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Characteristic classes of 2-fold symmetric products of spheres.

Dedicated to Professor Z. Suetuna on his 60th birthday.

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In this note we shall compute characteristic classes of $\boldsymbol{S}^2*\boldsymbol{S}^2$, $\boldsymbol{S}^4*\boldsymbol{S}^4$ and $\boldsymbol{S}^8*\boldsymbol{S}^8$ in the sense of my previous paper [3]. The notation and terminology of this note are the same as in [3].

1. Cohomology groups of $\boldsymbol{S}^n*\boldsymbol{S}^n$.

Let $\boldsymbol{S}^n*\boldsymbol{S}^n=\{S^n*S^n, S^n\times S^n, S^n\times S^n, \varphi, G\}$ be the $2n$-dimensional C^∞-M-space of the symmetric product of S^n mentioned in [3, Example 1.3].

In case where n is even, non-trivial cohomology groups of S^n*S^n are as follows (Liao [1]):

$$H^q(S^n*S^n; Z)\approx\begin{cases} Z & q=0, n, 2n; \\ Z_2 & q=n+2k+1 \qquad (0<k\leqq[(n-1)/2]). \end{cases}$$

Thus S^n*S^n is not topological manifold unless $n=2$, because the Poincaré duality relation does not hold for S^n*S^n. Let i' be the injection $S^n\to S^n\times S^n$ defined by

$$i'(x)=(x, x_0),$$

where x_0 is a point of S^n. Then a generator $\mu_n{}^{(n)}$ of $H^n(S^n*S^n; Z)\approx Z$ is given by

$$i'^*\varphi^*\mu_n{}^{(n)}=\{S^n\}, \tag{1.1}$$

where $\{S^n\}$ is the generator of $H^n(S^n; Z)\approx Z$ determined by the orientation of S^n. Furthermore we denote the generator of $H^{2n}(S^n*S^n; Z)\approx Z$ determined by the orientation of S^n*S^n by $\mu_{2n}{}^{(n)}$.

Now let $S_d{}^n$ be the diagonal sphere of $S^n\times S^n$ and let $NS_d{}^n$ be a sufficiently small (closed) normal tube neighbourhood of $S_d{}^n$ in $S^n\times S^n$. Let $\mathfrak{N}(S_d{}^n)=\{NS_d{}^n, p, S_d{}^n, \Sigma^n, GL(n; R)\}$ be the normal bundle. ($\mathfrak{N}(S_d{}^n)$ is equivalent to the tangent bundle of S^n.) We consider fibre bundles $\mathfrak{N}^\circ(S_d{}^n)=\{\mathrm{Int}\, NS_d{}^n, p, S_d{}^n, \mathrm{Int}\,\Sigma^n, GL(n; R)\}$, $\varphi(\mathfrak{N}^\circ(S_d{}^n))=\{\varphi(\mathrm{Int}\, NS_d{}^n), p, S_d{}^n, \mathrm{Int}(\Sigma^n/G), \Gamma\}$ and their spectral sequences [3, Section 5].

Since Σ^n/G is contractible, we obtain

$$H_{\mathcal{K}}{}^n(\mathrm{Int}(\Sigma^n/G);Z)\approx Z,$$
$$H_{\mathcal{K}}{}^1(\mathrm{Int}(\Sigma^n/G);Z)=0.$$

Therefore, we have ${}^{\varphi}d_s{}^{\varphi}\kappa_s{}^{2\varphi}e_2=0$ $(2\leqq s<\infty)$ for any element ${}^{\varphi}e_2$ of $H^q(S_d{}^n;Z)\otimes H_{\mathcal{K}}{}^n(\mathrm{Int}(\Sigma^n/G);Z)$. Moreover, as is easily verified, we have

$${}^{\varphi}J^{q+1,n-1}=0 \qquad (q=0,n),$$
$$i_\infty{}^*({}^{\varphi}E_\infty{}^{q,n})=H_{\mathcal{K}}{}^{q+n}(\varphi(\mathrm{Int}\,NS_d{}^n);Z) \qquad (q=0,n).$$

Hence we obtain the following lemma:

LEMMA 1.1. *S^n*S^n satisfies the assumption* (AVI) *of* [**3**, *Section* 5] *for* $r=n, 2n$.

