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On the theory of differentials in commutative rings

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Hitherto the concept of differentials is introduced usually as the dual notion of the derivations. For instance the differentials belonging to a function field K/k are the dual elements of the k-derivations of K into itself. If V is a differentiable manifold the local differentials at a point of V are nothing other than the elements of the dual space of the space of tangent vectors at the point. Recently P. Cartier introduced in [4][1)] the notion of differentials without referring to the special module of derivations. His theory was used to settle the problem of separability in the extension of fields and the theory of simple points. We shall present here a more general version of the theory of differentials in a commutative R-algebra. In § 1 we list the notations and terminologies which will be used throughout the rest of the paper. In § 2 we shall prove some properties of differentials in a commutative algebra. Among others we shall introduce the notion of the quasi-separability for an R-algebra[2)] S. When R is a field and S is a finite R-module the quasi-separability coincides with the separability of S/R. In § 3 we shall give a theorem by which we can determine the structure of the module of differentials. § 4 is devoted to the proof of the following theorem. Let P be a point of an algebraic variety V which is defined over k, and let R be the quotient ring of P in V/k. Then under suitable conditions it will be shown that P is a simple point of V if and only if the module of k-differentials $D_k(R)$ is a free module. We shall study in § 5 the contribution of our theory to the theory of ramifications. We shall introduce there the notion of d-different. Quasi-separability and unramifiedness will be shown to be equivalent under some additional assumptions. On the other hand we have another kind of different which is called the homological different in [2]. We can easily prove that the homological different is contained in the d-different and in some occasions they have the same radical. But the precise relation between them is yet unknown. The relation between the d-different and the differential index defined in our previous paper [7] will be discussed in the last paragraph. They give rise to the same results in the case where we are studying the covering of algebraic curves. But in the case

1) The numbers in the bracket refer to the bibliography at the end of the paper.
2) For the precise definitions see § 1.

of higher dimensional varieties the matter is not so simple. The study of their relation suggests us to introduce a series of differents associated with the differentials of higher degree. We shall show in some special case that there exists a non-increasing sequence of differents beginning from the d-different and terminating in the different defined by the differential index. It is now conjectured that the d-different has some geometric meaning related to the differential forms of degree 1, but nothing is yet known about it.

§ 1. Notations and terminologies

All rings considered in this paper are assumed to be commutative and contain the unity which will be denoted by 1. Let S and R be two rings. We shall say that S is an *R-algebra* if R is an operator domain of S and (i) there exists an R-homomorphism f from R into S such that $f(1)=1$; (ii) each element r of R operates on S by the rule $rs=f(r)s$ where s is in S and the right hand side is the multiplication in S. Let $\mathfrak{a}$ be an ideal of S, then we shall call the ideal $f^{-1}(\mathfrak{a}\frown f(R))$ the *contraction* of $\mathfrak{a}$ and will be denoted simply by $\mathfrak{a}\frown R$. In this case $R/(\mathfrak{a}\frown R)$ can be considered as a subring of $S/\mathfrak{a}$ in an obvious way. Let x be an element of R, we denote usually the element $f(x)$ in S by the same letter x if there is no danger of confusion. Let S and R be two R-algebra with the ring homomorphism $f: R\rightarrow S$ and $g: R\rightarrow T$. A ring homomorphism π from S into T will be called an *R-algebra homomorphism* if we have $g=\pi\circ f$.

Let S be an R-algebra, an R-derivation of S into an S-module V is an R-linear map from S into V such that $D(xy)=xD(y)+yD(x)$ for x and y in S. D is an R-derivation of S if, and only if, D is zero on $f(R)$. We shall denote by $D_R(S)$ *the module of R-differentials in* S, i. e. the S-module satisfying the following conditions:

(1) There is an R-derivation d_R from S into $D_R(S)$ (d_R will be called the *differential operater* over R).

(2) $D_R(S)$ is generated by $\{d_Rx, x\in S\}$ over S.

(3) For any R-derivation D of S into an S-module V, there exists an S-linear map h from $D_R(S)$ into V such that $D=hd_R$.

It is known that for a given R-algebra S, there exists a unique $D_R(S)$, up to an S-isomorphism. Let φ be the homomorphism from $S\otimes_R S$ into S defined by $\varphi(\sum_i a_i\otimes b_i)=\sum_i a_ib_i$. The kernel I of φ is generated by the element of the form $\{1\otimes x-x\otimes 1\}$. We make $S\otimes_R S$ into an S-module by the rule $a(b\otimes c)=ab\otimes c$. Then $S\otimes S=S\otimes 1+I$ (direct sum) as an S-module. Let $\mathfrak{N}$ be the submodule of $S\otimes S$ generated by the elements of the form $\{1\otimes ab-a\otimes b-b\otimes a\}$, then $D_R(S)$ is given by the difference module $(S\otimes S)/\mathfrak{N}$ and the differential $d_Ra(a\in S)$ is defined as the class of $1\otimes a$ mod R. Since $\mathfrak{N}=S\otimes 1+I^2$

(direct sum) as S-module, we have $D_R(S)$ is isomorphic to I/I^2 (cf. [**4**, Exposé 13]).

In the following we shall have occasions to consider several R-algebras S, $T, \cdots$ at the same time. To distinguish the differential operaters in $D_R(S)$, $D_R(T), \cdots$ we shall use the symbols $d_R^S, d_R^T, \cdots$. Thus for the same element x, $d_R^S x$ and $d_R^T x$ may have different meaning. We shall sometimes omit the subscript R if there is no danger of confusion. Let $a_1, \cdots, a_s$ be elements of S, then the differentials $da_i, \cdots, da_s$ will be called linearly independent if $\lambda_1 da_1 + \cdots + \lambda_s da_s = 0$ ($\lambda_i \in S$) implies necessarily $\lambda_1 = \cdots = \lambda_s = 0$. Differentials $da_1, \cdots, da_s$ are linearly independent if there exist s R-derivations D_i ($i = 1, \cdots, s$) of S such that $D_i(a_j) = \delta_{ij}$.

§ 2. Module of R-differentials

Let S be an R-algebra with the ring homomorphism $f: R \to S$, and let T be an S-algebra with the ring homomorphism $g: S \to T$. Then T is an R-algebra with the ring homomorphism $h = g \cdot f: R \to T$. Let x be an element of S, then $d_R^T x$ stand for the R-differential $dg(x)$ in $D_R(T)$. Then $Dx = d_R^T x$ is an R-derivation of S into a T-module $D_R(T)$, hence there exists an S homomorphism $\alpha: D_R(S) \to D_R(T)$ such that

$$Dx = d_R^T x = \alpha \cdot d_R^S x \qquad \text{for} \quad x \in S .$$

From this we can define a homomorphism $\varphi_{R;S,T}: T \otimes_S D_R(S) \to D_R(T)$ by $\varphi_{R\ S,T}(\sum t_i \otimes d_R^S x_i) = \sum_i t_i d_R^T x_i$, where $t_i \in T$, and $x_i \in S$. Let $N_{R;S,T}$ and $D_{S,T}$ be the kernel and cokernel of $\varphi_{R;S,T}$ respectively, then we have an exact sequence

$$\text{(A)} \qquad 0 \longrightarrow N_{R;S,T} \longrightarrow T \otimes_S D_R(S) \longrightarrow D_R(T) \xrightarrow{j} D_{S,T} \longrightarrow 0 .$$

We shall show that $D_{S,T}$ is isomorphic to $D_S(T)$. To prove this it is sufficient to show that $D_{S,T}$ has the universal mapping property with respect to the differential operator $\tilde{\alpha} = j d_R^T$. Let D be an S-derivation of T into a T-module V. Then D is, a fortiori, an R-derivation of T into V, and there exists a T-homomorphism $\beta: D_S(T) \to V$ such that $Dx = \beta d^T x$ for $x \in T$. Since D is an S-derivation, $Dx = 0$ holds for x in S, i. e. β vanishes on the image of $\varphi_{R;S,T}$. Taking the quotient we get a homomorphism $\tilde{\beta}$ from $D_{S,T}$ into V, and $Dx = \beta d^T x = \tilde{\beta} \cdot \tilde{\alpha} x$ which proves that $D_{S,T}$ has the property (3). Since it is easily seen that $D_{S,T}$ satisfies the properties (1) and (2), the uniqueness property of the module of differentials implies that $D_{S,T}$ is isomorphic to $D_S(T)$. The above results will be formulated in the

PROPOSITION 1. *Let S be an R-algebra and let T be an S-algebra. Then $D_S(T)$ is isomorphic to the difference module $D_R(T)/TD(S)$, where $D(S)$ is a submodule of $D_R(T)$ generated by the elements $\{d_R^T s, s \in S\}$. Moreover if $D_R(S)$ is zero, $D_S(T)$ is isomorphic to $D_R(T)$.*

THEOREM 1. *Let S be an R-algebra and let T be an S-algebra. Then any R-derivation of S into a T-module V can be extended to an R-derivation of T into V if, and only if,* (i) *$\varphi_{R;S,T}$ is injective and* (ii) *$T\otimes D_R(S)$ is the direct summand of $D_R(T)$ (as a T-module)*[3]. *Moreover if it is known that $D_R(S)$ is a finite module, then we'll have* (i) *and* (ii) *if any R-derivation of S into a finite T-module can be extended to an R-derivation of T.*

PROOF. Let $z=\sum_i t_i\otimes d^S x_i$ be such that $\varphi_{R;S,T}(z)=0$, i. e. $\sum_i t_i d^T x_i=0$. (We omit the subscript R for the sake of simplicity.) Since $D: x\to 1\otimes d^S x$ is an R-derivation of S into a (finite) T-module $T\otimes_S D_R(S)$, it can be extended to an R-derivation of T into $T\otimes D_R(S)$. We shall denote this extension by the same letter D. Let α be a T-homomorphism $D_R(T)\to T\otimes_S D_R(S)$ such that $Dx=\alpha d^T x$ for x in T. From $\sum_i t_i d^T x_i=0$, follows $0=\alpha(\sum_i t_i d^T x_i)=\sum_i t_i\alpha d^T x_i=\sum_i t_i D x_i$ $=\sum_i t_i(1\otimes d^S x_i)=\sum_i t_i\otimes d^S x_i$ since x's are in S. Thus $\varphi_{R;S,T}$ is injective. Moreover we see that $\alpha\varphi_{R;S,T}=1$ from the definitions. Hence the exact sequence

$$0\longrightarrow T\otimes D_R(S)\xrightarrow{\varphi_{R:S,T}} D_R(T)\longrightarrow D_{S,T}\longrightarrow 0$$

splits and we get the condition (ii).

