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Bonded groups

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§ 1. Introduction.

Historically, discrete flows and continuous flows have played the most important roles in topological dynamics (see [**1**] and [**2**]). A *discrete flow* is a transformation group whose phase group is the additive group I of all integers with the discrete topology. A *continuous flow* is a transformation group whose phase group is the additive group R of all real numbers with the usual topology. As we know, in either R or I, every non-trivial cyclic subgroup (i. e. a cyclic group generated by a non-identity element of the group) is syndetic (we call this Property S). A subset N of a topological group G is called left syndetic (see [**2**]) if there exists a compact subset K of G such that $N\cdot K=NK=\{xy\mid x\in N, y\in K\}=G$. Similarly, we can define *right syndetic subsets*. A set N is called *syndetic* if it is both left syndetic and right syndetic. A subgroup H of G which is left syndetic is also right syndetic, and vice versa. As we know, the almost periodicity properties of transformation groups are based on syndetic subsets of the phase group. It is interesting to consider the following problem:

"What is the structure of a topological group which has the Property S?" The author will show in this paper (see Theorem 4) that a group of this type is either (a) compact, (b) topologically isomorphic to I, (c) topologically isomorphic to R or (d) radical (see [**7**]) which is not locally compact (see Theorem 5).

We rarely consider compact transformation groups, ab initio, in topological dynamics, since under a compact phase group each point is always almost periodic (see [**2**]) and recurrent (see [**2**]) and any orbit is equal to its orbit closure. However, it is interesting we discover a new type group, the non-locally compact, radical group having the Property S.

In the present paper, a topological group will be denoted by G, and e (0 in the abelian case) will denote either the identity element of G or the trivial subgroup consisting of the identity only. The additive group of all integers with the discrete topology will be denoted by I, and the additive group of all real numbers with the usual topology will be denoted by R. A *topological isomorphism* between two topological groups, G_1 and G_2, is simultaneously an algebraic isomorphism and a homeomorphism and we denote its existence by $G_1 \cong G_2$. If G_1 and G_2 are two topological groups, their *direct product* (topological group) $G_1 \times G_2$ is the algebraic direct product group endowed with the cartesian product topology. All other terminologies and notations which are not mentioned here are taken from [**1**], [**5**], [**6**] and [**11**].

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§ 2. Fundamental properties.

Definition 1. A *bonded group* is a topological group such that each non-trivial cyclic subgroup of this group is syndetic (see § 1). In particular, if a group consists of the identity alone, it is a bonded group.

From this definition, it follows that every non-trivial subgroup of a bonded group is syndetic.

Lemma 1. *Every closed subgroup A of a bonded group G is bonded.*

Proof. Let x be any element different from e in A and let C be the cyclic subgroup of A (of course, of G) generated by x. Since G is a bonded group, there exists a compact subset K in G such that $CK = G$. It follows that $C(K \cap A) = A$, and since $K \cap A$ is compact, C is syndetic in A.

Lemma 2. *If G is a bonded group, H a topological group and $\phi : G \to H$ a continous homomorphism of G into H, then $\phi(G)$ is a bonded group.*

Proof. Let $\phi(x)$ be an element in $\phi(G)$ such that $\phi(x) \neq e$, where e is the identity of H and let C be the cyclic subgroup of $\phi(G)$ generated by $\phi(x)$. Since x is not the identity of G, the cyclic group B of G generated by x is syndetic, i. e. there exists a compact subset K of G such that $BK = G$. But $\phi(K)$ is compact and $\phi(G) = \phi(B)\phi(K) = C\phi(K)$. It follows that C is syndetic.

Corollary. *Let G be a bonded group and let N be a closed normal subgroup of G. Then the quotient group G/N is a bonded group.*

Lemma 3. *The additive group I of all integers with the discrete topology, the additive group R of all real numbers with the usual topology and all compact groups are bonded.*

Proof. Every non-trivial cyclic subgroup of each of them is syndetic.

