J. Math. Soc. Japan
Vol. 14, No. 4, 1962

# Dimension-theoretical structure of locally compact groups

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(Received April 6, 1962)

This paper is devoted to the study of dimension-theoretical structure of locally compact groups and their factor spaces. Montgomery-Zippin [9] proved that every finite-dimensional, locally compact group is a generalized Lie group and finally Yamabe [17] proved that every locally compact group is also a generalized Lie group. These are the most important results not only for the group-theoretical structure of locally compact groups but also for the dimension-theoretical structure of such groups. Montgomery [7] had proved also, before his fundamental theorem cited above was established, that the invariance theorem of a domain is true in finite-dimensional, locally connected, locally compact, separable metric groups. P. Alexandroff conjectured that the covering dimension of any locally compact group coincides with its inductive dimension. Recently this conjecture has been solved in the affirmative by Pasynkov [15]. His result will be generalized in § 2, after some preliminaries of § 1, for factor spaces of finite-dimensional locally compact groups by connected compact subgroups. It will also be proved that $\dim G = \dim H + \dim G/H$, where dim denotes the covering dimension, for any locally compact group $G$ and any closed subgroup $H$ of it. Montgomery-Zippin [8], Yamanoshita [18] and others have considered the dimension of factor spaces of locally compact groups and obtained the equality for some special cases. Our theorem seems to be a complete answer for the problem concerning the covering dimension of factor spaces of locally compact groups. In § 3 the decomposition theorem for locally compact groups will be proved. Both Pasynkov's theorem cited above and the author's decomposition theorem show that there are some analogy between the dimension-theoretical structure of locally compact groups and that of Euclidean spaces. In § 4 we shall point out a difference between the two by proving that the invariance theorem of a domain is not true in any finite-dimensional, locally compact, metric group which is not locally connected. Combining this with Montgomery's invariance theorem mentioned above, we know that a finite-dimensional, locally compact, metric group is locally connected (or equivalently a Lie group) if and only if the invariance theorem is valid in it.

In this paper a topological group means a $T_1$-group. Hence a locally compact group and its factor space by a closed subgroup are always normal

Hausdorff spaces (cf. Lemma 1.1 below). A homomorphism means a continuous one and an isomorphism means a homeomorphic one. A coset and a factor space mean respectively a left coset and a left factor space. Throughout this paper $n$ and $m$ denote integers which are not less than $-1$. We use three notions of dimension of a normal Hausdorff space $R$ defined as follows. $R$ has the covering dimension $\leqq n$, $\dim R \leqq n$, if every finite open covering of $R$ can be refined by an open covering whose order is at most $n+1$, where the order of a covering is the greatest number $r$ such that $r$ elements of it have a non-empty intersection. $\operatorname{Ind} R$ and $\operatorname{ind} R$ denote respectively the large and the small inductive dimension of $R$: For the empty set $\phi$, let $\operatorname{Ind}\phi = \operatorname{ind}\phi = -1$. We call $\operatorname{Ind} R \leqq n$, if for any pair $F \subset D$ of a closed set $F$ and an open set $D$ there exists an open set $E$ with $F \subset E \subset D$ such that $\operatorname{Ind}(\bar{E}-E) \leqq n-1$. We call $\operatorname{ind} R \leqq n$, if for any point $x$ of $R$ and any open set $D$ with $x \in D$ there exists an open set $E$ with $x \in E \subset D$ such that $\operatorname{ind}(\bar{E}-E) \leqq n-1$. When d is any one of dim, Ind, ind, we call $\mathrm{d}\, R = n$, if $\mathrm{d}\, R \leqq n$ is true and $\mathrm{d}\, R \leqq n-1$ is false. It is well known that for any separable metric space $R$ the equalities $\dim R = \operatorname{Ind} R = \operatorname{ind} R$ are valid [**6**].

## § 1. Preliminaries.

LEMMA 1.1. *Let $G$ be a locally compact group and $H$ a closed subgroup of $G$. Then the factor space $G/H = K$ is paracompact.*

PROOF. Let $\mathfrak{V} = \{V\}$ be the system of all neighborhoods of the identity of $G$. For any point $k$ of $K$ let

$$U_V(k) = \rho(V \cdot \rho^{-1}(k)),$$

where $\rho: G \to G/H = K$ is the natural projection. Then $\{U_V; V \in \mathfrak{V}\}$ forms a uniform structure which agrees with the preasigned natural topology of $K$. This is verified by a straight-forward computation and its proof is left to the reader. Let $V$ be a compact neighborhood of the identity, $k$ an arbitrary point of $K$ and $g$ an element of $G$ with $\rho(g) = k$. Since $U_V(k) = \rho(V \cdot \rho^{-1}(k)) = \rho(VgH) = \rho(Vg)$, $U_V(k)$ is compact. Thus $K$ is uniformly locally compact. Hence $K$ is paracompact by Morita [**11**].

REMARK 1.2. If a topological space $R$ admits a locally finite open covering $\{D_\delta; \delta \in \Delta\}$ such that $\bar{D}_\delta$ is compact for every $\delta \in \Delta$, then $R$ is the sum of mutually disjoint open sets each of which is $\sigma$-compact (i. e. expressible as the sum of a countable number of compact sets).

PROOF. Since $\{D_\delta; \delta \in \Delta\}$ is locally finite, $\{\bar{D}_\delta; \delta \in \Delta\}$ is locally finite. Suppose that $\{\bar{D}_\delta; \delta \in \Delta\}$ is not star-finite. ($\{\bar{D}_\delta; \delta \in \Delta\}$ is called star-finite if for any $\delta \in \Delta$, the number of indices $\delta'$ with $\bar{D}_\delta \cap \bar{D}_{\delta'} \neq \phi$ is finite.) Then it is easy to find an index $\delta \in \Delta$ and sequences $\{\delta_i; \delta_i \in \Delta\}$ and $\{p_i\}$ such that i)

$p_i \in \bar{D}_\delta \cap \bar{D}_{\delta_i}$, ii) $\delta_i \neq \delta_j$ if $i \neq j$, iii) $p_i \neq p_j$ if $i \neq j$. Since $\bar{D}_\delta$ is compact, there exists an accumulating point $p$ of $\{p_i\}$. Then $\{\bar{D}_\delta ; \delta \in \Delta\}$ cannot be locally finite at $p$, which is a contradiction. The assertion of the remark is a trivial consequence of the star-finiteness of $\{\bar{D}_\delta ; \delta \in \Delta\}$.

LEMMA 1.3 (Montgomery-Zippin [**10**, Theorem, p. 237]). *A locally compact group $G$ with $\dim G = n$ has a small neighborhood of the identity which is the direct product of a local Lie group $L$ with $\dim L = n$ and a compact group $N$ with $\dim N = 0$.*

LEMMA 1.4. *Let $G$ be a non-empty locally compact group and $N$ a compact normal subgroup of $G$ with $\dim N = 0$. If the factor group $G/N$ is a Lie group with $\dim G/N = n$, then there exists a neighborhood of the identity which is the direct product of a local Lie group $L$ with $\dim L = n$ and $N$.*

PROOF. Let $f$ be the natural projection of $G$ onto $G/N$ and $g^*(t)$ an arbitrary one-parameter subgroup of $G/N$. Then there exists a one-parameter subgroup $g(t)$ of $G$ such that $f(g(t)) = g^*(t)$ by Montgomery-Zippin [**10**, Theorem 1, p. 192]. Hence the method of the proof of Pontrjagin [**16**, Theorem 69] can be applied with no modification and we have the lemma.

LEMMA 1.5 (Nagami [**14**] or C. H. Dowker [**3**]). *If every point of a paracompact Hausdorff space $R$ has its neighborhood whose covering dimension is at most $n$, then we have $\dim R \leqq n$.*

LEMMA 1.6.[1] *Let $G$ be a non-empty locally compact group with $\dim G = n$ and $H$ a closed subgroup of $G$ with $\dim H = m$.*[2] *Then every point of $G/H$ has a neighborhood which is homeomorphic to the direct product of an $(n-m)$-dimensional Euclidean cube and a compact Hausdorff space whose covering dimension is* 0.