Conversely assume that $\varphi_{R;S,T}$ is injective and $T\otimes D_R(S)$ is a direct summand of $D_R(T)$. Let D be an R-derivation of S into a T-module V. Then there exists an S-homomorphism α from $D_R(S)$ into V such that $D=\alpha d^S$. Let α_0 be the T-homomorphism from $T\otimes D_R(S)$ into V which is induced in a natural way by α, i. e. $\alpha_0(t\otimes d^S x)=t\alpha(d^S x)$. Since $T\otimes D_R(S)$ is the direct summand of $D_R(T)$, α_0 can be extended at least in one way to a T-homomorphism $\tilde{\alpha}_0$ from $D_R(T)$ into V. Then the derivation $\tilde{D}=\tilde{\alpha}_0 d^T x$ will be the extension of D. In fact if x is an element of S, we have

$$\tilde{D}x=\tilde{\alpha}_0 d^T x=\alpha_0(1\otimes d^S x)=\alpha d^S x=Dx\,.$$

COROLLARY 1. *If T is a field, then any derivation of S into a T-module can be extended to an R-derivation of T if, and only if, the kernel N of $\varphi_{R;S,T}$ is zero.*

In the above situation $D_S(T)=0$, if the extension of the derivation (if exists) is unique, hence we have the

COROLLARY 2. *The homomorphism $\varphi_{R:S,T}$ is an isomorphism if, and only if, any derivation of S can be extended in a unique way to the derivation of T.*

If T is not a field, the vanishing of the kernel $\varphi_{R:S,T}$ does not necessarily imply that the extension is always possible.

EXAMPLE. Let R be a field of characteristic $\neq 2$, and let $S=R[X^2]$ and $T=R[X]$. The correspondence $1\otimes d(X^2)\to 2XdX$ gives clearly an isomorphism of $T\otimes D_R(S)$ into $D_R(T)$. On the other hand let D be a derivation of S into

3) The necessity of the condition (ii) is pointed out by H. Sato.

T defined by $\partial/\partial(X^2)$. This derivation cannot be extended to the derivation of T into T since $D(X^2)=2XD(X)=1$.

PROPOSITION 2. *Let S be an R-algebra and let U be a multiplicatively closed set in S. Assume that U does not contain any zero divisor. Then we have an isomorphism $D_R(S_U)\cong S_U\otimes D_R(S)$.*

PROOF. By the above Cor. 2 it is sufficient to show that any R-derivation of S into an S_U-module V can be extended in a unique way to an R-derivation of S_U into V. But it is seen immediately that $D(1/u)=-(1/u^2)Du$ is the unique extension of the given derivation D of S.

The examle mentioned before is a special case of the following proposition, in which $\varphi_{R;S,T}$ would be injective though the extension of the derivation might be impossible.

PROPOSITION 3. *Let S be a domain with the quotient field K and L be a separable extension of K. Let T be a subring of L such that L is the quotient field of T. Then if $D_R(S)$ is a free module over S, the homomorphism $\varphi_{R;S,T}$ is an injective map.*

PROOF. Let us consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & T\otimes_S D_R(S) & \xrightarrow{\varphi_{R;S,T}} & D_R(T) \\ & & & & \downarrow{\scriptstyle\varphi} & & \downarrow \\ & & 0 & \longrightarrow & L\otimes_K D_R(K) & \xrightarrow{\varphi_{R;K,L}} & D_R(L) \end{array}$$

where $\varphi_{R;K,L}$ is injective since any derivation of K can be extended to the derivation of L. Let z be an element of $T\otimes_S D_R(S)$ such that $\varphi_{R;S,T}(z)=0$. From the above diagram we see that $\varphi(z)=0$. Let $z=\sum_i t_i\otimes d^S s_i$, where $\{d^S s_i\}$ is a free base of $D_R(S)$. By Prop. 2, $D_R(K)=K\otimes_S D_R(S)$, hence $D_R(K)$ is also a free module with the base $\{d^K s_i\}$. From the general property of tensor product, $L\otimes_K D_R(K)$ is also a free module with the base $\{1\otimes d^K s_i\}$. Hence if $\varphi(z)=\sum_i t_i\otimes d^K s_i=0$, we have $t_i=0$ for all i, i.e. $z=0$. Thus the proof is complete.

PROPOSITION 4. *Let S be an R-algebra and assume that S is the direct sum of two ideals A and B. Then $D_R(S)=D_R(A)+D_R(B)$ (direct sum), and A annihilates $D_R(B)$ and B annihilates $D_R(A)$.*

PROOF. Since S is the direct sum of A and B, we have $S\otimes_R S=A\otimes_R A+B\otimes_R B+A\otimes_R B+B\otimes_R A$ (direct sum). Let $\mathfrak{R}_S$, $\mathfrak{R}_A$ and $\mathfrak{R}_B$ be respectively the relation deals of $S\otimes S$, $A\otimes A$ and $B\otimes B$ defining the modules of differentials. Then it is easily seen that $\mathfrak{R}_S=\mathfrak{R}_A+\mathfrak{R}_B+A\otimes B+B\otimes A$. Let a and b be elements of A and B respectively, then $\varphi_S(a\otimes b)=ab=0$. Hence $a\otimes b$ is in I_S for any elements a in A and b in B. From this we can see that $a\otimes b=(a\otimes e_B)(e_A\otimes b)$ is contained in I_S^2 where e_A and e_B are units of A and B respectively. Thus we get the equality $\mathfrak{R}_S=\mathfrak{R}_A+\mathfrak{R}_B+A\otimes B+B\otimes A$. From this we can see that $D_R(S)=(S\otimes S)/\mathfrak{N}_S=(A\otimes A)/\mathfrak{N}_A+(B\otimes B)/\mathfrak{N}_B=D_R(A)+D_R(B)$. $adb=0$ follows from the fact that $a\otimes b$ is contained in I_S^2.

DEFINITION 1. *Let S be an R-algebra, we shall say that S is a quasi-separable R-algebra if we have $D_R(S)=0$.*

The following propositions are immediate consequences of the exact sequence (A).

PROPOSITION 5. *If S is a quasi-separable R-algebra and if T is a quasi-separable S-algebra, then T is a quasi-separyble R-algebra. On the other hand if T is a quasi-separable R-algebra, then T is a quasi-separable S-algebra.*

PROPOSITION 6. *Let A be an arbitrary ring and let $\mathfrak{a}$ be an ideal of A, then $A/\mathfrak{a}$ is a quasi-separable A-algebra.*

The following proposition is a direct consequence of the definitions and Prop. 6.

PROPOSITION 7. *Let S be a quasi-separable R-algebra and let $\mathfrak{M}$ be an ideal of S. Let $\mathfrak{m}$ be an ideal of R such that $\mathfrak{m}\subseteqq\mathfrak{M}\cap R$. Then $S/\mathfrak{M}$ is a quasi-separable $(R/\mathfrak{m})$-algebra.*

PROOF. Let $\pi_S: S\rightarrow S/\mathfrak{M}$ and $\pi_R: R\rightarrow R/\mathfrak{m}$ be respectively the natural homomorphism. Then there exists a homomorphism $\tilde{f}: R/\mathfrak{m}\rightarrow S/\mathfrak{M}$ such that $\pi_S\cdot f=\tilde{f}\cdot\pi_R$. Then $S/\mathfrak{M}$ is a quasi-separable R-algebra and hence it is quasi-separable over $(R/\mathfrak{m})$ with the ring homomorphism $\tilde{f}$ by Prop. 5.

PROPOSITION 8. *Let S be a quasi-separable R-algebra and let U be a multiplicatively closed set in S. Let V be the set $U\cap R$ then the quotient ring S_U is quasi-separable over R and R_V respectively.*

This is an immediate consequence of the successive applications of Prop. 7, Prop. 2 and Prop. 5.

