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Some applications of the Weierstrass mean value theorem

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1. Introduction. Let $f(z)$ be an analytic function holomorphic in some domain D of the complex plane. Then $f(z)$ is said to be univalent in D if for any two points in D of affix z_1 and z_2 it is true that $f(z_1)=f(z_2)$ implies $z_1=z_2$. Now in order that $f(z)$ be univalent in D it is necessary, but not sufficient, that $f'(z) \neq 0$ for $z \in D$. If, however, we specialize D and require that D be convex then $\operatorname{Re}\{f'(z)\}>0$, $z \in D$, implies that $f(z)$ is univalent in D. This latter result was first proved by J. Wolff [7] for the case in which D is a half-plane and generalized by Noshiro and Warschawski [2], [6] to convex domains. On the other hand Herzog and Piranian [1] demonstrated that if a domain D has the property that $f(z)$ is univalent in D whenever $\operatorname{Re}\{f'(z)\}>0$ throughout D then the domain is not far from being convex. Thus if we propose to obtain conditions for univalency in non-convex domains, we must expect some restrictions on $f'(z)$ other than $\operatorname{Re}\{f'(z)\}>0$. In this note some conditions of this nature are given, bearing mostly on $\operatorname{Arg} f'(z)$ for $z \in D$. These conditions arise in a natural manner from an application of a theorem due to Weierstrass [5]. The results are similar to those obtained by Reade [3] whose results are based upon a criterion for univalency due to Umezawa [4]. The theorem of Weierstrass employed is the following

THEOREM (Weierstrass). *Let C be a rectifiable curve joining the points of affix $z=\alpha$ and $z=\beta$. Let $z=z(t)$ $(a \leqq t \leqq b)$ be a parametric representation of C. Then if $f(z)$ is continuous on C and $g(z)$ is continuous and positive on C*

$$\int_a^b f(z)g(z)dt = X\int_a^b g(z)dt$$

where the point of affix X is contained in the convex hull of the set of values which $f(z)$ assumes on C.

2. Applications of the Weierstrass theorem. For the sake of completeness we begin with a proof of the Noshiro-Warschawski Theorem.

THEOREM (Noshiro-Warschawski). *A function regular in a convex domain D is univalent in D if* $\operatorname{Re}\{f'(z)\}>0$ *for* $z \in D$.

PROOF. Let z_1 and z_2 be two distinct points of D. Then upon integrating the derivative of $f(z)$ along the line segment connecting z_1 and z_2 we have

$$f(z_2)-f(z_1)=\int_{z_2}^{z_1} f'(z)dz$$
$$=(z_2-z_1)\int_0^1 f'(z_2+(z_1-z_1)t)dt\,.$$

But by the Weierstrass theorem the latter integral has the value Z^*, where Z^* is the affix of a point lying in the convex hull C^* of the image of the line segment $\overline{z_1z_2}$ under the mapping effected by $f'(z)$. But since $\operatorname{Re}\{f'(z)\}>0$ for $z\in D$ it follows that the origin is not in C^* and hence $f(z_2)-f(z_1)=Z^*(z_2-z_1)\neq 0$ unless $z_2=z_1$.

REMARK. Clearly the theorem is still valid if for some $0\leqq\alpha\leqq 2\pi$, $\operatorname{Re}\{e^{-i\alpha}f'(z)\}>0$ for $z\in D$.

We next prove a theorem first given by Reade [**3**]. Our proof differs from Reade's in that whereas Reade employed a condition for univalency due to Umezawa we use the mean value theorem of Weierstrass.

THEOREM 2.1. *Let φ be fixed and $0\leqq\varphi\leqq\pi$. Let D be a domain in which it is possible to join each pair of distinct points z_1, z_3 by a pair of straight line segments $\overline{z_1z_2}$, $\overline{z_2z_3}$ lying in D such that*

(1)
$$\left|\operatorname{Arg}\frac{z_3-z_2}{z_2-z_1}\right|\leqq\varphi$$

φ being independent of z_1 and z_3 in D. Then if $f(z)$ is analytic in D and if for some $0\leqq\alpha\leqq 2\pi$

$$\alpha\leqq\operatorname{Arg}f'(z)\leqq\pi-\varphi+\alpha$$

for $z\in D$, $f(z)$ is univalent in D.

PROOF. Let z_1 and z_3 be two distinct points of D and let $z_2\neq z_1,\ \neq z_3$ be a point of D such that the line segments $\overline{z_1z_2}$ and $\overline{z_2z_3}$ lie in D and satisfy (1). Now we may write

$$f(z_3)-f(z_1)=\int_{z_2}^{z_1} f'(z)dz+\int_{z_2}^{z_3} f'(z)dz$$
$$=(z_2-z_1)\int_0^1 f'(z_1+(z_2-z_1)t)dt$$
$$+(z_3-z_2)\int_0^1 f'(z_2+(z_3-z_2)t)dt\,.$$

Hence by the Weierstrass mean value theorem we have

(2)
$$f(z_3)-f(z_1)=Z_1^*(z_2-z_1)+Z_1^*(z_3-z_2)$$
$$=Z_1^*(z_2-z_1)\Big[1+\frac{Z_2^*}{Z_1^*}\frac{z_3-z_2}{z_2-z_1}\Big],$$

since by hypothesis, $|\operatorname{Arg}(Z_2^*/Z_1^*)|<\pi-\varphi$ it follows from (1) that

$$\left|\operatorname{Arg}\frac{Z_2^*}{Z_1^*}\frac{z_3-z_2}{z_2-z_1}\right|<\pi.$$

Therefore the quantity in the brackets in the last member of (2) cannot vanish and, by (2), $f(z_3)\neq f(z_1)$.

