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On the recursive functions of ordinal numbers.

Dedicated to Professor Z. Suetuna for his 60th birthday.

By

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In a former paper [**6**], the author developed a theory of ordinal numbers independently of the set theory and then constructed the set theory in the theory of ordinal numbers.

In that theory, we used only the predicates $<, =$ and only the functions N, max, Iq, j, Min, Rec, and χ. (We used other special variables and functions $0, \omega, \delta$, represented by the above described functions.)

In this paper, we shall call a function *semi-recursive* if it is represented by N, max, Iq, j, Min and Rec, and a semi-recursive function *recursive*, if every Min in the function satisfies the well-known condition as in the case of the recursive functions of natural numbers (to be given precisely later). We shall define, moreover, $\mathfrak{M}_a$ as the model generated by N, max, Iq, j, Min, Rec and the ordinals less than a. $\mathfrak{M}_a$ is well-ordered by the original order and has the same order type as the ordinal $\mathfrak{m}(a)$. Then we shall prove that an interpretation of a recursive function f in the model of ordinals less than $\mathfrak{m}(a)$ is f itself and that the power of $f(a_1, \cdots, a_n)$ is not greater than the power of $\max(a_1, \cdots, a_n)$, if f is recursive and $\max(a_1, \cdots, a_n) \geqq \omega$. It seems very difficult to generalize this proposition to the case of semi-recursive functions, because the consistency of the set theory could be proved, if it is proved.

On the formalized system developed in [**6**], we shall prove that there exists a recursive function C such that we can replace the axiom of cardinal by the weaker axiom $\forall x \exists y \forall z (C(x, y, z) = 0 \wedge y > 0)$ to construct the set theory. We shall give further the condition for the ordinal a with countable power that the ordinals less than a constitute the model of the set theory.

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§ **1.** Let O ($>\omega$) be a cardinal number and O be the class of all the ordinal numbers $<\mathrm{O}$. We say simply 'a is an ordinal', if $a\in O$. (We may regard, in the following, O as the class of all the ordinal numbers.) We use the concepts on ordinals $0, \omega, <, =, a'$ (successor of a), $\max(a, b)$ as usual. Moreover we can define the following functions $N, \delta, \mathrm{Iq}, \mathrm{Eq}, j, g^1, g^2$ from ordinals to an ordinal.

$$N(a)=\begin{cases} 0 & \text{if } a>0, \\ 1\ (=0') & \text{otherwise.}\end{cases} \qquad \delta(a)=\begin{cases} b & \text{if } 0<a<\omega \text{ and } a=b', \\ a & \text{otherwise.}\end{cases}$$

$$\mathrm{Iq}(a, b)=\begin{cases} 0 & \text{if } a<b, \\ 1 & \text{otherwise.}\end{cases} \qquad \mathrm{Eq}(a, b)=\begin{cases} 0 & \text{if } a=b, \\ 1 & \text{otherwise.}\end{cases}$$

$$j(g^1(a), g^2(a))=a, \quad g^1(j(a, b))=a, \quad g^2(j(a, b))=b.$$

$$j(a, b)<j(c, d) \rightleftarrows \max(a, b)<\max(c, d)$$
$$\vee(\max(a, b)=\max(c, d)\wedge(b<d \vee (b=d\wedge a<c))).$$

(We use a logical symbol in this section as an abbreviation of a word or a phrase of English.) Moreover, we shall define Min and Rec as follows. If $f\in O^O$, that is, f is a function from O to O, we define

$$\mathrm{Min}(f)=\begin{cases} \text{the least number } a \text{ such that } f(a)=0, \text{ if } \exists x(f(x)=0), \\ 0 \quad \text{otherwise.}\end{cases}$$

and

$$\mathrm{Con}(f, a, b)=\begin{cases} f(b) & \text{if } b<a, \\ 0 & \text{otherwise.}\end{cases}$$

(We see easily that Con can be represented by Min, N, Iq, Eq and max.) Let $\boldsymbol{f}$ be a function from $O^O\times O$ to O. Then we can define by the transfinite induction Rec satisfying

$$\forall x(\mathrm{Rec}(\boldsymbol{f}, x)=\boldsymbol{f}(\{y\}\mathrm{Con}(\{z\}\mathrm{Rec}(\boldsymbol{f}, z), x, y), x)).$$

(We use the notation $\{x\}A$ instead of usual notations λxA or $\hat{x}A$. Rec is a function from

$$O^{(O^O\times O)}\times O$$

to O.)