The following diagram is commutative $(q=0,n)$:

$$\begin{array}{ccc} E_2{}^{q,n}=H^q(S_d{}^n;Z)\otimes H_{\mathcal{K}}{}^n(\mathrm{Int}\,\Sigma^n;Z) & \xrightarrow{i_\infty{}^*\kappa_\infty{}^2} & H_{\mathcal{K}}{}^{q+n}(\mathrm{Int}\,NS_d{}^n;Z) \\ \uparrow id.\otimes\varphi^* & & \uparrow \varphi^* \\ {}^{\varphi}E_2{}^{q,n}=H^q(S_d{}^n;Z)\otimes H_{\mathcal{K}}{}^n(\mathrm{Int}\,(\Sigma^n/G);Z) & \xrightarrow{i_\infty{}^{*\varphi}\kappa_\infty{}^2} & H_{\mathcal{K}}{}^{q+n}(\varphi(\mathrm{Int}\,NS_d{}^n);Z). \end{array} \tag{1.2}$$

Denote by $\mu_d{}^{(0)}$ the generator of $H^0(S_d{}^n;Z)\approx Z$. Then we have

$$i'^*\tilde{\iota}_1{}^*i_\infty{}^*\kappa_\infty{}^2(\mu_d{}^{(0)}\otimes\{\mathrm{Int}\,\Sigma^n\})=\{S^n\},$$

where $\tilde{\iota}_1$ denotes the map $S^n\times S^n\to(S^n\times S^n,\partial NS_d{}^n)$. Hence we obtain by (1.1) and (1.2)

$$\iota_1{}^*i_\infty{}^*\kappa_\infty{}^2(\mu_d{}^{(0)}\otimes\{\mathrm{Int}(\Sigma^n/G)\})=2\mu_n{}^{(n)}, \tag{1.3}$$

where ι_1 denotes the map $S^n*S^n\to(S^n*S^n,\varphi(\partial NS_d{}^n))$.

Furthermore the tangent D-bundle [**3**, Definition 3.3] of $\boldsymbol{S}^n*\boldsymbol{S}^n$ is given as the following collection (i), (ii):

(i) The tangent bundle $\mathfrak{T}(S^n\times S^n)=\{T(S^n\times S^n),p,S^n\times S^n,E^{2n},SO(2n;R)\}$.

(ii) An isomorphism ${}^1\alpha$ of G into the group of bundle maps of $\mathfrak{T}(S^n\times S^n)$ defined by ${}^1\alpha(g)=dg$.

2. C-classes of S^2*S^2.

Let us regard S^2 as the space of 2 homogeneous complex variables $[\alpha,\beta]$. Since $S^2\times S^2$ has the complex analytic structure and G operates analytically on $S^2\times S^2$, $\boldsymbol{S}^2*\boldsymbol{S}^2=\{S^2*S^2,S^2\times S^2,S^2\times S^2,G,\varphi\}$ is a 2-dimensional complex analytic M-space.

Consider the complex projective plane CP^2 as the space of 3 homogeneous complex variables $[\alpha,\beta,\gamma]$. We define the analytic map $\tilde{h}_0$ of $S^2\times S^2$ onto CP^2 by

$$\tilde{h}_0([\alpha,\beta],[\alpha',\beta'])=[\alpha\alpha',\alpha\beta'+\alpha'\beta,\beta\beta'].$$

Then $\tilde{h}_0$ gives the homeomorphic map h_0 of $S^2 * S^2$ onto CP^2 such that

$$\tilde{h}_0 = h_0 \circ \varphi .$$

Therefore $S^2 * S^2$ and CP^2 are homeomorphic.