PROOF. It suffices to construct a neighborhood of $\rho(e)$ satisfying the conditions of the lemma, where $\rho$ is the natural projection of $G$ onto $G/H$ and $e$ is the identity of $G$. Let $D$ be an arbitrary open neighborhood of $\rho(e)$. By Lemma 1.3 $\rho^{-1}(D)$ contains a neighborhood of $e$ which is the direct product of a connected local Lie group $L_1$ with $\dim L_1 = n$ and a compact subgroup $N$ with $\dim N = 0$ such that i) $L_1 = L_1^{-1}$, ii) $\overline{(L_1 N)^2}$ is compact. Let $P = H \cap N$ and $G_0 = (L_1 N)^\infty$ $(= \bigcup_{i=1}^{\infty} (L_1 N)^i)$. Then $G_0$ is an open subgroup of $G$ and $N$ is a normal subgroup of $G_0$. Hence $H_0 = H \cap G_0$ is a relatively open subgroup of $H$ and $P$ is a normal subgroup of $H_0$. By Lemmas 1.1 and 1.5 we have

$$\dim H_0 = \dim H = m .$$

1) This lemma generalizes the last half of Montgomery-Zippin [**10**, Theorem, p. 239], the first half of which will also be generalized in Theorem 2.1 below.

2) Since $H$ is closed, we have $n \geqq m$ at once.

Since $(\overline{L_1N})^2$ is compact, both $H_0$ and $H_0N$ are $\sigma$-compact and locally compact. Hence by Pontrjagin [16, G), § 20] $H_0/P$ is isomorphic to $H_0N/N$. Let $\rho_1$ be the natural projection of $G_0$ onto $G_0/N$. Since $H_0N$ is closed in $G_0$ and $H_0N/N = G_0/N-\rho_1(G_0-H_0N)$, we know that $H_0N/N$ is a closed subgroup of $G_0/N$. Since $G_0/N$ is evidently a Lie group, $H_0N/N$ is a Lie group and hence $H_0/P$ is so.

By Lemma 1.3 there exists a connected, compact, local Lie group $M_1$ with $\dim M_1=m$ such that

i) $M_1P$ is a relative neighborhood of the identity in $H_0$,

ii) $M_1P$ is the direct product of $M_1$ and $P$,

iii) $M_1P\subset L_1N$.

Since $M_1$ is connected and $N$ is totally disconnected, we have $M_1\subset L_1$. Since $M_1$ is compact, $M_1$ is a closed local subgroup of $L_1$. Therefore there exist subsets $L$ of $L_1$ and $M$ of $M_1$ with $\dim L=n$ and $\dim M=m$ such that we can introduce into $L$ a canonical coordinate system of the second kind which has the following properties:

i) $L$ is the totality of points whose coordinates are of the form

$$(t_1, \cdots, t_n), \qquad |t_i|\leqq 1,$$

ii) $M$ is the totality of points whose coordinates are of the form

$$(0, \cdots, 0, t_{n-m+1}, \cdots, t_n), \qquad |t_i|\leqq 1.$$

Let $\Lambda$ be the totality of points whose coordinates are of the form

$$(t_1, \cdots, t_{n-m}, 0, \cdots, 0), \qquad |t_i|\leqq 1.$$

Here we notice that every element of $LN$ can be expressed as $\lambda\mu\nu$, $\lambda\in\Lambda$, $\mu\in M$, $\nu\in N$, and every element of $MP$ can be so as $\mu p$, $\mu\in M$, $p\in P$. We continue to use this notion in the following of the present proof. Let $W_1$ be the totality of points whose coordinates are of the form

$$(t_1, \cdots, t_n), \qquad |t_i|\leqq \varepsilon, \quad 0<\varepsilon<1,$$

such that $W_1^{-1}W_1\subset L$. We set $W=W_1N$. Let $\lambda_1\mu_1\nu_1$ and $\lambda_2\mu_2\nu_2$ be two points of $W$ which are contained in the same coset by $H$; then there exists a point $\mu p$ of $MP$ such that $\lambda_1\mu_1\nu_1=\lambda_2\mu_2\nu_2\mu p$. Hence we have $\lambda_1=\lambda_2$. Since $\mu_1\nu_1 =\mu_2\nu_2\mu p=\mu_2\mu\nu_2 p$, we have $\mu_1=\mu_2\mu$ and hence $\nu_1=\nu_2 p$. Conversely if $\lambda_1\mu_1\nu_1$ and $\lambda_2\mu_2\nu_2$ of $W$ satisfy $\lambda_1=\lambda_2$ and $\nu_2^{-1}\nu_1\in P$, then these two points are contained in the same coset by $H$. Thus we know that $\lambda_1\mu_1\nu_1$ and $\lambda_2\mu_2\nu_2$ of $W$ fall in the same coset by $H$ if and only if $\lambda_1=\lambda_2$ and $\nu_2^{-1}\nu_1\in P$.

Let $f$ be the natural projection of $N$ onto $N/P$ and $g$ the mapping of $\rho(W)$ onto the product space $(W_1\cap\Lambda)\times(N/P)$ defined in such a way that

$$g(\rho(\lambda\mu\nu))=(\lambda, f(\nu));$$

then it can easily be seen by the above observation that $g$ is a homeomorphism. Let $\Lambda_\varepsilon$ be the totality of points whose coordinates are of the form

$$(t_1, \cdots, t_{n-m}, 0, \cdots, 0), \qquad |t_i| \leqq \varepsilon .$$

Then $\rho(\Lambda_\varepsilon N)$ is a neighborhood of $\rho(e)$ which is homeomorphic to $\Lambda_\varepsilon \times (N/P)$. $N/P$ is a compact Hausdorff space with $\dim N/P = 0$ by Pontrjagin [**16**, A), § 48] and the lemma is proved.

LEMMA 1.7. *Let $G$ be a non-empty, locally compact, projective limit of Lie groups with* $\dim G = n$, *and $H$ a compact subgroup of $G$. Then the factor space $G/H = K$ is the projective limit of $(n-m)$-manifolds $K_\alpha$, $\alpha \in A$, accompanied by the mappings $\omega_{\alpha\beta}: K_\alpha \to K_\beta$, $\beta < \alpha$, which are open continuous and locally homeomorphic.*

PROOF. Let $G_\alpha$, $\alpha \in A_1$, be Lie groups and $\pi_{\alpha\beta}: G_\alpha \to G_\beta$, $\beta < \alpha$, open homomorphism of $G_\alpha$ onto $G_\beta$ such that the projective limit of $\{G_\alpha, \pi_{\alpha\beta}; \alpha \in A_1\}$ is $G$. Let $\pi_\alpha$, $\alpha \in A_1$, be (open) homomorphism of $G$ onto $G_\alpha$. Let $N_\alpha^\beta$, $\beta < \alpha$, be the kernel of $\pi_{\alpha\beta}$ and $N_\alpha$ the kernel of $\pi_\alpha$. Since $G$ is locally compact, we can assume without loss of generality that every $N_\alpha$ and every $N_\alpha^\beta$ are compact. We set

$$\dim G_\alpha = n(\alpha) .$$

Since any small $n(\alpha)$-cell in $G_\alpha$ can be lifted to $G$ by Montgomery-Zippin [**10**, p. 194], we have

