$ is a point.

M-spaces of different categories other than C^∞, for example, C^r $(0\leqq r<\infty)$, *real analytic*, *complex analytic* or *almost complex M-space* will be defined in obvious manners.

The product of two M-spaces $\boldsymbol{M}=\{M, \tilde{M}, W, G, \varphi\}$, $\boldsymbol{M}'=\{M', \tilde{M}', W', G', \varphi'\}$ of the same category will become again an M-space $\boldsymbol{M}\times\boldsymbol{M}'=\{M\times M', \tilde{M}\times\tilde{M}', W\times\tilde{M}'\cup\tilde{M}\times W', G\times G', \varphi\times\varphi'\}$ of the same category.

Let $\{{}^iW\}$ $(i\in\Lambda)$ be the set of all ${}^iW$ on which G operates non-trivially. We put $\overline{m}=\max_{i\in\Lambda}\dim{}^iW$ and $\underline{m}=\min_{i\in\Lambda}\dim{}^iW$, and shall call them *max-V-dimension* and *min-V-dimension* of $\boldsymbol{M}$ respectively. Obviously an M-space becomes a quotient space if and only if $m=\overline{m}=\underline{m}$.

Let x be a point of M. Let us choose a point $\tilde{x}\in\tilde{M}$ such that $\varphi(\tilde{x})=x$. Then, as is easily verified, the structure of the isotropy group of G at $\tilde{x}$: $G_{\tilde{x}}=\{g\,;\,g(\tilde{x})=\tilde{x}, g\in G\}$ does not depend on the choice of $\tilde{x}$, and is uniquely determined by x. Hence we call $G_{\tilde{x}}$ simply the *isotropy group* of x and denote it by G_x. A point of M which has a non-trivial isotropy group will be called the *singular point* of M.

A C^r-M-space $\boldsymbol{M}=\{M, \tilde{M}, W, G, \varphi\}$ $(0\leqq r\leqq\infty)$ will be called *orientable* if $\tilde{M}$ and ${}^iW$ $(i=1,\cdots,l)$ are orientable and the action on ${}^iW$ of each $g\in{}^iG$ is orientation-preserving.

Let $\boldsymbol{M}$ be an almost complex M-space. Then we can regard $\boldsymbol{M}$ as a

C^∞-M-space with the induced C^∞-differentiable structure in a natural manner. By the same argument as in the case of almost complex manifolds, we see that the C^∞-M-space $\boldsymbol{M}$ is orientable.

Let $\tilde{M}$ be a paracompact connected C^∞-manifold (with or without boundary) and $W=\sum_{i=1}^{l} {}^iW$ a finite union of submanifolds of $\tilde{M}$, and let G be a finite group of C^∞-automorphisms of W which transform each ${}^iW$ onto itself. If we identify the points which are transformed by elements of G, we obtain a space M with the identification topology. A C^∞-M-space thus obtained will be denoted simply by $\boldsymbol{M}=\{\tilde{M}, W, G, \varphi\}$.

We give here some examples of M-spaces.

EXAMPLE 1.1. Let $\tilde{M}=W=R^m$ (m-dimensional Euclidean space), and let G be the group of order 2 with a generator g such that $g(x_1, x_2, \cdots, x_m)=(-x_1, x_2, \cdots, x_m)$, where $(x_1, x_2, \cdots, x_m)$ means the Cartesian coordinates of R^m. The underlying topological space M of the C^∞-M-space $\boldsymbol{M}=\{R^m, R^m, G, \varphi\}$ is the half plane of R^m. The isotropy group at $\varphi(0, x_2, \cdots, x_m)$ has order 2.

EXAMPLE 1.2. Let S^m be the m-sphere with the natural differentiable structure, that is, the set of points $(x_0, x_1, \cdots, x_m)$ of R^{m+1} satisfying $x_0^2+x_1^2+\cdots+x_m^2=1$. Let $\tilde{M}=W=S^m$ and let G be a finite group of C^∞-automorphisms of S^m of order n with a generator g such that $g(x_0, x_1, \cdots, x_m)=(x_0\cos(2\pi/n)-x_1\sin(2\pi/n),\ x_0\sin(2\pi/n)+x_1\cos(2\pi/n),\ x_2, \cdots, x_m)$. We denote the C^∞-M-space $\boldsymbol{M}=\{S^m, S^m, G, \varphi\}$ by $\boldsymbol{S}^m(1/n)$. The underlying topological space of $\boldsymbol{S}^m(1/n)$ is a topological m-sphere. The isotropy group at $\varphi(0, 0, x_2, x_3, \cdots, x_m)$ has order n.

EXAMPLE 1.3. Suppose that $\bar{M}^m$ (resp. $\bar{M}_m$) be an m-dimensional C^∞-manifold (resp. an m-dimensional complex analytic manifold) and that $\mathfrak{S}_n$ is the symmetric group of degree n. Denote $\tilde{M}^{mn}=\bar{M}^m\times\cdots\times\bar{M}^m$ (resp. $\tilde{M}_{mn}=\bar{M}_m\times\cdots\times\bar{M}_m$) ($n$-fold product of $\bar{M}^m$ resp. $\bar{M}_m$). $\mathfrak{S}_n$ operates on $\tilde{M}^{mn}$ (resp. $\tilde{M}_{mn}$) as C^∞-(resp. complex analytic) automorphism group in a natural way. We denote C^∞-M-space $\boldsymbol{M}=\{\tilde{M}^{mn}, \tilde{M}^{mn}, \mathfrak{S}_n, \varphi\}$ (resp. complex analytic M-space $\boldsymbol{M}=\{\tilde{M}_{mn}, \tilde{M}_{mn}, \mathfrak{S}_n, \varphi\}$) by $\mathfrak{S}_n(\bar{M}^m)$ (resp. $\mathfrak{S}_n(\bar{M}_m)$) and call it the *n-fold symmetric product* of $\bar{M}^m$ (resp. $\bar{M}_m$). We denote $\boldsymbol{M}$ simply $\bar{\boldsymbol{M}}^m*\bar{\boldsymbol{M}}^m$ (resp. $\bar{\boldsymbol{M}}_m*\bar{\boldsymbol{M}}_m$) in case $n=2$.

EXAMPLE 1.4. Let Σ^{m+1} be an $(m+1)$-dimensional closed cell with the natural differentiable structure, that is, the set of points $(x_0, x_1, \cdots x_m)$ of R^{m+1} satisfying $x_0^2+x_1^2+\cdots+x_m^2\leqq 1$. Let $\tilde{M}=\Sigma^{m+1}$, $W=S^m$, and let G be an arbitrary finite group of C^∞-automorphisms of S^m, for example, that of Example 1.2 or that which is used when we define a lens space in case m is odd. Then $\boldsymbol{M}=\{\Sigma^{m+1}, S^m, G, \varphi\}$ is a C^∞-M-space.

Now we shall define a C^∞-map between two C^∞-M-spaces. A C^r-map between two C^r-M-spaces etc. will be defined in a similar manner.

DEFINITION 1.2. Let $\boldsymbol{M}=\{M, \tilde{M}, W, G, \varphi\}$ and $\boldsymbol{M}'=\{M', \tilde{M}' W', G', \varphi'\}$ be two C^{∞}-M-spaces. We mean by a *C^{∞}-map* $\boldsymbol{h}=(\tilde{h}, h)$ of $\boldsymbol{M}$ into $\boldsymbol{M}'$ a continuous map $h: M \rightarrow M'$ for which there exists a C^{∞}-map $\tilde{h}$ of $\tilde{M}$ into $\tilde{M}'$ in the usual sense such that

(i) The following diagram is commutative:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{h}} & \tilde{M}' \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} \\ M & \xrightarrow{h} & M' \end{array} .$$

(ii) $\tilde{h}(\tilde{M}-W) \subset \tilde{M}'-W'$.

(iii) *One* of the following (a), (b) holds for each ${}^iW$:

(a) $\tilde{h}({}^iW) \cap W'=\phi$, $\tilde{h}(x)=\tilde{h}(g(x)) \qquad (x \in {}^iW, g \in {}^iG)$.

(b) $\tilde{h}({}^iW)$ is contained in a ${}^jW'$.

h will be called the *underlying map* of $\boldsymbol{h}$. We shall call $\boldsymbol{h}$ the *C^{∞}-map of the first* resp. *the second kind with respect to* ${}^iW$ according as (a) or (b) takes place.

REMARK 1.1. By regarding C^{∞}-M-spaces $\boldsymbol{M}$ and $\boldsymbol{M}'$ as C^0-M-spaces, we can define a C^0-map of $\boldsymbol{M}$ into $\boldsymbol{M}'$.

Obviously we have

PROPOSITION 1.1. *Let $\boldsymbol{h}: \boldsymbol{M} \rightarrow \boldsymbol{M}'$ and $\boldsymbol{h}': \boldsymbol{M}' \rightarrow \boldsymbol{M}''$ be two C^{∞}-maps, then $\boldsymbol{h}' \circ \boldsymbol{h}: \boldsymbol{M} \rightarrow \boldsymbol{M}''$ is a C^{∞}-map.*

DEFINITION 1.3. Two C^{∞}-M-spaces $\boldsymbol{M}$ and $\boldsymbol{M}'$ will be called *isomorphic* if there exist C^{∞}-maps $\boldsymbol{h}=(\tilde{h}, h): \boldsymbol{M} \rightarrow \boldsymbol{M}'$ and $\boldsymbol{h}'=(\tilde{h}', h'): \boldsymbol{M}' \rightarrow \boldsymbol{M}$ such that $h' \circ h$ and $h \circ h'$ are homeomorphic maps of M onto itself and of M' onto itself respectively.

The following proposition is a direct consequence of Proposition 1.1.

PROPOSITION 1.2. *If $\boldsymbol{M}$ is isomorphic to $\boldsymbol{M}'$ and $\boldsymbol{M}'$ is isomorphic to $\boldsymbol{M}''$, then $\boldsymbol{M}$ is isomorphic to $\boldsymbol{M}''$.*

EXAMPLE 1.5. $\boldsymbol{M}$ of Example 1.1 and the half plane of R^m are not isomorphic.

EXAMPLE 1.6. $\boldsymbol{S}^m(1/n)$ (Example 1.2) and the m-sphere S^m are not isomorphic unless $n=\pm 1$.

2. *D*-bundles.

For the sake of simplicity we consider the C^{∞} case mainly in this section. In C^r and other cases, corresponding definitions and properties can be established in similar manners.

We shall now define a C^{∞}-D-bundle over a C^{∞}-M-space $\boldsymbol{M}=\{M, \tilde{M}, W, G, \varphi\}$ $(W=\sum_{i=1}^{l} {}^iW)$.

Let F be a C^{∞}-manifold and let Γ be a group of C^{∞}-automorphisms of F.

Let furthermore ${}^iF$ be a submanifold of F and ${}^i\Gamma$ a subgroup of Γ which transforms ${}^iF$ onto itself $(i=1, \cdots, l)$. We shall write the restriction of ${}^i\Gamma$ on ${}^iF$ by the same notation ${}^i\Gamma$.

DEFINITION 2.1. A *C^∞-D-bundle* $\mathfrak{D}=\{\boldsymbol{M}, \mathfrak{B}, {}^i\mathfrak{B}, {}^i\alpha, {}^i\lambda\}$ is a collection as follows:

(i) $\mathfrak{B}=\{B, p, \tilde{M}, F, \Gamma\}$ is a fibre bundle over $\tilde{M}$ with the fibre F and the structural group Γ in the usual sense.

(ii) ${}^i\mathfrak{B}=\{{}^iB, {}^ip, {}^iW, {}^iF, {}^i\Gamma\}$ is a fibre bundle in the usual sense for $i=1, \cdots, l$.

(iii) For each ${}^i\mathfrak{B}$ $(i=1, \cdots, l)$, ${}^i\alpha$ is an isomorphism of ${}^iG$ into the group of C^∞-bundle maps of ${}^i\mathfrak{B}$ onto itself such that

$$ {}^ip({}^i\alpha(g)(x))=g({}^ip(x)) \qquad (g\in {}^iG, x\in {}^iB). $$

(iv) For each ${}^i\mathfrak{B}$ $(i=1, \cdots, l)$, ${}^i\lambda$ is a fibre-preserving C^∞-injection ${}^i\mathfrak{B}\to\mathfrak{B}|{}^iW$ which induces the identity map of base space. ($\mathfrak{B}|{}^iW$ denotes the restriction of $\mathfrak{B}$ on ${}^iW$.)

$\boldsymbol{M}$, $\mathfrak{B}$, F and Γ will be called the *base space*, the *total bundle*, the *fibre* and the *structural group* of $\mathfrak{D}$ respectively. We shall sometimes write $\mathfrak{D}=\{\boldsymbol{M}, \mathfrak{B}, F, \Gamma\}$ to emphasize these objects, although $\boldsymbol{M}$, $\mathfrak{B}$, F and Γ do not exhaust the data to define our $\mathfrak{D}$.

In the particular case $\tilde{M}=W$, D-bundles become V-bundles introduced by I. Satake (Baily [**1**], Satake [**4**]).

Now we define a notion of D-bundle map.

DEFINITION 2.2. Let $\mathfrak{D}=\{\boldsymbol{M}, \mathfrak{B}, F, \Gamma\}$ and $\mathfrak{D}'=\{\boldsymbol{M}', \mathfrak{B}', F, \Gamma\}$ be two C^∞-D-bundles with the same fibre and the same structural group, and let $\bar{\boldsymbol{h}}=(\tilde{\bar{h}}, \bar{h}):\boldsymbol{M}=\{M, \tilde{M}, W, G, \varphi\}\to\boldsymbol{M}'=\{M', \tilde{M}', W', G', \varphi'\}$ be a C^0-map (resp. C^∞-map). Then by a *D-bundle map* (resp. *C^∞-D-bundle map*) $h:\mathfrak{D}\to\mathfrak{D}'$ over $\bar{\boldsymbol{h}}$ we mean a collection as follows:

(i) We have a C^0-(resp. C^∞-) bundle map $h:\mathfrak{B}\to\mathfrak{B}'$ over $\tilde{\bar{h}}:\tilde{M}\to\tilde{M}'$ in the usual sense.

(ii) If $\bar{\boldsymbol{h}}$ is of the first kind with respect to ${}^iW$, then we have a fibre-preserving map (resp. fibre-preserving C^∞-map) ${}^ih:{}^i\mathfrak{B}\to\mathfrak{B}'$ over $\tilde{\bar{h}}|{}^iW:{}^iW\to\tilde{M}'$. If $\bar{\boldsymbol{h}}$ is of the second kind with respect to ${}^iW$, then we have a fibre-preserving map (resp. fibre-preserving C^∞-map) ${}^ih:{}^i\mathfrak{B}\to{}^j\mathfrak{B}'$ over $\tilde{\bar{h}}|{}^iW:{}^iW\to{}^jW'$ which satisfies the following condition:

$$ {}^ih({}^i\alpha(g)(x))={}^j\alpha'(g')({}^ih(x)), $$

where $x\in{}^iB, g\in{}^iG$ and g' is an element of ${}^jG'$ such that $\tilde{\bar{h}}(g(p(x)))=g'(\tilde{\bar{h}}(p(x)))$.