Next we shall consider the generalization of Prop. 2 in the case where U is an arbitrary multiplicatively closed set in S such that U does not contain 0 but contains 1. Let S be an R-algebra and let $\mathfrak{a}$ be an ideal of S. Then applying the functor $\otimes_S D_R(S)$ to the exact sequence

$$0\longrightarrow\mathfrak{a}\longrightarrow S\longrightarrow S/\mathfrak{a}\longrightarrow 0$$

we see easily that $(S/\mathfrak{a})\otimes_S D_R(S)$ is isomorphic to the difference module $D_R(S)/\mathfrak{a}D_R(S)$. Now we can define an S-homomorphism ρ from $\mathfrak{a}/\mathfrak{a}^2$ in $D_R(S)/\mathfrak{a}D_R(S)$ by the rule

$$\rho(\text{class of } a)=\text{class of } da,\quad a\in\mathfrak{a}.$$

We shall show the following

PROPOSITION 9. *Using the notations as above*

$$(\textit{kernel of } \varphi_{R;S,S/\mathfrak{a}})=(\textit{Image of } \rho)$$

i. e. the sequence

$$\mathfrak{a}/\mathfrak{a}^2\longrightarrow(S/\mathfrak{a})\otimes_S D_R(S)\longrightarrow D_R(S/\mathfrak{a})\longrightarrow 0$$

is exact.[4)]

Proof. For the sake of simplicity we shall use the notation d for d_R^S and d^* for $d_R^{(S/\mathfrak{a})}$. We shall also denote the class of elements x mod $\mathfrak{a}$ by $\bar{x}$. Then the map $h=\varphi_{R;S,S/\mathfrak{a}}$ is given by the rule

$$h(\sum_i \bar{t}_i \otimes dx_i)=\sum_i \bar{t}_i d^* \bar{x}_i .$$

Hence $D(\mathfrak{a})=\{S\text{-module generated by the differentials } da,\ a\in\mathfrak{a}\}$ and $\mathfrak{a}D_R(S)$ are contained in the kernel of h. Thus the homomorphism h induces the homomorphism $\tilde{h}$ of $D_R(S)/D(\mathfrak{a})+\mathfrak{a}D_R(S)$ into $D_R(S/\mathfrak{a})$. We shall show that $\tilde{h}$ is an isomorphism. Let δ be the map from $S/\mathfrak{a}$ into $D_R(S)/D(\mathfrak{a})+\mathfrak{a}D_R(S)$ defined by the following way. Let $\bar{a}$ be an element of $S/\mathfrak{a}$ and let a be one of its representative, then

$$\delta(\bar{a})=\{\text{the class of } da \bmod D(\mathfrak{a})+\mathfrak{a}D(S)\} .$$

It is not difficult to verify that the map δ gives actually an R-derivation of $S/\mathfrak{a}$ into $(S/\mathfrak{a})\otimes D_R(S)$. Then there exists an $(S/\mathfrak{a})$-homomorphism τ from $D_R(S/\mathfrak{a})$ into $(S/\mathfrak{a})\otimes D_R(S)$ such that $\delta=\tau d^*$. Let x be an arbitrary element of S, then $\tilde{h}\tau d^*(\bar{x})=\tilde{h}\delta(\bar{x})=\tilde{h}$ (class of dx) $=h(dx)=d^*\bar{x}$. From this we see that $\tilde{h}\tau=$ identity map. Similarly $\tau\tilde{h}$(class of $adx)=\tau(\bar{a}d^*\bar{x})=\bar{a}\tau d^*(\bar{x})=\bar{a}\delta(\bar{x})=\{$class of $adx\}$ and we see that $\tau h=$ identity. Thus h is an isomorphism. On the other hand it is seen at once that the image of ρ is equal to the class of $\mathfrak{a}D_R(S)+D(\mathfrak{a})$ mod $\mathfrak{a}D_R(S)$. The final assertion follows from the isomorphism theorem.

Proposition 10. *Let S be an R-algebra and let U be a multiplicatively closed set in S, then we have $D_R(S_U)\cong S_U\otimes D_R(S)$.*

Proof. Let $\mathfrak{n}$ be the ideal of S generated by the elements x such that $xu=0$ for some element u in U. Let $S^*=S/\mathfrak{n}$, then $S_U=S^*_{U^*}$ by definition where U^* is the image of U under the natural homomorphism $S\to S^*$. By Prop. 9 we have an exact sequence

$$\mathfrak{n}/\mathfrak{n}^2\longrightarrow S^*\otimes_S D_R(S)\longrightarrow D_R(S^*)\longrightarrow 0 .$$

Applying the exact functor $\otimes S^*_{U^*}$ $(=\otimes S_U)$ we get

$$(\mathfrak{n}/\mathfrak{n}^2)\otimes_{S^*} S^*_{U^*}\longrightarrow S^*_{U^*}\otimes_S D_R(S)\longrightarrow D_R(S^*)\otimes S^*_{U^*}\longrightarrow 0 .$$

Since U^* does not contain any zero divisor, $D_R(S^*)\otimes S^*_{U^*}$ is isomorphic to $D_R(S^*_{U^*})$ (Prop. 2)) and $(\mathfrak{n}/\mathfrak{n}^2)\otimes S^*_{U^*}=0$ which completes the proof.

Proposition 11. *Let S and T be R-algebras and assume that S is quasi-separable over R, then $S\otimes_R T$ is a quasi-separable T-algebra.*

Proof. Let us put $Y=S\otimes_R T$, then we have an exact sequence

$$Y\otimes_T D_R(T)\longrightarrow D_R(Y)\xrightarrow{h} D_T(Y)\longrightarrow 0 ,$$

4) The special case of this exact sequence is given in a more precise form in Exposé 17 in [4].

$$Y\otimes_S D_R(S)\longrightarrow D_R(Y)\xrightarrow{\rho} D_S(Y)\longrightarrow 0\,.$$

By our assumption we have $D_R(S)=0$, hence $D_R(Y)$ and $D_S(Y)$ are isomorphic by the isomorphism ρ defined by

$$\rho d_R^Y(y)=d_S^Y(y)\,,\qquad y\in Y\,.$$

Let y be an arbitrary element of Y and set $y=\sum_i s_i\otimes t_i$ where $s_i\in S, t_i\in T$. Then

$$\begin{aligned} d_T^Y(y) &= hd_R^Y(y)=h\rho^{-1}d_S^Y(\sum_i s_i\otimes t_i)=h\rho^{-1}\sum_i(s_i\otimes 1)d_S^Y(1\otimes t_i)\\ &=\sum_i(s_i\otimes 1)h\rho^{-1}d_S^Y(1\otimes t_i)=\sum_i(s_i\otimes 1)hd_R^Y(1\otimes t_i)\\ &=\sum_i(s_i\otimes 1)d_T^Y(1\otimes t_i)=0\,. \end{aligned}$$

This proves the assertion.

The following proposition is immediate.

PROPOSITION 12. *Let k be a field and let L be the perfect closure of k, or a separable extension of k, then $D_k(L)=0$.*

Let S be an R-algebra. If S is a field and R is a subfield of S such that $[S:R]<\infty$, then it is easily seen that $D_R(S)$ is equal to zero if, and only if, S is a separable extension of R. We can prove a generalization of this results in the following

PROPOSITION 13. *Let S be an algebra over a field R and assume that S is of finite rank over R. Then $D_R(S)=0$ if, and only if, S is a separable algebra*[5] *over R.*

PROOF. Assume first that S is a separable R-algebra. Then S is the direct sum of the fields K_i where K_i's are separable extensions of R. Hence $D_R(S)=D_R(K_1)+\cdots+D_R(K_h)=0$ (Prop. 4).

Conversely assume that $D_R(S)=0$, and let $S^e=S\otimes_R S$. Then we have an exact sequence of S^e-modules

$$0\longrightarrow I\longrightarrow S^e\xrightarrow{\varphi} S\longrightarrow 0\,.$$

Since S is of finite rank over R and R is a field, the ideal I has finite generators. By our assumption $D_R(S)=I/I^2=0$, we see that $I=I^2$. From this we can conclude that S is S^e-projective ([**2**, Th. 2.5]). Hence S is a separable R-algebra ([**5**, Th. 7.10]).

The following Proposition gives a sufficient condition for S to be a quasi-separable R-algebra.

PROPOSITION 14.[6] *Let S be an R-algebra and assume that S is S^e-projective,*

5) When R is a field, S is called a separable R-algebra if $S\otimes_R K$ is semi-simple for any extension K of R (cf. [**1**]).

6) This result is stronger than the analogous one in [**2**].

then S is quasi-separable over R.

PROOF. Assume that S is S^e-projective, then there exists an element z in $S\otimes S$ such that $zI=Iz$ and $\varphi(z)=1$ ([**5**, IX, Th. 7.7]). If $z=\sum_i a_i\otimes b_i$, then $\sum_i a_ib_i=1$. Let x be an arbitrary element of S. Since $z(1\otimes x)=z(x\otimes 1)$, we get the relation

$$\sum a_i\otimes b_ix=\sum_i a_ix\otimes b_i\,.$$

Hence

$$\begin{aligned}0&=\sum_i a_i\otimes b_ix-\sum_i a_ix\otimes b_i\\&=\sum_i a_i(1\otimes b_ix-x\otimes b_i)\\&\equiv\sum_i a_i(b_i\otimes x)=(\sum_i a_ib_i)\otimes x=1\otimes x \qquad (\mathrm{mod}\ \mathfrak{N})\end{aligned}$$

i.e. $1\otimes x\equiv 0 \pmod{\mathfrak{N}}$. Since $\mathfrak{N}=S\otimes 1+I^2$ we see that $1\otimes x-x\otimes 1\equiv 0 \pmod{I^2}$. Since I is generated by $\{1\otimes x-x\otimes 1\}$, the above argument shows that $I=I^2$, hence $D_R(S)=0$.

§ 3. The structure of $D_R(S)$

The following proposition is proved elsewhere (cf. [**4**]), but for the sake of convenience of readers we shall write down the proof.

PROPOSITION 15. *Let R be a ring and let Λ be an index set, and let $\{X_\lambda, \lambda\in\Lambda\}$ be a set of indeterminates. Let A be a polynomial ring in $\{X_\lambda\}$ with coefficients in R. Then $D_R(A)$ is a free module with the base $\{dX_\lambda\}$.*

PROOF. Let us introduce a new set of indeterminates $\{Y_\lambda\}$ with the same index set Λ. Then $A\otimes_R A$ is isomorphic to the polynomial ring in two sets of indeterminates $\{X_\lambda\}$ and $\{Y_\lambda\}$. The homomorphism φ from $A\otimes_R A$ into A defined in § 1 is given by the substitution of X_λ in the place of Y_λ. Let us put $Z_\lambda=X_\lambda-Y_\lambda$ and let us represent the elements of $A\otimes A$ as polynomials in X_λ and Z_λ. The kernal I of φ is given as the set of polynomials of degree $\geqq 1$ in Z_λ and I^2 is given as the set of polynomials of degree $\geqq 2$ in Z's. Since $D_R(A)$ is isomorphic (as a A-module) to I/I^2, we see immediately that $D_R(A)$ is ismorphic to the module of homogeneous elements of degree 1 in Z's. By definition $Z_\lambda=dX_\lambda$ and they are linearly independent over A. Thus the proof is complete.