$$n(\alpha) \leqq \dim G = n .$$

Let

$$\max \{n(\alpha); \alpha \in A_1\} = n_1$$

and $\alpha_0$ an element of $A_1$ such that $\dim G_{\alpha_0} = n_1$. Let $\beta$ be an arbitrary index with $\alpha_0 < \beta$. Since $G_{\alpha_0}$ and $G_\beta$ are Lie groups, it is well known that $\dim G_\beta = \dim G_{\alpha_0} + \dim N_\beta^{\alpha_0}$. Thus we have $\dim G_\beta = n_1$ and $\dim N_\beta^{\alpha_0} = 0$. Since $N_\beta^{\alpha_0}$ is a 0-dimensional compact Lie group, it is a finite group. Since $N_{\alpha_0}$ is isomorphic to the projective limit of $\{N_\beta^{\alpha_0}, \pi_{\gamma\beta}; \alpha_0 \leqq \beta < \gamma\}$, we have $\dim N_{\alpha_0} = 0$. Therefore there exists a neighborhood of the identity of $G$ which is the direct product of a local Lie group $L$ with $\dim L = n_1$ and $N_{\alpha_0}$ by Lemma 1.4. By Morita [**12**] we have $\dim LN_{\alpha_0} \leqq \dim L + \dim N_{\alpha_0} = n_1$. Since $G$ is, by Lemma 1.1, paracompact, we have $n = \dim G \leqq \dim LN_{\alpha_0} = n_1$ by Lemma 1.5. Therefore we have $n = n_1$. Let

$$A = \{\alpha; \alpha_0 \leqq \alpha\};$$

then we have $\dim G_\alpha = n$ and $\dim N_\alpha = 0$ for any $\alpha \in A$ and $G$ is the projective limit of

$$\{G_\alpha; \alpha \in A\} .$$

Let $\dim H = m$. Then $m \leqq n$. Let $H_\alpha$, $\alpha \in A$, be the image of $H$ under $\pi_\alpha$. Since $N_\alpha$ is compact, $HN_\alpha$ is a closed subgroup of $G$. Since $H_\alpha = G_\alpha - \pi_\alpha$

$(G-HN_\alpha)$, $H_\alpha$ is a closed subgroup of $G_\alpha$. Hence we know that $H_\alpha$ is a Lie group.

Let $p_\alpha$ be the natural projection of $H$ onto $H/H\cap N_\alpha$ and define $p_{\alpha\beta}$: $H/H\cap N_\alpha \to H/H\cap N_\beta$, $\beta<\alpha$, as follows:

$$p_{\alpha\beta}(r_\alpha)=p_\beta p_\alpha^{-1}(r_\alpha), \qquad r_\alpha \in H/H\cap N_\alpha .$$

Let

$$q_\alpha(r_\alpha)=\pi_\alpha p_\alpha^{-1}(r_\alpha), \qquad r_\alpha \in H/H\cap N_\alpha .$$

Since $H$ is compact, $q_\alpha$ is an isomorphism of $H/H\cap N_\alpha$ onto $H_\alpha$. Thus we obtain the following diagram:

$$\begin{tikzcd}
 & H \arrow[dl, "p_\alpha"'] \arrow[dr, "\pi_\alpha"] & \\
H/H\cap N_\alpha \arrow[rr, "q_\alpha"] \arrow[d, "p_{\alpha\beta}"'] & & H_\alpha = HN_\alpha/N_\alpha \arrow[d, "\pi_{\alpha\beta}"] \\
H/H\cap N_\beta \arrow[rr, "q_\beta"] & & H_\beta = HN_\beta/N_\beta
\end{tikzcd}$$

It is almost evident that $\{H/H\cap N_\alpha, p_{\alpha\beta}; \alpha\in A\}$ forms a spectrum. Let $\tilde{H}$ be the projective limit of $\{H/H\cap N_\alpha, p_{\alpha\beta}; \alpha\in A\}$ and define $p: H\to\tilde{H}$ as follows:

$$p(h)=\langle p_\alpha(h); \alpha\in A\rangle, \qquad h\in H.$$

Then $p$ is an isomorphism of $H$ onto $\tilde{H}$ by Gleason [4]. Since $q_\alpha p_\alpha=\pi_\alpha$, we know that $H$[3)] is the projective limit of $\{H_\alpha, \pi_{\alpha\beta}\}$. Since $\dim H\cap N_\alpha=0$, we have

$$\dim H_\alpha=\dim H=m, \qquad \alpha\in A,$$

as in the preceding argument.

Let $\rho_\alpha$ be the natural projection of $G_\alpha$ onto $K_\alpha=G_\alpha/H_\alpha$. For any pair $\beta<\alpha$ define $\omega_{\alpha\beta}: K_\alpha\to K_\beta$ in such a way that

$$\omega_{\alpha\beta}(k_\alpha)=\rho_\beta\pi_{\alpha\beta}\rho_\alpha^{-1}(k_\alpha), \qquad k_\alpha\in K_\alpha .$$

Then we obtain the following diagram:

$$\begin{tikzcd}
G_\alpha \arrow[r, "\rho_\alpha"] \arrow[d, "\pi_{\alpha\beta}"'] & K_\alpha = G_\alpha/H_\alpha \arrow[d, "\omega_{\alpha\beta}"] \\
G_\beta \arrow[r, "\rho_\beta"] & K_\beta = G_\beta/H_\beta
\end{tikzcd}$$

Let $g_\alpha$ and $g'_\alpha$ be arbitrary elements of $\rho_\alpha^{-1}(k_\alpha)$; then $g_\alpha^{-1}g'_\alpha\in H_\alpha$ and hence $\pi_{\alpha\beta}(g_\alpha)^{-1}\cdot\pi_{\alpha\beta}(g'_\alpha)\in\pi_{\alpha\beta}(H_\alpha)=H_\beta$, which implies $\rho_\beta\pi_{\alpha\beta}(g_\alpha)=\rho_\beta\pi_{\alpha\beta}(g'_\alpha)$. Thus $\omega_{\alpha\beta}$ is a mapping of $K_\alpha$ into $K_\beta$. It is almost evident that i) $\omega_{\alpha\beta}$ is an open con-

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3) When $H$ is not compact but $\sigma$-compact, we can also conclude that $H$ is the projective limit of Lie groups, since $q_\alpha$ are also isomorphisms in this case by virtue of Pontrjagin [**16**, G), § 20].

tinuous onto mapping, ii) $\{K_\alpha, \omega_{\alpha\beta}; \alpha \in A\}$ forms a spectrum, iii) the equality

$$\omega_{\alpha\beta}\rho_\alpha = \rho_\beta \pi_{\alpha\beta}$$

holds for any pair $\beta < \alpha$.

Let $\rho$ be the natural projection of $G$ onto $G/H$ and define mappings $\omega_\alpha : G/H = K \to G_\alpha/H_\alpha = K_\alpha$, $\alpha \in A$, as follows:

$$\omega_\alpha(k) = \rho_\alpha \pi_\alpha \rho^{-1}(k), \qquad k \in K.$$

Let $g$ and $g'$ be arbitrary elements of $G$ with $g^{-1}g' \in H$; then $\pi_\alpha(g)^{-1} \cdot \pi_\alpha(g') \in H_\alpha$ and hence $\omega_\alpha$ is well defined. Thus we have the following diagram:

$$\begin{array}{ccl} G & \xrightarrow{\rho} & K = G/H \\ \pi_\alpha \downarrow & & \downarrow \omega_\alpha \\ G_\alpha & \xrightarrow{\rho_\alpha} & K_\alpha = G_\alpha/H_\alpha \\ \pi_{\alpha\beta} \downarrow & & \downarrow \omega_{\alpha\beta} \\ G_\beta & \xrightarrow{\rho_\beta} & K_\beta = G_\beta/H_\beta \end{array}$$

It is almost evident that i) $\omega_\alpha$ is an open continuous onto mapping for any $\alpha \in A$, ii) the equality

$$\omega_\beta = \omega_{\alpha\beta}\omega_\alpha$$

holds for any pair $\beta < \alpha$.

Let $\tilde{K}$ be the projective limit of $\{K_\alpha, \omega_{\alpha\beta}; \alpha \in A\}$ and let

$$\omega(k) = \langle \omega_\alpha(k); \alpha \in A \rangle, \qquad k \in K.$$

Since $\omega_{\alpha\beta}\omega_\alpha(k) = \omega_\beta(k)$ for any pair $\beta < \alpha$, $\omega$ is a mapping of $K$ into $\tilde{K}$. Let us prove that $\omega$ is a homeomorphism of $K$ onto $\tilde{K}$. Since $\omega_\alpha$ is continuous for any $\alpha \in A$, $\omega$ is evidently continuous.