(iii) We have the following commutative diagram:

$$\begin{array}{ccc} {}^i\mathfrak{B} & \xrightarrow{{}^i h} & {}^j\mathfrak{B}' \text{ (or } \mathfrak{B}') \\ \downarrow {}^i\lambda & h & \downarrow {}^j\lambda' \\ \mathfrak{B} & \longrightarrow & \mathfrak{B}' \end{array} .$$

Obviously we have

PROPOSITION 2.1. *Let $\mathfrak{D}, \mathfrak{D}'$ and $\mathfrak{D}''$ be C^∞-D-bundles over C^∞-M-spaces $\boldsymbol{M}, \boldsymbol{M}'$ and $\boldsymbol{M}''$ respectively, and let $\bar{\boldsymbol{h}}: \boldsymbol{M} \to \boldsymbol{M}'$ and $\bar{\boldsymbol{h}}': \boldsymbol{M}' \to \boldsymbol{M}''$ be C^0-maps (resp. C^∞-maps). If $h: \mathfrak{D} \to \mathfrak{D}'$ and $h': \mathfrak{D}' \to \mathfrak{D}''$ are D-bundle maps (resp. C^∞-D-bundle maps) over $\bar{\boldsymbol{h}}$ and $\bar{\boldsymbol{h}}'$ respectively, then $h' \circ h: \mathfrak{D} \to \mathfrak{D}''$ is a D-bundle map (resp. C^∞-D-bundle map) over $\bar{\boldsymbol{h}}' \circ \bar{\boldsymbol{h}}$.*

DEFINITION 2.3. Two C^∞-D-bundles $\mathfrak{D}$ and $\mathfrak{D}'$ over the same base space $\boldsymbol{M}$ having the same fibre and the same structural group will be called *equivalent* (resp. *C^∞-equivalent*), if there exists a D-bundle map (resp. C^∞-D-bundle map) $h: \mathfrak{D} \to \mathfrak{D}'$ such that h induces the identity map of base space and that the restriction of each ${}^i h$ on fibre is an onto isomorphic map.

The following proposition is an immediate consequence of Proposition 2.1.

PROPOSITION 2.2. *The equivalence (resp. C^∞-equivalence) of C^∞-D-bundles defined as above is an equivalence relation.*

Now we consider the cross section of D-bundles.

DEFINITION 2.4. Let $\mathfrak{D} = \{\boldsymbol{M}, \mathfrak{B}, F, \Gamma\}$ be a C^∞-D-bundle over a C^∞-M-space $\boldsymbol{M} = \{M, \tilde{M}, W, G, \varphi\}$ and let N be a subset of M. Then by a *cross section* (resp. *C^∞-cross section*) f of $\mathfrak{D}$ over N we shall mean a continuous (resp. C^∞-) map $f: \varphi^{-1}(N) \to B$ which has the following properties:

(i) $p \circ f$ is the identity map of $\varphi^{-1}(N)$ onto itself.

(ii) If $x \in \varphi^{-1}(N) \cap {}^i W$, then $f(x) \in {}^i\lambda({}^i B)$ and $f(g(x)) = {}^i\lambda({}^i\alpha(g)({}^i\lambda^{-1}(f(x))))$ $(g \in {}^i G)$.

DEFINITION 2.5. Let $\mathfrak{D} = \{\boldsymbol{M}, \mathfrak{B}, F, \Gamma\}$ be a C^∞-D-bundle over $\boldsymbol{M}$ and let $\tilde{N}$ be a subset of $\tilde{M}$ (resp. of ${}^i W$). The cross section f of $\mathfrak{B}$ (resp. of ${}^i\mathfrak{B}$) over $\tilde{N}$ in the usual sense will be called *G-cross section* if f satisfies

$$\begin{cases} f(x) \in {}^i\lambda({}^i B) & \text{if } x \in {}^i W \cap \tilde{N}, \\ f(g(x)) = {}^i\lambda({}^i\alpha(g)({}^i\lambda^{-1}(f(x)))) & \text{if } x, g(x) \in {}^i W \cap \tilde{N}, \end{cases}$$

$$(\text{resp. } f(g(x)) = {}^i\alpha(g)(f(x)) \quad \text{if } x, g(x) \in \tilde{N}) .$$

Therefore the cross sections of $\mathfrak{D}$ over N correspond bijectively to the G-cross sections of $\mathfrak{B}$ over $\varphi^{-1}(N)$.

It should be noted that if $\varphi^{-1}(N)$ contains a singular point x of ${}^i W$ on which there exists a $g \in G$ such that ${}^i\alpha(g)(\tilde{x}) \neq \tilde{x}$ for any $\tilde{x} \in {}^i p^{-1}(x)$, we have no cross section of $\mathfrak{D}$ over N. For example, there is no cross section of tangent D-bundle of $\boldsymbol{S}^2(1/n)$ (see Section 3) over the points $(0, 0, \pm 1)$ if $n \neq \pm 1$.

Let us denote by $\mathfrak{F}$ a fibre bundle with the fibre F over one point.

Definition 2.6. A C^∞-D-bundle $\mathfrak{D}$ with the fibre F will be called the *product D-bundle* (resp. *C^∞-product D-bundle*) if there exists a D-bundle map (resp. C^∞-D-bundle map) $h:\mathfrak{D}\rightarrow\mathfrak{F}$.

Definition 2.7. A C^∞-D-bundle $\mathfrak{D}=\{\boldsymbol{M},\mathfrak{B},F,\Gamma\}$ will be called the *principal D-bundle* if $F=\Gamma$, ${}^iF={}^i\Gamma$ $(i=1,\cdots,l)$ and if Γ and each ${}^i\Gamma$ operate on F and on ${}^iF$ as the left translations respectively.

Definition 2.8. Let $\mathfrak{D}=\{\boldsymbol{M},\mathfrak{B},F,\Gamma\}$ be a C^∞-D-bundle and let Γ' be a subgroup of Γ. Suppose that there exists a C^∞-D-bundle $\mathfrak{D}'=\{\boldsymbol{M},\mathfrak{B}',F,\Gamma'\}$ which is equivalent (resp. C^∞-equivalent) to $\mathfrak{D}$ (considering the structural group of $\mathfrak{D}'$ as Γ), then the structural group Γ of D is calld *reducible* (resp. *C^∞-reducible*) *to* Γ'.

3. Vector D-bundles and tangent D-bundles of M-spaces.

In this section we shall consider D-bundles whose fibres are real or complex vector spaces and whose structural groups are real or complex linear groups. For the sake of simplicity we consider real cases mainly. Complex cases can be analogously treated.

We denote by E^n (resp. E_n) the n-dimensional real (resp. n-dimensional complex) vector space.

Definition 3.1. A C^∞-D-bundle $\mathfrak{D}=\{\boldsymbol{M},\mathfrak{B},F,\Gamma\}$ will be called the *C^∞-vector-D-bundle* if $F=E^n$, ${}^iF=E^{n_i}$ $(n_i\leqq n)$ $(i=1,\cdots,l)$ and Γ is the general linear group $GL(n;R)$ or its subgroup.

We denote $\min_i n_i$ by $\min_D \dim F$.

We now note two lemmas about fibre bundles which are used to define associated bundles of D-bundles.

Lemma 3.1. *Let $\mathfrak{B}=\{B,p,X,E^n,\Gamma\}$ be a C^∞-vector bundle over a C^∞-manifold X in the usual sense. Suppose that $\Gamma=GL(n;R)$ (resp. orthogonal group $O(n)$) and that $V(k)$ (resp. $V_0(k)$) $(k\leqq n)$ is a fixed k-frame (resp. orthogonal k-frame) in E^n, where k-frame means an ordered set of k independent vectors in E^n. By $GL(n,(n-k);R)$ we shall denote the subgroup of $GL(n;R)$ consisting of all elements of $GL(n;R)$ which operate as identity on the subspace of E^n spanned by $V(k)$. Then the associated bundle $\mathfrak{B}^{(k)}$ of $\mathfrak{B}$ with the fibre $GL(n;R)/GL(n,(n-k);R)$ (resp. $O(n)/O(n-k)$) can be regarded as the C^∞-fibre bundle over X whose fibre over x $(x\in X)$ is the set of all injections of $V(k)$ (resp. $V_0(k)$) into $p^{-1}(x)$.*

This lemma is clear, and so the proof is omitted. In the special case where $k=n$, above lemma is used by F. Hirzebruch [**3**, § 4].

The following lemma 3.1′ is the complex form of lemma 3.1.

Lemma 3.1′. *Let $\mathfrak{B}=\{B,p,X,E_n,U(n)\}$ be a C^∞-vector bundle over a C^∞-*

manifold X *with* E_n *as fibre and the unitary group* $U(n)$ *as structural group in the usual sense, and let* $V^C(k)$ $(k \leqq n)$ *be a fixed unitary* k*-frame in* E_n. *Suppose that* $\mathfrak{B}^{[k]}$ *is a* C^∞*-fibre bundle over* X *whose fibre over* x $(x \in X)$ *is the set of all injections of* $V^C(k)$ *into* $p^{-1}(x)$, *then* $\mathfrak{B}^{[k]}$ *is the associated bundle of* $\mathfrak{B}$ *with the fibre* $U(n)/U(n-k)$.

The following lemmas are immediate consequences of Lemmas 3.1 and 3.1′.

LEMMA 3.2. *Let* $\mathfrak{B}=\{B,p,X,E^n,\Gamma\}$ *and* $\mathfrak{B}'=\{B',p',X',E^{n'},\Gamma'\}$ *be two* C^∞*-vector bundles with structural group* $\Gamma=GL(n;R)$, $\Gamma'=GL(n',R)$ (*resp.* $\Gamma=O(n)$, $\Gamma'=O(n')$) *in the usual sense. If* $\lambda:\mathfrak{B}\to\mathfrak{B}'$ *is a fibre-preserving injection* $(n\leqq n')$, *then* λ *induces a fibre-preserving injection* $\lambda^{(k)}:\mathfrak{B}^{(k)}\to\mathfrak{B}'^{(k)}$ $(k\leqq n)$ *in an obvious manner, where* $\mathfrak{B}^{(k)},\mathfrak{B}'^{(k)}$ *denote associated bundles defined in Lemma* 3.1.

LEMMA 3.2′. *Let* $\mathfrak{B}=\{B,p,X,E_n,U(n)\}$ *and* $\mathfrak{B}'=\{B',p',X',E_{n'},U(n')\}$ *be two* C^∞*-vector bundles in the usual sense. If* $\lambda:\mathfrak{B}\to\mathfrak{B}'$ *is a fibre-preserving injection* $(n\leqq n')$, *then* λ *induces a fibre-preserving injection* $\lambda^{[k]}:\mathfrak{B}^{[k]}\to\mathfrak{B}'^{[k]}$ $(k\leqq n)$ *in an obvious manner, where* $\mathfrak{B}^{[k]},\mathfrak{B}'^{[k]}$ *denote associated bundles defined in Lemma* 3.1′.

Now we introduce the following definition.

DEFINITION 3.2. Let $\mathfrak{D}=\{\boldsymbol{M},\mathfrak{B},E^n,GL(n;R)\}$ (resp. $\mathfrak{D}=\{\boldsymbol{M},\mathfrak{B},E^n,O(n)\}$) be a C^∞-vector D-bundle over a C^∞-M-space $\boldsymbol{M}=\{M,\tilde{M},W,G,\varphi\}$ and let k be an integer $\leqq \min_D \dim F$. We mean by an *associated D-bundle* $\mathfrak{D}^{(k)}=\{\boldsymbol{M},\mathfrak{B}^{(k)}, GL(n;R)/GL(n,(n-k);R),GL(n;R)\}$ (resp. $\mathfrak{D}^{(k)}=\{\boldsymbol{M},\mathfrak{B}^{(k)},O(n)/O(n-k),O(n)\}$, $\mathfrak{D}^{[k]}=\{\boldsymbol{M},\mathfrak{B}^{[k]},U(n)/U(n-k),O(n)\}$) of $\mathfrak{D}$ with the fibre $GL(n;R)/GL(n,(n-k);R)$ (resp. $O(n)/O(n-k)$, $U(n)/U(n-k)$) a C^∞-D-bundle as follows:

(i) $\mathfrak{B}^{(k)}=\{B^{(k)},p,\tilde{M},GL(n,(n-k);R),GL(n;R)\}$ (resp. $\mathfrak{B}^{(k)}=\{B^{(k)},p,\tilde{M},O(n)/O(n-k),O(n)\}$, $\mathfrak{B}^{[k]}=\{B^{[k]},p,\tilde{M},U(n)/U(n-k),O(n)\}$) is the associated bundle of $\mathfrak{B}$ with the fibre $GL(n;R)/GL(n,(n-k);R)$ (resp. $O(n)/O(n-k)$, $U(n)/U(n-k)$) in the usual sense.

(ii) ${}^i\mathfrak{B}^{(k)}=\{{}^iB^{(k)},p,{}^iW,GL(n_i;R)/GL(n_i,(n_i-k);R),GL(n_i;R)\}$ (resp. ${}^i\mathfrak{B}^{(k)}=\{{}^iB^{(k)},p,{}^iW,O(n_i)/O(n_i-k),O(n_i)\}$, ${}^i\mathfrak{B}^{[k]}=\{{}^iB^{[k]},p,{}^iW,U(n_i)/U(n_i-k),O(n_i)\}$) is the associated bundle of ${}^i\mathfrak{B}$ with the fibre $GL(n_i;R)/GL(n_i,(n_i-k);R)$ (resp. $O(n_i)/O(n_i-k)$, $U(n_i)/U(n_i-k)$) in the usual sense $(i=1,\cdots,l)$.

(iii) For any $g\in G$, define ${}^i\alpha^{(k)}(g):{}^i\mathfrak{B}^{(k)}\to{}^i\mathfrak{B}^{(k)}$ (resp. ${}^i\alpha^{[k]}(g):{}^i\mathfrak{B}^{[k]}\to{}^i\mathfrak{B}^{[k]}$) as the C^∞-automorphism induced by ${}^i\alpha(g):{}^i\mathfrak{B}\to{}^i\mathfrak{B}$ (Lemmas 3.2, 3.2′). (Clearly the correspondence $g\to{}^i\alpha^{(k)}(g)$ (resp. $g\to{}^i\alpha^{[k]}(g)$) gives an isomorphism of G into the group of C^∞-bundle maps of ${}^i\mathfrak{B}^{(k)}$ (resp. ${}^i\mathfrak{B}^{[k]}$) onto itself.)

(iv) Define the fibre-preserving injection ${}^i\lambda^{(k)}:{}^i\mathfrak{B}^{(k)}\to\mathfrak{B}^{(k)}|{}^iW$ (resp. ${}^i\lambda^{[k]}:{}^i\mathfrak{B}^{[k]}\to\mathfrak{B}^{[k]}|{}^iW$) as that induced by ${}^i\mathfrak{B}\to\mathfrak{B}|{}^iW$ (Lemmas 3.2, 3.2′).

By the standard argument we obtain the following proposition.

PROPOSITION 3.1. *If* C^∞*-vector D-bundles* $\mathfrak{D}$ *and* $\mathfrak{D}'$ *are equivalent* (*resp.* C^∞*-equivalent*), *then their associated bundles* $\mathfrak{D}^{(k)}$ *and* $\mathfrak{D}'^{(k)}$, $\mathfrak{D}^{[k]}$ *and* $\mathfrak{D}'^{[k]}$ *are equi-*

valent (resp. C^∞-equivalent) respectively.

It is to be noticed that, $\mathfrak{D}^{[n]}$ can only be defined in case $n=\min_D \dim F$, and in this case $\mathfrak{D}^{[n]}$ is nothing other than the associated principal bundle in the usual sense if M is a usual manifold.

Now we consider tangent D-bundles of C^∞-M-spaces.

Let $\boldsymbol{M}=\{M, \tilde{M}, W, G, \varphi\}$ be an m-dimensional C^∞-M-space. Each connected component of ${}^iW$ has the same dimension m_i $(i=1,\cdots,l)$.

DEFINITION 3.3. By a *tangent D-bundle* $\mathfrak{T}(\boldsymbol{M})$ of $\boldsymbol{M}$ we mean a C^∞-D-bundle $\mathfrak{D}=\{\boldsymbol{M}, \mathfrak{T}(\tilde{M}), E^m, GL(m;R)\}$ as follows:

(i) $\mathfrak{T}(\tilde{M})=\{T(\tilde{M}), p, \tilde{M}, E^m, GL(m;R)\}$ is the tangent vector bundle of $\tilde{M}$ in the usual sense.

(ii) $\mathfrak{T}({}^iW)=\{T({}^iW), p, {}^iW, E^{m_i}, GL(m_i;R)\}$ is the tangent vector bundle of ${}^iW$ in the usual sense.

(iii) ${}^i\alpha(g):\mathfrak{T}({}^iW)\to\mathfrak{T}({}^iW)$ is the C^∞-automorphism dg of $\mathfrak{T}({}^iW)$ onto itself induced by $g: {}^iW\to{}^iW$.