Now we shall define $\{\mathfrak{A}_{m,n}\}(m, n=0, 1, 2, 3, \cdots)$. $\mathfrak{A}_{m,n}$ is a class of functions from $\underbrace{O^O\times\cdots\times O^O}_{m}\times\underbrace{O\times\cdots\times O}_{n}$ to O. (Especially $\mathfrak{A}_{0,0}\subset O$.) If $\alpha\in\mathfrak{A}_{m,n}$, then α is of the form

$$\{f_1, \cdots, f_m, x_1, \cdots, x_n\}\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n).$$

If no confusion is likely to occur, we use the notations like

$$\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n) \in \mathfrak{A}_{m,n}.$$

$\{\mathfrak{A}_{m,n}\}$ is defined as the least classes satisfying the following conditions.

1. $\{x\}x \in \mathfrak{A}_{0,1}$.
2. If $\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n) \in \mathfrak{A}_{m,n}$, then

$$\{f_1, \cdots, f_{m+k}, x_1, \cdots, x_{n+l}\}\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n) \in \mathfrak{A}_{m+k,n+l}.$$

3. If $\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n) \in \mathfrak{A}_{m,n}$ and $(k_1, \cdots, k_m)$ and $(l_1, \cdots, l_n)$ are any permutations of $(1, \cdots, m)$ and $(1, \cdots, n)$ respectively, then

$$\{f_{k_1}, \cdots, f_{k_m}, x_{l_1}, \cdots, x_{l_n}\}\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n) \in \mathfrak{A}_{m,n}.$$

4. If $\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n) \in \mathfrak{A}_{m,n}$, then

$$\{f_1, \cdots, f_m, x_1, \cdots, x_n\}f_1(\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n)) \in \mathfrak{A}_{m,n}.$$

5. If $\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n) \in \mathfrak{A}_{m,n}$ and $\boldsymbol{g}(f_1, \cdots, f_m, x_1, \cdots, x_n) \in \mathfrak{A}_{m,n}$ then
$N(\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n))$,
$j(\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n), \boldsymbol{g}(f_1, \cdots, f_m, x_1, \cdots, x_n))$,
$\max(\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n), \boldsymbol{g}(f_1, \cdots, f_m, x_1, \cdots, x_n))$ and
$\mathrm{Iq}(\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n), \boldsymbol{g}(f_1, \cdots, f_m, x_1, \cdots, x_n))$ belong to $\mathfrak{A}_{m,n}$.
6. If $\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n) \in \mathfrak{A}_{m,n}$, then

$$\mathrm{Min}(\{x\}\boldsymbol{f}(f_1, \cdots, f_m, x, x_2, \cdots, x_n)) \in \mathfrak{A}_{m,n-1}.$$

7. If $\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n) \in \mathfrak{A}_{m,n}$ and $\boldsymbol{g}(f_2, \cdots, f_m, x_2, \cdots, x_n) \in \mathfrak{A}_{m-1,n-1}$, then $\mathrm{Rec}(\{f, x\}\boldsymbol{f}(f, f_2, \cdots, f_m, x, x_2, \cdots, x_m), \boldsymbol{g}(f_2, \cdots, f_m, x_2, \cdots, x_n)) \in \mathfrak{A}_{m-1,n-1}$.

Clearly we have the following propositions.

Proposition 1. *Every $\mathfrak{A}_{m,n}$ is countable.*

Proposition 2. *If $\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n) \in \mathfrak{A}_{m,n}$,*

$\boldsymbol{g}_1(f_1, \cdots, f_k, x_0, x_1, \cdots, x_l) \in \mathfrak{A}_{k,l+1}, \cdots, \boldsymbol{g}_m(f_1, \cdots, f_k, x_0, x_1, \cdots, x_l) \in \mathfrak{A}_{k,l+1}$
and $\boldsymbol{h}_1(f_1, \cdots, f_k, x_1, \cdots, x_l) \in \mathfrak{A}_{k,l}$,
$\cdots, \boldsymbol{h}_n(f_1, \cdots, f_k, x_1, \cdots, x_l) \in \mathfrak{A}_{k,l}$, then

$$\boldsymbol{f}(\{x\}\boldsymbol{g}_1(f_1, \cdots, f_k, x, x_1, \cdots, x_l), \cdots, \{x\}\boldsymbol{g}_m(f_1, \cdots, f_k, x, x_1, \cdots, x_l), \boldsymbol{h}_1(f_1, \cdots, f_k, x_1, \cdots, x_l), \cdots, \boldsymbol{h}_n(f_1, \cdots, f_k, x_1, \cdots, x_l)) \in \mathfrak{A}_{k,l}.$$

Definition. We call a function α to be *semi-recursive,* if $\alpha \in \mathfrak{A}_{m,n}$ for some m, n.