To prove that $\omega$ is one-to-one, let $k$ and $k_1$ be different elements of $K$. Let $g$ and $g_1$ be elements of $G$ such that $gH = \rho^{-1}(k)$ and $g_1H = \rho^{-1}(k_1)$; then $g^{-1}g_1 \in H$. Since $H$ is the projective limit of $\{H_\alpha\}$, there exists an index $\alpha$ with $\pi_\alpha(g^{-1}g_1) \notin H_\alpha$. Then $\pi_\alpha(g)H_\alpha \cap \pi_\alpha(g_1)H_\alpha = \phi$ and $\rho_\alpha\pi_\alpha(g) \neq \rho_\alpha\pi_\alpha(g_1)$. Hence $\omega_\alpha(k) \neq \omega_\alpha(k_1)$ and we have $\omega(k) \neq \omega(k_1)$. Therefore $\omega$ is one-to-one.

To prove that $\omega$ is onto, let $\tilde{\omega}_\alpha$, $\alpha \in A$, be the projections of $\tilde{K}$ onto $K_\alpha$ and $\tilde{k}$ an arbitrary element of $\tilde{K}$. For any $\alpha \in A$, $\pi_\alpha^{-1}\rho_\alpha^{-1}\tilde{\omega}_\alpha(\tilde{k})$ is a coset of $G$ by the compact subgroup $HN_\alpha$ and hence a compact subset of $G$. It is obvious that $\{\pi_\alpha^{-1}\rho_\alpha^{-1}\tilde{\omega}_\alpha(\tilde{k}); \alpha \in A\}$ is the family of compact subsets of $G$ which has the finite intersection property. Hence $\cap \{\pi_\alpha^{-1}\rho_\alpha^{-1}\tilde{\omega}_\alpha(\tilde{k}); \alpha \in A\}$ is not empty and contains a point $g$. The image of $\rho(g)$ under $\omega$ is $\tilde{k}$. Therefore $\omega$ is onto.

To show that $\omega$ is open let $k$ be an arbitrary point of $K$, $D$ an arbitrary open neighborhood of $k$ and $g$ an element of $G$ with $\rho(g) = k$. Then there exist an index $\alpha$ and an open neighborhood $D_1$ of $g$ such that $\pi_\alpha^{-1}\pi_\alpha(D_1) \subset \rho^{-1}(D)$. $E = \tilde{\omega}_\alpha^{-1}\rho_\alpha\pi_\alpha(D_1)$ is an open neighborhood of $\omega(k)$. We have $\pi_\alpha^{-1}\rho_\alpha^{-1}\tilde{\omega}_\alpha(E)$

$=\pi_\alpha^{-1}\rho_\alpha^{-1}\rho_\alpha\pi_\alpha(D_1)=\pi_\alpha^{-1}(\pi_\alpha(D_1)H_\alpha)=\pi_\alpha^{-1}(\pi_\alpha(D_1)\pi_\alpha(H))=\pi_\alpha^{-1}\pi_\alpha(D_1H)=D_1HN_\alpha$ $=(\pi_\alpha^{-1}\pi_\alpha(D_1))H\subset\rho^{-1}(D)H=\rho^{-1}(D)$. Let $g_1$ be an element of $G$ with $\rho(g_1)\in\omega^{-1}(E)$. Then $\rho_\alpha\pi_\alpha\rho^{-1}\rho(g_1)=\rho_\alpha\pi_\alpha(g_1H)=\rho_\alpha(\pi_\alpha(g_1)H_\alpha)=\rho_\alpha\pi_\alpha(g_1)$ is an element of $\tilde{\omega}_\alpha(E)$ $=\tilde{\omega}_\alpha\tilde{\omega}_\alpha^{-1}\rho_\alpha\pi_\alpha(D_1)=\rho_\alpha\pi_\alpha(D_1)$. Hence $g_1$ is an element of $\pi_\alpha^{-1}\rho_\alpha^{-1}\rho_\alpha\pi_\alpha(D_1)=D_1HN_\alpha$. Therefore we have $\omega^{-1}(E)\subset\rho(D_1HN_\alpha)\subset D$ and know that $\omega$ is open.

By the above observation $\omega$ is a homeomorphism of $G/H=K$ onto $\tilde{K}$. Recall that $\dim H_\alpha=m$ for any $\alpha\in A$. Hence we have

$$\dim K_\alpha=n-m$$

for any $\alpha\in A$.

Finally let us show that the local restriction of $\omega_{\alpha\beta}$ is a homeomorphism. Let $\beta<\alpha$ be an arbitrary ordered pair and $k_\alpha$ an arbitrary element of $K_\alpha$. Let $k$ be an element of $K$ with $\omega_\alpha(k)=k_\alpha$ and $g$ an element of $G$ with $\rho(g)$ $=k$. Let $LN_\beta$ be a neighborhood of the identity of $G$ which is the direct product of a local Lie group $L$ with $\dim L=n$ and $N_\beta$ such that $L$ is the totality of points of the form $(t_1,\cdots,t_n)$, $|t_i|\leqq 1$, in some canonical coordinate system $\Sigma$ of the second kind. For any number $\delta$ with $0<\delta<1$, let $\Lambda_\delta$ be the totality of points of the form

$$(t_1,\cdots,t_{n-m},0,\cdots,0)\,,\qquad |t_i|\leqq\delta\,,$$

in $\Sigma$ and $M_\delta$ the totality of points of the form

$$(0,\cdots,0,t_{n-m+1},\cdots,t_n)\,,\qquad |t_i|\leqq\delta\,,$$

in $\Sigma$. Let $P_\beta$ be the intersection of $H$ and $N_\beta$. By the same argument as in the proof of Lemma 1.5 there exists a positive number $\varepsilon<1$ such that

i) $M_\varepsilon P_\beta$ is a relative neighborhood of the identity in $H$ which is the direct product of $M_\varepsilon$ and $P_\beta$,

ii) $\Lambda_\varepsilon M_\varepsilon N_\beta\cap H=M_\varepsilon P_\beta$,

iii) $(\Lambda_\varepsilon M_\varepsilon N_\beta)^{-1}\Lambda_\varepsilon M_\varepsilon N_\beta\subset LN_\beta$.

Let $\lambda_1$ and $\lambda_2$ be two elements of $\Lambda_\varepsilon$ and consider two elements $\rho_\alpha\pi_\alpha(g\lambda_1)$ and $\rho_\alpha\pi_\alpha(g\lambda_2)$ of $K_\alpha$. Suppose that $\omega_{\alpha\beta}\rho_\alpha\pi_\alpha(g\lambda_1)=\omega_{\alpha\beta}\rho_\alpha\pi_\alpha(g\lambda_2)$. Then $\rho_\beta\pi_{\alpha\beta}\pi_\alpha(g\lambda_1)$ $=\rho_\beta\pi_{\alpha\beta}\pi_\alpha(g\lambda_2)$ and hence $\rho_\beta\pi_\beta(g\lambda_1)=\rho_\beta\pi_\beta(g\lambda_2)$. We have $\pi_\beta(g\lambda_2)^{-1}\cdot\pi_\beta(g\lambda_1)$ $=\pi_\beta(\lambda_2^{-1}\lambda_1)\in H_\beta$ and hence $\lambda_2^{-1}\lambda_1\in HN_\beta$. On the other hand $\lambda_2^{-1}\lambda_1\in\Lambda_\varepsilon^{-1}\Lambda_\varepsilon\subset L$ and hence $\lambda_2^{-1}\lambda_1\in HN_\beta\cap LN_\beta\subset M_\varepsilon N_\beta$. Therefore there exist an element $\mu$ of $M_\varepsilon$ and an element $\nu$ of $N_\beta$ such that $\lambda_1=\lambda_2\mu\nu$. Since this expression is unique, we have $\lambda_1=\lambda_2$. Thus we can conclude by the compactness of $g\Lambda_\varepsilon$ that $\rho_\alpha\pi_\alpha(g\Lambda_\varepsilon)$ and $\rho_\beta\pi_\beta(g\Lambda_\varepsilon)$ are the homeomorphic image of $g\Lambda_\varepsilon$ under the mapping $\rho_\alpha\pi_\alpha$ and $\rho_\beta\pi_\beta$ respectively and that $\rho_\beta\pi_\beta(g\Lambda_\varepsilon)$ is the homeomorphic image of $\rho_\alpha\pi_\alpha(g\Lambda_\varepsilon)$ under the mapping $\omega_{\alpha\beta}$. Let $\Lambda'_\varepsilon$ be the totality of points of the form $(t_1,\cdots,t_{n-m},0,\cdots,0)$, $|t_i|<\varepsilon$, in $\Sigma$. If we replace $\Lambda_\varepsilon$ with $\Lambda'_\varepsilon$, then the above statements with this replacement is also valid. Since $g\Lambda'_\varepsilon$ is homeomorphic to an $(n-m)$-Euclidean space and $K_\alpha$ and $K_\beta$ are $(n-m)$-manifolds, we know that $\rho_\alpha\pi_\alpha(g\Lambda'_\varepsilon)$ and $\rho_\beta\pi_\beta(g\Lambda'_\varepsilon)$ are open sets of $K_\alpha$ and $K_\beta$ respectively, by a famous Brouwer's invariance theorem of a domain. $\rho_\alpha\pi_\alpha(g\Lambda'_\varepsilon)$ contains $\rho_\alpha\pi_\alpha(g)=\omega_\alpha\rho(g)=\omega_\alpha(k)=k_\alpha$. Thus the proof is completely finished