(iv) ${}^i\lambda:\mathfrak{T}({}^iW)\to\mathfrak{T}(\tilde{M})|{}^iW$ is the fibre-preserving injection determined by the injection ${}^iW\to\tilde{M}$ in an obvious manner.

Let us consider a Riemannian metric of $\tilde{M}$. In this paper, by a Riemannian metric of $\tilde{M}$ we always mean the Riemannian metric with respect to which every $g\in G$ is an isometric transformation. Then by the standard argument we can regard $\mathfrak{T}(\boldsymbol{M})$ has the orthogonal group $O(m)$ as structural group.

Associated bundles of $\mathfrak{T}(\boldsymbol{M})$ in the sense of Definition 3.2 will be denoted by $\mathfrak{T}^{(k)}(\boldsymbol{M})$ and $\mathfrak{T}^{[k]}(\boldsymbol{M})$.

As is easily verified we have

PROPOSITION 3.2. *If two C^∞-M-spaces $\boldsymbol{M}$ and $\boldsymbol{M}'$ are isomorphic, then tangent D-bundles $\mathfrak{T}(\boldsymbol{M})$ and $\mathfrak{T}(\boldsymbol{M}')$ (resp. $\mathfrak{T}^{(k)}(\boldsymbol{M})$ and $\mathfrak{T}^{(k)}(\boldsymbol{M}')$; $\mathfrak{T}^{[k]}(\boldsymbol{M})$ and $\mathfrak{T}^{[k]}(\boldsymbol{M}')$) are C^∞-equivalent.*

For the later use we now consider a generalization of the concept of associated D-bundles of $\mathfrak{T}(\boldsymbol{M})$.

By a *normal tube neighbourhood* N^iW of ${}^iW$ we mean the set of points of $\tilde{M}$ whose distance from ${}^iW$ are $\leqq\varepsilon$ (sufficiently small positive number). For each point x of N^iW there exists only one geodesic line l_x passing x and normal to ${}^iW$. Denote $l_x\cap{}^iW=p(x)$. Then we have a fibre bundle ${}^i\mathfrak{N}=\{N^iW, p, {}^iW, \Sigma^{m-m_i}, O(m-m_i)\}$ $(i=1,\cdots,l)$, where Σ^{m-m_i} denotes the closed interior of the unit sphere of E^{m-m_i}. ${}^i\mathfrak{N}$ will be called the *normal bundle* of ${}^iW$. Let ${}^i\mathfrak{N}^{(m_i')}$ (resp. ${}^i\mathfrak{N}^{[m_i']}$) $(m_i'\leqq m-m_i)$ be the associated bundle of ${}^i\mathfrak{N}$ with the fibre $O(m-m_i)/O(m-m_i-m_i')$ (resp. $U(m-m_i)/U(m-m_i-m_i')$).

Let us assume that ${}^i\mathfrak{N}^{(m_i')}$ (resp. ${}^i\mathfrak{N}^{[m_i']}$) $(i=1,\cdots,l)$ has a cross section ${}^if$ over ${}^iW$.

DEFINITION 3.4. Let k be an integer such that $m_i' \leqq k \leqq (m_i+m_i')$ $(i=1,\cdots,l)$. By a *generalized associated D-bundle* $\mathfrak{T}^{(k)}(\boldsymbol{M})({}^1f,\cdots,{}^lf)=\{\boldsymbol{M},\mathfrak{T}^{(k)}(\tilde{M}), O(m)/O(m-k), O(m)\}$ (resp. $\mathfrak{T}^{[k]}(\boldsymbol{M})({}^1f,\cdots,{}^lf)=\{\boldsymbol{M},\mathfrak{T}^{[k]}(\tilde{M}), U(m)/U(m-k), O(m)\}$) of tangent D-bundle $\mathfrak{T}(\boldsymbol{M})$ of a C^∞-M-space $\boldsymbol{M}$ we mean a C^∞-D-bundle as follows:

(i) $\mathfrak{T}^{(k)}(\tilde{M})=\{T^{(k)}(\tilde{M}), p, \tilde{M}, O(m)/O(m-k), O(m)\}$ (resp. $\mathfrak{T}^{[k]}(\tilde{M})=\{T^{[k]}(\tilde{M}), p, \tilde{M}, U(m)/U(m-k), O(m)\}$) is the associated bundle of the tangent vector bundle $\mathfrak{T}(\tilde{M})$ of $\tilde{M}$ in the usual sense.

(ii) $\mathfrak{T}^{(k)}({}^iW)_f=\{T^{(k)}({}^iW), p, {}^iW, O(m_i)/O(m_i+m_i'-k), O(m_i)\}$ (resp. $\mathfrak{T}^{[k]}({}^iW)_f=\{T^{[k]}({}^iW), p, {}^iW, U(m_i)/U(m_i+m_i'-k), O(m_i)\}$) is the associated bundle of the tangent vector bundle $\mathfrak{T}({}^iW)$ in the usual sense.

(iii) ${}^i\alpha^{(k)}(g): \mathfrak{T}^{(k)}({}^iW)_f \rightarrow \mathfrak{T}^{(k)}({}^iW)_f$ (resp. ${}^i\alpha^{[k]}(g): \mathfrak{T}^{[k]}({}^iW)_f \rightarrow \mathfrak{T}^{[k]}({}^iW)_f$) is the C^∞-automorphism induced by $dg: T({}^iW)\rightarrow T({}^iW)$ (Lemma 3.2.).

(iv) Every point of the fibre $p^{-1}(x)$ of $\mathfrak{T}^{(k)}({}^iW)_f$ (resp. $\mathfrak{T}^{[k]}({}^iW)_f$) can be regarded as a $(k-m_i')$-frame and ${}^if(x)$ as an m_i'-frame (Lemma 3.1.) Let $V(k-m_i')_x$ (resp. $V^C(k-m_i')_x$) be a point on the fibre $p^{-1}(x)$ of $\mathfrak{T}^{(k)}({}^iW)_f$ (resp. $\mathfrak{T}^{[k]}({}^iW)_f$). We define the fibre-preserving injection ${}^i\lambda^{(k)}: \mathfrak{T}^{(k)}({}^iW)_f \rightarrow \mathfrak{T}^{(k)}(\tilde{M})$ (resp. ${}^i\lambda^{[k]}: \mathfrak{T}^{[k]}({}^iW)_f \rightarrow \mathfrak{T}^{[k]}(\tilde{M})$) by

$$ {}^i\lambda^{(k)}(V(k-m_i')_x)={}^i\lambda(V(k-m_i')_x)\vee{}^i\lambda_N({}^if_x) $$
$$ (\text{resp. } {}^i\lambda^{[k]}(V^C(k-m_i')_x)={}^i\lambda(V^C(k-m_i')_x)\vee{}^i\lambda_N({}^if_x)), $$

where ${}^i\lambda, {}^i\lambda_N$ are natural injections and the right hand side means the k-frame obtained as the union of $(k-m_i')$-frame ${}^i\lambda(V(k-m_i')_x)$ (resp. ${}^i\lambda(V^C(k-m_i')_x)$) and m_i-frame ${}^i\lambda_N({}^if_x)$.

In the above definition, we have regarded $\mathfrak{T}(\boldsymbol{M})$ as having the orthogonal group as structural group and what we have hitherto termed "frames" means more exactly "orthogonal frames". Of course, $\mathfrak{T}(\boldsymbol{M})$ can be also regarded as having the general linear group as structural group. The definition of the generalized associated D-bundles can be then correspondingly modified in replacing orthogonal frames by affine frames, but we shall not need these considerations in the sequel.

Part II. Obstruction theory of *D*-bundles.

4. Admissible cellular decompositions of C^∞-M-spaces.

We mean by a *cellular decomposition* K of a compact Hausdorff space X a collection, called the *cell complex*, $K=\{\sigma_i^q\}$ $(i=1,\cdots,\alpha_q; q=0,1,\cdots,m)$ of closed subsets of X such that

(i) Each σ_i^q is homeomorphic to the closed q-dimensional Euclidean simplex. σ_i^q is called *q-dimensional cell* or briefly *q-cell*.

(ii) We denote by p-section K^p the subset of K consisting of all q-cells for $q \leqq p$ and by $|K^p|$ the union of all cells of K^p. Then $X = |K^m|$.

(iii) $\sigma_i^q \cap |K^{q-1}|$ is the boundary $\partial\sigma_i^q$ of σ_i^q, and it is an exact union of cells called the faces of σ_i^q.

(iv) If $i \neq j$, the interiors of σ_i^q and σ_j^q have no point in common.

If we replace (i) by the weaker condition (i)′ below, K will be called the *cell complex in the wider sense.*

(i)′ The set of interior points $\sigma_i^q - \partial\sigma_i^q$ of each σ_i^q is homeomorphic to the open q-dimensional Euclidean simplex.

From now on we shall always consider a compact m-dimensional C^∞-M-space $\boldsymbol{M} = \{M, \tilde{M}, W, G, \varphi\}$ which is subject to the condition

$$(*) \quad \begin{cases} \boldsymbol{M} \text{ is satisfying } \textit{one} \text{ of the following (a), (b):} \\ \text{(a) Each } {}^iG \text{ has no fixed point.} \\ \text{(b) } W = \tilde{M} \text{ (i. e. } M \text{ is a quotient space).} \end{cases}$$

In the case of (*) (b), let W_j ($j = 1, \cdots, w$) be connected components of the set of all fixed points of G (i. e. the point $x \in \tilde{M}$ such that $x = g(x)$ for some $g \in G - \{e\}$). Then we assume

$$(**) \quad \begin{cases} \text{Each } W_j \ (j = 1, \cdots, w) \text{ is an orientable submanifold of } \tilde{M} \text{ on which} \\ \text{every element of } G \text{ operates as identity map, and } W_j \cap W_{j'} = \phi \text{ unless} \\ W_j = W_{j'}. \end{cases}$$

Let NW_j be a sufficiently small closed normal tube neighbourhood of W_j in $W = \tilde{M}$ such that $NW_j \cap NW_{j'} = \phi$ if $W_j \neq W_{j'}$, and that $NW_j = NW_{j'}$ if $W_j = W_{j'}$. The normal bundle $\mathfrak{N}(W_j) = \{NW_j, p, W_j, \Sigma^{m-n_j}, GL(m-n_j; R)\}$ is defined in an obvious manner, where n_j denotes the dimension of W_j. Clearly every element of G operates on $p^{-1}(x)$ ($x \in W_j$) as an automorphism.

Definition 4.1. A cellular decomposition $\tilde{K}$ of the covering manifold $\tilde{M}$ of $\boldsymbol{M}$ satisfying the conditions (*), (**), will be called *admissible* if the following conditions are fulfilled:

(i) For each ${}^iW$ ($i = 1, \cdots, l$), a subcomplex ${}^i\tilde{K}$ of $\tilde{K}$ gives a cellular decomposition of ${}^iW$. We denote $\tilde{K}_W = \sum_{i=1}^{l} {}^i\tilde{K}$.

(ii) Each $g \in G$ operates on $\tilde{K}_W$ as a cellular map.

(iii) For each W_j ($j = 1, \cdots, w$), there exists a cellular decomposition $\tilde{K}_j$ of W_j such that $p^{-1}(\tilde{\sigma})$ ($\tilde{\sigma} \in \tilde{K}_j$) is a cell of $\tilde{K}$ (in case where (*) (b) holds).

We donote by ${}_N\tilde{K}_j$ the subcomplex of $\tilde{K}$ which gives a cellular decomposition of NW_j, and by $\tilde{K}_C$ the set of cells contained in $\tilde{M} - \sum_j \operatorname{Int} NW_j$ which is a subcomplex of $\tilde{K}$ in virtue of the conditions (*), (**). Clearly, in case where (*) (a) holds, we have $\tilde{K}_C = \tilde{K}$.

As is easily verified, the covering manifold $\tilde{M}$ of $\boldsymbol{M}$ satisfying the conditions (*), (**) always has an admissible differentiable cellular decomposition such that $\tilde{K}_C$ and $\tilde{K}_j$ are simplicial.

EXAMPLE 4.1. M-spaces $\boldsymbol{M}=\{M, \tilde{M}, W, G, \varphi\}$ of Examples 1.1, 1.2, 1.4 (in case where G has no fixed point) and Example 1.3 for $n=2$ have admissible cellular decompositions. But M-spaces of Example 1.3 for $n\geqq 3$ cannot possess such a cellular decomposition, because the condition (**) is not satisfied.

The chain (resp. cochain) homomorphism $C_q(\tilde{K}_W)\to C_q(\tilde{K}_W)$ (resp. $C^q(\tilde{K}_W)\to C^q(\tilde{K}_W)$) $(q=0,1,\cdots,m)$ determined by the cellular map g will be denoted by $g_\#$ (resp. $g^\#$).

Now we define the first regular subdivision of an admissible cellular decomposition $\tilde{K}$ of $\tilde{M}$. It is constructed as follows. First, inductively with respect to the dimension of cells of each $\tilde{K}_j$, we introduce one new vertex on each $\tilde{\sigma}\in\tilde{K}_j$ $(j=1,\cdots,w)$ and divide $\tilde{\sigma}$ by the join of this vertex with the subdivision of the boundary of $\tilde{\sigma}$. We denote the complex thus obtained by $'\tilde{K}_j$ $(j=1,\cdots,w)$. Let $\Sigma_1^{m-n_j}$ and $\Sigma_2^{m-n_j}$ be subsets of Σ^{m-n_j} consisting of points whose distances from the origin are $\leqq 1/2$, $\geqq 1/2$ respectively, and let $\mathfrak{N}_1(W_j)=\{N_1W_j, p_1, W_j, \Sigma_1^{m-n_j}, GL(m-n_j;R)\}$ and $\mathfrak{N}_2(W_j)=\{N_2W_j, p_2, W_j, \Sigma_2^{m-n_j}, GL(m-n_j;R)\}$ be associated bundles of $\mathfrak{N}(W_j)$ with fibres $\Sigma_1^{m-n_j}$ and $\Sigma_2^{m-n_j}$ respectively. Then natural injections $\mathfrak{N}_1(W_j)\to\mathfrak{N}(W_j)$, $\mathfrak{N}_2(W_j)\to\mathfrak{N}(W_j)$ are defined. We denote their images by the same notations $\mathfrak{N}_1(W_j)$ and $\mathfrak{N}_2(W_j)$ respectively. For each NW_j, there exists, as is easily verified, a G-invariant cellular decomposition $_N'\tilde{K}_j$ of NW_j such that $p_1^{-1}('\tilde{\sigma})$ $('\tilde{\sigma}\in{}'\tilde{K}_j)$ is a cell of $_N'\tilde{K}_j$, and that $_N'\tilde{K}_j$ coincides with $\tilde{K}$ on the boundary of NW_j. We can assume without loss of generality that $_N'\tilde{K}_j$ is simplicial on $N_1W_j\cap N_2W_j$.

Secondly we construct a G-invariant cellular subdivision $'(\tilde{K}_C)$ of $\tilde{K}_C$ as follows. For each $\tilde{\sigma}$ of $\tilde{K}_C$, we introduce one new vertex $v_{\tilde{\sigma}}$ inductively with respect to the dimension of $\tilde{\sigma}$ and we divide $\tilde{\sigma}$ by the join of $v_{\tilde{\sigma}}$ with the subdivision of the boundary of $\tilde{\sigma}$, whereby we always choose $v_{g_\#\tilde{\sigma}}=g(v_{\tilde{\sigma}})$ $(g\in G)$ in case $\tilde{\sigma}\in\tilde{K}_W$.

From $_N'\tilde{K}_j$ $(j=1,\cdots,w)$ and $'(\tilde{K}_C)$ thus obtained we get a cellular decomposition of $\tilde{M}$ which we denote by $'\tilde{K}$. This is the *first regular subdivision* of $\tilde{K}$. Obviously $'\tilde{K}$ is an admissible cellular decomposition of $\tilde{M}$. Arbitrarily fine subdivision can be found by repeated regular subdivisions. We denote the ν-th regular subdivision of $\tilde{K}$ by $^{(\nu)}\tilde{K}$.