Let S be a ring containing a ring R and $\{z_\lambda, \lambda\in\Lambda\}$ be a set of generators of S over R. Let $\{Z_\lambda, \lambda\in\Lambda\}$ be a set of indeterminates with the index set Λ and let A be a polynomial ring in $\{Z_\lambda\}$ with coefficients in R. Then there exists an R-homomorphism $\varphi: A\to S$ such that

$$\varphi(Z_\lambda)=z_\lambda, \qquad \lambda\in\Lambda\,.$$

The kernel $\mathfrak{M}$ of φ will be called the defining ideal of S with respect to the generator system $\{z_\lambda\}$. Applying Prop. 9 to $S=A/\mathfrak{M}$, we get the exact sequence

$$\mathfrak{M}/\mathfrak{M}^2 \xrightarrow{\rho} S\otimes_A D_R(A) \xrightarrow{\varphi_{R;A,S}} D_R(S) \longrightarrow 0\,.$$

Since $D_R(A)$ is a free module, $S\otimes D_R(A)$ is also a free module over S with the base $\{1\otimes d_R^A Z_\lambda, \lambda\in\Lambda\}$. In the following we shall write simply dZ_λ for $1\otimes d_R^A Z_\lambda$. Then $S\otimes D_R(A)$ will be written as $D(S^*)=\sum SdZ_\lambda$. Let $\{F_\mu, \mu\in M\}$ be a set of generators of the defining ideal $\mathfrak{M}$, and let N be the submodule of $D(S^*)$ generated by the forms

$$\{\sum_\lambda(\partial F_\mu/\partial z_\lambda)dZ_\lambda, \mu\in M\}\,.$$

Then the image of the homomorphism ρ is given by N. From this we get the following

THEOREM 2. *Using the same notations and assumptions as above, the module of differentials $D_R(S)$ is isomorphic to the difference module $D(S^*)/N$.*

COROLLARY 1. *In the above notations assume that S is an integral domain of positive characteristic p and that the defining ideal $\mathfrak{M}$ of S is generated by the polynomials $\{F_\mu\}$ such that $\{F_\mu\}$ are polynomials in $\{Z_\lambda^p, \lambda\in\Lambda\}$. Then $D_R(S)$ is a free module over S.*

COROLLARY 2. *Let R be a normal domain with the quotient field K and let L be a separably algebraic extension of K. Let z be an element of L integral over R and let $f(X)$ be an irreducible monic polynomial for z over R. Then $D_R(R[z])$ is isomorphic (as R-module) to $R/(f'(z))$.*[7]

PROOF. In this case the defining ideal of $R[z]$ is the principal ideal $(f(X))$. Hence the Cor. follows directly from the Theorem.

§ 4. Differentials in local rings

Let R be a local ring and let M be the maximal ideal of R. Assume that R contains a field k. Let us denote by L the residue class field of R, then k can be considered in a natural way as a subfield of L. By Prop. 9 we have an exact sequence

$$M/M^2 \longrightarrow L\otimes_R D_k(R) \longrightarrow D_k(L) \longrightarrow 0\,.$$

Moreover if L is a separable extension of k, then the sequence

$$\text{(G)} \qquad 0 \longrightarrow M/M^2 \longrightarrow L\otimes_R D_k(R) \longrightarrow D_k(L) \longrightarrow 0$$

is known to be exact (Th. 5 in Exposé 17 in [4]).

Let R be a local ring with the maximal ideal M and let E be a finite R-module. A set of elements $u_1,\cdots,u_n$ of E is a system of generators of E if, and only if, $1\otimes u_i$ $(i=1,2,\cdots,n)$ form a system of generators of $(R/M)\otimes_R E$.

7) The analogous result is obtained also in [6] and [8].

Moreover $(u_1, \cdots, u_n)$ is a minimal set of generators of E if, and only if, $1 \otimes u_1, \cdots, 1 \otimes u_n$ form a base of the vector space $(R/M) \otimes E$ over the field R/M. Thus the number of the minimal set of generators of E (over R) is a well defined integer independent of the choice of the generator system of E. We shall call this integer the *rank of* E and it will be denoted by $\mathrm{rank}_R E$.

PROPOSITION 16. *Let R be a local ring containing a field k such that the residue class field of R is a finite separable extension of k, and that $D_k(R)$ is a finite module. Let $u_1, \cdots, u_n$ be a set of elements in the maximal ideal M of R. Then the differentials $du_1, \cdots, du_n$ (where d stands for d_k^R) form a minimal set of generators of $D_k(R)$ if, and only if, $u_1, \cdots, u_n$ form a minimal set of generators of M.*

THEOREM 3. *Let A be an affine ring*[8] *over a field k and let R be the quotient ring of A with respect to a prime ideal $\mathfrak{P}$. Assume that* (i) *A is an integral domain;* (ii) *the quotient field K of A is a separable extension of k;* (iii) *the residue class field L of R is separably algebraic over k. Then R is a regular local ring if, and only if, the module of k-differentials $D_k(R)$ is a free module.*

PROOF. By our assumption $\mathfrak{P}$ is a maximal ideal of A, hence the rank of R(=the rank of $\mathfrak{P}$) is equal to the transcendence degree of A over k. Let us put $r = \mathrm{rank}\, R$. By Prop. 2 we have $D_k(K) = K \otimes D_k(R)$. Moreover this isomorphism is given by the rule $1 \otimes d^R u \rightarrow d^K u$ (where we omit the subscript k in the differential operators). Now assume that R is a regular local ring and let $u_1, \cdots, u_r$ be a regular system of parameters of R. By the preceding Proposition we see that $\{d^R u_1, \cdots, d^R u_r\}$ is a minimal set of generators of $D_k(R)$. Hence by the above isomorphism we see that $\{d^K u_1, \cdots, d^K u_r\}$ is a system of generators of $D_k(K)$. On the other hand K is a separable extension of dimension r over k, hence $D_k(K)$ is a free module of rank r. This implies that $d^K u_1, \cdots, d^K u_r$ are linearly independent over K. That $d^R u_1, \cdots, d^R u_r$ are linearly independent over R follows immediately from this.

Conversely assume that $D_k(R)$ is a free module over R and let $\{u_1, \cdots, u_n\}$ be a minimal set of generators of M (=the maximal ideal of R). Then $\{d^R u_1, \cdots, d^R u_n\}$ form a base of $D_k(R)$. Applying again Proposition 2 we see that $D_k(K) = K \otimes_R D_k(R)$ is also a vector space of dimension n. From this we get immediately the equality $n = r$, proving that R is a regular local ring.

As an application of the preceding Proposition we can give an alternative proof of the Jacobian criterion for simple points (cf. [**4**], [**9**]).

PROPOSITION 17 *Let V^r be an affine variety defined over k and let $\mathfrak{P}$ be the defining ideal of V in $k[x_1, \cdots, x_n]$. Let P be a point of V such that $k(P)$ is separably algebraic over k and let $\mathfrak{O}$ be the quotient ring of P in V/k. Then $\mathfrak{O}$ is a regular local ring if, and only if, there exist $(n-r)$-polynomials $f_1, \cdots, f_{n-r}$ in*

8) An affine ring over k is a homomorphic image of a polynomial ring in a finite number of variables over k.

$\mathfrak{P}$ *such that the rank of the matrix* $(\partial f_i/\partial x_j)$ *is* $n-r$ *at* P.

PROOF. Let $\mathfrak{m}$ be the maximal ideal of $A=k[x_1,\cdots,x_n]$ defining the point P and let us put $R=A_{\mathfrak{m}}$. Then $\mathfrak{O}=R/\mathfrak{P}R$. By Prop. 9 we have the exact sequence

$$\mathfrak{P}^*/\mathfrak{P}^{*2}\xrightarrow{\ \rho\ }(P/\mathfrak{P}^*)\otimes_R D_k(R)\xrightarrow{\ i\ }D_k(R/\mathfrak{P}^*)\longrightarrow 0$$

where $\mathfrak{P}^*=\mathfrak{P}R$. Now assume that $R/\mathfrak{P}^*$ is a regular local ring. Applying Theorem 3 we see that $D_k(R/\mathfrak{P}^*)$ is a free module of rank $n-r$. Let φ be an $(R/\mathfrak{P}^*)$-homomorphism: $D_k(R/\mathfrak{P}^*)\to(R/\mathfrak{P}^*)\otimes D_k(R)$ such that

$$i\circ\varphi=\text{the identity}.$$

Then the above sequence implies the direct sum decomposition

$$(R/\mathfrak{P}^*)\otimes D_k(R)=\mathrm{Im}\,\rho+\mathrm{Im}\,\varphi\,.$$

Since $D_k(R)$ is a free module with the base $\{dx_1,\cdots,dx_n\}$, $(R/\mathfrak{P}^*)\otimes_R D_k(R)$ is also a free module with the base $\{1\otimes dx_1,\cdots,1\otimes dx_n\}$. Since $R/\mathfrak{P}^*$ is a local ring, each term in the right hand side is also a free module, and the rank of $\mathrm{Im}\,\varphi=n-r$. From this we can find $(n-r)$-polynomials $f_1,\cdots,f_{n-r}$ in $\mathfrak{P}$ such that $\{df_1,\cdots,df_{n-r}\}$ form a subsystem of free base of $D_k(R)$ module $\mathfrak{P}^*D_k(R)$ (where $(R/\mathfrak{P}^*)\otimes_R D_k(R)$ is canonically isomorphic to the residue class module $D_k(R)/\mathfrak{P}^*D_k(R)$). On the other hand we have

$$(R/\mathfrak{m}^*)\otimes_R D_k(R)=(R/\mathfrak{m}^*)\otimes_{(R/\mathfrak{P}^*)}((R/\mathfrak{P}^*)\otimes_R D_k(R))\,.$$