COROLLARY 1.8.[4] *A $\sigma$-compact closed subgroup of a locally compact group which is the projective limit of Lie groups is also the projective limit of Lie groups.*

Cf. the footnote 3).

COROLLARY 1.9.[4] *A locally compact group $G$ has a $\sigma$-compact open subgroup which is the projective limit of Lie groups.*

PROOF. By Glushkov [**5**] there exists an open subgroup $G_1$ of $G$ which is the projective limit of Lie groups. Let $U$ be a symmetric open neighborhood of the identity of $G$ such that i) $\overline{U^2}$ is compact and ii) $\overline{U^2}\subset G_1$. Then $U^\infty$ is a $\sigma$-compact open subgroup with $U^\infty\subset G_1$. By Corollary 1.8 $U^\infty$ is also the projective limit of Lie groups and the corollary is proved.

COROLLARY 1.10. *Let $G$ be a locally compact group with* $\dim G=n$ *and $H$ a connected compact subgroup of $G$ with* $\dim H=m$. *Then $K=G/H$ is the projective limit of $(n-m)$-manifolds $K_\alpha$ accompanied with projections $\omega_{\alpha\beta}$ which are open continuous and locally topological.*

PROOF. By Glushkov [**5**] there exists an open subgroup $G_0$ of $G$ which is the projective limit of Lie groups. Decompose $G$ into cosets $g_\xi G_0$, $\xi\in\Xi$, such that $G=\cup\{g_\xi G_0\,;\,\xi\in\Xi\}$ and $g_{\xi_1}G_0\cap g_{\xi_2}G_0=\phi$ for any $\xi_1$ and $\xi_2$ of $\Xi$ with $\xi_1\neq\xi_2$. For any $g\in G$ and any $g_0\in G_0$ we have $gG_0\supset gg_0H$ by virtue of the connectedness of $H$. Therefore $\rho(g_{\xi_1}G_0)\cap\rho(g_{\xi_2}G_0)=\phi$ whenever $\xi_1\neq\xi_2$, where $\rho$ is the natural projection of $G$ onto $G/H=K$. If we set

$$\varphi_\xi(k)=\rho(g_\xi\cdot\rho^{-1}(k)),\qquad k\in\rho(G_0),$$

we have a mapping $\varphi_\xi$ of $\rho(G_0)$ into $\rho(g_\xi G_0)$. By a straight-forward argument it can easily be seen that $\varphi_\xi$ is a homeomorphism of $\rho(G_0)$ onto $\rho(g_\xi G_0)$. Thus $K$ is the sum of mutually disjoint open sets $\rho(g_\xi G_0)$, $\xi\in\Xi$, any of which is homeomorphic to $\rho(G_0)$.

By Lemma 1.7 we can consider $G_0/H$ as the projective limit of $(n-m)$-manifolds $K_\alpha^0$, $\alpha\in A$, accompanied with open continuous mappings $\omega_{\alpha\beta}^0:K_\alpha^0\to K_\beta^0$, $\beta<\alpha$, which are locally topological. For any $\xi\in\Xi$ and any $\alpha\in A$, let $K_\alpha^\xi$ be a copy of $K_\alpha^0$ (as a topological space) and $\varphi_\alpha^\xi:K_\alpha^0\to K_\alpha^\xi$ a copy-mapping. For any $\xi\in\Xi$ and any pair $\beta<\alpha$ let $\omega_{\alpha\beta}^\xi:K_\alpha^\xi\to K_\beta^\xi$ be a mapping defined by

$$\omega_{\alpha\beta}^\xi=\varphi_\beta^\xi\omega_{\alpha\beta}^0(\varphi_\alpha^\xi)^{-1}.$$

Then it is evident that $\{K_\alpha^\xi,\omega_{\alpha\beta}^\xi\,;\,\alpha\in A\}$ forms a spectrum. For any $\alpha$ let $K_\alpha$ be the disjoint sum of $K_\alpha^\xi$, $\xi\in\Xi$, whose topology is defined as follows:

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4) Corollaries 1.8 and 1.9 were proved by Pasynkov [**15**].

A subset $D_\alpha$ of $K_\alpha$ is open if and only if $D_\alpha \cap K_\alpha^\xi$ is open for every $\xi \in \Xi$. For any pair $\beta < \alpha$ let $\omega_{\alpha\beta}: K_\alpha \to K_\beta$ be a mapping defined as follows: The restriction of $\omega_{\alpha\beta}$ to $K_\alpha^\xi$ coincides with $\omega_{\alpha\beta}^\xi$ for any $\xi$. It is almost evident that $\{K_\alpha, \omega_{\alpha\beta}; \alpha \in A\}$ forms a spectrum which has the following properties:

i) For any $\alpha \in A$, $K_\alpha$ is an $(n-m)$-manifold.

ii) For any pair $\beta < \alpha$, $\omega_{\alpha\beta}$ is an open continuous mapping which is locally topological.

iii) The projective limit of $\{K_\alpha\}$ is homeomorphic to $G$.

Thus the corollary is essentially proved.

LEMMA 1.11 (Pasynkov's criterion [**15**, Lemma 3]). *Let a locally compact Hausdorff space $K$ be the projective limit of the spectrum $\{K_\alpha, \omega_{\alpha\beta}\}$ which satisfies the following conditions:*

i) *For any $\alpha$, the sum theorem for the large inductive dimension is valid.*

ii) *For any $\alpha$,* $\operatorname{Ind} K_\alpha \leqq r$.

iii) *For any pair $\beta < \alpha$, $\omega_{\alpha\beta}$ is locally topological.*

iv) *$K$ is covered by a countable number of compact sets $F_i$, $i = 1, 2, \cdots$, with* $\operatorname{Ind} F_i \leqq r$ *for any $i$.*

*Then we have* $\operatorname{Ind} K \leqq r$.

## § 2. Dimension of factor spaces.

THEOREM 2.1. *Let $G$ be a locally compact group and $H$ a closed subgroup of $G$. Then*

$$\dim G = \dim H + \dim G/H.$$ [5)]

PROOF. First we consider the case when $\dim G < \infty$. Let $\dim G = n$ and $\dim H = m$. By Lemma 1.6 an arbitrary point $k$ of $G/H$ has a neighborhood $U$ which is homeomorphic to the direct product of an $(n-m)$-dimensional Euclidean cube $E$ and a compact Hausdorff space $C$ with $\dim C = 0$. By Morita [**12**] we have $\dim E \times C \leqq n-m$. On the other hand $E \times C$ contains a closed subset which is homeomorphic to $E$. Hence we have $\dim E \times C \geqq n-m$. Since $G/H$ is paracompact by Lemma 1.1, we have $\dim G/H \leqq n-m$ by Lemma 1.5. Since $E \times C$ is compact and hence $U$ is closed in $G/H$, we have $\dim G/H \geqq \dim U = n-m$. Thus we have $\dim G/H = n-m$ and the equality $\dim G = \dim H + \dim G/H$ is valid.