Let $K = \{\sigma^0, \sigma^1, \sigma_1{}^2, \sigma_2{}^2\}$ be a cellular decomposition of S^2 such that

$$\sigma^0 = [1, 1], \quad \sigma^1 = \{[1, \beta]; \ |\beta| = 1\},$$
$$\sigma_1{}^2 = \{[1, \beta]; \ |\beta| \leqq 1\}, \ \sigma_2{}^2 = \{[\alpha, 1]; \ |\alpha| \leqq 1\} .$$

Then $K \times K$ gives a cellular decomposition of $S^2 \times S^2$.

Since $S^2 - [0, 1] = \{[1, \beta]\}$ (resp. $S^2 - [1, 0] = \{[\alpha, 1]\}$) is diffeomorphic to the complex line (β) (resp. (α)), there exists a continuous orthonormal vector field $\{V_{[1,\beta]}\}$ on $S^2 - [0, 1]$ (resp. $\{'V_{[\alpha,1]}\}$ on $S^2 - [1, 0]$) such that each $V_{[1,\beta]}$ (resp. $'V_{[\alpha,1]}$) is the vector at $[1, \beta]$ (resp. $[\alpha, 1]$) parallel to the vector $\overrightarrow{0\,1} = V_e$ in (β) (resp. (α)). Denote by $(\beta/|\beta|) V_{[1,\beta]}$ $(\beta \neq 0)$ the vector at $[1, \beta]$ which is parallel to $(\beta/|\beta|) V_e$, where the multiplication of $\beta/|\beta|$ means the orthogonal transformation determined by $\beta/|\beta|$. It is easily shown that

$$(\beta/|\beta|) V_{[1,\beta]} = -(|\beta|/\beta)' V_{[1/\beta,1]} \qquad (\beta \neq 0) . \tag{2.1}$$

Now let K_d be a cellular decomposition of $S_d{}^2$ consisting of $\{\sigma_d{}^0, \sigma_d{}^1, \sigma_{d1}{}^2, \sigma_{d2}{}^2\}$ such that

$$\sigma_d{}^0 = (\sigma^0, \sigma^0), \quad \sigma_d{}^1 = \{(x, x); \ x \in \sigma^1\}, \quad \sigma_{d1}{}^2 = \{(x, x); \ x \in \sigma_1{}^2\},$$
$$\sigma_{d2}{}^2 = \{(x, x); \ x \in \sigma_2{}^2\} .$$

Then $\{p^{-1}(\sigma_d{}^0), p^{-1}(\sigma_d{}^1), p^{-1}(\sigma_{d1}{}^2), p^{-1}(\sigma_{d2}{}^2)\}$ gives a cellular decomposition of $\mathrm{Int}\, NS_d{}^2$. Denote by $(\sigma \times \sigma')^{\sim}$ $(\sigma, \sigma' \in K)$ the set $\sigma \times \sigma' - \mathrm{Int}\, NS_d{}^2$. Then the collection of $p^{-1}(\sigma_d)$ $(\sigma_d \in K_d)$ and suitable cellular subdivisions of $(\sigma \times \sigma')^{\sim}$ $(\sigma, \sigma' \in K)$ define an admissible cellular decomposition $\tilde{K}$ of $S^2 \times S^2$. We can assume without loss of generality that

$$|\tilde{K}^2| - \mathrm{Int}\, NS_d{}^2 - (\sigma_2{}^2 \times \sigma^0) - (\sigma^0 \times \sigma_2{}^2) \subset (S^2 - [0, \beta]) \times (S^2 - [0, \beta]) . \tag{2.2}$$

Denote the subcomplex of $\tilde{K}$ which gives a cellular decomposition of $NS_d{}^2$ (resp. $S^2 \times S^2 - \mathrm{Int}\, NS_d{}^2$) by ${}_N\tilde{K}$ (resp. $\tilde{K}_C$).

In the following let us denote by V_N the normalized vector of non-zero vector V.

First we compute the first C-class of $\boldsymbol{S^2 * S^2}$.