Clearly a semi-recursive function is constructed from N, max, Iq, j, Min and Rec.

Definition. $\mathfrak{M}_a = \{f(b) | f \in \mathfrak{A}_{0,1}$ and $b < a\}$.

Clearly we have the following propositions.

Proposition 3. *If $a \geqq \omega$, then the power of $\mathfrak{M}_a$ is equal to the power of a. (The power of a means the power of $\{x | x < a\}$.)*

PROPOSITION 4. *$f(x_1, \cdots, x_n) \in \mathfrak{A}_{0,n}$ and $a_1 \in \mathfrak{M}_a, \cdots, a_n \in \mathfrak{M}_a$, then $f(a_1, \cdots, a_n)$ $\in \mathfrak{M}_a$.*

Definition of the *bounded minimum.*

$$\mathrm{Bm}(f, a) = \begin{cases} \text{the least number } b \text{ such that } f(b) = 0 \wedge b < a, \text{ if exists such } b, \\ 0 \quad \text{otherwise.} \end{cases}$$

The bounded minimum $\mathrm{Bm}(f, x)$ clearly belongs to $\mathfrak{A}_{1,1}$ and is represented by $\mathrm{Min}(\{y\}\max(f(y), \mathrm{Iq}(y, x)))$.

DEFINITION. Min in $\mathrm{Min}(f)$ is called *recursive,* if $\exists x(f(x) = 0)$.

DEFINITION. A semi-recursive function is called *recursive,* if every Min contained in this function is recursive or is bounded.

As usual, it is easily proved that there exists an ordinal $\mathfrak{m}(a)$ and one to one mapping τ_a from $\mathfrak{M}_a$ onto $\{x | x < \mathfrak{m}(a)\}$ satisfying

$$a_1 \in \mathfrak{M}_a,\ a_2 \in \mathfrak{M}_a,\ a_1 < a_2 \rightarrow \tau_a(a_1) < \tau_a(a_2)\,.$$

DEFINITION. $\mathfrak{A}^a_{m,n}$ is defined by

$$\{\{f_1, \cdots, f_m, x_1, \cdots, x_n\}\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n, b) \mid \boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_{n+1}) \in \mathfrak{A}_{m,n+1} \text{ and } b \in \mathfrak{M}_a\}\,.$$

PROPOSITION 5. *If $\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_{n+k}) \in \mathfrak{A}^a_{m,n+k}$, $a_1 \in \mathfrak{M}_a, \cdots, a_k \in \mathfrak{M}_a$, then $\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n, a_1, \cdots, a_k) \in \mathfrak{A}^a_{m,n}$.*

DEFINITION. If $f \in \mathfrak{A}^a_{0,1}$ and $\tau_a(f(b)) = g(\tau_a(b))$ for every $b \in \mathfrak{M}_a$, then we say 'g is an f^{τ_a}'.

LEMMA 1. *If $a_1 \in \mathfrak{M}_a$ and $a_2 \in \mathfrak{M}_a$, then $\tau_a(N(a_1)) = N(\tau_a(a_1))$, $\tau_a(\max(a_1, a_2)) = \max(\tau_a(a_1), \tau_a(a_2))$, $\tau_a(\mathrm{Iq}(a_1, a_2)) = \mathrm{Iq}(\tau_a(a_1), \tau_a(a_2))$ and $\tau_a(j(a_1, a_2)) = j(\tau_a(a_1), \tau_a(a_2))$.*

LEMMA 2. *If g is an f^{τ_a} and* Min *in* $\mathrm{Min}(f)$ *is recursive, then* $\tau_a(\mathrm{Min}(f)) = \mathrm{Min}(g)$.