Next we consider the case when $\dim G = \infty$. When $\dim H = \infty$, the equality $\dim G = \dim H + \dim G/H$ is trivially true. Hence we consider the case when $\dim H < \infty$. Let $\dim H = m$. In this case we shall prove that dim

5) The author's colleague Dr. Y. Katuta proved this equality for the case when $G$ is a compact group.

$G/H=\infty$, which is not trivial at all.

Let $r$ be an arbitrary positive integer. By Corollary 1.9 there exists an open $\sigma$-compact subgroup $G_0$ of $G$ which is the projective limit of a spectrum $\{G_\alpha, \pi_{\alpha\beta}; \alpha \in A\}$ where $G_\alpha$ are Lie groups. Let $\pi_\alpha : G_0 \to G_\alpha$ be the projections. If

$$\sup\{\dim G_\alpha; \alpha \in A\} = r_1$$

is finite,

$$A_1 = \{\alpha; \dim G_\alpha = r_1\}$$

is equifinal in $A$. Hence it can easily be seen that the kernel $N_\alpha$ of some $\pi_\alpha$ is of covering dimension 0. By Lemma 1.4 there exists a neighborhood $U$ of the identity in $G_0$ which is the direct product of a Euclidean $r_1$-cube $E$ and $N_\alpha$. Since

$$\dim U = \dim E \times N_\alpha \leqq \dim E + \dim N_\alpha = r_1 + 0$$

by Morita [**12**], we have $\dim G = \dim G_0 \leqq r_1$ by Lemma 1.5, which is a contradiction. Hence there exists a compact normal subgroup $N$ of $G_0$ such that

i) $\dim G_0/N > r+m$,

ii) $G_0/N$ is a Lie group.

Since $H \cap G_0$ is $\sigma$-compact and $(H \cap G_0)N$ is closed, we know that $(H \cap G_0)N/N = Q$ is isomorphic to $H \cap G_0/H \cap N$ by Pontrjagin [**16**, G), § 20]. Since every small cell of $H \cap G_0/H \cap N$ can be lifted to $H \cap G_0$ by Montgomery-Zippin [**10**, p. 194], we have

$$\dim Q = \dim H \cap G_0/H \cap N \leqq \dim H \cap G_0 = \dim H = m.$$

Let $\pi$ be the natural projection of $G_0$ onto $G_0/N = P$ and let

$$\dim P = p.$$

Let $g_1(t), \cdots, g_p(t)$, $|t| \leqq \delta_1$, be one-parameter subgroups of $P$ which generate a canonical coordinate system of the second kind of $P$ such that $g_{p-q+1}(t), \cdots, g_p(t)$ generate a canonical coordinate system of the second kind of $Q$, where

$$q = \dim Q.$$

By Montgomery-Zippin [**10**, Theorem 1, p. 192], we can find one-parameter subgroups $g_1^*(t), \cdots, g_p^*(t)$, $|t| \leqq \delta_2$ $(\leqq \delta_1)$, of $G_0$ such that

i) $\pi(g_i^*(t)) = g_i(t)$ for any $t$ with $|t| \leqq \delta_2$ and $i = 1, \cdots, p$,

ii) $g_{p-q+1}^*(t), \cdots, g_p^*(t)$ are in $H \cap G_0$,

iii) $\pi(LL^{-1}L) \subset \{g_1(t_1) \cdots g_p(t_p); |t_i| \leqq \delta_1\}$, where

$$L = \{g_1^*(t_1) \cdots g_p^*(t_p); |t_i| \leqq \delta_2\},$$

$$\Lambda = \{g_1^*(t_1) \cdots g_{p-q}^*(t_{p-q}); |t_i| \leqq \delta_2\},$$

$$M = \{g_{p-q+1}^*(t_{p-q+1}) \cdots g_p^*(t_p); |t_i| \leqq \delta_2\}.$$

Let $\lambda_1\mu_1$ and $\lambda_2\mu_2$ be two elements of $L$, where $\lambda_1, \lambda_2 \in \Lambda$ and $\mu_1, \mu_2 \in M$. If $\lambda_1\mu_1$ and $\lambda_2\mu_2$ are elements of the same coset by $H \cap G_0$, then we have $\pi(\lambda_2)^{-1} \cdot \pi(\lambda_1) \in \pi(H \cap G_0) = Q$. Hence we have $\lambda_1 = \lambda_2$.[6] Conversely if $\lambda_1 = \lambda_2 \in \Lambda$ and $\mu_1, \mu_2$ be arbitrary elements of $M$, then $(\lambda_2\mu_2)^{-1} \cdot \lambda_1\mu_1 = \mu_2^{-1}\mu_1 \in M^{-1}M \subset H \cap G_0$. Thus we know that $\Lambda$ is homeomorphic to $\pi(\Lambda)$ under $\pi$. Since $\pi(\Lambda)$ is homeomorphic to a Euclidean $(p-q)$-cube, we have

$$\dim \Lambda = p-q .$$

Similarly we can know that $\Lambda$ is homeomorphic to $\rho_0(\Lambda)$, where $\rho_0$ is the natural projection of $G_0$ onto $G_0/H \cap G_0$. Hence

$$\dim \rho_0(\Lambda) = p-q .$$

Let $\rho$ be the natural projection of $G$ onto $G/H$; then $G_0/H \cap G_0$ is homeomorphic to $\rho(G_0)$ under the mapping $\rho\rho_0^{-1}$. We have $\dim G/H = \dim \rho(G_0)$ by Lemmas 1.1 and 1.5. Thus we have

$$\dim G/H \geqq \dim \rho(\Lambda) = \dim \rho_0(\Lambda) = p-q > r+m-m = r .$$

Since $r$ was an arbitrary positive integer, we have $\dim G/H = \infty$ and the theorem is completely proved.

THEOREM 2.2. *Let $G$ be a locally compact group with* $\dim G = n$ *and $H$ a connected compact subgroup of $G$ with* $\dim H = m$. *Then*

$$\dim G/H = \operatorname{Ind} G/H = \operatorname{ind} G/H = n-m .$$

PROOF. By Lemma 1.6 any point $k$ of $G/H$ has a neighborhood $U(k)$ which is homeomorphic to the direct product of a Euclidean $(n-m)$-cube $E$ and a compact Hausdorff space $C$ with $\dim C = 0$. Hence we have

$$\dim U(k) = n-m$$

as we see in the proof of Theorem 2.1. Since $G/H$ is paracompact by Lemma 1.1, we have

$$\dim G/H = n-m$$

by Lemma 1.5.

Since $n-m = \operatorname{ind} E \leqq \operatorname{ind} U(k) \leqq \operatorname{ind} G/H$, we have

$$\operatorname{ind} G/H \geqq n-m .$$

In general it can easily be seen by an easy induction on $\operatorname{Ind} R$ that $\operatorname{Ind} R \times S \leqq \operatorname{Ind} R$ for compact Hausdorff spaces $R$ and $S$ with $\operatorname{Ind} S = 0$. Hence we have

$$\operatorname{Ind} U(k) \leqq n-m .$$

On the other hand $n-m = \operatorname{Ind} E \leqq \operatorname{Ind} U(k)$. Therefore we have

$$\operatorname{Ind} U(k) = n-m .$$

6) Cf. Pontrjagin [**16**, A), § 44].