This implies that a free base of $(R/\mathfrak{P}^*)\otimes D_k(R)$ is also a free base of $(R/\mathfrak{m}^*)\otimes D_k(R)$. Hence $df_1,\cdots,df_{n-r}$ must also be linearly independent module $\mathfrak{m}^*D_k(R)$. This is the required result. Conversely assume that $\mathfrak{P}$ contains $(n-r)$-polynomials $f_1,\cdots,f_{n-r}$ such that the rank of the matrix $(\partial f_i/\partial x_j)$ modulo $\mathfrak{m}$ is equal to $n-r$. Then $\{df_1,\cdots,df_{n-r}\}$ is a part of minimal base of $D_k(R)$ modulo $\mathfrak{m}^*D_k(R)$. Hence we can find a free base of $D_k(R)$ containing $df_1,\cdots,df_{n-r}$ as members. Since $\mathrm{Im}(\rho)$ is contained in $D_k(R)df_1+\cdots+D_k(R)df_{n-r}+\mathfrak{P}^*D_k(R)$, $\mathrm{Im}(\rho)=(R/\mathfrak{P}^*)\cdot 1\otimes df_1+\cdots+(R/\mathfrak{P}^*)\cdot 1\otimes df_{n-r}$. Moreover $\{1\otimes df_1,\cdots,1\otimes df_{n-r}\}$ is a part of free base of $(R/\mathfrak{P}^*)\otimes D_k(R)$, hence $D_k(R/\mathfrak{P}^*)$ is a free module over k. Now the assertion follows from Th. 3.

COROLLARY 1. *Retaining the natations and assumptions as in Prop.* 17, P *is a simple point of* V *if, and only if,* $D_k(\mathfrak{O})$ *is a free module.*

COROLLARY 2. *Retaining the notations and assumptions as in Prop.* 17 *and assume that* $\mathfrak{O}$ *is a regular local ring. Then, any regular system of parameters is a separating transcendent base of the function field* $k(V)$ *over* k.

REMARK. We shall give here some examples which show that the assumption made in these propositions are inevitable.

EXAMPLE 1. Let k be a non-perfect field and let u be an element of k such that $u^{\frac{1}{p}} \notin k$. Let $A = k[X, Y]/(X^p + aY^p)$ and let $R = A_{(x,y)}$, where x and y are residue classes of X and Y respectively. Then $D_k(R)$ is a free module with the base dx and dy (cf. Cor. 1 of Th. 2), but R is not a regular local ring. In this example the quotient field K of R is not a separable extension of k.

EXAMPLE 2. Let V be a plane curve defined over a non-perfect field k by the equation $X^2 + Y^p = u$, where u is an element of k such that $u^{\frac{1}{p}}$ is not contained in k. As is easily seen V is a normal curve, and in particular the point $P = (0, u^{\frac{1}{p}})$ is a normal point of V and the quotient ring R of P in V/k is a regular local ring. Let A be the affine coordinate ring of V. By Th. 2 we see easily that $D_k(A)$ is generated by dx and dy with the relation $2xdx = 0$. Hence $D_k(R) = R \otimes D_k(A)$ is also generated by $d^R x$ and $d^R y$ with the relation $2xd^R x = 0$ $(d^R x \neq 0)$. Thus $D_k(R)$ is not a free module. In this example the assumption (iii) in Th. 3 is not satisfied.

In the same example the coordinate function x is a regular system of parameter of R. But $k(V)$ is not separable over $k(X)$, showing the Cor. 2 does not hold in general if $k(P)$ is not separable over k.

Let R be a local domain containing a field k and let M be the maximal ideal of R. Let L be the residue field of R and assume that L is a separable extension of k. Let $\{u_1, \cdots, u_s\}$ be a set of separating transcendence basis of L over k and let $\alpha_1, \cdots, \alpha_s$ be representatives of $u_1, \cdots, u_s$ in R. The elements $\alpha_1, \cdots, \alpha_s$ are algebraically independent over k, hence the field $k(\alpha_1, \cdots, \alpha_s) = F$ is contained in the ring R. Let us denote by K the quotient field of R and assume that it is possible to find elements u's and α's in such a way that K is a seperable extension of the field F. In this case we shall say that the local ring R has a *separating residue field.* Now we can generalize Theorem 3 in the following

THEOREM 3′. *Let A be an affine ring over a field k and let R be the quotient ring of A with respect to a prime ideal. Assume that* (i) *A is an integral domain,* (ii) *the quotient field K of A is a separable extension of k,* (iii) *the residue class field of R is a separable extension of k,* (iv) *R has a separating residue field. Then the module of differentials $D_k(R)$ is a free module if, and only if, R is a regular local ring.*

PROOF. Let L be the residue field of R. By our assumptions there exists a field F in R such that L is separably algebraic over F and the quotient field K of R is a separable extension of F, and such that F is a purely transcendental extension of k of dimension r $(r = \dim_k L)$. Since K is a separable extension over F, the module of differentials $D_F(K)$ is a free module of rank $n-r$ where $n = \dim_k K$. Now assume that R is a regular local ring. By Th. 3 we see that $D_F(R)$ is a free module. On the other hand, applying the Prop. 3 and the exact sequence (A) we see that the sequence

$$0 \longrightarrow R \otimes_F D_k(F) \longrightarrow D_k(R) \longrightarrow D_F(R) \longrightarrow 0$$

is exact. Since $D_F(R)$ is a free module the above exact sequence splits and $D_k(R)$ is isomorphic to the direct sum of $R \otimes D_k(F)$ and $D_F(R)$, both of them are free modules over R. Thus we have proved that $D_k(R)$ is a free module. Conversely assume that $D_k(R)$ is a free module of rank m. Then $D_k(R) = K \otimes_R D_k(R)$ is also a free module of rank m, hence $m=n$. Let M be the maximal ideal of R. Then R/M is a separable extension of dimension r over k and $D_k(R/M)$ is a vector space of dimension r over the field R/M. From this and the exact sequence

$$0 \longrightarrow M/M^2 \longrightarrow (R/M) \otimes D_k(R) \longrightarrow D_k(R/M) \longrightarrow 0\,,$$

we see that M/M^2 is a vector space of dimension $n-r$ over R/M. Since the rank of R is n$-r$, R must be a regular local ring.

COROLLARY. *Let V^n be a variety defined over a field k and let P be a point of V (not necessarily algebraic over k) and let R be the quotient ring of P in V/k. Now assume that the local ring R has a separating residue field, then the module of differentials $D_k(R)$ is a free module if, and only if, P is a simple point of V. Moreover the rank of $D_k(R)$ is equal to n.*

§ 5. Ramification theory and quasi-separability

Let S be an R-algebra and let $\mathfrak{P}$ be a prime ideal of S and let $\mathfrak{p}$ be the contraction of $\mathfrak{P}$ in R. We shall say that $\mathfrak{P}$ *is unramified* if

(U_1) $\mathfrak{p}S_{\mathfrak{P}} = \mathfrak{P}S_{\mathfrak{P}}$,

(U_2) $S_{\mathfrak{P}}/\mathfrak{P}S_{\mathfrak{P}}$ is a finite separable extension of $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

Let $\mathfrak{p}$ be a prime ideal of R, then S will be said to be *unramified over* $\mathfrak{p}$ if

(U_1') every prime ideal $\mathfrak{P}$ of S such that $\mathfrak{P} \cap R = \mathfrak{p}$ is unramified;

(U_2') There exist only a finite number of primes in S such that $\mathfrak{P} \cap R = \mathfrak{p}$.

We shall say that S *is unramified* (over R) if S is unramified over every prime ideal $\mathfrak{P}$ of R.

The following proposition is convenient in further discussions.

PROPOSITION 18.[9] *Let S be an R-algebra and let $\mathfrak{p}$ be a prime ideal of R. Then S is unramified over $\mathfrak{p}$ if, and only if, $S_U/\mathfrak{p}S_U$ is a separable $(R_U/\mathfrak{p}R_U)$-algebra, where $U = R - \mathfrak{p}$.*

PROOF. Let $\mathfrak{P}$ be a prime ideal of S. Then we have $\mathfrak{P} \cap R = \mathfrak{p}$ if, and only if, $\mathfrak{P}S_U/\mathfrak{p}S_U$ is the prime ideal of $S^* = S_U/\mathfrak{p}S_U$. Now assume that S^* is a separable $R^* = R_U/\mathfrak{p}R_U$ algebra. Then there are only a finite number of prime

9) This formulation is due to [**2**].

ideals $\mathfrak{Q}_1, \cdots, \mathfrak{Q}_h$ in S^* and we have $\bigcap_{i=1}^{h} \mathfrak{Q}_i = 0$. Let $\tilde{\mathfrak{P}}_i$ be the complete inverse image of $\mathfrak{Q}_i$ in S_U, and let $\mathfrak{P}_i = \tilde{\mathfrak{P}}_i \cap S$. Then $\tilde{\mathfrak{P}}_i = \mathfrak{P}_i S_U$, $\bigcap_{i=1}^{h} \mathfrak{P}_i S_U = \bigcap \tilde{\mathfrak{P}}_i = \mathfrak{p} S_U$. Hence we have $\mathfrak{P}_i S_{U(\mathfrak{P}_i S_U)} = \mathfrak{p} S_{U(\mathfrak{P}_i S_U)}$. Since $S_{U(\mathfrak{P}_i S_U)} = S_{\mathfrak{P}_i}$, we get the relation ($\mathrm{U}_1$). Moreover $S_U/\tilde{\mathfrak{P}}_i = (S_U/\mathfrak{p} S_U)/\mathfrak{Q}_i$ is a separable extension of the field R^*, and (U_2) is also satisfied.

