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A note on predicates of ordinal numbers

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We shall assume the axiom of constructibility (cf. [1], [3]) throughout this paper. In [8], we considered the hierarchy of predicates of ordinal numbers in the first or the second number-class. In this paper we shall consider ordinal numbers of higher number-classes and we shall define the notions of primitive and general recursive functions of those ordinal numbers similarly as in [8] by introducing $\omega_1, \cdots, \omega_n$ (where ω_i is the initial ordinal number of the $i+2$ nd number class) as initial functions of primitive recursive functions of ordinal numbers in the $n+2$ nd class. (We shall simply say an ordinal number is in the n th class if it is in the m th number-class for some $m \leqq n$.) Then the arguments given in §§ 1-6 in [8] will be available with only slight modifications. The main result of this paper states:

A predicate in Kleene hierarchy of predicates with variables of types $\leqq n+1$ (for $n \geqq 1$) is in $\Sigma_1^{n+1} \cap \Pi_1^{n+1}$ if and only if it is expressible by a general recursive predicate of ordinal numbers in the $n+2$ nd class.

By the way we shall define the classical hierarchy (cf. [2]) of ordinal numbers in the third class and classically expressible ordinal numbers. We shall denote the least ordinal number not classically expressible by ω_1^* and show the analogous properties of ω_1^* to those of ω^*. It seems to suggest some analogies between the classical hierarchy of predicates with variables of type-2 and Kleene hierarchy of predicates with variables of type-1.

In the following we shall show how to extend the considerations in [8] to the third class and we shall only show the outline for the extension to higher number classes. Some acquaintance with [8] is assumed throughout this paper. We shall often cite definitions, propositions and theorems concerning with ordinal numbers in the $n+1$ st class ($n \geqq 2$) by putting the superscript n to the corresponding ones in [8].

§ 1. Primitive recursive^2 functions.

We say simply 'a is an ordinal number', if a is an ordinal number in the third class. We follow [8] for most of notions and notations on ordinal numbers and use ω_1 as the initial ordinal number in the third number class.

DEFINITION. A function is said to be *primitive recursive*^{2}, if it can be defined by a series of applications of the schemata (I)-(XII) of [8, § 1] and the following

(II′) $$f(a)=\omega_1 .$$

DEFINITION. A function is said to be *primitive recursive*^{2} *in the narrow sense*, if it can be defined by a series of applications of the schemata (I)-(VII), (IX)-(XII) and (II′).

DEFINITION. A predicate is said to be *primitive recursive*^{2} (*in the narrow sense*), if it has a primitive recursive^2 representing function (in the narrow sense).

We can prove Propositions 1-4 in § 1 of [8] by regarding 'primitive recursive' there as 'primitive recursive^2' defined above.

§ 2. General recursive^2 functions.

DEFINITION. A function is said to be *general recursive*^{2} if it can be defined by a series of applications of the schemata (I)-(XIII) of [8] and (II′).

A function is said to be *general recursive*^{2} *in the narrow sense*, if it can be defined by a series of applications of the schemata (I)-(VII), (IX)-(XIII) and (II′). A predicate is said to be *general recursive*^{2} (*in the narrow sense*), if it has a general recursive^2 representing function (in the narrow sense).

§ 3. Construction of a model of set theory.

We can construct a model of set theory in the theory of primitive recursive^2 functions quite similarly as in [8, § 3]. (Cf. also [5] and [6].)

§ 4. Elementary^2 functions.

DEFINITION. A function is said to be *elementary*^{2}, if it can be defined by a series of applications of the schemata (I)-(XI), (XIV)-(XVII) and (II′). A predicate is said to be *elementary*^{2}, if it has an elementary^2 representing function.

We have Proposition 5 in [8] by regarding 'elementary' there 'elementary^2' defined above.

§ 5. Relations among elementary^2, primitive recursive^2 and general recursive^2 predicates and their quantified forms.