By Lemma 1.9 there exists an open $\sigma$-compact subgroup $G_0$ of $G$ which is the projective limit of Lie groups. Since $H$ is connected, we have $G_0 \supset H$. Since $G_0/H$ is $\sigma$-compact, $G_0/H$ is covered by a countable number of compact sets $F_i$ with $\operatorname{Ind} F_i = n-m$, $i=1, 2, \cdots$.

By Lemma 1.7 $G_0/H$ is the projective limit of $\{K_\alpha, \omega_{\alpha\beta}\}$ where $K_\alpha$ are $(n-m)$-manifolds and $\omega_{\alpha\beta}$ are open continuous and locally topological. Since $G_0/H$ is $\sigma$-compact and hence $K_\alpha$ are $\sigma$-compact, $K_\alpha$ are separable metric. Hence for any $\alpha$

$$\dim K_\alpha = \operatorname{Ind} K_\alpha = n-m .$$

Thus all conditions in Lemma 1.11 are satisfied and we conclude that

$$\operatorname{Ind} G_0/H \leqq n-m .$$

Since $n-m = \operatorname{Ind} F_i \leqq \operatorname{Ind} G_0/H$, we have

$$\operatorname{Ind} G_0/H = n-m .$$

Since $G/H$ is, by an analogous argument as in the proof of Corollary 1.10, the sum of mutually disjoint open sets each of which is homeomorphic to $G_0/H$, we conclude that

$$\operatorname{Ind} G/H = \operatorname{Ind} G_0/H = n-m$$

by an easy induction on $\operatorname{Ind} G_0/H$. Since $\operatorname{ind} G/H \leqq \operatorname{Ind} G/H$, we have also

$$\operatorname{ind} G/H = n-m .$$

Thus the proof is completed.

COROLLARY 2.3. *Let $G$ be a locally compact group with $\dim G = n$ which is the projective limit of Lie groups and $H$ a compact subgroup of $G$ with $\dim H = m$. Then*

$$\dim G/H = \operatorname{Ind} G/H = \operatorname{ind} G/H = n-m .$$

PROOF. There exists an open subgroup $G_0$ which is $\sigma$-compact. Then $G_1 = G_0 H$ is also an open subgroup which is $\sigma$-compact. By an analogous argument to the proof of Theorem 2.2, we have

$$\dim G/H = \operatorname{ind} G/H = \operatorname{Ind} G_1/H = \operatorname{Ind} G/H = n-m ,$$

which proves the corollary.

## § 3. Decomposition theorem.

THEOREM 3.1. *Let $G$ be a locally compact group with $\dim G = n$. Then there exist $n+1$ subspaces $B_i$, $i=1, \cdots, n+1$, such that for any $i$ $B_i$ is a paracompact space with $\dim B_i \leqq 0$.*

PROOF. Let $V$ be an open symmetric neighborhood of the identity of $G$ such that $\overline{V^2}$ is compact. Let $G_0 = V^\infty$; then $G_0$ is an open $\sigma$-compact subgroup of $G$. By Lemma 1.3 there exists an open neighborhood of the identity of $G$ which is the direct product of a local Lie group $L$ and a compact group $N$ such that

i) $LN \subset V$,

ii) $L$ is homeomorphic to a Euclidean $n$-space,

iii) $\dim N = 0$.

Let $W$ be a relatively open neighborhood of the identity in $L$ such that

i) the closure of $W$ in $L$, say $F$, is homeomorphic to a Euclidean $n$-cube,

ii) $FF^{-1}F \subset L$.

Let $x_1, x_2, \cdots$ be a sequence of points of $G_0$ and $t(1), t(2), \cdots$ be a sequence of positive integers which satisfies the following conditions:

i) $1 \leqq t(1) \leqq t(2) \leqq \cdots$.

ii) $x_i \in \overline{V^m}$ for $i = 1, \cdots, t(m)$, $m = 1, 2, \cdots$.

iii) $x_i \notin \overline{V^m}$ for $i > t(m)$, $m = 1, 2, \cdots$.

iv) $\cup \{x_i WN; i = 1, \cdots, t(m)\} \supset \overline{V^m}$, $m = 1, 2, \cdots$.

Then $\{x_i WN; i = 1, 2, \cdots\}$ is a star-finite open covering of $G_0$. $\{x_i FN; i = 1, 2, \cdots\}$ is therefore a star-finite closed covering of $G$.[7] Since $F$ is separable metric, there exist $n+1$ subsets $F_i$, $i = 1, \cdots, n+1$, of $F$ with $\dim F_i = 0$ for any $i$ (cf. Hurewicz-Wallman [**6**]).

We set

$$H_i = x_1 F_i N \cup \left( \bigcup_{j=2}^{\infty} (x_j F_i N - \bigcup_{k<j} x_k FN) \right), \qquad i = 1, \cdots, n+1 .$$

It is evident that $G_0 = \bigcup_{i=1}^{n+1} H_i$. Let us prove that every $H_i$ is paracompact. Set for every $i$

$$H_{i1} = x_1 F_i N ,$$
$$H_{ij} = x_j F_i N - \bigcup_{k<j} x_k FN , \qquad j = 2, 3, \cdots ;$$

then $H_i = \bigcup_{j=1}^{\infty} H_{ij}$ and $\bigcup_{j=1}^{k} H_{ij}$ is relatively closed in $H_i$ for $k = 1, 2, \cdots$. Set

$$J_j = \{k ; x_j FN \cap x_k FN \neq \phi\} , \qquad j = 1, 2, \cdots ;$$

then $J_j$ is a finite set of indices from the star-finiteness of $\{x_i FN; i = 1, 2, \cdots\}$. It is evident that

$$x_j^{-1} x_k FN \subset FF^{-1}FN \subset LN \quad \text{for any} \quad k \in J_j,\ j = 1, 2, \cdots .$$

Therefore if we set

$$E_{ij} = \cup \{H_{ik} ; k \in J_j\} , \qquad i = 1, \cdots, n+1,\ j = 1, 2, \cdots ,$$

then we have $x_j^{-1} E_{ij} \subset LN$ for $i = 1, \cdots, n+1$, $j = 1, 2, \cdots$.

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7) Cf. the argument in the proof of Remark 1.2.

Write an element of $LN$ as the product of $\lambda\in L$ and $\nu\in N$ and define a mapping $\pi$ of $LN$ onto $L$ in such a way that

$$\pi(\lambda\nu)=\lambda\,.$$

Now let us prove the equalities:

$$x_j^{-1}E_{ij}=\pi(x_j^{-1}E_{ij})N\,,\qquad i=1,\cdots,n+1,\ j=1,2,\cdots.$$

Evidently $x_j^{-1}E_{ij}\subset\pi(x_j^{-1}E_{ij})N$. To show that $x_j^{-1}E_{ij}\supset\pi(x_j^{-1}E_{ij})N$, let $\lambda$ be an arbitrary element of $\pi(x_j^{-1}E_{ij})$. Then there exists an index $k\in J_j$ such that $\lambda\in\pi(x_j^{-1}H_{ik})$, and hence $\lambda\in\pi(x_j^{-1}x_kF_iN-\cup\{x_j^{-1}x_{k'}FN;\,k'<k\})$. Therefore we have

$$\begin{aligned}\lambda N&\subset\pi(x_j^{-1}x_kF_iN-\cup\{x_j^{-1}x_{k'}FN;\,k'<k\})N\\&=\pi(\pi(x_j^{-1}x_kF_i)N-\pi(\cup\{x_j^{-1}x_{k'}F;\,k'<k\})N)N\\&=\pi(\pi(x_j^{-1}x_kF_i)-\pi(\cup\{x_j^{-1}x_{k'}F;\,k'<k\})N)N\\&=(\pi(x_j^{-1}x_kF_i)-\pi(\cup\{x_j^{-1}x_{k'}F;\,k'<k\}))N\\&=\pi(x_j^{-1}x_kF_i)N-\pi(\cup\{x_j^{-1}x_{k'}F;\,k'<k\})N\\&=x_j^{-1}x_kF_iN-\cup\{x_j^{-1}x_{k'}FN;\,k'<k\}\\&=x_j^{-1}(x_kF_iN-\cup\{x_{k'}FN;\,k'<k\})\\&=x_j^{-1}H_{ik}\subset x_j^{-1}E_{ij}.\end{aligned}$$

Thus we know that $E_{ij}$ is homeomorphic to the product space of $\pi(x_j^{-1}E_{ij})$ and $N$. Since $\pi(x_j^{-1}E_{ij})$ is separable metric and $N$ is compact (Hausdorff), the product space $\pi(x_j^{-1}E_{ij})\times N$ is paracompact by Dieudonné [**2**]. Therefore we can conclude that $E_{ij}$ is paracompact.