Conversely assume that S is unramified over R, then we see easily that any two prime ideals $\mathfrak{P}$ and $\mathfrak{Q}$ in S such that $\mathfrak{P} \cap R = \mathfrak{Q} \cap R = \mathfrak{p}$ cannot satisfy the inclusion relations. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_h$ be all the prime ideals of S such that $R \cap \mathfrak{P}_i = \mathfrak{p}$, then $\bigcap_{i=1}^{h} \mathfrak{P}_i S_U = \mathfrak{p} S_U$ and $S_U/\mathfrak{p} S_U = S_U/\mathfrak{P}_1 S_U + \cdots + S_U/\mathfrak{P}_h S_U$ (direct sum). Since $S_U/\mathfrak{P}_i S_U = S_{\mathfrak{P}_i}/\mathfrak{P}_i S_{\mathfrak{P}_i}$ is a separable field extension of R^*, S^* is also a separable R^*-algebra, and the proof is complete.

In the following we shall discuss the relation between quasi-separability condition and the unramifiedness condition.

Definition 2. *Let S be an R-algebra and let $\mathfrak{P}$ be a prime ideal of S. We shall denote by $\mathfrak{p}$ the contraction of $\mathfrak{P}$ in R. We shall say that $\mathfrak{P}$ satisfies the finiteness condition if $S_{\mathfrak{P}}/\mathfrak{P} S_{\mathfrak{P}}$ is a finite extension of $R_{\mathfrak{p}}/\mathfrak{p} R_{\mathfrak{p}}$.*

S will be said to satisfy the finiteness condition over the prime ideal $\mathfrak{p}$ of R, if $S_U/\mathfrak{p} S_U$ is a finite $(R_U/\mathfrak{p} R_U)$-algebra, where $U = R - \mathfrak{p}$.

An R-algebra S will be said to satisfy the finiteness condition if S satisfies the finiteness condition over any prime ideal of R.

Theorem 4. *Let S be an R-algebra and let $\mathfrak{P}$ be a prime ideal of S satisfying the finiteness conditions, and let $\mathfrak{p}$ be its contraction in R. Then if the quotient ring $S_{\mathfrak{P}}$ is noetherian and quasi-separable over $R_{\mathfrak{p}}$, then $\mathfrak{P}$ is unramified.*

Proof. By our assumption $S_{\mathfrak{P}}$ is quasi-separable over $R_{\mathfrak{p}}$ and $\mathfrak{p} R_{\mathfrak{p}}$ is contained in $\mathfrak{P} S_{\mathfrak{P}}$. Hence $S_{\mathfrak{P}}/\mathfrak{P} S_{\mathfrak{P}}$ is quasi-separable over the field $R_{\mathfrak{p}}/\mathfrak{p} R_{\mathfrak{p}}$ by Prop. 7. This and the finiteness condition implies the condition (U_2). Let $S^* = S_{\mathfrak{P}}/\mathfrak{p} S_{\mathfrak{P}}$, then S^* is a local ring with the maximal ideal $\mathfrak{P}^* = \mathfrak{P} S_{\mathfrak{P}}/\mathfrak{p} S_{\mathfrak{P}}$, and S^* contains the field R^* such that the residue field $S^*/\mathfrak{P}^*$ $(= S_{\mathfrak{P}}/\mathfrak{P} S_{\mathfrak{P}})$ is separably algebraic over R^*. From (G) in § 3 we get an exact sequence

$$0 \longrightarrow \mathfrak{P}^*/\mathfrak{P}^{*2} \longrightarrow (S^*/\mathfrak{P}^*) \otimes_{S^*} D_{R^*}(S^*) .$$

From $D_{R_{\mathfrak{p}}}(S_{\mathfrak{P}}) = 0$ we get $D_{R^*}(S^*) = 0$ by virtue of Prop. 9. Hence the above exact sequence implies that $\mathfrak{P}^* = \mathfrak{P}^{*2}$. Since $\mathfrak{P}^*$ has a finite set of generators we must have $\mathfrak{P}^* = 0$, i. e. $\mathfrak{P} S_{\mathfrak{P}} = \mathfrak{p} S_{\mathfrak{P}}$, proving ($\mathrm{U}_1$).

Theorem 5. *Let S be an R-algebra and assume that S satisfy the finiteness condition over a prime ideal $\mathfrak{p}$ of R. Then if S is quasi-separable over R, S is unramified over $\mathfrak{p}$.*

Proof. Let $U = R - \mathfrak{p}$, then from $D_R(S) = 0$ we get $D_{R_U}(S_U) = 0$ by Prop. 8.

Applying Prop. 7 to the last relation we see that $S_U/\mathfrak{p}S_U$ is a quasi-separable $(R_U/\mathfrak{p}R_U)$-algebra. The theorem will follow from Prop. 13, Prop. 18 and the finiteness condition for S.

COROLLARY. *Let S be an R-algebra and assume that S is a finite R-module. Then if S is quasi-separable, S is unramified.*

In the next place we shall study when the quasi-separability will follow from unramifiedness. For this purpose it is convenient to introduce the notion of d-different.

DEFINITION 3. *Let S be an R-algebra and let $D_R(S)$ be the module of R-differentials in S. Let $\mathfrak{D}$ be the ideal of S generated by the elements x in S such that $xD_R(S)=0$. We shall call $\mathfrak{D}$ d-different of S over R and will be denoted by $\mathfrak{D}_d(S/R)$, or simply by $\mathfrak{D}_d$ if it is clear from the context.*

THEOREM 6. *Let S be an R-algebra and assume that the module $D_R(S)$ of R-differentials in S is a finite S-module. Let $\mathfrak{P}$ be a prime ideal of S which is unramified over R, then $\mathfrak{P}$ does not contain the d-different $\mathfrak{D}$.*

PROOF. Let $S^*=S_{\mathfrak{P}}$ and $R^*=R_{\mathfrak{p}}$, where $\mathfrak{p}=R\cap\mathfrak{P}$. Then S^* is unramified over R^*, i.e. if we denote by $\mathfrak{P}^*$ and $\mathfrak{p}^*$ the maximal ideals of S^* and R^* respectively, $S^*/\mathfrak{P}^*$ is a finite separable extension of $R^*/\mathfrak{p}^*$ and $\mathfrak{P}^*=\mathfrak{p}^*S^*$. Hence $D_{R^*}(S^*/\mathfrak{P}^*)=0$ and $(S^*/\mathfrak{P}^*)\otimes_{S^*}D_{R^*}(S^*)$ is generated by the elements $da_1,\cdots,da_n$, where $a_1,\cdots,a_n$ are any set of generators of $\mathfrak{P}^*$ (Prop. 9) and d stands for $d^{S^*}_{R^*}$. Since $\mathfrak{P}^*$ is generated by the elements in $\mathfrak{p}^*$ we see that $D_{R^*}(S^*)\otimes(S^*/\mathfrak{P}^*)=0$. On the other hand $D_{R^*}(S^*)=D_R(S^*)$ is a finite module since it is a homomorphic image of $S^*\otimes D_R(S)$, hence we must have $D_{R^*}(S^*)=0$. By Prop. 10, $D_R(S^*)=S^*\otimes D_R(S)$, and the annihilator of $D_R(S^*)$ is given by $\mathfrak{D}\otimes_S S^*=\mathfrak{D}^*$[10]. The above results implies that the annihilator $\mathfrak{D}^*$ of $D_R(S^*)$ must be a unit ideal, hence $\mathfrak{P}$ cannot contain the d-different $\mathfrak{D}$.

COROLLARY. *Let S be an R-algebra and assume that $D_R(S)$ is a finite S-module. Then if S is unramified over R, S is quasi-separable over R.*

PROOF. Assume that $D_R(S)$ is not zero. Then the d-different $\mathfrak{D}$ is not a unit ideal, and there exists a maximal ideal $\mathfrak{M}$ containing $\mathfrak{D}$. $\mathfrak{M}$ must be the one which is ramified over R.

THEOREM 7. *Let S be a noetherian R-algebra and let $\mathfrak{D}$ be the d-different. Let $\mathfrak{P}$ be a prime ideal of S satisfying the finiteness condition. Then if $\mathfrak{P}\not\supseteq\mathfrak{D}$, $\mathfrak{P}$ is unramified.*

PROOF. By Prop. 10 we eave $D_R(S_{\mathfrak{P}})=S_{\mathfrak{P}}\otimes_S D_R(S)$, hence if we have $\mathfrak{P}\not\supseteq\mathfrak{D}$, then $D_R(S_{\mathfrak{P}})=0$. Applying Prop. 8 we see that $D_{R_{\mathfrak{p}}}(S_{\mathfrak{P}})\neq 0$. The assertion follows from Th. 4.

Let R be a noetherian normal domain with the quotient field K and let L

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10) In general if E is a finite S-module and if $\mathfrak{a}$ is the annihilater of E, then the annihilater of $E\otimes_S S_U$ is given by $\mathfrak{a}\otimes_S S_U$.

be a finite separable extension of K. Let S be the integral closure of R in L, then as is well known S is a finite R-module and every prime ideal of S satisfies the finiteness condition. Then we can apply Theorem 6 and 7 to this case and we get the

THEOREM 8. *Under the same notations and assumptions as above let* $\mathfrak{D}_d$ *be the d-different of S over R. Then any prime ideal* $\mathfrak{P}$ *of S is unramified if, and only if,* $\mathfrak{P}$ *does not contain the d-different* $\mathfrak{D}_d$.

E. Noether and Auslander-Buchsbaum defined another kind of different for an R-algebra S (cf. [8] and [2]). The different is called by the name of "homological different" in [2]. Homological different is defined in the following way. Let S be an R-algebra. As we mentioned before, there exists an exact sequence of $S^e=S\otimes_R S$ modules

$$0\longrightarrow I\longrightarrow S^e\xrightarrow{\varphi} S\longrightarrow 0\,.$$

The homomorphism φ is defined by $\varphi(\sum_i x_i\otimes y_i)=\sum_i x_i y_i$ and S is made into a S^e-module by the rule $(a\otimes b)x=abx$, where a, b and x are elements of S. Let $\mathfrak{N}$ be the annihilater of the ideal I, then the homological different $\mathfrak{h}$ is given by $\varphi(\mathfrak{N})$.