Lemma 4. *Let G_1 and G_2 be topological groups such that $G_1 \neq e$, $G_2 \neq e$. If one of them is not compact, then $G = G_1 \times G_2$ is not bonded.*

Proof. Suppose G is bonded. Let G_1 be non-compact and let e_1 and e_2 be the identities of G_1 and G_2 respectively. Let $x_2 \in G_2$, $x_2 \neq e_2$ and let C be the cyclic subgroup of G generated by (e_1, x_2). Since C is syndetic and $\{e_1\} \times G_2 \supset C$, $\{e_1\} \times G_2$ is syndetic. Then $G/\{e_1\} \times G_2 \cong G_1$ should be compact, which contradicts the assumption. Hence G is not bonded.

Lemma 5. *Let G be a topological group. If G contains a non-trivial compact* *subgroup H, then G is bonded if and only if G is compact.*

PROOF. If G is compact, then, by Lemma 3, it is bonded. If G is bonded, then H is syndetic and G would be compact.

LEMMA 6. *Let G be a σ-compact group. Then G is of second category if and only if G is locally compact.*

PROOF. Let $G=\bigcup_{i=1}^{\infty} K_i$, where the K_i are compact subsets of G. If G is of second category, then K_n°, the interior of K_n, is non-empty for some n. Hence G is locally compact at some point of K_n°. The conclusion follows from the homogeneity of G. The converse is well-known.

COROLLARY. *Let G be a bonded group. Then G is of second category if and only if G is locally compact.*

LEMMA 7. *Let G be a locally compact topological group and let H be a subgroup of G. If H is bonded, then so is $\bar{H}$.*

PROOF. Let $x \in H$, $x \neq e$. Let C be the cyclic group generated by x. Since $\bar{C}$ is a locally compact monothetic group in $\bar{H}$, we know that either $C \cong I$ or $\bar{C}$ is compact. But C is syndetic in H, and hence $H=C\cdot K$ for some compact subset in H. If $\bar{C}$ is compact, then $\bar{H}$ is compact and is a bonded group. If $C\cong I$, then C is closed in $\bar{H}$ and $\bar{H}=\bar{C}\cdot K=C\cdot K=H$. Hence $\bar{H}$ is a bonded group.

COROLLARY. *Let G_1 and G_2 be topological groups and let G_1 be bonded and G_2 be locally compact. If $\phi: G_1 \to G_2$ is a continuous homomorphism of G_1 into G_2, then $\overline{\phi(G_1)}$ is bonded.*

DEFINITION 2. A semigroup S in a topological group G is said to be *e-proper* if S does not contain the identity element e of G.

DEFINITION 3. A topological group G is said to be a *radical group*, if G contains no e-proper open semigroup. A subgroup H of a topological group G is said to be a radical subgroup of G if H considered as a topological group with the induced topology is a radical group (see [7]).

Every compact group is radical, since each non-empty open semigroup of a compact group is a subgroup (see [8]).

A simple example of non-radical groups is the locally compact, non-compact abelian group, generated by a compact neighborhood of the identity.

LEMMA 8. *Let G be a topological group and C a subgroup of G which is discrete with respect to the relative topology of G and such that $G=CL$ where L is a compact subset of G. Then G is locally compact.*

PROOF. Let G/C be the (left) coset space of G. Since L is compact, G/C is compact. Let $P: G\to G/C$ be the natural projection. We know C is discrete, P is a local homeomorphism and G/C is compact, it follows that G is locally compact.

LEMMA 9. *Every bonded group G is either a compact group, a non-compact* *radical group such that each non-trivial cyclic subgroup of G is not discrete, or a locally compact, non-compact, group such that each non-trivial cyclic subgroup of G is discrete.*

Proof. Assume G is not compact. Then, by Lemma 5, G is torsion-free (i. e. each non-identity element of G is of infinite order).