THEOREM 2.2. *Let D be a domain any two points of which may be joined by the arc (lying in D) of an ellipse* $z=z_0+e^{i\beta}(a\cos t+ib\sin t)$, *for which* $t_1\leqq t\leqq t_2$, $0\leqq t_1,\ t_2\leqq\frac{\pi}{2}$, *and where* $a,\ b>0$, β *real.* (a, b, β *may depend on the two points to be connected.*) *Then if* $f(z)$ *is analytic in D and if for some* $0\leqq\alpha\leqq2\pi$

$$\alpha<\operatorname{Arg}f'(z)<\alpha+\frac{\pi}{2}$$

for all $z\in D$, $f(z)$ *is univalent in D.*

PROOF. Let z_1 and z_2 be two points of D then

$$\begin{aligned}f(z_2)-f(z_1)&=\int_{z_1}^{z_2}f'(z)dz\\&=e^{i\beta}\int_{t_1}^{t_2}f'(z)(-a\sin t+ib\cos t)dt,\end{aligned}$$

where $z_k=z_0+e^{i\beta}\ (a\cos t_k+ib\sin t_k)$, $k=1,2$.

Applying the Weierstrass theorem and simplifying yields

$$\begin{aligned}f(z_2)-f(z_1)&=e^{i\beta}\Big[-a\int_{t_1}^{t_2}f'(z)\sin t\,dt+ib\int_{t_1}^{t_2}f'(z)\cos t\,dt\Big]\\&=Z_2^*e^{i\beta}\Big[\frac{Z_1^*}{Z_2^*}\operatorname{Re}\{e^{-i\beta}(z_2-z_1)\}+i\operatorname{Im}\{e^{-i\beta}(z_2-z_1)\}\Big],\end{aligned}\tag{3}$$

where Z_1^* and Z_2^* are points in the convex hull of the sets which are images of the elliptic arcs from z_1 to z_2 under the mapping effected by $f'(z)$. It results from the hypothesis that $|\operatorname{Arg}(Z_1^*/Z_2^*)|<\frac{\pi}{2}$ and hence, since $Z_2^*\neq0$, the right hand member of (3) is never zero unless $z_2=z_1$ so that $f(z)$ is univalent in D.

DEFINITION. A domain D is said to have property U with constant $\theta=\theta(D)$ if when z_1 and z_2 in D are given there exists a constant θ (independent of z_1 and z_2), $\theta<\pi$, and a sequence of points $z_1=\zeta_0,\zeta_1,\zeta_2,\cdots,\zeta_n=z_2$ with the property that the segments $\overline{\zeta_0\zeta_1},\overline{\zeta_1\zeta_2},\cdots,\overline{\zeta_{n-1}\zeta_n}$ all belong to D and

$$\left|\operatorname{Arg}\frac{\zeta_{k+1}-\zeta_k}{z_2-z_1}\right|\leqq\theta,\qquad k=0,1,\cdots,n-1.$$

REMARK. A convex domain has the property U with constant $\theta=0$.

THEOREM 2.3. *Let D be a domain having the property* U. *Then if for some* $0\leqq\alpha\leqq2\pi$, $\operatorname{Arg}f'(z)$ *satisfies* $\alpha+\theta<\operatorname{Arg}f'(z)<\pi+\alpha-\theta$, $\theta<\frac{\pi}{2}$, *for z in D*; $f(z)$

is univalent in D.

PROOF. We may write

$$f(z_2)-f(z_1)=\sum_{k=0}^{n-1}\int_{\zeta_k}^{\zeta_{k+1}} f'(z)dz$$

$$=(z_2-z_1)\sum_{k=0}^{n-1}\lambda_k\frac{\zeta_{k+1}-\zeta_k}{z_2-z_1}$$

where the vectors λ_k satisfy $\alpha+\theta<\operatorname{Arg}\lambda_k<\pi+\alpha-\theta$ for some $0\leqq\alpha\leqq 2\pi$. The domain possesses the property U and we have

$$\alpha<\operatorname{Arg}\lambda_k\frac{\zeta_{k+1}-\zeta_k}{z_2-z_1}<\alpha+\pi\,, \qquad k=0,1,\cdots,n-1\,,$$

and consequently

$$\sum_{k=0}^{n-1}\lambda_k\frac{\zeta_{k+1}-\zeta_k}{z_2-z_1}\neq 0$$

which implies $f(z)$ is univalent.

The following interesting theorem is obtained if $\theta(D)=\frac{\pi}{2}$.

THEOREM 2.4. *Let D be a domain any two points of which may be joined by a finite number of line segments (lying in D) and parallel to the coordinate axis such that the x and y coordinates of the end points form either a non-decreasing or non-increasing sequence. Then if for some* $0\leqq\alpha\leqq 2\pi$

$$\alpha<\operatorname{Arg}f'(z)<\alpha+\frac{\pi}{2}$$

for $z\in D$, *$f(z)$ is univalent in D.*

The proof follows along the same lines of Theorem 2.3 and 2.1.

3. Examples. We can easily construct examples for which Theorem 2.2 and 2.3 apply but for which Reade's theorem does not apply. For example consider the region in the first quadrant bounded by two confocal ellipses with center at the origin and with foci at ± 1. If the difference between their corresponding semi axes is sufficiently small the region does not satisfy the criterion of Reade's theorem. Theorem 2.2 or 2.3 applied to the function $f(z)=z\sin^{-1}z+\sqrt{1-z^2}$ in this region shows that $f(z)$ is univalent in the region.

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