PROPOSITION 19. *The homological different* $\mathfrak{h}$ *is contained in the d-different* $\mathfrak{D}$. *Moreover if the kernel I has a finite number of generaters, then the radicals of* $\mathfrak{h}$ *conincides with that of* $\mathfrak{D}$[11].

PROOF. Let $\mathfrak{S}=I^2:I$, i.e. the set of elements s in S^e such that $sI\subseteqq I^2$. We shall show that $\varphi(\mathfrak{S})=\mathfrak{D}$. Let a be an element of $\mathfrak{D}$. Since $D_R(S)$ is isomorphic (as a S-module) to I/I^2 we see that $aI\subseteqq I^2$. Hence $a\otimes 1$ is contained in $\mathfrak{S}$, and $a=\varphi(a\otimes 1)$ is contained in $\varphi(\mathfrak{S})$. Conversely let a be an element of $\varphi(\mathfrak{S})$. Then there exists an element α in S^e such that $\varphi(\alpha)=a$. Let us put $\alpha=\sum_{i=1}^{s} c_i\otimes d_i$. Let x be an arbitrary element of S. Using the relation $\Sigma c_i d_i=a$, we get

$$\begin{aligned} a(1\otimes x-x\otimes 1)&=(a\otimes 1)(1\otimes x-x\otimes 1)\\ &=((\Sigma c_i d_i)\otimes 1)(1\otimes x-x\otimes 1)\\ &=[(\Sigma c_i\otimes d_i)-\Sigma(c_i\otimes 1)(1\otimes d_i-d_i\otimes 1)](1\otimes x-x\otimes 1)\\ &=\alpha(1\otimes x-x\otimes 1)\equiv 0 \qquad (\mathrm{mod}\, I^2)\,. \end{aligned}$$

Since I is generated by $1\otimes x-x\otimes 1$, the above relation implies that a is contained in $\mathfrak{D}$. By definition $\mathfrak{h}=\varphi(\mathfrak{N})$ and $\mathfrak{N}=(0):I\subset I^2:I=\mathfrak{S}$. Hence $\mathfrak{h}=\varphi(\mathfrak{N})$ is contained in $\mathfrak{D}=\varphi(\mathfrak{S})$.

Now assume that I has a finite number of generaters and let $\{\alpha_1,\cdots,\alpha_n\}$

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11) We owe the last part of the proposition to H. Sato.

be a set of generaters of I. Let s be an arbitrary element of $\mathfrak{S}$. From $sI \subseteqq I^2$ we get the relation

$$s\alpha_i = \sum_{j=1}^{n} a_{ij}\alpha_j \qquad (i=1, \cdots, n)$$

where a's are in I. If we put $\delta = \det|s\delta_{ij} - a_{ij}|$, then $\delta\alpha_i = 0$ for $i=1,2,\cdots,n$. Hence δ is contained in $\mathfrak{N}$. Applying the homomorphism φ to the relation $\delta = \det|s\delta_{ij} - a_{ij}|$, we see that $\varphi(s)^n$ is contained in $\varphi(\mathfrak{N}) = \mathfrak{h}$. Since $\mathfrak{D} = \varphi(\mathfrak{S})$ and s is an arbitrary element of $\mathfrak{S}$, the above relation implies that the d-different $\mathfrak{D}$, hence the radial of $\mathfrak{D}$, is contained in the radical of $\mathfrak{h}$.

§ 6. The module of differentials of higher degree

Let R be a local ring and let E be a finite R-module. Then the number of minimal set of generators $u_1, \cdots, u_n$ of E is a well determined integer. It is equal to the dimension of the vector space $E \otimes_R (R/M)$ over the field R/M, where M is the maximal ideal of R. We shall call this integer the *rank* of E and denote it by $\operatorname{rank}_R E$.

LEMMA. *Let R be a local ring and let E be a finite R-module over R. Let $n = \operatorname{rank} E$, then the rank of the exterior algebra $\Lambda^t E$ is equal to ${}_nC_t$. Moreover if E is a free module, then $\Lambda^t E$ is also a free module.*

PROOF. Let M be the maximal ideal of R and let $K = R/M$. If $\bar{E} = E \otimes_R K$, then the dimension of $\Lambda^t \bar{E}$ is, as is well known, equal to ${}_nC_t$, Hence if we can show that $\Lambda^t \bar{E}$ is isomorphic to $(\Lambda^t E) \otimes_R K$ the proof will be complete. For the sake of simplicity we shall denote $1 \otimes x$ by $\bar{x}$, where x is an element of E. Let f be a map from $\underbrace{E \times \cdots \times E}_{t}$ into $\Lambda^t \bar{E}$ defined by

$$f(x_1, \cdots, x_t) = \bar{x}_1 \wedge \cdots \wedge \bar{x}_t\,.$$

Then f is multilinear and skew symmetric. Hence there exists an R-homomorphism $g : \Lambda^t E \rightarrow \Lambda^t \bar{E}$ such that

$$g(x_1 \wedge \cdots \wedge x_t) = \bar{x}_1 \wedge \cdots \wedge \bar{x}_t\,.$$

Since $\Lambda^t \bar{E}$ is a K-vector space, g induces a homomorphism $\varphi : (\Lambda^t E) \otimes_R K \rightarrow \Lambda^t \bar{E}$ defined by

$$\varphi[1 \otimes (x_1 \wedge \cdots \wedge x_t)] = \bar{x}_1 \wedge \cdots \wedge \bar{x}_t\,.$$

On the other hand

$$\psi(\bar{x}_1 \wedge \cdots \wedge \bar{x}_t) = 1 \otimes (x_1 \wedge \cdots \wedge x_t)$$

is a well defined homomorphism from $\Lambda^t \bar{E}$ into $(\Lambda^t E) \otimes_R K$. Since $\varphi \cdot \psi = 1$ and $\psi \circ \varphi = 1$, we see that $\Lambda^t \bar{E}$ and $(\Lambda^t E) \otimes K$ are isomorphic. The last assertion is proved in [**3**, Chap. III].

In this occassion we would like to give a refinement of Th. 2 in our previous paper [7].

THEOREM 9. *Let V^r be a variety defined over a field k and let P be a point of V such that $k(P)$ is separably algebraic over k. Let R be the quotient ring of P in V/k, and assume that $\Lambda^r D_k(R)$ is a free R-module with the base $dt_1 \wedge \cdots \wedge dt_r$. Then P is a simple point of V and $\{t_1, \cdots, t_r\}$ is a set of uniformizing parameters at P.*

PROOF. Let M be the maximal ideal of R and let $\{u_1, \cdots, u_n\}$ be a set of minimal base of M. Then $D_k(R)$ is of rank n by Prop. 16. Hence by the above Lemma the rank of $\Lambda^r D_k(R)$ is of rank ${}_nC_r$. From this we see that the rank of $\Lambda^r D_k(R)$ is 1 if, and only if, $n=r$. Since rank of R is r and n is equal to the number of minimal base of M, we have $n=r$ if, and only if, R is a regular local ring. This proves that R is a regular local ring. Moreover P is separably algebraic over the ground field, P must be a simple point of V. Let $u_1, \cdots, u_r$ be a set of uniformizing parameters at P, then by Prop. 17 $D_k(R)$ is a free module with the base $du_1, \cdots, du_r$, hence $\Lambda^r D_k(R)$ is a free module of rank 1 with the base $du_1, \cdots, du_r$. Combining the assumptions in the Theorem we see that $dt_1 \wedge \cdots \wedge dt_r$ and $du_1 \wedge \cdots \wedge du_r$ differ by a unit of R. Then by Prop. 1 in [7], $t_1, \cdots, t_r$ are a set of uniformizing parameters at P on V.

REMARK. The assumptions made in Th. 2 in [7] are superfluous since we have assumed that $\Lambda^r D_k(R)$ is of rank 1.

Let V be a normal variety defined over a field k and let $K=k(V)$ is the function field of V over k. Let L be a finite separable extension of K and let U be the normalization of V in L. Then there exists a natural rational map $\pi: U \rightarrow V$ called a covering map. Let P be a point of V and let Q be a point of U lying above P. We shall denote by R and S respectively the quotient rings of P and Q in V/k and U/k. We shall denote by $t_1, \cdots, t_n$ a minimal set of generators of the maximal ideal M of R. To avoid the confusion in the notations we shall denote d_k^S simply by d. Let E be the submodule of $D_k(S)$ generated by $dt_1, \cdots, dt_n$, Then $D_R(S)$ is isomorphic to the difference module $D_k(S)/E$ by Prop. 1. Let $\mathfrak{a}_t$ be the annihilaters of the difference modules $\Lambda^t D_k(S)/\Lambda^t E$ $(t=1, 2, \cdots)$. $\mathfrak{a}_1$ is nothing other than the d-different of S/R.