Let C_x be the free cyclic subgroup of G which is generated by a non-identity element x of G and let $G=C_xK_x$, where K_x is a compact subset of G. Then, by a theorem of F. B. Wright (here C_x is a regular one-parameter group in the sense of Wright. See [**9**, Theorem 1]), C_x is either a radical group or topologically isomorphic to I.

If some free cyclic subgroup C_a, generated by a, of G is not radical, then $C_a \cong I$ i. e. C_a is discrete with respect to the relative topology of G. Since G is a group, C_a is closed. By Lemma 8, G is locally compact. Since C_a is closed, G cannot be compact. Let C_b be another free cyclic subgroup of G, we shall show that C_b is discrete and, therefore, is closed. Assume that C_b is not discrete. Since $\overline{C}_b$ is a monothetic group and C_b is not topologically isomorphic to I, it follows that C_b is compact. Then G should be compact which contradicts the assumption. Hence, if some C_a is discrete then all C_x are discrete and G is locally compact.

If all C_x are radical, we shall show that G itself is radical. Suppose not, then there exists an e-proper non-empty open semigroup M of G. Let $a\in M$, then $M \cap C_a = B \neq \phi$ and B is an e-proper open semigroup of C_a. This gives a contradiction to the fact that C_a is radical. This proves that G is radical.

§ 3. Discrete bonded groups.

Lemma 10. *Let G be a discrete group. Then the following statements are equivalent:*

(1) *G is a bonded group and of infinite order.*

(2) *G is torsion-free and every free cyclic subgroup of G is syndetic.*

(3) *G is torsion-free and every free cyclic normal subgroup of G is syndetic.*

Proof. Clearly, (1) is equivalent to (2) and (2) implies (3).

By a well-known result in group theory, we know that if any subgroup H is of finite index to the group G, then there exists a subgroup H' of H such that H' is a normal subgroup and of finite index to G. Hence (3) implies (2).

Lemma 11. *Let G be a discrete torsion-free abelian* (*additive*) *group and let Z be a free-cyclic syndetic subgroup of G. Then G is free-cyclic.*

Proof. Let Z be generated by some non-zero $a_0\in G$. The quotient group $G/Z=\{Z, a_1+Z, \cdots, a_s+Z\}$, where each $a_i\in G$, $i=1,2,\cdots,s$, is finite.

Here $L=\{a_0, a_1, \cdots, a_s\}$ generates G. Since G is a finitely generated and torsion-free abelian group, $G=G_1\oplus\cdots\oplus G_r$, where G_i's are infinite cyclic groups. If $r>1$, then the cyclic subgroup Z has an infinite index, contrary to the hypothesis. Hence $r=1$, namely G is infinite cyclic.

Lemma 12. *Let G be a torsion-free group and let Z be a free cyclic normal subgroup of G. If the order of G/Z is a prime number p, then G is free cyclic.*

Proof. Let Z be generated by some non-identity element a in G. Clearly Z is a proper subgroup of G. We may write $G/Z=\{Z, bZ, b^2Z, \cdots, b^{p-1}Z \mid b\in G, b\neq e\}$. It follows that $b^p=a^s$ for some non-zero integer s. Since $bZ=Zb$, we have $ba=a^{r_1}b$ and $ab=ba^{r_2}$ for some non-zero integers r_1 and r_2. It follows that

$$ab=a^{r_1r_2}b \qquad \text{or} \qquad a=a^{r_1r_2},$$

and we have $r_1r_2=1$. Hence either $r_1=r_2=1$ or $r_1=r_2=-1$, i.e. either $ba=ab$, or $ba=a^{-1}b$.

Suppose $ba=a^{-1}b$. Then $ba^s=(a^{-1}b)a^{s-1}=a^{-s}b$ and, by $a^s=b^p$, we have $b^{p+1}=b^{-p+1}$ or $b^{2p}=e$. It is impossible since G is torsion-free. Hence $ab=ba$ and G is abelian. By Lemma 11, we have the required result.