DEFINITION 4.2. Let $\boldsymbol{M}=\{M, \tilde{M}, W, G, \varphi\}$, $\boldsymbol{M}'=\{M', \tilde{M}', W', G', \varphi'\}$ be two C^∞-M-spaces satisfying the conditions (*), (**), and let $\tilde{K}$, $\tilde{K}'$ be admissible cellular decompositions of $\tilde{M}$, $\tilde{M}'$ respectively. A C^0-map $\boldsymbol{h}=(\tilde{h},h):\boldsymbol{M}\to\boldsymbol{M}'$ will be called the *admissible cellular map* if the following conditions are satisfied:

(i) $\tilde{h}:\tilde{K}\to\tilde{K}'$ is a cellular map.

(ii) $\tilde{h}$ maps $\tilde{K}_C$ into $\tilde{K}_C'$.

(iii) In case where $\boldsymbol{M}$ satisfies (*) (b) and $\tilde{h}(W_j)\cap(\sum_{j'} W_{j'}')=\phi$, there exists a fibre bundle $\mathfrak{N}(\tilde{h}(W_j))=\{\tilde{h}(NW_j), p, \tilde{h}(W_j), \Sigma^{m-n_j}, GL(m-n_j; R)\}$ such that $\tilde{h}: \mathfrak{N}(W_j)\to\mathfrak{N}(\tilde{h}(W_j))$ is a bundle map. In case where both $\boldsymbol{M}, \boldsymbol{M}'$ satisfy (*) (b) and $\tilde{h}(W_j)\subset W_{j'}'$, $\tilde{h}$ is a cellular map of $\tilde{K}_j$ into $\tilde{K}_{j'}'$ and of ${}_N\tilde{K}_j$ into ${}_N\tilde{K}_{j'}'$.

5. The obstruction cocycle.

In this section we always consider a compact m-dimensional C^∞-M-space $\boldsymbol{M}=\{M, \tilde{M}, W, G, \varphi\}$ satisfying the conditions (*), (**). Let $\tilde{K}$ be an admissible cellular decomposition of $\tilde{M}$.

Let $\mathfrak{D}=\{\boldsymbol{M}, \mathfrak{B}, F, \Gamma\}$ be a C^0-D-bundle over $\boldsymbol{M}$. We shall use notations of Section 2 about $\mathfrak{D}$ without special references.

Suppose that $\mathfrak{D}$ is satisfying the following assumptions (A I)—(A VI).

(A I) $\pi_q(F)=0 \qquad (0\leqq q<r-1)$,
$\pi_{r-1}(F)\approx\bar{Z}$, (and $\pi_1(F)$ is 1-simple in case $r=2$),

where $\bar{Z}$ denotes the group of integers Z or Z mod 2.

(A II) ${}^iW$ $(i=1,\cdots,l)$ are disjoint.

(A III) For each ${}^i\mathfrak{B}$ $(i=1,\cdots,l)$, *one* of the following conditions (a), (b) holds:

(a) ${}^i\lambda: {}^i\mathfrak{B}\to\mathfrak{B}$ induces $({}^i\lambda\,|\,{}^iF)_*: \pi_q({}^iF)\approx\pi_q(F) \qquad (0\leqq q\leqq r-1)$.

(b) $\begin{cases} {}^iG \text{ operates on } {}^iW \text{ without fixed points.} \\ \pi_q({}^iF)=0 \qquad 0\leqq q<\min(\dim {}^iW, r). \end{cases}$

We write $i\in$ (a) (resp. $i\in$ (b)) when $\mathfrak{B}$ satisfies (a) (resp. (b)).

Now we define a C^0-D-bundle $\mathfrak{D}(\pi_{r-1})=\{\boldsymbol{M}, \mathfrak{B}(\pi_{r-1}), \bar{Z}, \pm 1\}$ as follows:

(i) $\mathfrak{B}(\pi_{r-1})=\{B(\pi_{r-1}), p, \tilde{M}, \bar{Z}, \pm 1\}$ is defined in the usual sense (Steenrod [5, p. 152]).

(ii) In case $i\in$ (a), ${}^i\mathfrak{B}(\pi_{r-1})=\{{}^iB(\pi_{r-1}), p, {}^iW, \bar{Z}, \pm 1\}$ is defined in the usual sense. ${}^i\lambda_\pi: {}^i\mathfrak{B}(\pi_{r-1})\to\mathfrak{B}(\pi_{r-1})\,|\,{}^iW$ is the injection defined by ${}^i\lambda$ in an obvious manner.

In case $i\in$ (b), ${}^i\mathfrak{B}(\pi_{r-1})$ is the product bundle ${}^iW\times 0$ and ${}^i\lambda_\pi: {}^i\mathfrak{B}(\pi_{r-1})\to \mathfrak{B}(\pi_{r-1})\,|\,{}^iW$ is defined by $(x, 0)\to(x, 0)\in B(\pi_{r-1})\,|\,{}^iW$ $(x\in{}^iW)$.

(iii) ${}^i\alpha(g): {}^i\mathfrak{B}\to{}^i\mathfrak{B}$ induces the bundle map ${}^i\alpha_\pi(g): {}^i\mathfrak{B}(\pi_{r-1})\to{}^i\mathfrak{B}(\pi_{r-1})$ in an obvious manner. We define the isomorphic map of ${}^iG$ into the group of bundle maps of ${}^i\mathfrak{B}(\pi_{r-1})$ onto itself by ${}^i\alpha_\pi$.

Then we assume

(A IV) $\mathfrak{D}(\pi_{r-1})$ is the product bundle.

(A V) Fixed cross sections $f_N{}^j$ of $\mathfrak{D}$ over the r-section of the boundary of ${}_N\tilde{K}_j$ $(j=1,\cdots,w)$ are given. (They will be introduced in the following according to circumstances in relation to the singular points of $\boldsymbol{M}$.) $f_N=\{f_N{}^j\}$ will be called the *standard cross section* of $\mathfrak{D}$.

For each $\mathfrak{N}(W_j)$ $(j=1,\cdots,w)$, a fibre bundle $\varphi(\mathfrak{N}(W_j))=\{\varphi(NW_j), p, \varphi(W_j), \Sigma^{m-n_j}/G, \Gamma_j\}$ is defined in the natural way. The fibre bundle $\mathfrak{N}^\circ(W_j)=\{\mathrm{Int}\, NW_j, p, W_j, \mathrm{Int}\,\Sigma^{m-n_j}, GL(m-n_j\,; R)\}$ (resp. $\varphi(\mathfrak{N}^\circ(W_j))=\{\varphi(\mathrm{Int}\, NW_j), p, \varphi(W_j), \mathrm{Int}(\Sigma^{m-n_j}/G), \Gamma_j\}$) is an associated bundle of $\mathfrak{N}(W_j)$ (resp. $\varphi(\mathfrak{N}(W_j))$).

In this paper we denote by H^*, H^p, H^q etc. the singular cohomology groups and by $H_{\mathcal{K}}{}^*, H_{\mathcal{K}}{}^p, H_{\mathcal{K}}{}^q$ etc. the singular cohomology groups with compact carriers.

There exists a spectral sequence $\{E_s\}$ of $\mathfrak{N}^\circ(W_j)$ (resp. $\{{}^\varphi E_s\}$ of $\varphi(\mathfrak{N}^\circ(W_j))$) such that

$$E_2{}^{p,q}=H^p(W_j\,; H_{\mathcal{K}}{}^q(\mathrm{Int}\,\Sigma^{m-n_j}\,; \bar{Z}))\,,$$

$$(\text{resp. } {}^\varphi E_2{}^{p,q}=H^p(\varphi(W_j)\,; H_{\mathcal{K}}{}^q(\mathrm{Int}(\Sigma^{m-n_j}/G))))\,,$$

$$E_\infty{}^{p,q}=J^{p,q}/J^{p+1,q-1}\,,$$

$$(\text{resp. } {}^\varphi E_\infty{}^{p,q}={}^\varphi J^{p,q}/{}^\varphi J^{p+1,q-1})\,,$$

where $J^{p,q}$ (resp. ${}^\varphi J^{p,q}$) is the submodule of $H_{\mathcal{K}}{}^{p+q}(\mathrm{Int}\, NW_j\,; \bar{Z})$ (resp. $H_{\mathcal{K}}{}^{p+q}(\varphi(\mathrm{Int}\, NW_j)\,; \bar{Z})$) determined by the filtration (H, Cartan et al., Séminaire de topologie algébrique, 1950-51, XXI). Let $d_s: E_s\to E_s$ (resp. ${}^\varphi d_s: {}^\varphi E_s\to {}^\varphi E_s$) be the differentiation of E_s (resp. ${}^\varphi E_s$) and $\kappa_t{}^s$ (resp. ${}^\varphi\kappa_t{}^s$) be the homomorphism of the set of elements e_s of E_s (resp. ${}^\varphi e_s$ of ${}^\varphi E_s$) satisfying $d_u\kappa_u{}^\varphi e_s=0$ (resp. ${}^\varphi d_u{}^\varphi\kappa_u{}^\varphi e_s=0$) for $s\leqq u<t$ onto E_t (resp. ${}^\varphi E_t$).

We assume

(A VI) (i) For any element ${}^\varphi e_2$ of ${}^\varphi E_2{}^{r-m+n_j,m-n_j}$, ${}^\varphi\kappa_t{}^2({}^\varphi e_2)$ $(2\leqq t<\infty)$ is a cocycle of ${}^\varphi d_t$.

(ii) There is a canonical direct sum decomposition

$${}^\varphi J^{r-m+n_j,m-n_j}={}^\varphi E_\infty{}^{r-m+n_j,m-n_j}+{}^\varphi J^{r-m+n_j+1,m-n_j-1}\,.$$

Let us denote the natural injection of (A VI) (ii) by

$$i_\infty{}^*: {}^\varphi E_\infty{}^{r-m+n_j,m-n_j}\longrightarrow {}^\varphi J^{r-m+n_j,m-n_j}\subset H_{\mathcal{K}}{}^r(\varphi(\mathrm{Int}\, NW_j)\,; \bar{Z})\,.$$

If $\mathfrak{N}(W_j)$ is trivial, the assumption (A VI) is always fulfilled. Furthermore we have

LEMMA 5.1. *If Σ^{m-n_j}/G is a topological $(m-n_j)$-dimensional closed cell, then* (A VI) (i), (ii) *are trivially satisfied. And we have a natural onto isomorphism*

$$i_\infty{}^*\circ{}^\varphi\kappa_\infty{}^2: {}^\varphi E_2{}^{r-m+n_j,m-n_j}\approx H_{\mathcal{K}}{}^r(\varphi(\mathrm{Int}\, NW_j)\,; \bar{Z})\,.$$

This is a direct consequence of the fact that cohomology groups with compact carriers of $(m-n_j)$-dimensional open cell vanish except $(m-n_j)$-dimension.

Under the above assumptions, let us consider the primary obstruction of

$\mathfrak{D}$. First, for each ${}^i\mathfrak{B}$ $(i=1,\cdots,l)$, we define a G-cross section f of ${}^i\mathfrak{B}$ over ${}^i\tilde{K}\cap\tilde{K}_C$ in the sense of Definition 2.5 as follows.

Define a G-cross section ${}^if$ of ${}^i\mathfrak{B}$ over ${}^i\tilde{K}\cap\tilde{K}_C{}^0$ such that ${}^if$ agrees with $f_N{}^j$ on ${}_N\tilde{K}_j\cap\tilde{K}_C{}^0$ $(j=1,\cdots,w)$. Since ${}^iG$ operates on ${}^i\tilde{K}\cap\tilde{K}_C$ without fixed points, such a G-cross section always exists. Suppose that a G-cross section ${}^if$ of ${}^i\mathfrak{B}$ over ${}^i\tilde{K}\cap\tilde{K}_C{}^q$ which agrees with $f_N{}^j$ on ${}_N\tilde{K}_j\cap\tilde{K}_C{}^q$ $(j=1,\cdots,w)$ is constructed. If $\pi_q({}^iF)=0$, we can extend ${}^if$ to a G-cross section of ${}^i\mathfrak{B}$ over ${}^i\tilde{K}\cap\tilde{K}_C{}^{q+1}$ which agrees with $f_N{}^j$ on ${}_N\tilde{K}_j\cap\tilde{K}_C{}^{q+1}$ $(j=1,\cdots,w)$. Therefore, in case where ${}^i\mathfrak{B}$ satisfies the assumption (A III) (a) (resp. (A III) (b)), we obtain a G-cross section ${}^if$ of ${}^i\mathfrak{B}$ over ${}^i\tilde{K}\cap\tilde{K}_C{}^{r-1}$ (resp. ${}^i\tilde{K}^r$) which agrees with $f_N{}^j$ on ${}_N\tilde{K}_j\cap\tilde{K}_C{}^{r-1}$ by the stepwise extensions. By the assumption (A II), $f=\sum_{i=1}^{l}{}^i\lambda({}^if)$ is a G-cross section of $\mathfrak{B}$ over $(\sum_{i\in(a)}{}^i\tilde{K}\cap\tilde{K}_C{}^{r-1})\cup(\sum_{i\in(b)}{}^i\tilde{K}^r)$

Next we extend f to a cross section of $\mathfrak{B}$ on $\tilde{K}^{r-1}\cup\sum_{i\in(b)}{}^i\tilde{K}^r$ in the usual way. Let us denote it by the same notation f. f determines the r-dimensional primary obstruction cocycle $\tilde{c}(f)\in C^r(\tilde{K};\mathfrak{B}(\pi_{r-1}))$. According to the assumption (A IV) we have

$$\tilde{c}(f)\in C^r(\tilde{K};\bar{Z})$$

by choosing an isomorphism $\mathfrak{B}(\pi_{r-1})\approx\tilde{M}\times\bar{Z}$. Moreover by the assumption (A V), we have

$$\tilde{c}(f)(\tilde{\sigma}^r)=0 \qquad (\tilde{\sigma}^r\in\sum{}_N\tilde{K}_j\cap\tilde{K}_C)\,.$$

Hence we can regard

$$\tilde{c}(f)\in C^r(\tilde{K},\sum_j{}_N\tilde{K}_j\cap\tilde{K}_C;\bar{Z})\,.$$

Let $\iota_C:(\tilde{K}_C,\sum_j{}_N\tilde{K}_j\cap\tilde{K}_C)\to(\tilde{K},\sum_j{}_N\tilde{K}_j\cap\tilde{K}_C)$, $\iota_j:({}_N\tilde{K}_j,{}_N\tilde{K}_j\cap\tilde{K}_C)\to(\tilde{K},\sum_j{}_N\tilde{K}_j\cap\tilde{K}_C)$ be inclusion maps and let $\iota_C{}^\#$, $\iota_j{}^\#$ be cochain homomorphisms induced by ι_C, ι_j respectively. Then we have

$$\iota_C{}^\#(\tilde{c}(f))\in C^r(\tilde{K}_C,\sum_j{}_N\tilde{K}_j\cap\tilde{K}_C;\bar{Z})\,,$$

$$\iota_j{}^\#(\tilde{c}(f))\in C^r({}_N\tilde{K}_j,{}_N\tilde{K}_j\cap\tilde{K}_C;\bar{Z})\,.$$

Since the restriction of f on $\tilde{K}_C$ is a G-cross section, we have

$$\iota_C{}^\#(\tilde{c}(f))(\tilde{\sigma}^r)=\iota_C{}^\#(\tilde{c}(f))(g_\#(\tilde{\sigma}^r)) \quad (\tilde{\sigma}^r\in\tilde{K}_W\cap\tilde{K}_C,\ g\in G) \tag{5.1}$$

by the assumption (A III).

Obviously $\varphi(\tilde{K}_C)$ gives a cellular decomposition in the wider sense of $\varphi(|\tilde{K}_C|)$. We define the r-dimensional cochain

$$c'(f)\in C^r(\varphi(\tilde{K}_C),\varphi(\sum_j{}_N\tilde{K}_j\cap\tilde{K}_C);\bar{Z})$$

by

$$c'(f)(\varphi(\tilde{\sigma}^r)) = \iota_C^{\#}(\tilde{c}(f))(\tilde{\sigma}^r) \quad (\tilde{\sigma}^r \in \tilde{K}_C).$$

Then $c'(f)$ is well defined by (5.1).