PROOF. We set $b = \mathrm{Min}(f)$. Then $f(b) = 0$, so $g(\tau_a(b)) = 0$. If there exists c satisfying $c < \tau_a(b)$ and $g(c) = 0$, then $\exists d(c = \tau_a(d) \wedge d \in \mathfrak{M}_a)$. Clearly $\tau_a(f(d)) = g(\tau_a(d)) = 0$ and $d < b$, which is a contradiction.

LEMMA 3. *If g is an f^{τ_a} and $b \in \mathfrak{M}_a$, then* $\tau_a(\mathrm{Bm}(f, b)) = \mathrm{Bm}(g, \tau_a(b))$.

PROOF. If $\exists x(f(x) = 0 \wedge x < b)$, then the proof is done in the same way as in the proof of Lemma 2. We consider therefore the case when $\forall x(x < b \vdash f(x) > 0)$. Then we have only to prove $\mathrm{Bm}(g, \tau_a(b)) = 0$. Let $c = \mathrm{Bm}(g, \tau_a(b))$ > 0. Then $c < \tau_a(b) \wedge g(c) = 0$, so there exists d such that $d \in \mathfrak{M}_a$ and $c = \tau_a(d)$, which is a contradiction.

THEOREM 1. *If $\{f_1, \cdots, f_m, x_1, \cdots, x_n\}\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n) \in \mathfrak{A}_{m,n}$ is recursive and g_i is an $f_i^{\tau_a}$ and $a_i \in \mathfrak{M}_a$ for each i ($i \leqq m$ and $i \leqq n$ respectively), then*

$$\tau_a(\boldsymbol{f}(f_1, \cdots, f_m, a_1, \cdots, a_n)) = \boldsymbol{f}(g_1, \cdots, g_m, \tau_a(a_1), \cdots, \tau_a(a_n))\,.$$

PROOF. We prove this by induction on the number of stages to construct

$\boldsymbol{f}$. If the outermost function of $\boldsymbol{f}$ is other than Rec, then the theorem is clearly proved by Lemmas 1-3 and the hypothesis of the induction. Therefore, we have only to prove the theorem in the case, when $\boldsymbol{f}$ is of the form

$$\mathrm{Rec}(\{f, x\}\boldsymbol{g}_1(f, f_1, \cdots, f_m, x, a_1, \cdots, a_n), \boldsymbol{g}_2(f_1, \cdots, f_m, a_1, \cdots, a_n)) .$$

We set $c = \boldsymbol{g}_2(f_1, \cdots, f_m, a_1, \cdots, a_n)$. From the hypothesis of the induction follows $\tau_a(c) = \boldsymbol{g}_2(g_1, \cdots, g_m, \tau_a(a_1), \cdots, \tau_a(a_n))$. Therefore we have only to prove

$$\tau_a(\mathrm{Rec}(\{f, x\}\boldsymbol{g}_1(f, f_1, \cdots, f_m, x, a_1, \cdots, a_n), c))$$
$$= \mathrm{Rec}(\{f, x\}\boldsymbol{g}_1(f, g_1, \cdots, g_m, x, \tau_a(a_1), \cdots, \tau_a(a_n)), \tau_a(c)) .$$

We prove this by the transfinite induction on c with the condition $c \in \mathfrak{M}_a$. We set

$$\{y\}f_0(y) = \{y\}\mathrm{Con}(\{z\}\mathrm{Rec}(\{f, x\}\boldsymbol{g}_1(f, f_1, \cdots, f_m, x, a_1, \cdots, a_n), z), c, y) ,$$

then $\mathrm{Rec}(\{f, x\}\boldsymbol{g}_1(f, f_1, \cdots, f_m, x, a_1, \cdots, a_n), c) = \boldsymbol{g}_1(f_0, c)$.

Clearly we have $f_0 \in \mathfrak{A}^a_{0,1}$.

However, if $c_0 \in \mathfrak{M}_a$, then

$$\tau_a(f_0(c_0)) = \begin{cases} \tau_a(\mathrm{Rec}(\{f, x\}\boldsymbol{g}_1(f, f_1, \cdots, f_m, x, a_1, \cdots, a_n), c) & \text{if } c_0 < c , \\ 0 \quad \text{otherwise.} \end{cases}$$

By the hypothesis of the transfinite induction, we have

$$\tau_a(f_0(c_0)) = \begin{cases} \mathrm{Rec}(\{f, x\}\boldsymbol{g}_1(f, g_1, \cdots, g_m, x, \tau_a(a_1), \cdots, \tau_a(a_n)), \tau_a(c_0)) & \text{if } c_0 < c , \\ 0 \quad \text{otherwise.} \end{cases}$$

Hence we have

$$\{y\}g_0(y) = \{y\}\mathrm{Con}(\{z\}\mathrm{Rec}(\{f, x\}\boldsymbol{g}_1(f, g_1, \cdots, g_m, x, \tau_a(a_1), \cdots, \tau_a(a_n)), z), \tau_a(c), y)$$

is an $f_0{}^{\tau_a}$.