Since

$$\begin{aligned}H_{ij}\subset x_jFN\cap H_i&\subset H_i-\cup\{x_kFN;\,k\notin J_j\}\\&=(\cup\{H_{ik};\,k\in J_j)\cup(\cup\{H_{ik};\,k\notin J_j\})-\cup\{x_kFN;\,k\notin J_j\}\\&\subset\cup\{H_{ik};\,k\in J_j\}=E_{ij}\,,\end{aligned}$$

the relative closure of $H_{ij}$ in the space $H_i$, say $\widetilde{H}_{ij}$, is contained in $x_jFN\cap H_i$ and hence in $E_{ij}$. Since $\widetilde{H}_{ij}$ is considered as the relative closure of $H_{ij}$ in the space $E_{ij}$, $\widetilde{H}_{ij}$ is paracompact by the paracompactness of $E_{ij}$. Since $\widetilde{H}_{ij}\subset x_jFN$ for $j=1,2,\cdots$,

$$\{\widetilde{H}_{ij}\,;j=1,2,\cdots\}$$

is as can easily be seen a locally finite relatively closed covering of $H_i$. Hence the paracompactness of $H_i$ is established by Morita [**13**].

Next let us prove that $\dim H_i\leqq 0$ for $i=1,\cdots,n+1$. Since $F$ is compact, there exists a sequence of open sets $D_r$, $r=1,2,\cdots$, of $L$ such that

$$F=\bigcap_{i=1}^{\infty}D_i\,.$$

We set

$$H_{ijr}=x_jF_iN-\cup\{x_kD_rN\,;\,k<j\}\,,\qquad i=1,\cdots,n+1\,,\ j=1,2,\cdots,\ r=1,2,\cdots;$$

then $H_{ijr}$ is relatively closed in $H_i$ and contained in $H_{ij}$. Since $\{\tilde{H}_{ij}\,;\,j=1,2,\cdots\}$ is locally finite in the space $H_i$,

$$\{H_{ijr}\,;\,j=1,2,\cdots\}$$

is also locally finite in the space $H_i$. Hence

$$K_{ir}=\bigcup_{j=1}^{\infty}H_{ijr}$$

is relatively closed in $H_i$. Since $H_{ijr}$ is a relatively closed subset of a paracompact space $x_jF_iN$, we have

$$\dim H_{ijr}\leqq\dim x_jF_iN=\dim F_iN=\dim F_i\times N\leqq\dim F_i+\dim N=0\,.$$

Hence by the sum theorem we have

$$\dim K_{ir}\leqq 0\,.$$

Since it is almost evident that $H_i=\bigcup_{r=1}^{\infty}K_{ir}$, we have

$$\dim H_i\leqq 0\,,\qquad i=1,\cdots,n+1\,,$$

by the sum theorem again.

Let $\{g_\xi G_0\,;\,\xi\in\Xi\}$ be a collection of all cosets by $G_0$ such that $g_\xi G_0\cap g_\eta G_0=\phi$ whenever $\xi$ and $\eta$ are different indices of $\Xi$. Setting

$$B_i=\cup\{g_\xi H_i\,;\,\xi\in\Xi\}\,,\qquad i=1,\cdots,n+1\,,$$

$B_i$ is evidently a paracompact space with $\dim B_i\leqq 0$ for every $i$. Thus the theorem is completely proved.

REMARK 3.2. It is to be noted that $B_i$, $i=1,\cdots,n+1$, constructed above satisfy the following condition: If $I=\{i_1,\cdots,i_j\}$ is any subset of $\{1,\cdots,n+1\}$, then $\cup\{B_i\,;\,i\in I\}$ is a paracompact space with $\dim\cup\{B_i\,;\,i\in I\}\leqq j-1$.

## § 4. Invariance theorem of a domain.

LEMMA 4.1 (Alexandroff-Hopf [**1**, Theorem IV′, p. 121]). *A compact metric space $R$ with $\dim R=0$ which has no isolated point is homeomorphic to a Cantor discontinuum.*

THEOREM 4.2. *The invariance theorem of a domain does not hold in any locally compact, metric group $G$ with $\dim G<\infty$ which is not locally connected.*

PROOF. Let $\dim G=n$; then by Lemma 1.3 there exists an open neighborhood of the identity of $G$ which is the direct product of a local Lie group $L$ which is homeomorphic to a Euclidean $n$-space and a compact metric group

$N$ with $\dim N=0$. Since $G$ is not locally connected, $N$ must be infinite. Hence we can consider $N$ as the projective limit of a spectrum $\{N_i, \pi_{ij}: N_i \to N_j;\ i=1,2,\cdots\}$ such that

i) $N_i$ is a finite group for every $i$,

ii) $\pi_{ij}$ are onto homomorphisms,

iii) for any $i$ the order of the kernel of $\pi_{i+1,i}$ is not less than 3.

Set $M_1=N_1$. By an easy application of the induction we can construct a sequence of finite subsets $M_i$ of $N_i$, $i=1,2,\cdots$, such that

$$|\pi_{i+1,i}^{-1}(\mu) \cap M_{i+1}|=2 \quad \text{for any} \quad \mu \in M_i,\ i=1,2,\cdots.$$

Let $M$ be the projective limit of $\{M_i, \pi_{ij}\}$. Since both $N$ and $M$ are compact metric spaces with $\dim N=\dim M=0$ which have no isolated point, there exists by Lemma 4.1 a homeomorphism $\varphi$ of $N$ onto $M$.

Here we notice that $M$ contains no non-empty open set of $N$. Suppose that a non-empty open set $D$ of $N$ is contained in $M$; then there exist a point $x$ of $D$ and a positive integer $i$ such that $\pi_i^{-1}\pi_i(x) \subset D$, where $\pi_j$ is the projection of $N$ onto $N_j$, $j=1,2,\cdots$. We have

$$|\pi_{i+1}\pi_i^{-1}\pi_i(x)| \geqq 3.$$

Since $\pi_i(x) \in M_i$ and $\pi_{i+1}\pi_i^{-1}\pi_i(x)=\pi_{i+1,i}^{-1}\pi_i(x)$, we have

$$|\pi_{i+1}\pi_i^{-1}\pi_i(x) \cap M_{i+1}|=2.$$

Since $\pi_{i+1}\pi_i^{-1}\pi_i(x) \subset \pi_{i+1}(M)=M_{i+1}$, we have

$$|\pi_{i+1}\pi_i^{-1}\pi_i(x) \cap M_{i+1}|=|\pi_{i+1}\pi_i^{-1}\pi_i(x)| \geqq 3,$$

which is a contradiction. Thus $M$ contains no non-empty open set of $N$.

Define a mapping $\psi: LN \to LM$ in such a way that

$$\psi(yx)=y\cdot\varphi(x), \qquad y \in L,\ x \in N.$$

Then $\psi$ is a homeomorphism of $LN$ onto $LM$. To prove that $LM$ contains no non-empty open set of $G$, assume the contrary. Then there exist a non-empty open set $L_1$ of $L$ and a non-empty open set $N_1$ of $N$ such that $L_1N_1 \subset LM$. Define a mapping $f: LN \to N$ in such a way that

$$f(yx)=x, \qquad y \in L,\ x \in N.$$

Then $f$ is an open continuous mapping. Hence $f(L_1N_1)=N_1$ is open in $N$ and is contained in $f(LM)=M$, which is a contradiction. Thus we know that the invariance theorem of a domain does not hold in $G$ and the proof is completed.

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