Let $\mathfrak{v}_1$ be the continuous field of normalized normal vectors with the outer direction defined on ${}_N\tilde{K} \cap \tilde{K}_C{}^2$. We can regard $\mathfrak{v}_1$ as an extension of the set of vectors

$$\{((\beta - \beta') V_{[1,\beta]}, (\beta' - \beta) V_{[1,\beta']})_N ; \quad ([1, \beta], [1, \beta']) \in \partial p^{-1}(S_d{}^2 - U([0, \beta]))\} ,$$

where $U([0, \beta])$ is a sufficiently small neighbourhood of $[0, \beta]$. Let $\mathfrak{v}_2$ be the continuous field of normalized vectors defined on ${}_N\tilde{K}\cap\tilde{K}_C{}^2$ which is the restriction of the set of vectors

$$\{(V_{[1,\beta]},\ V_{[1,\beta']})_N;\quad ([1,\beta],[1,\beta'])\in(S^2-U([0,\beta]))\times(S^2-U([0,\beta]))\}\,.$$

$\mathfrak{v}_1$ and $\mathfrak{v}_2$ are two non-zero mutually orthogonal vector fields, and define a continuous field of orthonormal 2-frames $f^{(2)}$ on ${}_N\tilde{K}\cap\tilde{K}_C^2$. Obviously $f^{(2)}$ is invariant under dg, that is, $f^{(2)}$ is a G-cross section of $\mathfrak{T}^{(2)}(S^2\times S^2)$ on ${}_N\tilde{K}\cap\tilde{K}_C^2$. We take $f^{(2)}$ as the standard cross section of $\mathfrak{T}^{(2)}(\boldsymbol{S}^2*\boldsymbol{S}^2)$. The continuous field of normalized vectors defined on $(S^2-U([0,\beta]))\times(S^2-U([0,\beta]))-\mathrm{Int}\,NS_d{}^2$

$$\bar{\mathfrak{v}}_1=\{((\beta-\beta')V_{[1,\beta]},\ (\beta'-\beta)V_{[1,\beta']})_N;\ ([1,\beta],[1,\beta'])\in$$
$$(S^2-U([0,\beta]))\times(S^2-U([0,\beta]))-\mathrm{Int}\,NS_d{}^2\}$$

is an extension of $\mathfrak{v}_1$. The continuous field of normalized vectors defined on $(S^2-U([0,\beta]))\times(S^2-U([0,\beta]))$

$$\bar{\mathfrak{v}}_2=\{(V_{[1,\beta]},\ V_{[1,\beta']})_N;\ ([1,\beta],[1,\beta'])\in(S^2-U([0,\beta]))\times(S^2-U([0,\beta]))\}$$

is an extension of $\mathfrak{v}_2$. The vectors of $\bar{\mathfrak{v}}_1$ and $\bar{\mathfrak{v}}_2$ are mutually orthogonal at each point of $(S^2-U[0,\beta])\times(S^2-U[0,\beta])-\mathrm{Int}\,NS_d{}^2$. Therefore the restrictions of $\bar{\mathfrak{v}}_1$ and $\bar{\mathfrak{v}}_2$ on $\tilde{K}_C{}^1$ define a continuous field $\bar{f}^{(2)}$ of orthonormal 2-frames which is invariant under dg.

Let us consider the obstruction $\tilde{c}(\bar{f}^{(2)})$. By (2.1) the restrictions of $\bar{\mathfrak{v}}_1,\bar{\mathfrak{v}}_2$ on $(\sigma^1\times\sigma^1)^\sim$ are

$$\{((1/\alpha')-(1/\alpha))\alpha^{2\,\prime}V_{[\alpha,1]},\ ((1/\alpha)-(1/\alpha'))\alpha'^{2\,\prime}V_{[\alpha',1]})_N;$$
$$([\alpha,1],[\alpha',1])\in(\sigma^1\times\sigma^1)^\sim\}\,,$$
$$\{(-\alpha^{2\,\prime}V_{[\alpha,1]},\ -\alpha'^{2\,\prime}V_{[\alpha',1]})_N;\ ([\alpha,1],[\alpha',1])\in(\sigma^1\times\sigma^1)^\sim\}\,.$$