In virtue of this and the hypothesis of the induction we have

$$\tau_a(\boldsymbol{g}_1(f_0, c)) = \boldsymbol{g}_1(g_0, \tau_a(c))$$
$$= \mathrm{Rec}(\{f, x\}\boldsymbol{g}_1(f, g_1, \cdots, g_m, x, \tau_a(a_1), \cdots, \tau_a(a_n)), \tau_a(c)) .$$

THEOREM 2. *If* $f(x_1, \cdots, x_n)$ *is recursive and* $a_1 \in \mathfrak{M}_a, \cdots, a_n \in \mathfrak{M}_a$, *then* $\tau_a(f(a_1, \cdots, a_n)) = f(\tau_a(a_1), \cdots, \tau_a(a_n))$.

THEOREM 3. *If* $f(x_1, \cdots, x_n)$ *is recursive, then the power of* $f(a_1, \cdots, a_n)$ *is not greater than the power of* $\max(a_1, \cdots, a_n)$ *provided that* $\max(a_1, \cdots, a_n) \geqq \omega$.

PROOF. Let $a = \max(a_1, \cdots, a_n)$. Then $\tau_a(f(a_1, \cdots, a_n)) = f(a_1, \cdots, a_n)$, $\tau_a(f(a_1, \cdots, a_n)) < \mathfrak{m}(a)$ and the power of $\mathfrak{m}(a)$ is equal to the power of a.

Let Ω and Ω_0 be two cardinal numbers and $\Omega_0 < \Omega$. It follows from Theorem 3 that $\exists x(x < \Omega \wedge g(x, a_1, \cdots, a_n) = 0)$ implies $\exists x(x < \Omega_0 \wedge g(x, a_1, \cdots, a_n) = 0)$ for $a_1 < \Omega_0, \cdots, a_n < \Omega_0$ and a recursive function g. We see, therefore, a recursive function f of the ordinal numbers $< \Omega$ remains recursive, even if the domain of f is restricted to Ω_0.

§ **2.** In this section, we shall prove that the functions defined in [**6**] are almost recursive. Some acquaintance with [**6**] is assumed in this section.

In this section we confine ourselves to the system of axioms obtained from the system of axioms in [**6**] by removing the axiom of cardinal. But instead of the axioms II 1-7 we shall use the following axioms:

II. 1. $\forall p \forall x \forall y (x = y \vdash p(x) = p(y))$.

2. a) $\forall p \forall x (p(x) = 0 \vdash p(\mathrm{Min}(p)) = 0 \wedge x \geqq \mathrm{Min}(p))$,

b) $\forall p (p(\mathrm{Min}(p)) = 0 \vee \mathrm{Min}(p) = 0)$.

3′. $\forall p \forall x \exists z \forall y (y < x \vdash p(y) < z)$.

7. $\forall p \forall x (\mathrm{Rec}(p, x) = p(\{y\}\mathrm{Con}(\{z\}\mathrm{Rec}(p, z), x, y), x))$,

where $\mathrm{Con}(f, a, b)$ is the abbreviation of a composition of Min, N, Iq, Eq and max, and satisfying the axiom of contraction in [**6**]. We see easily $\{f, x, y\}\mathrm{Con}(f, x, y)$ is recursive. We can easily define $S(f, g, \mathfrak{b}, a)$ satisfying the axiom of sum in [**6**] as a recursive function and also a recursive function $T(f_1, \cdots, f_m, a_1, \cdots, a_n)$ for every primitive formula $F(f_1, \cdots, f_m, a_1, \cdots, a_n)$, whose quantifiers are all bounded, satisfying the following formula:

$$\forall p_1 \cdots \forall p_m \forall x_1 \cdots \forall x_n (T(p_1, \cdots, p_m, x_1, \cdots, x_n) = 0 \vdash\dashv F(p_1, \cdots, p_m, x_1, \cdots, x_n)) .$$