Lemma 13. *Let G be a discrete, torsion-free group and let Z be a free cyclic subgroup of G. If Z is syndetic then G is free cyclic.*

Proof. Let Z_1 be a free-cyclic normal subgroup of G such that $Z_1\subset Z$ and Z_1 is syndetic. The quotient group G/Z_1 is finite. We shall prove the lemma by induction on the order of G/Z_1.

By Lemma 12, we know the lemma is true, when $O(G/Z_1)$, the order of G/Z_1, is prime.

Suppose the lemma is true for $O(G/Z_1)\leqq n-1$.

Let $\pi: G\to G/Z_1$ be the natural projection, by $\pi(g)=gZ_1\in G/Z_1$, where $g\in G$.

We shall divide the proof into two cases: G/Z_1 is not simple, and G/Z_1 is simple.

If $F=G/Z_1$ is not simple, then it contains a proper normal subgroup $N_1\subset F$. Consequently, $G_1=\pi^{-1}(N_1)$ is a proper normal subgroup of G. Since Z_1 is a normal subgroup of G_1 and $O(G_1/Z_1)<n$ by the induction hypothesis, G_1 is free cyclic. On the other hand, G_1 is a free cyclic normal subgroup of G and $O(G/G_1)<n$. Again, by the induction hypothesis, G is free cyclic.

If $F=G/Z_1$ is simple, we shall consider the following two cases: one is when F is a p-group and the other is when F is not a p-group.

If F is a p-group, then F must be of order p, for some prime number p, since otherwise F would contain a proper normal subgroup and would not be simple. Then, by Lemma 12, we know G is free cyclic.

If F is not a p-group, let F_1 be a sylow group of F. Clearly, $F_1\neq F$ and

$G_1 = \pi^{-1}(F_1)$ is a proper subgroup of G. Since Z_1 is a normal subgroup of G_1 and $O(G_1/Z_1) < n$, by the induction hypothesis, we know that G_1 is free cyclic. Hence $\pi(G_1) = F_1$ is a proper cyclic subgroup of F. This shows that every sylow group of F is cyclic. By a well-known result, we know that if every sylow group of a finite group is cyclic, then the group itself is solvable (see [**11**, Theorem 10, Ch. V]). Hence, F is solvable. From the fact that F is simple and solvable it follows that F is a cyclic group of prime order. This contradicts our assumption, namely that F is not a p-group.

Hence if this lemma is true for $O(F) \leqq n-1$ and if $O(F) = n$ and F is simple, then F is a cyclic group of prime order and, consequently, G is free cyclic.

Hence, by induction, the lemma is always true.

REMARK: Lemma 13 can also be proved, using the following group-theoretic theorem of Baer-Neumann-Witt: If the center of a group G is of finite index, then the commutator subgroup of G is finite.

By Lemma 11 and Lemma 13, we have the following result immediately.

THEOREM 1. *Every discrete bonded group is either finite or isomorphic to I.*

§ 4. Connected, locally compact bonded groups.

THEOREM 2. *Any connected locally compact bonded group G is either compact or topologically isomorphic to the additive group R of all real numbers with the usual topology.*

PROOF. It is known that for any connected, locally compact group G, there exists a maximal compact subgroup K, such that the quotient-space G/K is homeomorphic to n-dimensional Euclidean space R^n for some positive integer n (see [**3**, Theorem 13], together with [**10**, Theorem 5]).

Since G is not compact, by Lemma 5, $K = e$. Hence G is a Lie group. Let us introduce a canonical coordinate of the second kind in G and let $x_1(t), \cdots,$ $x_n(t)$ be n one-parameter groups of G such that all the x_i's are topologically isomorphic to R (see [**3**, Theorem 13]) and for each $g \in G$, there exist t_i, $-\infty < t_i < \infty$, $i = 1, 2, \cdots, n$ such that

$$g = x_1(t_1)x_2(t_2) \cdots x_n(t_n)\,.$$

Let $D = \{x_1(m) \mid m \in I\}$. Then $G/D \approx R^{n-1} \times T$, where T is the circle group and " $\approx$ " means "homeomorphism". Since D is a free cyclic subgroup of G and G is bonded, it follows that G/D and hence $R^{n-1} \times T$ is compact. Hence, $n-1 = 0$ or $n = 1$ and G is topologically isomorphic to R.