THEOREM 10. *Assume that P and Q are simple points of V and U respectively, then we have*

$$\mathfrak{a}_1 \supseteqq \mathfrak{a}_2 \supseteqq \cdots \supseteqq \mathfrak{a}_r \supseteqq \mathfrak{a}_1{}^r$$

where $r=\dim V$. Moreover $\mathfrak{a}_r$ is a principal ideal.[12]

PROOF. Let $(t_1, \cdots, t_r)$ and $(u_1, \cdots, u_r)$ be a regular system of parameters of R and S respectively. Then, since $D_k(R)$ is a free module with the base $\{d^R t_1, \cdots, d^R t_r\}$, $d^S t_1, \cdots, d^S t_r$ are also linearly independent over S (Prop. 3). For the sake of simplicity we shall use the symbol d for d_k^S. Let us express dt's in

12) Here we again have the proof of purity of branch loci in this special case.

terms of the free base $du_1, \cdots, du_r$ of $D_k(S)$,

$$(1)\qquad dt_i = \sum_{j=1}^{r} a_{ij} du_j\,, \qquad 1 \leqq i \leqq r\,, \qquad a_{ij} \in S\,.$$

Let A_{ij} be the cofacter of a_{ij} and $A = \det|a_{ik}|$. Then we can solve (10) in the form

$$(2)\qquad du_j = \sum_{k=1}^{r} (A_{jk}/A) dt_k \qquad 1 \leqq j \leqq r\,.$$

From (20) we get

$$\begin{aligned} du_{j_1} \wedge \cdots \wedge du_{j_t} &= \sum (A_{j_1k_1} \cdots A_{j_tk_t}/A) dt_{k_1} \wedge \cdots \wedge dt_{k_t} \\ &= \sum\nolimits_2 \{\sum_{(\alpha)}\nolimits_1 (A_{j_1k_1} \cdots A_{j_tk_t}/A) dt_{\alpha_1} \wedge \cdots \wedge dt_{\alpha_t}\} \end{aligned}$$

where the sum $\sum_{(\alpha)1}$ is extended over all the $t\,!$ permutations $k_1, \cdots, k_t$ of $\alpha_1, \cdots, \alpha_t$ and the sum $\sum_2$ is extended over all set of indices $\alpha_1 < \cdots < \alpha_t$ taken from $1, 2, \cdots, r$. Now assume that an element a of S is in $\mathfrak{a}_t$. Then $a du_{j_1} \wedge \cdots \wedge du_{j_t}$ is contained in $\sum_2 S(dt_{\alpha_1} \wedge \cdots \wedge dt_{\alpha_t})$ for all $\alpha_1 < \cdots < \alpha_t$ and vice versa. Taking into account that $dt_1, \cdots, dt_r$ are linearly independent over S we get the conclusion: *a is contained in* $\mathfrak{a}_t$ *if, and only if,*

$$a \sum_{(\alpha)}\nolimits_1 A_{j_1k_1} \cdots A_{j_tk_t} \in (A^t)$$

for all pairs of t-tuples $(j_1, \cdots, j_t)$ *and* $(\alpha_1, \cdots, \alpha_t)$. We shall calculate

$$\sum_{(\alpha)}\nolimits_1 A_{j_1k_1} \cdots A_{j_tk_t} = \det \begin{vmatrix} A_{j_1\alpha_1} \cdots A_{j_1\alpha_t} \\ \vdots \qquad \vdots \\ A_{j_t\alpha_1} \cdots A_{j_t\alpha_t} \end{vmatrix}.$$

Let $s_1 < \cdots < s_{n-t}$ and $\beta_1 < \cdots < \beta_{n-t}$ be the set of integers such that $(j_1, \cdots, j_t, s_1, \cdots, s_{r-t})$ and $(k_1, \cdots, k_t, \beta_1, \cdots, \beta_{r-t})$ are even permutations of $(1, 2, \cdots, r)$ respectively. By an easy calculation we get

$$\begin{pmatrix} A_{j_1\alpha_1} & \cdots & A_{j_1\alpha_t} & A_{j_1\beta_1} & \cdots & A_{j_1\beta_{r-t}} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{j_t\alpha_1} & \cdots & A_{j_t\alpha_t} & A_{j_t\beta_1} & \cdots & A_{j_t\beta_{r-t}} \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} a_{j_1\alpha_1} & \cdots & a_{j_t\alpha_1} & a_{s_1\alpha_1} & \cdots & a_{s_{r-t}\alpha_1} \\ \vdots & & \cdots & \cdots & & \vdots \\ a_{j_1\alpha_t} & \cdots & a_{j_t\alpha_t} & a_{s_1\alpha_t} & \cdots & a_{s_{r-t}\alpha_t} \\ a_{j_1\beta_1} & \cdots & a_{j_t\beta_1} & a_{s_1\beta_1} & \cdots & a_{s_{r-t}\beta_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{j_1\beta_{r-t}} & \cdots & a_{j_t\beta_{r-t}} & a_{s_1\beta_{r-t}} & \cdots & a_{s_{r-t}\beta_{r-t}} \end{pmatrix}$$

$$= \begin{pmatrix} A & & & & \\ & \ddots & & 0 & \\ & & A & & \\ & & & a_{s_1\beta_1} \cdots\cdots a_{s_{r-t}\beta_1} & \\ & * & & \vdots \qquad\quad \vdots & \\ & & & a_{s_1\beta_{r-t}} \cdots a_{s_{r-t}\beta_{r-t}} & \end{pmatrix}.$$

Taking the determinant of both sides we get the equation[13]

$$\begin{vmatrix} A_{j_1\alpha_1} \cdots A_{j_t\alpha_t} \\ \vdots \quad\quad \vdots \\ A_{j_t\alpha_1} \cdots A_{j_t\alpha_t} \end{vmatrix} = A^{t-1}B, \qquad \text{where} \qquad B = \begin{vmatrix} a_{s_1\beta_1} \cdots\cdots a_{s_1\beta_{r-t}} \\ \vdots \quad\quad\quad \vdots \\ a_{s_{r-t}\beta_1} \cdots a_{s_{r-t}\beta_{r-t}} \end{vmatrix}.$$

From these calculation we can conclude the following: *$a \in \mathfrak{a}_t$ if, and only if, $aB \in (A)$ for any minor determinant B of order $r-t$ of A* (where it should be understood that $B=1$ if $t=r$). From this assertion the inclusion relation

$$\mathfrak{a}_1 \supseteqq \mathfrak{a}_2 \supseteqq \cdots \supseteqq \mathfrak{a}_r$$

follows immediately. In particular $a \in \mathfrak{a}_r$ if, and only if, a is contained in the principal ideal (A), and A is itself contained in $\mathfrak{a}_r$. From this we get $\mathfrak{a}_r = (A)$. Now assume that $a_1, \cdots, a_r$ are elements in $\mathfrak{a}_1$, then $a_i A_{\alpha\beta} \in (A)$ for any triplet (i, α, β), $1 \leqq i, \alpha, \beta \leqq r$. Hence

$$a_1 \cdots a_r \begin{vmatrix} A_{11} \cdots A_{1r} \\ \vdots \quad\quad \vdots \\ A_{r1} \cdots A_{rr} \end{vmatrix} = a_1 \cdots a_r A^{r-1} \in (A^r)$$

i. e. $a_1 \cdots a_r \in (A)$. This proves that $\mathfrak{a}_1^r$ is contained in $\mathfrak{a}_r$. Thus the proof is complete.

Let V and U be as before and let C be an irreducible subvariety of codimension 1 on U and let D be a subvariety of V lying under C. Let Q and P be respectively the points of C and D such that $\pi(Q)=P$. It is possible to find such a pair among the simple points of U and simple points of V. Let R and S be the quotient rings of P and Q in V/k and U/k respectively and let $\{t_1, \cdots, t_r\}$ and $\{u_1, \cdots, u_r\}$ be sets of uniformizing parameters at P and Q. Let $\omega = d^R t_1 \wedge \cdots \wedge d^R t_r$. Then the multiplicity of C in the divisor of the differential form $\pi * \omega = d^S t_1 \wedge \cdots \wedge d^S t_r$[14] is called the differential index[15] of C and will be denoted by $e(C)$. If we represent $d^S t$'s in terms of $d^S u$'s, we can write

$$d^S t_1 \wedge \cdots \wedge d^S t_r = \alpha d^S u_1 \wedge \cdots \wedge d^S u_r.$$

Then the differential index $e(C)$ is given by the multiplicity of C in the divisor (α). On the other hand let $\mathfrak{p}$ be the prime ideal in S defining the divisor C, then $S_\mathfrak{p}$ is a discrete valuation ring. Let $\tilde{\omega}$ be its prime element, and let $\mathfrak{D}$ be the d-different of S over R. Then the above Theorem implies that $(\tilde{\omega}^{e(C)})$ is contained in $\mathfrak{D}S_\mathfrak{p}$. Hence if we define e_1 by $\mathfrak{D}S_\mathfrak{p} = (\tilde{\omega}^{e_1})$, we get the inequality

$$e_1 \leqq e(C). \tag{3}$$

13) This is a generalization of the Theorem of Sylvester.

14) In [7], we did not distinguish d^R and d^S. But it was reasonable since $S \otimes D(R) \to D(S)$ is injective (Prop. 3). But here we adopt the precise notations to make the matters clear.

15) Cf. [7], § 6.

From Th. 14 we get at once the

COROLLARY. $e_1 = e(C)$ *if* $\dim V = 1$.

The inequality cannot be replaced by an equality in general as is shown in the following example.

EXAMPLE. Let V be an affine space with coordinates x and y, and let U be an affine space with coordinates u and v. We shall assume that they satisfy the following relation

$$x = u^3 + u^2 v\,, \qquad y = u^2 v\,.$$

Let P and Q be the points with the coordinates $(0, 0)$ and $(0, 0)$. Let $A = k[x, y]$ and $B = k[u, v]$, then the local rings R and S of P and Q are given by $k[x, y]_{(x,y)}$ and $k[u, v]_{(u,v)}$ respectively. By a simple calculation we see that d-different of S/R is given by (u^3), but the ideal $\mathfrak{a}_2$ in the Theorem is given by (u^4). Hence $e_1 = 3$, $e(C) = 4$, where C is the straight line defined by the equation $u = 0$ in U.

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Added in proof: In the course of the proof I found that the recent paper of *Ernst Kunz, Die Primidealteiler der Differenten in allgemeinen Ringen*, J. Reine Angew Math., **204** (1960), 166–182, have considerable overlap with our present paper. But our method are quite independent of his.