We have the following lemma:

LEMMA 5.2. *$c'(f)$ is a cocycle.*

PROOF. Let σ^{r+1} be an $(r+1)$-cell of $\varphi(\tilde{K}_C - \sum_j {}_N\tilde{K}_j \cap \tilde{K}_C)$. If $\sigma^{r+1} = \varphi(\tilde{\sigma}^{r+1})$, then

$$\delta c'(f)(\sigma^{r+1}) = \delta(\iota_C^{\#}(\tilde{c}(f)))(\tilde{\sigma}^{r+1}) = \iota_C^{\#}(\tilde{c}(f))(\partial\tilde{\sigma}^{r+1}) = 0$$

q. e. d.

Therefore the cohomology class

$$\{c'(f)\} \in H^r(\varphi(\tilde{K}_C), \varphi(\sum_j {}_N\tilde{K}_j \cap \tilde{K}_C); \bar{Z})$$

is defined.

On the other hand, as is easily verified, $\iota_j^{\#}(\tilde{c}(f))$ is a cocycle. We define the correspondence

$${}_p\iota_j : \tilde{K}_j \to ({}_N\tilde{K}_j, {}_N\tilde{K}_j \cap \tilde{K}_C) \tag{5.2}$$

by

$${}_p\iota_j(\tilde{\sigma}) = p^{-1}(\tilde{\sigma}) - (p^{-1}(\tilde{\sigma}) \cap \tilde{K}_C).$$

We fix now an orientation of Σ^{m-n_j} and give to ${}_p\iota_j(\tilde{\sigma})$ the orientation which is the product of the orientations of $\tilde{\sigma}$ and of Σ^{m-n_j}. Then ${}_p\iota_j$ is one to one and induces an onto isomorphism

$${}_p\iota_j^{\#} : C^q(\tilde{K}_j; \bar{Z}) \approx C^{q+(m-n_j)}({}_N\tilde{K}_j, {}_N\tilde{K}_j \cap \tilde{K}_C; \bar{Z}).$$

Since ${}_p\iota_j^{\#}$ and the differentiation are commutative, ${}_p\iota_j^{\#}$ induces an onto isomorphic map

$${}_p\iota_j^{*} : H^q(\tilde{K}_j; \bar{Z}) \approx H^{q+(m-n_j)}({}_N\tilde{K}_j, {}_N\tilde{K}_j \cap \tilde{K}_C; \bar{Z}) \qquad (q = 0, 1, \cdots, n_j). \tag{5.3}$$

Define

$$\{\tilde{c}_j(f)\} = {}_p\iota_j^{*-1}(\{\iota_j^{\#}(\tilde{c}(f))\}),$$

then we have

$$\{\tilde{c}_j(f)\} \in H^{r-m+n_j}(\tilde{K}_j; \bar{Z}).$$

Let S^{m-n_j-1} be the $(m-n_j-1)$-sphere which is the boundary of Σ^{m-n_j}. Since S^{m-n_j-1}/G is an $(m-n_j-1)$-dimensional manifold and Σ^{m-n_j}/G is contractible, we have

$$H_{\mathcal{K}}^{m-n_j}(\mathrm{Int}(\Sigma^{m-n_j}/G); \bar{Z}) \approx H^{m-n_j}(\Sigma^{m-n_j}/G, S^{m-n_j-1}/G; \bar{Z})$$
$$\approx H^{m-n_j-1}(S^{m-n_j-1}/G; \bar{Z}) \approx \bar{Z}.$$

The fixed orientation of Σ^{m-n_j} determines a generator $\{\mathrm{Int}(\Sigma^{m-n_j}/G)\}$ of $H_{\mathcal{K}}^{m-n_j}(\mathrm{Int}(\Sigma^{m-n_j}/G); \bar{Z}) \approx \bar{Z}$.

Making use of the assumption (A VI), we define

$$\{c_j(f)\} \in H_{\mathcal{K}}^r(\varphi(\mathrm{Int}\, NW_j); \bar{Z}) = H^r(\varphi(NW_j), \varphi(NW_j \cap |\tilde{K}_C|); \bar{Z})$$

by

$$\{c_j(f)\} = i_\infty{}^*({}^{\varphi}\kappa_\infty{}^2(\{\tilde{c}_j(f)\}\otimes\{\mathrm{Int}(\Sigma^{m-n_j}/G)\}))\,.$$

Now we have

$$H^*(M, \varphi(|\sum_j {}_N\tilde{K}_j \cap \tilde{K}_G|); \bar{Z}) = H^*(\varphi(|\tilde{K}_G|), \varphi(|\sum_j {}_N\tilde{K}_j \cap \tilde{K}_G|); \bar{Z})$$
$$\oplus \sum_j H^*(\varphi(|{}_N\tilde{K}_j|), \varphi(|{}_N\tilde{K}_j \cap \tilde{K}_G|); \bar{Z})\,.$$

Let $\iota : M \to (M, \varphi(|\sum_j {}_N\tilde{K}_j \cap \tilde{K}_G|))$ be the inclusion map. Define

$$\{c(f)\} = \iota^*(\{c'(f)\} \oplus \sum_j \{c_j(f)\})\,,$$

then we have

$$\{c(f)\} \in H^r(M; \bar{Z})\,.$$

$\{c(f)\}$ is defined after a stepwise extension of f_N to a cross section on $\tilde{K}^{r-1}$ whose restriction on $\tilde{K}_G$ is a G-cross section. Finally we shall show in this section that an alteration of the extension f of f_N does not alter $\{c(f)\}$.

More generally, suppose that f_N and f_N' are *G-homotopic* standard cross sections of $\mathfrak{D}$, that is to say, there is a family of G-cross sections $f_N(t) = \{f_N{}^j(t)\}$ of $\mathfrak{B}$ on $\sum_j {}_N\tilde{K}_j \cap \tilde{K}_G{}^r$ with a continuous parameter $0 \leqq t \leqq 1$ satisfying $f_N(0) = f_N, f_N(1) = f_N'$ (considering f_N, f_N' as G-cross section of $\mathfrak{B}$). Let f and f' be two cross sections of $\mathfrak{B}$ defined on $\tilde{K}^{r-1} \cup \sum_{i\in(\mathrm{b})} {}^i\tilde{K}^r$ such that they are extensions of f_N and f_N' respectively, and that their restrictions on $\tilde{K}_G$ are G-cross sections.

The closed interval $I = [0, 1]$ is decomposed by the cell complex K_I consisting of two 0-cells $[0]$, $[1]$ and the 1-cell I. By the suitable orientations of cells, the coboundary formula is given as follows:

$$\delta[0] = -I, \quad \delta[1] = I\,.$$

Let $\tilde{K} \times K_I$ be the product complex of $\tilde{K}$ and K_I. Clearly $\tilde{K} \times K_I$ gives an admissible cellular decomposition of the product C^∞-M-space $\boldsymbol{M} \times \boldsymbol{I} = \{M \times I, \tilde{M} \times I, W \times I, G, \varphi\}$. Let $\mathfrak{D} \times I = \{\boldsymbol{M} \times \boldsymbol{I}, \mathfrak{B} \times I, F, \Gamma\}$ be a C^∞-D-bundle over $\boldsymbol{M} \times \boldsymbol{I}$ defined in an obvious manner.

We define a cross section F of $\mathfrak{B} \times I$ on $((\tilde{K}^{r-1} \cup \sum_{i\in(\mathrm{b})} {}^i\tilde{K}^r) \times ([0] \cup [1])) \cup ((\sum_j {}_N\tilde{K}_j \cap \tilde{K}_G{}^r) \times I)$ by

$$F(x, 0) = f, \quad F(x, 1) = f'\,,$$
$$F(x, t) = (f_N(t))(x)\,.$$

We can extend F to a cross section $\tilde{F}$ defined on $(\tilde{K} \times K_I)^{r-1}$ whose restriction on $\tilde{K}_G \times K_I$ is a G-cross section.

Then the obstruction cocycle

$$\tilde{c}(\tilde{F}) \in C^r(\tilde{K} \times K_I; \bar{Z})$$

is defined. $\tilde{c}(\tilde{F})$ coincides with $\tilde{c}(f)$ on $\tilde{K} \times [0]$ and with $\tilde{c}(f')$ on $\tilde{K} \times [1]$. Using the natural isomorphism $I^* : C^r(\tilde{K} \times K_I, \tilde{K} \times [0] \cup \tilde{K} \times [1]; \bar{Z}) \approx C^{r-1}(\tilde{K}; \bar{Z})$, we define

$$\tilde{d}(f, f') \in C^{r-1}(\tilde{K}; \bar{Z})$$

by

$$\tilde{d}(f, f') = I^*(\tilde{c}(\tilde{F}) - \tilde{c}(f) \times [0] - \tilde{c}(f') \times [1]).$$

Obviously $\tilde{d}(f, f')(\tilde{\sigma}^{r-1}) = 0$ for any $\tilde{\sigma}^{r-1} \in \sum_j {}_N\tilde{K}_j \cap \tilde{K}_C$. Hence we can regard

$$\tilde{d}(f, f') \in C^{r-1}(\tilde{K}, \sum_j {}_N\tilde{K}_j \cap \tilde{K}_C; \bar{Z}).$$

Since the restriction of $\tilde{F}$ on $\tilde{K}_C \times K_I$ is a G-cross section, we have

$$\iota_C^{\#}(\tilde{d}(f, f'))(\tilde{\sigma}^{r-1}) = \iota_C^{\#}(\tilde{d}(f, f'))(g_{\#}(\tilde{\sigma}^{r-1})) \qquad (\tilde{\sigma}^{r-1} \in \tilde{K}_W \cap \tilde{K}_C, g \in G).$$

Then we define

$$d'(f, f') \in C^{r-1}(\varphi(\tilde{K}_C), \varphi(\sum_j {}_N\tilde{K}_j \cap \tilde{K}_C); \bar{Z})$$

by

$$d'(f, f')(\varphi(\tilde{\sigma}^{r-1})) = \iota_C^{\#}(d'(f, f'))(\tilde{\sigma}^{r-1}) \qquad (\tilde{\sigma}^{r-1} \in \tilde{K}_C).$$

Clearly we have

$$\delta(d'(f, f')) = c'(f) - c'(f'),$$

which implies

$$\{c'(f)\} = \{c'(f')\}. \tag{5.4}$$

On the other hand we have

$$\delta(\iota_j^{\#}(\tilde{d}(f, f'))) = \iota_j^{\#}(\tilde{c}(f)) - \iota_j^{\#}(\tilde{c}(f')).$$

Therefore we obtain

$$\{\tilde{c}_j(f)\} = \{\tilde{c}_j(f')\},$$
$$\{c_j(f)\} = \{c_j(f')\}. \tag{5.5}$$

Combining (5.4) and (5.5), we have

$$\{c(f)\} = \{c(f')\}.$$

Thus we have proved the following proposition.

PROPOSITION 5.1. *Let $\{f_N\}$ be a family of standard cross sections which are G-homotopic to each other. Then the cohomology class $\{c(f)\} \in H^r(M; \bar{Z})$ is independent of the choice of the extension of f_N.*

DEFINITION 5.1. The cohomology class $\{c(f)\} \in H^r(M; \bar{Z})$ will be called the *primary obstruction class* $\bar{c}_K(\mathfrak{D})$ of $\mathfrak{D}$ with respect to $\{f_N\}$.

REMARK 5.1. In case where $\mathfrak{D}$ is the fibre bundle, $\bar{c}_K(\mathfrak{D})$ becomes the primary obstruction class in the usual sense.

6. Invariance of $\bar{c}_K(\mathfrak{D})$.

The primary obstruction class $\bar{c}_K(\mathfrak{D})$ is defined by means of a fixed admissible cellular decomposition $\tilde{K}$ of $\tilde{M}$. In this section we shall prove that $\bar{c}_K(\mathfrak{D})$ is independent of the choice of the admissible cellular decomposition of $\tilde{M}$.

Let $\boldsymbol{M}=\{M, \tilde{M}, W, G, \varphi\}$, $\boldsymbol{M}'=\{M', \tilde{M}', W', G', \varphi'\}$ be compact m-dimensional and m'-dimensional C^∞-M-spaces satisfying the conditions (*), (**) (Section 4), and let $\tilde{K}, \tilde{K}'$ be admissible cellular decompositions of $\tilde{M}, \tilde{M}'$ respectively.

PROPOSITION 6.1. *Let $\mathfrak{D}=\{\boldsymbol{M}, \mathfrak{B}, F, \Gamma\}$, $\mathfrak{D}'=\{\boldsymbol{M}', \mathfrak{B}', F, \Gamma\}$ be C^0-D-bundles with the same fibre and the same structural group, satisfying the assumptions* (A I)-(A VI). *Let $h:\mathfrak{D}\to\mathfrak{D}'$ be a D-bundle map over a C^0-map $\bar{\boldsymbol{h}}=(\tilde{\bar{h}}, \bar{h}):\boldsymbol{M}\to\boldsymbol{M}'$ satisfying the following conditions:*

(i) *If ${}^ih:{}^i\mathfrak{B}\to{}^j\mathfrak{B}'$ (resp. ${}^i\mathfrak{B}\to\mathfrak{B}'$), then their fibres ${}^iF$ and ${}^jF'$ (resp. ${}^iF$ and F') are the same, and ${}^ih|{}^iF$ is the identity map.*

(ii) *$\bar{\boldsymbol{h}}$ is an admissible cellular map with respect to $\tilde{K}, \tilde{K}'$.*

(iii) *h maps the standard cross section $f_N{}^j$ of $\mathfrak{D}$ into the standard cross section $f_N{}'^{j'}$ of $\mathfrak{D}'$ in case $\tilde{\bar{h}}(W_j)\subset W_{j'}{}'$.*

(iv) *If $\tilde{\bar{h}}(W_j)\subset W_{j'}{}'$, then $m-n_j=m'-n_{j'}{}'$ and the following diagram is commutative:*

$$\begin{array}{ccc}
H^{r-m+n_j}(\varphi(W_j);\bar{Z})\otimes H_{\mathcal{K}}^{m-n_j}(\mathrm{Int}(\Sigma^{m-n_j}/G);\bar{Z}) & \xrightarrow{i_\infty{}^*\circ\varphi_{K_\infty}{}^2} & H_{\mathcal{K}}{}^r(\varphi(\mathrm{Int}\,NW_j);\bar{Z}) \\
\uparrow (\bar{h}\,|\,\varphi(W_j))^*\otimes id. & & \uparrow (\bar{h}\,|\,\varphi(NW_j))^* \\
H^{r-m'+n_{j'}{}'}(\varphi(W_{j'}{}');\bar{Z})\otimes H_{\mathcal{K}}^{m'-n_{j'}{}'}(\mathrm{Int}(\Sigma^{m'-n_{j'}{}'}/G');\bar{Z}) & \xrightarrow{i_\infty{}^*\circ\varphi_{K_\infty}{}^2} & H_{\mathcal{K}}{}^r(\varphi(\mathrm{Int}\,NW_{j'}{}');\bar{Z}).
\end{array}$$

Then we have

$$\bar{c}_K(\mathfrak{D})=\bar{h}^*(\bar{c}_{K'}(\mathfrak{D}')),$$

provided that isomorphisms $\mathfrak{B}(\pi_{r-1})\approx\tilde{M}\times\bar{Z}$, $\mathfrak{B}'(\pi_{r-1})\approx\tilde{M}'\times\bar{Z}$ are suitably chosen.