We call the predicates constructed from elementary^2 (or primitive recursive^2 or general recursive^2) predicates, propositional connectives and quantifiers (er^2)-(or (pr^2)- or (gr^2)-) *predicates*. If a predicate is obtained from an elementary^2 (or primitive recursive^2 or general recursive^2) predicate by prefixing a sequence of alternating k quantifiers, we call it a *k-er^2-(or k-pr^2- or k-gr^2-) predicate.* A k-$*^2$-predicate is said a $\Sigma_k^{2,*}$- or *$\Pi_k^{2,*}$-predicate,* according as the outermost quantifier is existential or universal, where $*$ stands for *er, pr* or *gr.* We use $\Sigma_k^{2,*}$ or $\Pi_k^{2,*}$ to denote the class of $\Sigma_k^{2,*}$ or $\Pi_k^{2,*}$-predicates (resp.). If a predicate is in both $\Sigma_k^{2,*}$ and $\Pi_k^{2,*}$, it is called a *$\Sigma_k^{2,*} \cap \Pi_k^{2,*}$-predicate.* A predicate is said to be *expressible in the $\Sigma_k^{2,*}$-* (or *$\Pi_k^{2,*}$-*) *form,* if it is equivalent to a $\Sigma_k^{2,*}$- (or $\Pi_k^{2,*}$-) predicate. A predicate is said to be *expressible in the $\Sigma_k^{2,*} \cap \Pi_k^{2,*}$-form,* if it is equivalent to a $\Sigma_k^{2,*} \cap \Pi_k^{2,*}$-predicate. If we consider the similar concept concerning with predicates primitive (or general) recursive^2 in the narrow sense, we write prn^2 (or grn^2) instead of pr^2 (or gr^2).

We have the propositions, theorems and corollary obtained from Propositions 6-8, Theorems 1-3 and Corollary in [8, §5] by replacing (er) by (er^2); (pr) by (pr^2); (gr) by (gr^2); *er*-form by er^2-form; *pr*-form by pr^2-form; *gr*-form by gr^2-form; Σ_k^{er}, Π_k^{er} by $\Sigma_k^{2,er}$, $\Pi_k^{2,er}$ (resp.); Σ_k^{pr}, Π_k^{pr} by $\Sigma_k^{2,pr}$, $\Pi_k^{2,pr}$ (resp.); Σ_k^{gr}, Π_k^{gr} by $\Sigma_k^{2,gr}$, $\Pi_k^{2,gr}$ (resp.); Σ_k^{grn}, Π_k^{grn} by $\Sigma_k^{2,grn}$, $\Pi_k^{2,grn}$ (resp.). The proofs are performed in the same way as in [8]. Then we use $\Sigma_k^{2,ord}$ or $\Pi_k^{2,ord}$ to denote $\Sigma_k^{2,er}$ $(=\Sigma_k^{2,pr}=\Sigma_k^{2,gr})$ or $\Pi_k^{2,er}$ $(=\Pi_k^{2,pr}=\Pi_k^{2,gr})$ $(k \geqq 1)$ and say 'a predicate is expressible in k-2-quantifier form' if it is expressible in $\Sigma_k^{2,ord}$-form or in $\Pi_k^{2,ord}$-form.

§ 6. The enumeration theorem and hierarchy theorem.

We can prove the enumeration theorem for elementary^2 functions, the normal form theorem for general recursive^2 functions and the hierarchy theorem for any of (er^2)-, (pr^2)- and (gr^2)-predicates.

Let C_1 be a class of functions of one variable satisfying the following conditions:

(1) a', 0, ω, ω_1, a, $\mathrm{Iq}(g^1(a), g^2(a))$, $\max(g^1(a), g^2(a))$, $g^1(a)+g^2(a)$, $J(g^1(a), g^2(a))$, $fn(a)$ and $\mathrm{u}(a)$ belong to C_1.

(2) If $f(a)$ and $g(a)$ belong to C_1, then $f(g(a))$, $j(f(a), g(a))$, $g^1(f(a))$, $g^2(f(a))$ and $\mu x_{x<g^2(a)}\, f(j(g^1(a), x))$ belong to C_1.

Let $T_1(x, y)$ be a primitive recursive function possessing the properties of $T(x, y)$ defined in [8, §6] and the further property

$$T_1(2^{10}, a) = \omega_1 .$$

Then we have the proposition, theorems and corollaries obtained from Proposition 9, Theorems 4-7 and Corollaries of Theorem 4 in [8, §6] by replacing C by C_1, T by T_1, Σ_k^{ord} or Π_k^{ord} by $\Sigma_k^{2,ord}$ or $\Pi_k^{2,ord}$ (resp.), h-quantifier-forms by h-2-quantifier-forms and elementary, primitive recursive or general recursive by considering elementary^2, primitive^2 or general recursive^2 (resp.).

§ 7. Expression of our k-2-quantifier forms in Kleene hierarchy.

DEFINITION. An ordinal number a is said to be *closed*[2] with respect to functions $f_1, \cdots, f_n$, if the following conditions are satisfied:

1) $\omega_1 < a$.