§ 5. Locally compact bonded groups.

THEOREM 3. *Every locally compact bonded group G is one of the following three types:* (I) *compact,* (II) *topologically isomorphic to the additive group I of all* *integers with the discrete topology, or* (III) *topologically isomorphic to the additive group R of all real numbers with the usual topology.*

PROOF. Since every compact group is bonded, we need only prove the case for a locally compact but not compact group.

We divide this case into three parts: (a) G is connected, (b) G is discrete and (c) G is neither connected nor discrete. (a) If G is connected, then by Theorem 2, $G \cong R$. (b) If G is discrete, then by Theorem 1, $G \cong I$. The only case left to discuss is case (c) for which G is neither compact, connected, nor discrete.

Let G_0 be the component subgroup of G. Then G/G_0 is a totally disconnected, locally compact group. Since G_0 is a closed subgroup of G, by Lemma 1, G_0 is bonded. Since G_0 is connected, by Theorem 1, we know that G_0 is either compact or isomorphic to R. By Lemma 5, we know that G_0 cannot be a non-trivial compact subgroup of G since otherwise G would be compact. G_0 cannot be the identity either, since then G would be totally disconnected and since not discrete, would contain a non-trivial open compact subgroup. But by Lemma 5 G would then be compact. This gives a contradiction. Hence $G_0 \cong R$.

Since G/G_0 is compact, by a result of Iwasawa (see [**3**, Lemma 3.8]), there exists a compact subgroup K of G, such that $G = K \cdot G_0$ and $K \cap G_0 = e$. By Lemma 5, we have $K = e$ and $G = G_0$, which contradicts the assumption. This proves the groups of case (c) cannot be bonded. The proof is completed.

COROLLARY 1. *Any locally compact solenoidal group G (i. e. there is a continuous homomorphism ϕ from R into G and $\overline{\phi(R)} = G$) (see* [**2**], [**4**], [**6**] *or* [**8**]) *is either topologically isomorphic to R or is compact.*

PROOF. By Lemma 3, Theorem 3, Corollary to Lemma 7, and the fact that the closure of a continuous image of a connected group in a topological group is again connected, we have the required result.

THEOREM 4. *Every bonded group is one of the following four types.* (I) *compact,* (II) *topologically isomorpoic to I,* (III) *topologically isomorphic to R or* (IV) *radical which is not compact.*

PROOF. By Theorem 3, Lemma 9, and the fact that I and R are not radical, we have the required result.

§ 6. Non-locally-compact bonded groups.

THEOREM 5. *Every non-locally-compact bonded group G has the following properties.*

(1) *It is radical.*

(2) *Every closed subgroup of G is radical.*

(3) *It is forsion free, indecomporable and every non-trivial cyclic subgroup*

is not discrete.

(4) *For every non-trivial cyclic subgroup C there exists a compact subset K_c of G, such that $G = C \cdot K_c$. Here, $(K_c)^\circ$, the interior of K_c, is always empty.*

(5) *If some K_c in (4) is finite, then G is algebraically isomorphic to I.*

PROOF. Property (1) is the direct consequence of Theorem 3 and Theorem 4. Property (2) is a result of Theorem 3 and Lemma 1. Property (3) is a result of Lemma 4, Lemma 5 and Lemma 9. Property (4) is a simple application of the proof of Lemma 6. Property (5) is the direct consequence of Theorem 1. A simple example of (5) is a free cyclic subgroup in the circle group with the induced topology.

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