(See [**6**], Chapter II, § 2 for the definition of primitive formula.) From this and the axiom II. 3′ follows that $\{f, x\}\sup(f, x)$ is recursive. In the same way, we see, every function defined in [**6**], Chapter II, § 3 and Chapter III, § 1 is recursive. ($G(a, b, c)$, $a - b$, $\mathrm{Od}(a)$ and $C(a)$ are also defined by $\mathrm{Min}(z)(z < c \wedge K(z, c) = j(a, b))$, $\mathrm{Min}(x)(b + x = a \wedge x < a)$, $\mathrm{Min}(x)(x < a' \wedge x = a)$ and $\mathrm{Min}(x)(x < a \wedge x \in a)$ respectively.)

Now, we shall prove that every function in [**6**], Chapter III, § 4 is recursive.

$C_0(a, b)$ is also defined by $\mathrm{Min}(x)\,(x < \omega \wedge a < B_0(x', b))$.

The following function $H(a, b)$ is clearly recursive:

$$H(0, b) = b \wedge \forall x (0 < x \wedge x < \omega \vdash H(x, b) = H(\delta(x), b) + \tilde{j}(0, H(\delta(x), b), 0))$$
$$\wedge \forall x (\omega \leqq x \vdash H(x, b) = 0) .$$

$A_1(n, a, b)$ is also defined by

$$A_1(0, a, b) = a \wedge \forall x (\omega \leqq x \vdash A_1(x, a, b) = 0)$$
$$\wedge \forall x (0 < x \wedge x < \omega \vdash A_1(x, a, b) = A_0(A_1(\delta(x), a, b), H(x, b))) .$$

$\mathrm{Cp}(a, b)$ is also defined by $\mathrm{Min}(x)\,(B_1(x, b) > a \wedge \omega < x)$. Therefore we have that $A(a, b, c)$ is recursive.

Now we define recursive functions $A(b,c)$ and $C(b,x,c)$ by $\sup(\{x\}A(x,b,c),b)$ and the following formula respectively:

$$C(b,0,c)=\mathrm{Min}(z)\,(z<A(b,c)\wedge\exists y(y<b\wedge z=A(y,b,c)))$$
$$\wedge\forall x(x>0\vdash C(b,x,c)=\mathrm{Min}(z)\,(z<A(b,c)\wedge\exists y(y<b\wedge z=A(y,b,c))$$
$$\wedge\forall u(u<A(b,c)\vdash \mathrm{Con}(\{v\}C(b,v,c),x,u)<z)))\,.$$

Then from [**6**], Chapter III, § 2 follows the following theorem:

THEOREM 4. *We can construct the set theory in the system of axioms* I. 1-24 *in* [**6**] *and* II. 1, 2, 3′, 7 *and* $\forall x\exists y\forall z(C(x,y,z)=0\wedge y>0)$, *where* C *is a recursive function.* (*the axiom of cardinal is unnecessary.*)

NOTE. We use the axiom II. 3′ to prove the recursivity of C but for the recursivity of C we necessitate only the weaker axiom, which states that ordinals run over the domain less than a certain ordinal number.

Now, we define the semi-recursive function $D(b)$ by $\mathrm{Min}(y)\,(\forall z(C(b,y,z)=0\wedge y>0))$. Then $\forall x\exists y\forall z(C(x,y,z)=0\wedge y>0)$ is equivalent to $\forall x\forall z(C(x,D(x),z)=0\wedge D(x)>0)$.

THEOREM 5. *If* $\exists p\forall x(x<D(a)\vdash\exists y(y<a\wedge x=p(y)))$ *for every* $a\geqq\omega$, *then the consistency of the set theory holds.*

PROOF. Now, we shall consider the ordinal less than Ω_2. If the power of b is countable, then we have clearly $\exists y\forall x(C(b,y,x)=0\wedge y>0)$, whence follows $\forall x(C(b,D(b),x)=0\wedge D(b)>0)$. From the hypothesis of the theorem we see that the power of $D(b)$ is countable. Therefore, if we confine ourselves to consider the ordinals less than Ω_1, then $\forall' x\exists' y\forall' z(C(x,y,z)=0\wedge y>0)$ holds, where $\forall' x$ (or $\exists' y$) means that x (or y) runs over the ordinals less than Ω_1. The axioms I. 1-24 and II. 1, 2, 3′, 7 also hold in Ω_1, if we replace $\forall x$ (or $\exists y$) by $\forall' x$ (or $\exists' y$), and we interpret p as 'for all functions from Ω_1 to Ω_1' etc. Hence we have the consistency of the set theory from the hypothesis of the theorem and the axiom on the ordinals less than Ω_2.