PROOF. In case where $\tilde{\bar{h}}(W_j)\cap(\sum_{j'}W_{j'}{}')=\phi$ we can find a G'-cross section $f_{h(j)}{}'$ of $\mathfrak{B}'$ on $\tilde{\bar{h}}(NW_j\cap|\tilde{K}_G{}^r|)$. We extend $f_N{}'\cup(\bigcup_j f_{h(j)}{}')$ to a cross section f' of $\mathfrak{B}'$ over $\tilde{K}'^{r-1}\cup\sum_{i\in(\mathrm{b})}{}^i\tilde{K}'^r$ whose restriction on $\tilde{K}_G$ is a G-cross section as in Section 5. Let us define the cross section f of $\mathfrak{B}$ over $\tilde{K}^{r-1}\cup\sum_{i\in(\mathrm{b})}{}^i\tilde{K}^r$ by the map $\tilde{\bar{h}}$ and the cross section f' of $\mathfrak{B}'$. Obviously f is an extension of f_N whose restriction on $\tilde{K}_G$ is a G-cross section. Then we have

$$\tilde{c}(f)(\tilde{\sigma}^r)=\tilde{c}(f')(\tilde{\bar{h}}(\tilde{\sigma}^r)) \qquad (\tilde{\sigma}^r\in\tilde{K}).$$

We can regard $\tilde{c}(f')$ as an r-dimensional cochain

$$\tilde{c}_1(f') \in C^r(\tilde{K}', (\sum_{j'} {}_N\tilde{K}_{j'}' \cap \tilde{K}_C') \cup \tilde{\bar{h}}(\sum_j {}_N\tilde{K}_j \cap \tilde{K}_C); \bar{Z}).$$

Let $\iota_{C_h}' : (\tilde{K}_C', (\sum_{j'} {}_N\tilde{K}_{j'}' \cap \tilde{K}_C') \cup \tilde{\bar{h}}(\sum_j {}_N\tilde{K}_j \cap \tilde{K}_C)) \to (\tilde{K}', (\sum_{j'} {}_N\tilde{K}_{j'}' \cap \tilde{K}_C') \cup \tilde{\bar{h}}(\sum_j {}_N\tilde{K}_j \cap \tilde{K}_C))$ be the inclusion map. Since the restriction of f' on $\tilde{K}_C'$ is a G'-cross section, $\iota_{C_h}'^{\#}(\tilde{c}_1(f'))$ determines an r-dimensional cochain

$$c_1'(f') \in C^r(\varphi'(\tilde{K}_C'), \varphi'((\sum_{j'} {}_N\tilde{K}_{j'}' \cap \tilde{K}_C') \cup \tilde{\bar{h}}(\sum_j {}_N\tilde{K}_j \cap \tilde{K}_C)); \bar{Z})$$

in an obvious manner. $c_1'(f')$ is a cocycle (Lemma 5.2) and so the cohomology class

$$\{c_1'(f')\} \in H^r(\varphi'(\tilde{K}_C'), \varphi'((\sum_{j'} {}_N\tilde{K}_{j'}' \cap \tilde{K}_C') \cup \tilde{\bar{h}}(\sum_j {}_N\tilde{K}_j \cap \tilde{K}_C)); \bar{Z})$$

is defined. We have

$$c'(f') = \iota_h'^{\#}(c_1'(f')),$$

where $\iota_h' : (\varphi(\tilde{K}_C'), \varphi(\sum_{j'} {}_N\tilde{K}_{j'}' \cap \tilde{K}_C')) \to (\varphi(\tilde{K}_C'), \varphi((\sum_{j'} {}_N\tilde{K}_{j'}' \cap \tilde{K}_C') \cup \tilde{\bar{h}}(\sum_j {}_N\tilde{K}_j \cap \tilde{K}_C)))$ is the inclusion map.

By the commutative diagram

$$\begin{array}{ccc}
(\tilde{K}_C, \sum_j {}_N\tilde{K}_j \cap \tilde{K}_C) & \xrightarrow{\tilde{\bar{h}} | \tilde{K}_C} & (\tilde{K}_C', (\sum_{j'} {}_N\tilde{K}_{j'}' \cap \tilde{K}_C') \cup \tilde{\bar{h}}(\sum_j {}_N\tilde{K}_j \cap \tilde{K}_C)) \\
\downarrow \varphi & & \downarrow \varphi' \\
(\varphi(\tilde{K}_C), \varphi(\sum_j {}_N\tilde{K}_j \cap \tilde{K}_C)) & \xrightarrow{\bar{h} | \varphi(\tilde{K}_C)} & (\varphi'(\tilde{K}_C'), \varphi'(\sum_{j'} {}_N K_{j'}' \cap \tilde{K}_C') \\
 & & \cup \bar{h}(\varphi(\sum_j {}_N\tilde{K}_j \cap \tilde{K}_C))),
\end{array}$$

we have

$$c'(f) = (\bar{h} | \varphi(\tilde{K}_C))^{\#}(c_1'(f')). \tag{6.1}$$

In case where $\tilde{\bar{h}}(W_j) \subset W_{j'}'$, we have

$$\{c_j(f)\} = (\bar{h} | \varphi(NW_j))^*(\{c_{j'}(f')\}) \tag{6.2}$$

by the condition (iv) and the following equation:

$$\iota_j^{\#}(\tilde{c}(f)) = (\tilde{\bar{h}} | NW_j)^{\#}(\iota_{j'}'^{\#}(\tilde{c}(f'))).$$

In case where $\tilde{\bar{h}}(W_j) \cap (\sum_{j'} W_{j'}') = \phi$, we have

$$\{c_j(f)\} = (\bar{h} | \varphi(NW_j))^*(\{c_1'(f')\}), \tag{6.3}$$

by the following lemma:

LEMMA 6.1. *Let $\{{}^hE_s\}$ be the spectral sequence of the fibre bundle* $\mathfrak{N}(\tilde{\bar{h}}(W_j))$

(*Definition* 4.2. (iii)) *and let* ${}^{h}_{p}\iota_j$ *be the map of* (5.2) *with respect to* $\mathfrak{N}(\tilde{\bar{h}}(W_j))$. *Then we have*

$$i_\infty{}^*({}^h\kappa_\infty{}^2(\{{}^h_p\iota_j{}^*(\tilde{c}(f_{h(j)}{}'))\}\otimes\{\mathrm{Int}\,\Sigma^{m-n_j}\}))=\iota_{h(j)}{}^*(\{c_1{}'(f')\})\,,$$

where $\iota_{h(j)}$ *is the inclusion map* $(\varphi'(\tilde{\bar{h}}(|{}_N\tilde{K}_j|)),\ \varphi'(\tilde{\bar{h}}(|{}_N\tilde{K}_j\cap\tilde{K}_C|))\to(\varphi'(|\tilde{K}_C{}'|),$ $\varphi'(|\sum_{j'}{}_N\tilde{K}_{j'}{}'\cap\tilde{K}_C{}'|\cup\tilde{\bar{h}}(|\sum_j{}_N\tilde{K}_j\cap\tilde{K}_C|)))$.

This lemma is a direct consequence of the definition of the obstruction cocycle and Lemma 5.1. So the proof is omitted.

Combining (6.1), (6.2) and (6.3), we have

$$\begin{aligned}\bar{h}^*(\{c(f')\})&=\bar{h}^*(\iota'^*(\{c'(f')\}\oplus\sum_{j'}\{c_{j'}(f')\}))\\&=\bar{h}^*(\iota'^*(\iota_h{}'^*(\{c_1{}'(f')\})))\oplus\bar{h}^*(\iota'^*(\sum_{j'}\{c_{j'}(f')\}))\\&=\iota^*(\{c'(f)\}\oplus\sum_j\{c_j(f)\})=\{c(f)\}\,.\end{aligned}$$

Hence we obtain

$$\bar{c}_K(\mathfrak{D})=\bar{h}^*(\bar{c}_{K'}(\mathfrak{D}))\,. \qquad \text{q. e. d.}$$

PROPOSITION 6.2. *Let* $'\tilde{K}$ *be the first regular subdivision of* $\tilde{K}$, *then we have*

$$\bar{c}_K(\mathfrak{D})=\bar{c}_{'K}(\mathfrak{D})\,,$$

provided that the standard cross section $'f_N$ *with respect to* $'\tilde{K}$ *is given by* (6.4).

PROOF. Let $\tilde{i}:|'\tilde{K}|\to|\tilde{K}|$ be the identity map. Clearly, for each $\tilde{K}_j$, there exists a cellular map $\tilde{h}_j:'\tilde{K}_j\to\tilde{K}_j$ which is homotopic to the identity map. Let $\tilde{h}_j(t):'\tilde{K}_j\to\tilde{K}_j$ $(0\leqq t\leqq 1)$ be a homotopy satisfying $\tilde{h}_j(0)=\text{identity}$, $\tilde{h}_j(1)=\tilde{h}_j$. Let $\tilde{x}$ be a point of N_2W_j which is expressed by (x,s) $(x\in W_j,\ 1/2\leqq|s|\leqq 1)$. Define the map $\tilde{h}_j{}^{(2)}:N_2W_j\to NW_j\cap|\tilde{K}_C|$ by

$$\tilde{h}_j{}^{(2)}((x,s))=(x,s/|s|)\,.$$

Furthermore let us define the map $\tilde{i}^{(2)}:('\tilde{K})_C\to\tilde{K}_C$ by

$$\begin{aligned}&\tilde{i}^{(2)}\,|\,'(\tilde{K}_C)=\tilde{i}\,|\,'(\tilde{K}_C)\,,\\&\tilde{i}^{(2)}\,|\,N_2W_j=\tilde{h}_j{}^{(2)}\,.\end{aligned}$$

Then there is a homotopy $\tilde{h}_C(t):('\tilde{K})_C\to\tilde{K}_C$ such that $\tilde{h}_C(0)=\tilde{i}^{(2)}$ and $\tilde{h}_C(1)$ is a cellular map, and that

$$\tilde{h}_C(t)({}^iW)\subset{}^iW,\ \ \tilde{h}_C(t)(N_2W_j)\subset NW_j\cap|K_C|,\ \ (\tilde{h}_C(t))(g(\tilde{x}))=g((\tilde{h}_C(t))(\tilde{x}))$$

for any $\tilde{x}\in W, g\in G$. For each W_j, let us consider the fibre bundles $\varphi(\mathfrak{N}(W_j))=\{\varphi(NW_j),p,\varphi(W_j),\Sigma^{m-n_j}/G,\Gamma_j\}$, $\varphi(\mathfrak{N}_1(W_j))=\{\varphi(N_1W_j),p,\varphi(W_j),\Sigma^{m-n_j}/G,\Gamma_j\}$ and their associated bundles $\varphi(\dot{\mathfrak{N}}(W_j))=\{\varphi(NW_j\cap|\tilde{K}_C|),p,\varphi(W_j),S^{m-n_j-1}/G,\Gamma_j\}$, $\varphi(\dot{\mathfrak{N}}_1(W_j))=\{\varphi(N_1W_j\cap N_2W_j),p,\varphi(W_j),S^{m-n_j-1}/G,\Gamma_j\}$. Since $\varphi(\dot{\mathfrak{N}}_1(W_j))$ and

$\varphi(\dot{\mathfrak{N}}(W_j))$ are equivalent, there is a homotopy $h_j'(t): \varphi(N_1 W_j \cap N_2 W_j) \to \varphi(NW_j \cap |\tilde{K}_C|)$ which is covering $h_j(t): \varphi(W_j) \to \varphi(W_j)$ with respect to the equivalent bundle map $\varphi(\dot{\mathfrak{N}}_1(W_j)) \to \varphi(\dot{\mathfrak{N}}(W_j))$. Clearly $h_j'(t)$ defines a homotopy $\tilde{h}_j'(t): N_1 W_j \cap N_2 W_j \to NW_j \cap |\tilde{K}_C|$ covering $h_j(t)$ with respect to the equivalent bundle map $\mathfrak{N}_1(W_j) \to \mathfrak{N}(W_j)$ such that $(\tilde{h}_j'(t))(g(\tilde{x})) = g(\tilde{h}_j'(t)(\tilde{x}))$ for any $\tilde{x} \in N_1 W_j \cap N_2 W_j$, $g \in G$. Obviously $\tilde{h}_j'(1)$ is homotopic to $\tilde{h}_C(1)|(N_1 W_j \cap N_2 W_j)$, and $\tilde{h}_j'(1)$ maps $p^{-1}(\tilde{\sigma}_j^q) \cap \tilde{K}_C$ $(\tilde{\sigma}_j^q \in ('\tilde{K}_j)^q)$ into ${}_N\tilde{K}_j \cap \tilde{K}_C{}^{q+m-n_j-1}$. Therefore we can find a homotopy $\tilde{h}_j''(t): {}_N('\tilde{K})_j \cap ('\tilde{K})_C \to {}_N\tilde{K}_j \cap \tilde{K}_C$ satisfying

(i) $(\tilde{h}_j''(t))(g(\tilde{x})) = g(\tilde{h}_j''(t)(\tilde{x}))$ for any $\tilde{x} \in |{}_N('\tilde{K})_j \cap ('\tilde{K})_C|$, $g \in G$.

(ii) $\tilde{h}_j''(0) = \tilde{h}_j'(1)$, $\tilde{h}_j''(1) = \tilde{h}_C(1)|({}_N('\tilde{K})_j \cap ('\tilde{K})_C)$.

(iii) $\tilde{h}_j''(t)(p^{-1}(\tilde{\sigma}_j^q) \cap \tilde{K}_C) \subset {}_N\tilde{K}_j \cap \tilde{K}_C{}^{q+m-n_j}$ $\quad (\tilde{\sigma}_j^q \in ('\tilde{K}_j)^q)$.

For a point $\tilde{x}$ of NW_j expressed by (x, s) $(x \in W_j, 0 < |s| \leqq 1)$, we denote the point (x, ts) $(0 \leqq t \leqq |s|^{-1})$ by $\tilde{x}_t$. Define the map $\tilde{h}: '\tilde{K} \to \tilde{K}$ by

(i) $\tilde{h}|('\tilde{K})_C = \tilde{h}_C(1)$,

(ii) $\tilde{h}|'\tilde{K}_j = \tilde{h}_j$,

(iii) $\tilde{h}((x, s)) = ((\tilde{h}_j'(1))(x, s/|s|))_{4|s|} \qquad x \in '\tilde{K}_j, 0 < s \leqq 1/4$;
$\tilde{h}((x, s)) = (\tilde{h}_j''(4s-1))(x, s/(2|s|)) \qquad x \in '\tilde{K}_j, 1/4 \leqq s \leqq 1/2$.

Then, as is easily verified, $\tilde{h}$ determines the admissible cellular map $\boldsymbol{h} = (\tilde{h}, h)$ of $\boldsymbol{M}$ into itself with respect to $'\tilde{K}$ and $\tilde{K}$ such that h is homotopic to the identity map $M \to M$.

Assumption (i), (iii) and (iv) of Proposition 6.1 are trivially satisfied in this case, if we take the standard cross section $'f_N$ such that

$$\{'f_N\} = \{\tilde{h}^{-1}(f_N)\}. \tag{6.4}$$

Hence Proposition 6.2 is a direct consequence of Proposition 6.1. q. e. d.

Now we prove the following theorem:

THEOREM 6.1. *Let* $\mathfrak{D} = \{\boldsymbol{M}, \mathfrak{B}, F, \Gamma\}$, $\mathfrak{D}' = \{\boldsymbol{M}', \mathfrak{B}', F, \Gamma\}$ *be C^0-D-bundles with the same fibre and the same structural group, satisfying the assumptions* (A I)–(A VI) *and* (*), (**). *Let* $h: \mathfrak{D} \to \mathfrak{D}'$ *be a D-bundle map over a C^0-map* $\bar{\boldsymbol{h}} = (\tilde{\bar{h}}, \bar{h})$: $\boldsymbol{M} \to \boldsymbol{M}'$ *satisfying the conditions* (i), (iv) *of Proposition* 6.1. *Suppose furthermore that the following conditions holds*:

(i) *If* $\boldsymbol{M}$ *satisfies* (*) (b) *and* $\tilde{\bar{h}}(W_{j_1}) \cap (\sum_{j'} W_{j'}') = \phi$, $\tilde{\bar{h}}(W_{j_2}) \cap (\sum_{j'} W_{j'}') = \phi$, *then* $\tilde{\bar{h}}(W_{j_1})$ *and* $\tilde{\bar{h}}(W_{j_2})$ *are submanifolds of* $\tilde{M}'$ *and they are satisfying* $\tilde{\bar{h}}(W_{j_1}) \subseteqq \tilde{\bar{h}}(W_{j_2})$ *or* $\tilde{\bar{h}}(W_{j_1}) \supseteqq \tilde{\bar{h}}(W_{j_2})$ *unless* $\tilde{\bar{h}}(W_{j_1}) \cap \tilde{\bar{h}}(W_{j_2}) = \phi$.