Let f_0 be the map $(\sigma^1\times\sigma^0)^\sim\to U(2)$ defined by

$$f(\alpha,\sigma^0)=\begin{pmatrix}\alpha^2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}(1-(1/\alpha))/|\sqrt{2}-(\sqrt{2}/\alpha)| & 1/\sqrt{2}\\ ((1/\alpha)-1)/|\sqrt{2}-(\sqrt{2}/\alpha)| & 1/\sqrt{2}\end{pmatrix}.$$

The map $(\sigma^1\times\sigma^0)^\sim\to|f(\alpha,\sigma^0)|=\alpha^2(1-(1/\alpha))/|1-(1/\alpha)|$ is homotopic to a generator of $\pi_1(U(2))$, because the maps from $\partial(\sigma^1\times\sigma^0)^\sim$ (Fig. 1) into S^1 defined by

$$(\alpha,1)\to\alpha^2\,,$$
$$(\alpha,1)\to(1-(1/\alpha))/|1-(1/\alpha)|$$

have degree 2, -1 respectively. Thus we have

$$\tilde{c}(\bar{f}^{(2)})((\sigma_2{}^2\times\sigma^0)^\sim)=\tilde{c}(\bar{f}^{(2)})((\sigma^0\times\sigma_2{}^2)^\sim)=1\,.$$

Moreover it is obvious by (2.2) that

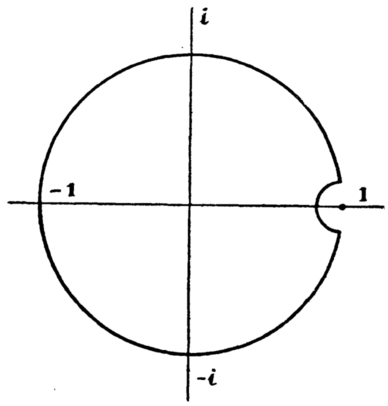

Fig. 1.

$$\tilde{c}(\bar{f}^{(2)})(\tilde{\sigma})=0 \qquad (\tilde{\sigma}\in\tilde{K}_C^2,\ \tilde{\sigma}\neq(\sigma_2{}^2\times\sigma^0)^\sim,\ (\sigma^0\times\sigma_2{}^2)^\sim)$$

Therefore, as is easily verified, the cohomology class $\{c'(f)\}$ ([**3**, Section 5]) satisfies

$$i'^*\tilde{\iota}_1{}^*\varphi^*(\{c'(f)\})=\{S^2\}\,.$$

Using (1.1), we obtain

$$\iota_1{}^*(\{c'(f)\})=\mu_2{}^{(2)}\,. \qquad (2.3)$$

On the other hand, we have

$$\tilde{c}(\bar{f}^{(2)})(p^{-1}(\sigma_d{}^0))=1\,.$$

Denote $\{c_j(f)\}$ in [**3**, Section 5] simply by $\{c_N(f)\}$. Then we have by (1.3)

$$\iota^*(\{c_N(\bar{f}^{(2)})\})=2\mu_2{}^{(2)}\,.$$

Therefore $\bar{c}(\mathfrak{T}^{(2)}(\boldsymbol{S}^2*\boldsymbol{S}^2))$ is given by

$$\bar{c}(\mathfrak{T}^{(2)}(\boldsymbol{S}^2*\boldsymbol{S}^2))=\{c(\bar{f}^{(2)})\}=3\mu_2{}^{(2)}\,.$$

Furthermore the second C-class of $\boldsymbol{S}^2*\boldsymbol{S}^2$ is $3\mu_4{}^{(2)}$ by [**3**, Theorem 8.1].

Hence we obtain the following theorem:

THEOREM 2.1. *C-classes of the 2-dimensional complex analytic M-space $\boldsymbol{S}^2*\boldsymbol{S}^2$ are as follows:*

$$C_1(\boldsymbol{S}^2*\boldsymbol{S}^2)=3\mu_2{}^{(2)}\,,$$

$$C_2(\boldsymbol{S}^2*\boldsymbol{S}^2)=3\mu_4{}^{(2)}\,.$$

REMARK 2.1. In a forthcoming paper [**4**] this theorem will be reproved by considering the invariance of characteristic classes of M-spaces under the isomorphic in the wider sense.