2) $a_1 < a, \cdots, a_{r_i} < a \rightarrow f_i(a_1, \cdots, a_{r_i}) < a \qquad (1 \leqq i \leqq n)$.

We can prove, for given $f_1, \cdots, f_n$, the existence of an ordinal number closed[2] with respect to $f_1, \cdots, f_n$. Now we can define 'system of equations' and 'system of equations restricted by a_0' in the same way as in [8]. We have the propositions corresponding to Propositions 10 and 11 in [8].

To translate our predicates in Kleene hierarchy of finite types, we shall first explain some notions and notations in Kleene's theory which we are to use in the translation. In [4], Kleene defined type-$n+1$ objects to be the 1-place functions from type-n objects to natural numbers. But we take here as type-$n+1$ objects the i-places ($i \geqq 0$) functions from type-n objects to type-m-objects ($m \leqq n$). By the help of the λ-notation and type-$n+1$ objects in Kleene's sense we can express these type-$n+1$ objects in Kleene's theory; e.g. if H is a type-2 object which mapps type-1 objects to type-1 objects, $\mathrm{H}(\alpha)$ is expressed as $\lambda x \mathrm{H}'(\lambda y < \alpha(y), x >)$, H' being a type-2 object in Kleene's sense. Especially type-$n+1$ variables in this sense are primitive recursive in the sense of 1.5 of [4], because they can be constructed from the λ-notation, type-$n+1$ variables in Kleene's sense and certain primitive recursive functions in Kleene's sense. In the following we shall use $\alpha, \beta, \cdots, \alpha_1, \alpha_2, \cdots$ to denote type-1 variables and $\alpha^2, \beta^2, \cdots, \alpha_1^2, \alpha_2^2, \cdots$ to denote type-2-variables in this sense. The work of these variables can be understood by the usage. Types of objects used in the rest of this section are less than 3.

In [8] we make an ordinal number a correspond to a function α from natural numbers to natural numbers which gives a well-ordering of natural numbers and whose order-type is a', and each ordinal number b less than a to a natural number $\hat{b}$ such that it is in the domain of α (i.e. $D(\alpha, \hat{b})$) and the order-type of $\alpha \upharpoonright \hat{b}$ (cf. [8] for the notation) is b'. Here we use a 2-places type-2 object from type-1 objects to $\{0, 1\}$ which gives a well-ordering of type-1 objects (i.e. a function from $N^N \times N^N$ to $\{0, 1\}$ which gives a well-ordering of N^N, N being the set of natural numbers) to express an ordinal number in the third class by means of the axiom of constructibility.

Let $\alpha = \beta$ be $\forall x(\alpha(x) = \beta(x))$;

$\alpha \neq \beta$ be $\neg \alpha = \beta$;

$D^2(\alpha^2, \alpha, \beta)$ be $\alpha^2(\alpha, \beta) = 0 \vee \alpha^2(\beta, \alpha) = 0$;

$D^2(\alpha^2, \alpha)$ be $\exists \beta D^2(\alpha^2, \alpha, \beta)$;

$$W^2(\alpha^2) \text{ be } \forall\alpha\forall\beta(D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)\vdash D^2(\alpha^2,\alpha,\beta))$$
$$\wedge\ \forall\alpha\forall\beta(\alpha^2(\alpha,\beta)=0\wedge\alpha^2(\beta,\alpha)=0\vdash\alpha=\beta)$$
$$\wedge\ \forall\alpha\forall\beta\forall\gamma(\alpha^2(\alpha,\beta)=0\wedge\alpha^2(\beta,\gamma)=0\vdash\alpha^2(\alpha,\gamma)=0)$$
$$\wedge\ \forall\psi\exists x(\alpha^2(\lambda u\psi(u,x),\lambda u\psi(u,x+1))=0\vee\lambda u\psi(u,x)$$
$$=\lambda u\psi(u,x+1))$$

which means that α^2 is a well-ordering and which is of order 2 ([4], §7).