THEOREM 6. *If* $\tau_a(D(b))=D(\tau_a(b))$ *for every* a *and* $b\in\mathfrak{M}_a$, *then the consistency of the set theory holds.*

PROOF. From the hypothesis of the theorem follows

$$D(b)=\tau_b(D(b))<\mathfrak{m}(b)\,.$$

Since the power of $\mathfrak{m}(b)$ is equal to the power of b for every $b\geqq\omega$, the theorem follows from Theorem 5.

§ 3. We now define τ, $\mathfrak{M}$, ω_∞ by τ_ω, $\mathfrak{M}_\omega$, $\mathfrak{m}(\omega)$ and shall consider them in this section. We assume the axiom of cardinal or the axiom $\forall x\exists y\forall z(C(x,y,z)=0\wedge y>0)$ in this section.

Clearly the power of ω_∞ is countable and the system of ordinals less than ω_∞ is a model of the set theory. Theorem 2 shows that the interpretation of a recursive function f in this model is f itself. Especially for every $a, b, c \in \mathfrak{M}$, $\tau(C(b, a, c)) = C(\tau(b), \tau(a), \tau(c))$ holds, so we have

$$\forall x(x < \omega_\infty \vdash \exists y(y < \omega_\infty \wedge y > 0 \wedge \forall z(z < \omega_\infty \vdash C(x, y, z) = 0))) .$$

In the same way as in Lemma 2, we have easily the following lemma.

Lemma 4. *Let $\{x\}f(x)$ be semi-recursive. We define $\{x\}f^*(x)$ from $\{x\}f(x)$ by replacing every* Min( ) *in f by the bounded minimum* Bm(, ω_∞). *Then* $\tau(\mathrm{Min}(f)) = \mathrm{Bm}(f^*, \omega_\infty)$.

In virtue of this lemma we have easily

Theorem 7. *There exists an ordinal ω_∞ satisfying the following conditions:*

(1) *The power of ω_∞ is countable.*

(2) $\forall x(x < \omega_\infty \vdash \exists y(y < \omega \wedge y > 0 \wedge \forall z(z < \omega_\infty \vdash C(x, y, z) = 0)))$.

(3) *For every recursive function f,*
$\forall x_1 \cdots \forall x_n(x_1 < \omega_\infty \wedge \cdots \wedge x_n < \omega_\infty \vdash f(x_1, \cdots, x_n) < \omega_\infty)$.

(4) *For every recursive function f,*
$\forall x(x < \omega_\infty \vdash \exists z(z < \omega_\infty \wedge \forall y(y < x \vdash f(y) < z)))$.

(5) *For every semi-recursive function f,*
$\forall x_1 \cdots \forall x_n(x_1 < \omega_\infty \wedge \cdots \wedge x_n < \omega_\infty \vdash f^*(x_1, \cdots, x_n) < \omega_\infty)$.

(6) *For every semi-recursive function f,*
$\forall x(x < \omega_\infty \vdash \exists z(z < \omega_\infty \wedge \forall y(y < x \vdash f^*(y) < z)))$.

Moreover the conditions (2), (6) *mean that the ordinals $< \omega_\infty$ constitute the model of the set theory.*

Theorem 8. *ω_∞ is greater than Church-Kleene's ω_1.* (See [**1**].)

Proof. If f is a recursive function from the natural numbers to the natural numbers, then there exists a recursive function $\tilde{f}$ in our sense such that $\tilde{f}$ is equal to f on the domain of natural numbers. In this sense, $\tilde{U}, \tilde{T}_2$ is recursive in our sense, where U, T_2 are Kleene's function and predicate (see [**3**, pp. 278 and 281]). We define a new recursive function $\varphi(c, a, b)$ by $\tilde{U}(\mathrm{Min}(y)\,(y < \omega \wedge \tilde{T}_2(c, a, b, y)))$ and $B(c, x)$ by the following formula.