(ii) $\tilde{\bar{h}}^{-1}(W_{j'}') \subset \sum_j W_j \qquad (j = 1, \cdots, w)$.

Then we have

$$\tilde{c}_K(\mathfrak{D}) = \bar{h}^*(\tilde{c}_{K'}(\mathfrak{D}')),$$

provided that the standard cross sections f_N of $\mathfrak{D}$ and f_N' of $\mathfrak{D}'$ are suitably chosen.

Before proceeding to the proof, we mention an important special case of this theorem:

THEOREM 6.2. *Let $\mathfrak{D}=\{\boldsymbol{M}, \mathfrak{B}, F, \Gamma\}$ be a C^0-D-bundle over a C^∞-M-space satisfying the assumptions* (A I)-(A VI) *and* (*), (**). *Then the primary obstruction class $\bar{c}_K(\mathfrak{D})$ of $\mathfrak{D}$ is independent of the choice of the admissible cellular decomposition $\tilde{K}$, provided that the standard cross section is suitably chosen.*

In fact, all assumptions on $\bar{\boldsymbol{h}}$ in Theorem 6.1 are obviously satisfied if $\bar{\boldsymbol{h}}$ is the identity map.

PROOF OF THEOREM 6.1. If $\boldsymbol{M}$ satisfies (*) (b) and $\tilde{\bar{h}}(W_j)\cap(\sum_{j'} W_{j'}')=\phi$, we take a sufficiently small normal tube neighbourhood $N(\tilde{\bar{h}}(W_j))$. Let $\mathfrak{N}(\tilde{\bar{h}}(W_j))=\{N(\tilde{\bar{h}}(W_j)), p, \tilde{\bar{h}}(W_j), \Sigma^{m-n_j}, \Gamma_j'\}$ be the normal bundle. We can find $\mathfrak{N}(\tilde{\bar{h}}(W_j))$ satisfying $\mathfrak{N}(\tilde{\bar{h}}(W_{j_1}))=\mathfrak{N}(\tilde{\bar{h}}(W_{j_2}))|\tilde{\bar{h}}(W_{j_1})$ in case $\tilde{\bar{h}}(W_{j_1})\subset\tilde{\bar{h}}(W_{j_2})$ and $N(\tilde{\bar{h}}(W_{j_1}))\cap N(\tilde{\bar{h}}(W_{j_2}))=\phi$ in case $\tilde{\bar{h}}(W_{j_1})\cap\tilde{\bar{h}}(W_{j_2})=\phi$.

First assume that the following condition [C] holds:

$$[\mathrm{C}]\quad\begin{cases}\tilde{K}' \text{ is an admissible cellular decomposition of } \tilde{M}' \text{ which has } p^{-1}(\tilde{\sigma}')\\ (\tilde{\sigma}'\in{}_h\tilde{K}_j') \text{ as a cell, where } {}_h\tilde{K}_j' \text{ is a cell complex giving a cellular}\\ \text{decomposition of } \tilde{\bar{h}}(W_j).\end{cases}$$

Then, for a sufficiently large integer ν, ${}^{(\nu)}\tilde{K}$ satisfies the following conditions:

$$\tilde{\bar{h}}({}^{(\nu)}\tilde{\sigma})\ ({}^{(\nu)}\tilde{\sigma}\in({}^{(\nu)}\tilde{K})_C) \text{ and } \tilde{\bar{h}}({}^{(\nu)}\tilde{\sigma}_j)\ ({}^{(\nu)}\tilde{\sigma}_j\in{}^{(\nu)}\tilde{K}_j) \text{ have acyclic carriers,}$$

$$\tilde{\bar{h}}({}^{(\nu)}NW_j)\subset NW_{j'}' \qquad \text{if } \bar{h}(W_j)\subset W_{j'}',$$

$$\tilde{\bar{h}}({}^{(\nu)}NW_j)\subset N(\tilde{\bar{h}}(W_j)) \quad \text{otherwise.}$$

Let $\tilde{\bar{h}}_j$: ${}^{(\nu)}\tilde{K}_j\to\tilde{K}_{j'}'$ (or ${}^{(\nu)}\tilde{K}_j\to{}_h\tilde{K}_j'$) be a cellular map which is homotopic to the map $\tilde{\bar{h}}|{}^{(\nu)}\tilde{K}_j$, and let $\tilde{\bar{h}}_j(t):{}^{(\nu)}\tilde{K}_j\to\tilde{K}_{j'}'$ (or ${}^{(\nu)}\tilde{K}_j\to{}_h\tilde{K}_j'$) be a homotopy satisfying $\tilde{\bar{h}}_j(0)=\tilde{\bar{h}}|{}^{(\nu)}\tilde{K}_j$, $\tilde{\bar{h}}_j(1)=\tilde{\bar{h}}_j$. Let us define the map $\tilde{\bar{h}}_C:({}^{(\nu)}\tilde{K})_C\to\tilde{K}_C'$ by

$$\tilde{\bar{h}}_C|(\tilde{\bar{h}}^{-1}(\tilde{K}_C'-\sum_j N(\tilde{\bar{h}}(W_j)))=\tilde{\bar{h}}|\tilde{\bar{h}}^{-1}(\tilde{K}_C'-\sum_j N(\tilde{\bar{h}}(W_j)))\,,$$

$$\tilde{\bar{h}}_C(\tilde{x})=(x, s/|s|)\ (\tilde{x}\notin\tilde{\bar{h}}^{-1}(\tilde{K}_C'-\sum_j N(\tilde{\bar{h}}(W_j))),$$

$$\tilde{\bar{h}}(\tilde{x})=(x, s),\ x\in\sum_j\tilde{\bar{h}}(W_j),\ 0\leqq|s|\leqq 1)\,.$$

Then by the same method as used in the proof of Proposition 6.2, making use of $\tilde{\bar{h}}_j(t)$, $\tilde{\bar{h}}c$, we obtain an admissible cellular map $\bar{\boldsymbol{h}}'=(\tilde{\bar{h}}', \bar{h}'):\boldsymbol{M}\to\boldsymbol{M}'$ with respect to ${}^{(\nu)}\tilde{K}$ and $\tilde{K}'$ such that $\bar{h}'$ is homotopic to $\bar{h}$. Therefore, by Propositions 6.1 and 6.2, we have

$$\bar{c}_K(\mathfrak{D}) = \bar{h}^*(\bar{c}_{K'}(\mathfrak{D}')) .$$

Now the condition [C] is satisfied if $\boldsymbol{h}$ is the identity map. So Theorem 6.2 is completely proved.

We can finally get rid of the condition [C] applying Theorem 6.2 to $\mathfrak{D}'$.

q. e. d.

Part III. Characteristic classes.

7. Definition of characteristic classes.

First we shall define characteristic classes of vector-D-bundles.

Let $\boldsymbol{M} = \{M, \tilde{M}, W, G, \varphi\}$ be a compact m-dimensional C^∞-M-space satisfying the conditions (*), (**) of Section 4, and let $\tilde{K}$ be an admissible cellular decomposition of $\tilde{M}$. Let $\mathfrak{D}_1 = \{\boldsymbol{M}, \mathfrak{B}_1, E^n, O(n)\}$ be a C^∞-vector D-bundle over $\boldsymbol{M}$ with the n-dimensional real vector space E^n as fibre and the orthogonal group $O(n)$ as structural group, and $\mathfrak{D}_2 = \{\boldsymbol{M}, \mathfrak{B}_2, E_n, U(n)\}$ a C^∞-vector D-bundle over $\boldsymbol{M}$ with the n-dimensional complex vector space E_n as fibre and the unitary group $U(n)$ as structural group. In the following we shall use notations in Section 3 without special references.

Let $\mathfrak{D}_1{}^{(k)} = \{\boldsymbol{M}, \mathfrak{B}_1{}^{(k)}, O(n)/O(n-k), O(n)\}$, $\mathfrak{D}_1{}^{[k]} = \{\boldsymbol{M}, \mathfrak{B}_1{}^{[k]}, U(n)/U(n-k), O(n)\}$ $(k \leqq \min_D \dim F)$ be the associated D-bundles of $\mathfrak{D}_1$, and $\mathfrak{D}_2{}^{(k)} = \{\boldsymbol{M}, \mathfrak{B}_2{}^{(k)}, U(n)/U(n-k), U(n)\}$ $(k \leqq \min_D \dim F)$ the associated D-bundle of $\mathfrak{D}_2$.

As is well-known we have

$$\pi_j(O(n_i)/O(n_i-k)) = \begin{cases} O & j < n_i - k\,, \\ Z & j = n_i - k, \quad n_i - k:\ \text{even or } k = 1\,, \\ Z_2 & j = n_i - k \quad \text{otherwise}, \end{cases}$$

$$\pi_j(U(n_i)/U(n_i-k)) = \begin{cases} O & j < 2(n_i-k)+1\,, \\ Z & j = 2(n_i-k)+1\,. \end{cases}$$

Suppose that, for each ${}^i\mathfrak{B}_1{}^{(k)}$ (resp. ${}^i\mathfrak{B}_1{}^{[k]}$, ${}^i\mathfrak{B}_2{}^{(k)}$), n_i satisfies *one* of the following conditions (i), (ii):

(i) $n_i = n$,

(ii) $\begin{cases} {}^iG \text{ operates on } {}^iW \text{ without fixed points;} \\ \dim {}^iW \leqq n_i - k \ (\text{resp. } \dim {}^iW \leqq 2(n_i-k)+1). \end{cases}$

Then the assumption (A III) of Section 5 is fulfilled for $\mathfrak{D}_1{}^{(k)}$ (resp. $\mathfrak{D}_1{}^{[k]}$, $\mathfrak{D}_2{}^{(k)}$). Moreover let us assume the conditions (A IV), (A V) and (A VI) of Section 5 for $\mathfrak{D}_1{}^{(k)}$ (resp. $\mathfrak{D}_1{}^{[k]}$, $\mathfrak{D}_2{}^{(k)}$). Then we have the primary obstruction classes

$$\bar{c}(\mathfrak{D}_1{}^{(k)}) \in H^{n-k+1}(M; \bar{Z})\,,$$

$$\bar{c}(\mathfrak{D}_1^{[k]}) \in H^{2(n-k)+2}(M; Z),$$

$$\bar{c}(\mathfrak{D}_2^{(k)}) \in H^{2(n-k)+2}(M; Z),$$

respectively. ($\bar{c}(\mathfrak{D}_1^{(k)})$, $\bar{c}(\mathfrak{D}_1^{[k]})$ and $\bar{c}(\mathfrak{D}_2^{(k)})$, as defined in Section 4, are determined respectively by $\mathfrak{D}_1^{(k)}$, $\mathfrak{D}_1^{[k]}$, $\mathfrak{D}_2^{(k)}$ and their standard cross sections. These cross sections exist in virtue of the assumtion (A V). We suppose in the following that these standard cross sections are given.)

DEFINITION 7.1. $\bar{c}(\mathfrak{D}_1^{(k)})$ $(1 \leqq k \leqq \min_D \dim F)$ will be called the $(n-k+1)$-*th SW-class* $SW_{n-k+1}(\mathfrak{D}_1)$ *of* $\mathfrak{D}_1$.

DEFINITION 7.2. $\bar{c}(\mathfrak{D}_1^{[k]})$ $(1 \leqq k \leqq \min_D \dim F)$ will be called the $((n-k+1)/2)$-*th P-class* $P_{(n-k+1)/2}(\mathfrak{D}_1)$ of $\mathfrak{D}_1$

DEFINITION 7.3. $\bar{c}(\mathfrak{D}_2^{(k)})$ $(1 \leqq k \leqq \min_D \dim F)$ will be called the $(n-k+1)$-*th C-class* $C_{n-k+1}(\mathfrak{D}_2)$ *of* $\mathfrak{D}_2$.

Obviously in particular case where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are fibre bundles in the usual sense, SW-classes, P-classes and C-classes coincide with the Stiefel-Whitney classes, the Pontrjagin classes and the Chern classes of fibre bundles respectively.

Now we shall define characteristic classes of M-spaces.

Let $\boldsymbol{M}_1 = \{M_1, \tilde{M}_1, W_1, G_1, \varphi_1\}$ (resp. $\boldsymbol{M}_2 = \{M_2, \tilde{M}_2, W_2, G_2, \varphi_2\}$) be a compact m-dimensional C^∞-M-space (resp. $2m$-dimensional almost complex M-space) satisfying the following conditions:

(B I) ${}^iW_1$ (resp. ${}^iW_2$) $(i=1,\cdots,l)$ are disjoint.

(B II) $\boldsymbol{M}_1$ (resp. $\boldsymbol{M}_2$) satisfies the conditions (*), (**) of Section 4.

(B III) $\tilde{M}_1$ and ${}^iW_1$ $(i=1,\cdots,l)$ are orientable.

(B IV) ${}^i\alpha(g)$ $(g \in G)$ is an orientation-preserving isometric transformation with respect to the Riemannian metric of $\tilde{M}_1$ (resp. $\tilde{M}_2$).

Then the tangent D-bundle $\mathfrak{T}(\boldsymbol{M}_1) = \{\boldsymbol{M}_1, \mathfrak{T}(\tilde{M}_1), E^m, O(m)\}$ (resp. $\mathfrak{T}(\boldsymbol{M}_2) = \{\boldsymbol{M}_2, \mathfrak{T}(\tilde{M}_2), E_m, U(m)\}$) is defined (Section 3).

Suppose that ${}^i\mathfrak{N}_1 = \{{}^iN_1, p, {}^iW_1, E^{m-m_i}, O(m-m_i)\}$ (resp. ${}^i\mathfrak{N}_2 = \{{}^iN_2, p, {}^iW_2, E_{m-m_i}, U(m-m_i)\}$) $(i=1,\cdots,l)$ be the normal bundles of ${}^iW_1$ in $\tilde{M}_1$ (resp. ${}^iW_2$ in $\tilde{M}_2$). We introduce now the following assumptions (***), (***)':

(***) The associated bundle ${}^i\mathfrak{N}_1^{(m-m_i)} = \{{}^iN_1^{(m-m_i)}, p, {}^iW_1, O(m-m_i), O(m-m_i)\}$ of ${}^i\mathfrak{N}_1$ (resp. ${}^i\mathfrak{N}_2^{(m-m_i)} = \{{}^iN_2^{(m-m_i)}, p, {}^iW_2, U(m-m_i), U(m-m_i)\}$ of ${}^i\mathfrak{N}_2$) has a cross section ${}^if$ over ${}^iW_1$ (resp. ${}^iW_2$) for $1 \leqq i \leqq l$.

(***)' The associated bundle ${}^i\mathfrak{N}_1^{[k_i]} = \{{}^iN_1^{[k_i]}, p, {}^iW, U(m-m_i)/U(m-m_i-k_i), O(m-m_i)\}$ $(m-m_i \geqq k_i \geqq k-[(m_i+1)/2])$ of ${}^i\mathfrak{N}_1$ has a cross section ${}^if$ over ${}^iW_1$ for $1 \leqq i \leqq l$, where k_i must be equal to $m-m_i$ in case ${}^iG$ operates on ${}^iW$ with fixed points.

Then the generalized associated D-bundle $\mathfrak{T}^{(k)}(\boldsymbol{M}_1)({}^1f,\cdots,{}^lf) = \{\boldsymbol{M}_1, \mathfrak{T}^{(k)}(\tilde{M}_1), O(m)/O(m-k), O(m)\}$ $(\max_i(m-m_i) \leqq k \leqq m)$ (resp. $\mathfrak{T}^{[k]}(\boldsymbol{M}_1)({}^1f,\cdots,{}^lf) = \{\boldsymbol{M}_1, \mathfrak{T}^{[k]}(\tilde{M}_1),$

$U(m)/U(m-k), O(m)\}\ \max_i(k_i \leqq k \leqq m))$ and the generalized associated D-bundle $\mathfrak{T}^{(k)}(\boldsymbol{M}_2)({}^1f,\cdots,{}^lf)=\{\boldsymbol{M}_2, \mathfrak{T}^{(k)}(\tilde{M}_2), U(m)/U(m-k), U(m)\}$ $(\max_i(m-m_i)\leqq k\leqq m)$ satisfies the assumption (A III) (Section 5). Furthermore the assumption (AIV) is fulfilled by virtue of the conditions (B III) and (BIV).