3. P-classes of $\boldsymbol{S}^4*\boldsymbol{S}^4$, $\boldsymbol{S}^8*\boldsymbol{S}^8$.

First, we shall compute P-classes of $\boldsymbol{S}^4*\boldsymbol{S}^4$.

Let o and o' be two poles of S^4. We can regard the open set S^4-o' (resp. S^4-o) of S^4 as the set of quaternions $\{(q);\ \|q\|<1\}$ (resp. $\{(q)';\ \|q\|<1\}$). By means of the identification (q) and $((1-\|q\|)q/\|q\|)'$ $(q\neq 0)$, we obtain the natural differentiable structure of S^4.

Let $K=\{\sigma^0,\sigma^3,\sigma_1{}^4,\sigma_2{}^4\}$ be a cellular decomposition of S^4 consisting of cells

$$\sigma^0=(1/2),\quad \sigma^3=\{(q);\ \|q\|=1/2\}\,,$$

$$\sigma_1{}^4=\{(q);\ \|q\|\leqq 1/2\},\quad \sigma_2{}^4=\{\{(q)';\ \|q\|\leqq 1/2\}\,.$$

The quaternion space (q) (resp. $(q)'$) has a continuous field of normalized vectors $\{V_1(q)\}$ (resp. $\{V_1{}'(q)\}$), where $V_1(q)$ (resp. $V_1{}'(q)$) is the parallel translation of the normalized vectors at the origin defined by real axis of (q) (resp. $(q)'$).

It is obvious that

$$V_1(q)=-V_1'((1-\|q\|)q/\|q\|) \qquad (q\neq 0)\,. \tag{3.1}$$

Since the first Pontrjagin class of S^4 is zero, there exists a continuous field of unitary 4-frames $\{(V_1^c(x), V_2^c(x), V_3^c(x), V_4^c(x));\, x\in S^4\}$, i.e., a cross section of the fibre bundle $\mathfrak{T}^{[4]}(S^4)$.

Now let $K_d=\{\sigma_d{}^0, \sigma_d{}^3, \sigma_{d1}{}^4, \sigma_{d2}{}^4\}$ be a cellular decomposition of $S_d{}^4$ such that

$$\sigma_d{}^0=(\sigma^0,\sigma^0),\quad \sigma_d{}^3=\{(x,x);\, x\in\sigma^3\}\,,$$
$$\sigma_{d1}{}^4=\{(x,x);\, x\in\sigma_1{}^4\},\quad \sigma_{d2}{}^4=\{(x,x);\, x\in\sigma_2{}^4\}\,.$$

Then $p^{-1}(K_d)$ gives a cellular decomposition of $\mathrm{Int}\, NS_d{}^4$. Denote by $(\sigma\times\sigma')^\sim$ $(\sigma,\sigma'\in K)$ the set $\sigma\times\sigma'-\mathrm{Int}\, S_d{}^4$. Then the collection of $p^{-1}(\sigma_d)$ $(\sigma_d\in K_d)$ and suitable cellular subdivisions of $(\sigma\times\sigma')^\sim$ $(\sigma,\sigma'\in K)$ define an admissible cellular decomposition $\tilde{K}$ of $S^4\times S^4$. We can assume without loss of generality that

$$|\tilde{K}^4|-\mathrm{Int}\, NS_d{}^4-(\sigma_2{}^4\times\sigma^0)-(\sigma^0\times\sigma_2{}^4)\subset(S^4-o')\times(S^4-o')\,. \tag{3.2}$$

We denote the subcomplex of $\tilde{K}$ which gives a cellular decompotion of $NS_d{}^2$ (resp. $S^4\times S^4-\mathrm{Int}\, NS_d{}^4$) by ${}_N\tilde{K}$ (resp. $\tilde{K}_C$).