$$B(c, 0) = \mathrm{Min}(z)\,(z < \omega \wedge \varphi(c, z, z) = 0 \wedge \forall x(x < \omega \wedge \varphi(c, x, x) = 0 \vdash \varphi(c, z, x) = 0)$$

$$\wedge \forall x(x > 0 \vdash B(c, x) = \mathrm{Min}(z)\,(z < \omega \wedge \varphi(c, z, z) = 0 \wedge \forall y(y < x \vdash \neg(z = B(c, y)))$$

$$\wedge \forall u(u < \omega \wedge \forall y(y < x \vdash \neg(u = B(c, y))) \wedge \varphi(c, u, u) = 0 \vdash \varphi(c, z, u) = 0))) .$$

$B(c)$ is defined by $\mathrm{Min}(z)\,(B(c, z) = 0 \wedge B(c, z') = 0)$. We see $\omega_1 = \sup(B, \omega)$ by [**5**], whence follows that all the ordinals not greater than ω_1 belong to $\mathfrak{M}$ and $\omega_1 < \omega_\infty$.

Note. Until now, we have considered the model of Gödel's set theory. We can consider the model of the set theory with the axiom of inaccessible number by the same method. Let S_a, e. g., be the set theory with Tarski's axiom $\mathfrak{A}$ (cf. [**7**]) and without the axiom of replacement. Then by the above method we see the existence of an ordinal a_∞ satisfying the following conditions:

(1) The power of a_∞ is countable.

(2) $\forall x(x < a_\infty \vdash \exists y(0 < y \wedge y < a_\infty \wedge \forall z(z < a_\infty \vdash C(x, y, z) = 0)))$.

(3) $\forall x(x < a_\infty \vdash \exists y(y < a_\infty \wedge x < y \wedge \forall z(z < y \vdash \bar{D}(z) < y)$
$\wedge \forall u(u < y \vdash \forall v(v < a_\infty \vdash u \cdot v \in y))))$,

where $\bar{D}(b)$ is the abbreviation of $\mathrm{Min}(u)\,(\forall v(v < a_\infty \vdash C(b, u, v) = 0) \wedge u > 0)$ and $b \cdot c$ and $b \in c$ are recursive function and predicate defined in [**6**, p. 106]. Conversely the conditions (2), (3) mean that the ordinals $< a_\infty$ constitute the model of S_a.

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References

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Correction of 'On the theory of ordinal numbers' ([6]).

p.	l.	Errata	Corrections
96	8	$\forall x \forall y \neg(x < y \wedge y < x)$	$\forall x \forall y \forall z(x = y \wedge y = z \vdash x = z)$
96	26	$\vdash$	$\vdash\dashv$
97	1	2.4.	24.
97	4	$((x = y$	$(x = y$
98	30	$T(f, \cdots, g, a, \cdots, b)$.	$T(f, \cdots, g, a, \cdots, b)$
98	32	$V(f, \cdots, g, a, \cdots, b))$	$V(f, \cdots, g, a, \cdots, b))$.
100	1	.	,
100	19	$\mathrm{Con}(\{v\}\, U(u), x, z)$	$\mathrm{Con}(\{u\}\, U(u), x, z)$
101	3	$T(x)$	$T(a)$
103	5	$(\mathrm{Con}(K, a, T_2(a)))))$	$\mathrm{Con}(K, a, T_2(a))))$
103	8	$K(j(a, b))$	$K(j(a, b))$.

104	3	$\forall\exists xy\exists z(J(y, b)+z = x$	$0 < b \rightarrow \forall x\exists y\exists z(J(y, b)+z = x$
105	2	$g_1(\tilde{K}(x, a)) < a \wedge g_2(\tilde{K}$	$\tilde{g}_1(\tilde{K}(x, a)) < a \wedge \tilde{g}_2(\tilde{K}$
107	19	LEMMA 2	THEOREM 2
108	31	$x(b)$	$\chi(b)$
109	16	$p(p)$	$p(y)$
111	21	$\sup((y)A_1(\delta(x), y, b), b)$	$\sup(\{y\}A_1(\delta(x), y, b), b)$
111	29	$j(0, B_1(\delta(x), b), 0)$	$\tilde{j}(0, B_1(\delta(x), b), 0)$
112	2	$A_3(x, b)$	$A_2(x, b)$
112	6	$B_0(b)$	$B_1(b)$
112	8	$B(\delta(\mathrm{Cp}(x, b)), b)$	$B_1(\delta(\mathrm{Cp}(x, b)), b)$
112	9	$j(u, y, z)$	$\tilde{j}(u, y, z)$