Concerning $\mathfrak{T}^{(1)}(\boldsymbol{M}_1)({}^1f,\cdots,{}^lf)$, $\mathfrak{T}^{[k]}(\boldsymbol{M}_1)({}^1f,\cdots,{}^lf)$ and $\mathfrak{T}^{(k)}(\boldsymbol{M}_2)({}^1f,\cdots,{}^lf)$, we define correspondences $\mathfrak{T}^{(1)}(\tilde{M}_1)(\pi_{m-1})\approx \tilde{M}_1\times Z$, $\mathfrak{T}^{[k]}(\tilde{M}_1)(\pi_{2(m-k)+1})\approx \tilde{M}_1\times Z$ and $\mathfrak{T}^{(k)}(\tilde{M}_2)(\pi_{2(m-k)+1})\approx \tilde{M}_2\times Z$ by the canonical generators (Borel-Hirzebruch [**2**, Appendix I]).

Moreover suppose that $\mathfrak{T}^{(k)}(\boldsymbol{M}_1)({}^1f,\cdots,{}^lf)$, $\mathfrak{T}^{[k]}(\boldsymbol{M}_1)({}^1f,\cdots,{}^lf)$ and $\mathfrak{T}^{(k)}(\boldsymbol{M}_2)({}^1f,\cdots,{}^lf)$ satisfy the assumption (A V), and that $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ satisfy the assumption (A VI). Then we have the primary obstruction classes

$$\bar{c}(\mathfrak{T}^{(k)}(\boldsymbol{M}_1)({}^1f,\cdots,{}^lf))\in H^{m-k+1}(M;\tilde{Z}),$$
$$\bar{c}(\mathfrak{T}^{[k]}(\boldsymbol{M}_1)({}^1f,\cdots,{}^lf))\in H^{2(m-k)+2}(M;Z),$$
$$\bar{c}(\mathfrak{T}^{(k)}(\boldsymbol{M}_2)({}^1f,\cdots,{}^lf))\in H^{2(m-k)+2}(M;Z)$$

respectively. Here again we suppose that the standard cross sections are given.

DEFINITION 7.4. $\bar{c}(\mathfrak{T}^{(k)}(\boldsymbol{M}_1)({}^1f,\cdots,{}^lf))$ $(\max_i(m-m_i)\leqq k\leqq m)$ will be called the $(m-k+1)$*-th SW-class* $SW_{m-k+1}(\boldsymbol{M}_1)$ of $\boldsymbol{M}_1$.

DEFINITION 7.5. $\bar{c}(\mathfrak{T}^{[k]}(\boldsymbol{M}_1)({}^1f,\cdots,{}^lf))$ $(\max_i k_i\leqq k\leqq m)$ will be called the $((m-k+1)/2)$*-th P-class* $P_{(m-k+1)/2}(\boldsymbol{M}_1)$ of $\boldsymbol{M}_1$.

DEFINITION 7.6. $\bar{c}(\mathfrak{T}^{(k)}(\boldsymbol{M}_2))({}^1f,\cdots,{}^lf)$ $(\max_i(m-m_i)\leqq k\leqq m)$ will be called the $(m-k+1)$*-th C-class* $C_{m-k+1}(\boldsymbol{M}_2)$ of $\boldsymbol{M}_2$.

In particular case where $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ become manifolds, SW-classes, P-classes and C-classes coincide with the Stiefel-Whitney classes, the Pontrjagin classes and the Chern classes of manifolds respectively.

The following Proposition is an immediate consequence of the definition of characteristic classes and Theorem 6.2.

PROPOSITION 7.1. *Let* $\boldsymbol{M}_1$ *and* $\boldsymbol{M}_1'$ (*resp.* $\boldsymbol{M}_2$ *and* $\boldsymbol{M}_2'$) *be isomorphic compact* C^∞*-M-spaces* (*resp. almost complex M-spaces*) *satisfying the assumptions* (B I)-(B IV), (***) (*or* (***)') *and the assumption* (A VI) *of Section* 5. *Then we have*

$$SW_k(\boldsymbol{M}_1)=SW_k(\boldsymbol{M}_1') \qquad (k=1,2,\cdots),$$
$$P_{k/2}(\boldsymbol{M}_1)=P_{k/2}(\boldsymbol{M}_1') \qquad (k=1,2,\cdots),$$
$$C_k(\boldsymbol{M}_2)=C_k(\boldsymbol{M}_2') \qquad (k=1,2,\cdots).$$

(*We suppose that the standard cross sections are suitably given.*)

8. The Euler-Poincaré characteristic.

Let $\boldsymbol{M}=\{M, \tilde{M}, \tilde{M}, G, \varphi\}$ be a compact m-dimensional C^∞-M-space such that M is the quotient space $\tilde{M}/G$ (i.e. $\bar{m}=\underline{m}=m$).

Suppose that $\boldsymbol{M}$ satisfies the assumptions (B I)–(B IV) of Section 7. Then $\boldsymbol{M}$ has an admissible cellular decomposition $\tilde{K}$ of $\tilde{M}$.

Let $\mathfrak{T}(\boldsymbol{M})=\{\boldsymbol{M}, \mathfrak{T}(\tilde{M}), E^m, SO(m)\}$ be the tangent D-bundle of $\boldsymbol{M}$ and $\mathfrak{T}^{(1)}(\boldsymbol{M})=\{\boldsymbol{M}, \mathfrak{T}^{(1)}(\tilde{M}), S^{m-1}, SO(m)\}$ the associated D-bundle of $\mathfrak{T}(\boldsymbol{M})$. Take the normal vector field with the outer direction on the boundary of NW_j $(j=1,\cdots,w)$ as the standard cross section f_N of $\mathfrak{T}^{(1)}(\boldsymbol{M})$, then f_N has an obvious geometrical meaning and its homotopy class is defined independently of the choice of the admissible cellular decomposition. Moreover, as is easily verified, the assumption (A VI) of Section 5 is always satisfied in this case.

Hence we obtain the following proposition:

Proposition 8.1. *If* $\boldsymbol{M}$ *is a compact quotient space* $\tilde{M}/G$ $(\dim \tilde{M}=m)$ *satisfying the assumptions* (B I)–(B IV) *of Section* 7, *then the m-th SW-class* $SW_m(M)$ *is always defined.*

Furthermore we assume that $\boldsymbol{M}$ is oriented and the set of all fixed points of $\tilde{M}$ under any $g\in G$ forms a submanifold of $\tilde{M}$ whose dimension is $\leqq m-2$. Then, as is easily verified, we have $H_m(M;Z)\approx Z$. $H_m(M;Z)$ has a generator $[M]$ satisfying $\varphi_*[\tilde{M}]=(\text{ord } G)\,[M]$, where $[\tilde{M}]$ is the fundamental homology class of $H_m(\tilde{M};Z)\approx Z$. $[M]$ will be called the *fundamental homology class* of M.

Definition 8.1. $SW_m(\boldsymbol{M})[M]\in Z$ will be called the *Euler-Poincaré characteristic of* $\boldsymbol{M}$, and will be denoted by $\chi(\boldsymbol{M})$.

Clearly $\chi(\boldsymbol{M})$ becomes the Euler-Poincaré characteristic in the usual sense in case where $\boldsymbol{M}$ is a C^∞-manifold.

Now we shall prove the following theorem which is a generalization of the well-known theorem on C^∞-manifolds.

Theorem 8.1. *Let* $\boldsymbol{M}=\tilde{M}/G$ *be a compact connected m-dimensional* C^∞-*quotient space satisfying above assumptions. Suppose that M has a simplicial decomposition K such that the set of singular points is a subcomplex of K and* $\varphi^{-1}(\sigma)$ $(\sigma\in K)$ *is differentiable. Then we have*

$$\chi(\boldsymbol{M})=\sum_{i=0}^{m}(-1)^i \dim H^i(M;R)\,.$$

Proof. K and φ give a simplicial decomposition $\hat{K}$ of $\tilde{M}$. A subcomplex $\hat{K}_j$ of $\hat{K}$ gives a simplicial decomposition of W_j. Let $'K, ''K$ denote the first and the second regular subdivisions of K respectively. φ and $'K$ (resp. φ and $''K$) give the first (resp. the second) regular subdivision $'\hat{K}$ (resp. $''\hat{K}$) of $\hat{K}$.

A vertex $''\hat{\sigma}^0$ of $''\hat{K}$ lies in the interior of just one simplex $\hat{\sigma}$ of $\hat{K}$. Denote by $b(''\hat{\sigma}^0)$ the barycenter of $\hat{\sigma}$. This vertex assignment b determines a unique simplicial map $b: ''\hat{K}\to '\hat{K}$ in an obvious manner. The fixed points of b are the barycenters of $\hat{K}$.

Now $\tilde{x}$ and $b(\tilde{x})$ $(\tilde{x}\in\tilde{M})$ lie on a single simplex of $'\hat{K}$, and are joined by a unique line segment of the simplex. Since $\hat{K}$ is differentiable, the segment has a tangent direction at each point. And we can define the unit tangent vector at $\tilde{x}$. We denote it by $f(\tilde{x})$. It follows that $f(\tilde{x})$ is defined and is continuous except at the barycenters of simplexes of $\hat{K}$. Obviously $f(\tilde{x})$ is G-invarient. For each q-simplex $'\hat{\sigma}^q$ of $'\hat{K}$, there is a dual $(m-q)$-cell $*'\hat{\sigma}^q$ which is the union of those simplexes of $''\hat{K}$ having the barycenter of $\hat{\sigma}^q$ as vertex of last order. The set of $*'\hat{\sigma}$ gives a cellular decomposition $*'\hat{K}$ of $\tilde{M}$. Clearly $*'\hat{K}$ is G-invariant. In a similar manner, we obtain a cellular decomposition $*'\hat{K}_j$ which is dual to $'\hat{K}_j$, where $'\hat{K}_j$ is the first regular subdivision of $\hat{K}_j$. By a suitable modification, we can regard $*'\hat{K}$ as an admissible cellular decomposition $\tilde{K}$ of $\tilde{M}$. The singularities of f occur at the centers of the m-celles of $\tilde{K}$. Obviously f is (homotopic to) the normal vector field with outer direction on $\sum_j {}_N\tilde{K}_j\cap\tilde{K}_C$. That is to say, f is an extension of the standard cross section f_N.

Clearly we have

$$\{c'(f)\}\,[\varphi(\tilde{K}_C)]=\sum_{\sigma\in K-\varphi(\sum_j \hat{K}_j)}(-1)^{\dim\sigma}\,, \tag{8.1}$$

where $[\varphi(\tilde{K}_C)]$ is the fundamental homology class of $H_m(\varphi(\tilde{K}_C), \varphi(\sum_j {}_N\tilde{K}_j\cap\tilde{K}_C); Z)\approx Z$.

Moreover we have the commutative diagram:

$$\begin{array}{ccc} H^{n_j}(\tilde{K}_j; Z)\otimes H^{m-n_j}(\Sigma^{m-n_j}, S^{m-n_j-1}; Z) & \xrightarrow{i_\infty{}^*\circ\kappa_\infty{}^2} & H^m({}_N\tilde{K}_j, {}_N\tilde{K}_j\cap\tilde{K}_C; Z) \\ \uparrow id.\otimes\varphi^* & & \uparrow\varphi^* \\ H^{n_j}(\tilde{K}_j; Z)\otimes H^{m-n_j}(\Sigma^{m-n_j}/G, S^{m-n_j-1}/G; Z) & \xrightarrow{i_\infty{}^*\circ{}^{\varphi}\kappa_\infty{}^2} & H^m(\varphi(|{}_N\tilde{K}_j|), \varphi(|{}_N\tilde{K}_j\cap\tilde{K}_C|); Z) \end{array}$$

Therefore we have

$$\begin{aligned} \{c_j(f)\}\,[\varphi({}_N\tilde{K}_j)] &= \frac{1}{\operatorname{ord}(G/G_j)}\,(\varphi^*\{c_j(f)\})\,[{}_N\tilde{K}_j] \\ &= \frac{1}{\operatorname{ord}(G/G_j)}\,((id.\otimes\varphi^*)(\{\tilde{c}_j(f)\}\otimes\{\Sigma^{m-n_j}/G\}))([\tilde{K}_j]\otimes[\Sigma^{m-n_j}]) \\ &= \{\tilde{c}_j(f)\}\,[\tilde{K}_j] \\ &= \sum_{\sigma\in\hat{K}_j}(-1)^{\dim\sigma}\,, \end{aligned} \tag{8.2}$$

where $[\varphi({}_N\tilde{K}_j)]$, $[{}_N\tilde{K}_j]$ and $[\tilde{K}_j]$ denote the fundamental homology classes of $H_m(\varphi(|{}_N\tilde{K}_j|), \varphi(|{}_N\tilde{K}_j \cap \tilde{K}_C|); Z) \approx Z$, $H_m({}_N\tilde{K}_j, {}_N\tilde{K}_j \cap \tilde{K}_C; Z) \approx Z$ and $H_{n_j}(\tilde{K}_j; Z) \approx Z$ respectively, and G_j denotes the subgroup of G whose elements act on NW_j as identity.

(8.1) and (8.2) enable us to compute $\chi(\boldsymbol{M})$, i.e.

$$\begin{aligned}\{c(f)\}[M] &= (\iota^*(\{c'(f)\} \oplus \sum_j \{c_j(f)\}))[M] \\ &= (\{c'(f)\} \oplus \sum_j \{c_j(f)\})(\iota_*[M]) \\ &= \sum_{\sigma \in K} (-1)^{\dim \sigma} = \sum_{i=0}^{m} (-1)^i \dim H^i(M; R) \qquad \text{q.e.d.}\end{aligned}$$

Let $\boldsymbol{M}$ be a compact $2m$-dimensional almost complex M-space. Suppose that C^∞-M-space $\boldsymbol{M}'$ induced from $\boldsymbol{M}$ satisfies the above assumptions. Then by the standard argument, we have

$$C_m(\boldsymbol{M})[M] = SW_{2m}(\boldsymbol{M}')[M']. \tag{8.3}$$

The following theorem is an immediate consequence of Theorem 8.1 and (8.3).

THEOREM 8.2. *Let $\boldsymbol{M} = \tilde{M}/G$ be a compact connected m-dimensional C^∞-quotient space satisfying the same assumptions of Theorem 8.1 (resp. a compact connected $2m$-dimensional almost complex quotient space). Then the Euler-Poincaré characteristic of $\boldsymbol{M}$ and the m-th SW-class (resp. the m-th C-class) of $\boldsymbol{M}$ are homotopy type invariants of M.*

EXAMPLE 8.1.

$$\chi(\boldsymbol{S}^m(1/n)) = \begin{cases} 0 & m: \text{odd} \\ 2 & m: \text{even}. \end{cases}$$

REMARK 8.1. I. Satake [4] introduced the Euler-Poincaré characteristic $\chi_V(M)$ for V-manifolds M. $\chi_V(\boldsymbol{M})$ is different from $\chi(\boldsymbol{M})$. $\chi_V(\boldsymbol{M})$ is not necessarily an integer and depends on the V-differentiable structure of $\boldsymbol{M}$, and is not topological invariant.

University of Tokyo.

Bibliography

[1] W. L. Baily, The decomposition theorem for V-manifolds, Amer. J. Math., **78** (1956), 862-888.

[2] A. Borel and F. Hirzebruch, Characteristic classes and homogeneous spaces II, Amer. J. Math., **81** (1959), 315-382.

[3] F. Hirzebruch, Neue topologische Methoden in der algebraischen Geometrie, Ergebnisse der Mathematik, 1956.

[4] I. Satake, The Gauss-Bonnet Theorem for *V*-manifolds, J. Math. Soc. Japan, **9** (1957), 464-492.
[5] N. Steenrod, The topology of fibre bundles, Princeton, 1951.
[6] I. Tamura, A diffeomorphy invariant of quotient manifolds, J. Math. Soc. Japan, **11** (1959), 343-353.
[7] I. Tamura, Characteristic classes of 2-fold symmetric products of spheres (to appear in J. Math. Soc. Japan).