$$\mathfrak{v}^c=\{(V_1^c(x), V_1^c(x'))_N, (V_2^c(x), V_2^c(x'))_N, (V_3^c(x), V_3^c(x'))_N, (V_4^c(x), V_4^c(x'))_N;$$
$$(x,x')\in S^4\times S^4\}$$

is a continuous field of unitary 4-frames. Also

$$\mathfrak{v}_1{}^c=\{((q-q')V_1^c((q)), (q'-q)V_1^c((q')))_N, ((q-q')V_2^c((q)), (q'-q)V_2^c((q')))_N\,,$$
$$((q-q')V_3^c((q)), (q'-q)V_3^c((q')))_N;\, ((q),(q'))\in(S^4-o')\times(S^4-o')-S_d{}^4\}$$

is a continuous field of unitary 3-frames defined on $(S^4-o')\times(S^4-o')-S_d{}^4$, where the left multiplication of $q-q'$ means the transformation determined by $q-q'$ in an obvious way. Both $\mathfrak{v}^c$ and $\mathfrak{v}_1{}^c$ are invariant under the operation of dg. Hence $\mathfrak{v}^c$ and $\mathfrak{v}_1{}^c$ define a G-cross section $f^{[7]}$ of $\mathfrak{T}^{[7]}$ $(S^4\times S^4)$ over $(S^4-o')\times(S^4-o')-S_d{}^4$. We take the restriction of $f^{[7]}$ on ${}_N\tilde{K}\cap\tilde{K}_C{}^4$ as the standard cross section of $\mathfrak{T}^{[7]}$ $(\boldsymbol{S}^4*\boldsymbol{S}^4)$.

Now we compute $\tilde{c}(f^{[7]})$. Obviously we have by (3.2)

$$\tilde{c}(f^{[7]})(\tilde{\sigma})=0\ \ (\tilde{\sigma}\in\tilde{K}^4,\ \tilde{\sigma}\neq(\sigma_2{}^4\times\sigma^0)^\sim, (\sigma^0\times\sigma_2{}^4)^\sim)\,. \tag{3.3}$$

Furthermore $f^{[7]}$ defines the map $f_1{}^{[7]}:\partial((\sigma_2{}^4\times\sigma^0)^\sim)\to U(7)$ such that

$$f_1{}^{[7]}((q)', (1/2))=j_4(Q_1)\,,$$
$$Q_1\in SO(4),\ Q_1(q')=((((1-\|q\|)q/\|q\|)-(1/2))_N)q'\,,$$

where $j_4: SO(4)\to U(7)$ is the composition of inclusion maps $SO(4)\to U(4)$ and $U(4)\to U(7)$. Hence we obtain (Tamura [**2**])

$$\tilde{c}(f^{[7]})((\sigma_2{}^4\times\sigma^0)^\sim)=\tilde{c}(f^{[7]})((\sigma^0\times\sigma_2{}^4)^\sim)=-2\,. \tag{3.4}$$

On the other hand, since $f^{[7]}$ defines the map $f_2^{[7]}: \partial p^{-1}(\sigma_d^0) \to U(7)$ such that

$$f_2^{[7]}(q) = j_4(Q_2),\ Q_2 \in SO(4),\ Q_2(q') = qq',$$

we have

$$\tilde{c}(f^{[7]})(p^{-1}(\sigma_d^0)) = 2. \tag{3.5}$$

Therefore, by (1.3), (3.3), (3.4) and (3.5), we have

$$\tilde{c}(\mathfrak{T}^{[7]}(\boldsymbol{S}^4 * \boldsymbol{S}^4)) = \{\tilde{c}(f^{[7]})\} = 2\mu_4^{(4)}.$$

Next we consider the second P-class of $\boldsymbol{S}^4 * \boldsymbol{S}^4$. Let

$$\begin{aligned}\mathfrak{v}_1 = \{&((1-\|q_1\|)(1-\|q_2\|)(q_1-q_2)V_1(q_1)+J(1-\|q_1'\|)(1-\|q_2'\|)(q_1'-q_2')V_1'(q_1'),\\ &(1-\|q_1\|)(1-\|q_2\|)(q_2-q_1)V_1(q_2)+J(1-\|q_1'\|)(1-\|q_2'\|)(q_2'-q_1')V_1'(q_2'))_N;\\ &(q_1)=(q_1')',\ (q_2)=(q_2')'\}\end{aligned}$$

be a continuous field of complex vectors defined on $S^4 \times S^4 - (S_d^4 \cup (o, o') \cup (o', o))$. As is easily verified, $\mathfrak{v}^c$ and $\mathfrak{v}_1$ define a continuous field of unitary 5-frames, which is invariant under dg, i.e., a G-cross section $f^{[5]}$ of $\mathfrak{T}^{[5]}(S^4 \times S^4)$ over $S^4 \times S^4 - (S_d^4 \cup (o, o') \cup (o', o))$. We take the restriction of $f^{[5]}$ on ${}_N\tilde{K} \cap \tilde{K}_C$ as the standard cross section.

Let us compute $\{c(f^{[5]})\}$. Let $U(o, o')$ be a sufficiently small ball containing (o, o') whose orientation agrees with that of $S^4 \times S^4$. Let $f_1^{[5]}$: $\partial U(o, o') \to S^7$ be the map defined by

$$f((q_1, q_2')) = ((1-\|q_1\|)(\|q_2'\|q_1-(1-\|q_2'\|)q_2'),\ (1-\|q_2'\|)((1-\|q_1\|)q_1-\|q_1\|q_2'))_N.$$

Since, as is easily verified, $f_1^{[5]}$ has degree 1, we have

$$\tilde{c}(f^{[5]})((\sigma_1^4 \times \sigma_2^4)^\sim) = -1. \tag{3.6}$$

By the way, we have

$$\begin{aligned}&\tilde{c}(f^{[5]})((\sigma_1^4 \times \sigma_2^4)^\sim)+\tilde{c}(f^{[5]})((\sigma_2^4 \times \sigma_1^4)^\sim)\\ &\quad +\tilde{c}(f^{[5]})(p^{-1}(\sigma_{d1}^4))+\tilde{c}(f^{[5]})(p^{-1}(\sigma_{d2}^4)) = 0,\end{aligned} \tag{3.7}$$

because the Pontrjagin classes of $S^4 \times S^4$ vanish.

(1.2), (3.6) and (3.7) enable us to compute $\{c(f^{[5]})\}$:

$$\begin{aligned}\{c(f^{[5]})\}[S^4 * S^4] &= \tilde{c}(f^{[5]})((\sigma_1^4 \times \sigma_2^4)^\sim)+\tilde{c}(f^{[5]})(p^{-1}(\sigma_{d1}^4))+\tilde{c}(f^{[5]})(p^{-1}(\sigma_{d2}^4))\\ &= -\tilde{c}(f^{[5]})((\sigma_1^4 \times \sigma_2^4)^\sim) = 1.\end{aligned}$$

Hence we obtain the following theorem:

THEOREM 3.1. *P-classes of the 8-dimensional C^∞-M-space $\boldsymbol{S}^4 * \boldsymbol{S}^4$ are as follows:*

$$P_1(\boldsymbol{S}^4 * \boldsymbol{S}^4) = 2\mu_4^{(4)},$$
$$P_2(\boldsymbol{S}^4 * \boldsymbol{S}^4) = \mu_8^{(4)}.$$

In a similar way we obtain the following theorem, making use of the Cayley numbers (Tamura [**2**]):

THEOREM 3.2. *P-classes of the* 16-*dimensional* $C^{\times}$-*M-space* $S^8 * S^8$ *are as follows:*

$$P_2(S^8 * S^8) = 6\mu_8{}^{(8)},$$
$$P_4(S^8 * S^8) = \mu_{16}{}^{(8)}.$$

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Bibliography

[1] S. D. Liao, On the topology of cyclic products of spheres, Trans. Amer. Math. Soc., **77** (1954), 520-551.
[2] I. Tamura, On Pontrjagin classes and homotopy types of manifolds, J. Math. Soc. Japan, **9** (1957), 250-262.
[3] I. Tamura, Characteristic classes of *M*-spaces I, J. Math. Soc. Japan, **11** (1959), 312-342.
[4] I. Tamura, Characteristic classes of *M*-spaces II (to appear).