$$\overset{2}{=}(\alpha^2,\beta^2) \text{ be } \exists\gamma^2(\forall\alpha(D^2(\alpha^2,\alpha)\vdash D^2(\beta^2,\gamma^2(\alpha)))$$
$$\wedge\ \forall\beta\exists\alpha(D^2(\beta^2,\beta)\vdash D^2(\alpha^2,\alpha)\wedge\beta=\gamma^2(\alpha))$$
$$\wedge\ \forall\alpha\forall\beta(D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)$$
$$\vdash(\alpha^2(\alpha,\beta)=0\wedge\alpha\neq\beta$$
$$\vdash\dashv\beta^2(\gamma^2(\alpha),\gamma^2(\beta))=0\wedge\gamma^2(\alpha)\neq\gamma^2(\beta)))),$$

which means under the assumption of $W^2(\alpha^2)$ and $W^2(\beta^2)$, that α^2 and β^2 are isomorphic and which is expressible in the Σ_1^2-form;

$$\alpha^2\restriction\alpha \text{ be } \lambda\beta\gamma(\alpha^2(\beta,\gamma)+\alpha^2(\beta,\alpha)+\alpha^2(\gamma,\alpha));$$

$Cl(\alpha^2;\beta_1^2,\cdots,\beta_m^2)$ be

$$\forall\alpha_1\cdots\forall\alpha_{i_1}(D^2(\alpha^2,\alpha_1)\wedge\cdots\wedge D^2(\alpha^2,\alpha_{i_1})\vdash D^2(\alpha^2,\beta_1^2(\alpha_1,\cdots,\alpha_{i_1})))$$
$$\wedge\cdots$$
$$\wedge\ \forall\alpha_1\cdots\forall\alpha_{i_m}(D^2(\alpha^2,\alpha_1)\wedge\cdots\wedge D^2(\alpha^2,\alpha_{i_m})\vdash D^2(\alpha^2,\beta_m^2(\alpha_1,\cdots,\alpha_{i_m}))),$$

which means the domain of α^2 is closed with respect to functions $\beta_1^2,\cdots,\beta_m^2$ and which is of order 2;

$$L(\alpha,\alpha^2) \text{ be } W(\alpha)\wedge W^2(\alpha^2)$$
$$\wedge\ \forall\varphi(\forall x(D(\alpha,x)\vdash D^2(\alpha^2,\lambda u\varphi(u,x)))$$
$$\wedge\ \forall x\forall y(D(\alpha,x)\wedge D(\alpha,y)$$
$$\vdash(\alpha(x,y)=0\wedge x\neq y\vdash\dashv\alpha^2(\lambda u\varphi(u,x),\lambda u\varphi(u,y))=0$$
$$\wedge\ \lambda u\varphi(u,x)\neq\lambda u\varphi(u,y)))$$
$$\vdash\exists\beta(D^2(\alpha^2,\beta)\wedge\forall x(D(\alpha,x)\vdash\alpha^2(\lambda u\varphi(u,x),\beta)=0\wedge\lambda u\varphi(u,x)\neq\beta))),$$

which means that the order-type of α is less than that of α^2 and which is of order 2;

$$N(\alpha^2) \text{ be } W^2(\alpha^2)\wedge\forall\alpha(W(\alpha)\vdash L(\alpha,\alpha^2)),$$

which means that the order-type of α^2 is not in the second number class and which is of order 2.

We define further auxiliary notions corresponding to the definitions of initial functions of primitive recursive² functions of ordinal numbers. These notions are used only under the assumption that $W^2(\alpha^2)$:

(Ǐ) $\forall\alpha\forall\beta(D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)\vdash(\beta=\psi_0^2(\alpha)\mapsto\alpha^2(\alpha,\beta)=0\wedge\alpha\neq\beta$

$\wedge\forall\gamma(\alpha^2(\alpha,\gamma)=0\wedge\alpha\neq\gamma\vdash\alpha^2(\beta,\gamma)=0)))$

$\wedge\forall\alpha(D^2(\alpha^2,\alpha)\vdash D^2(\alpha^2,\psi_0^2(\alpha)))$

(abbr. $M_0^2(\alpha^2;\psi_0^2)$), where $\psi_0^2(\alpha)$ corresponds to the successor of α in the sense of α^2.

(ǏI$_1$) $\forall\alpha\forall\beta(D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)\vdash(\beta=\psi_1^2(\alpha)\mapsto\neg\exists\gamma(\alpha^2(\gamma,\beta)=0\wedge\gamma\neq\beta)))$

$\wedge\forall\alpha(D^2(\alpha^2,\alpha)\vdash D^2(\alpha^2,\psi_1^2(\alpha)))$

(abbr. $M_1^2(\alpha^2;\psi_1^2)$), where $\psi_1^2(\alpha)$ stands for the first element of the domain of α^2.

(ǏI$_2$) $\forall\alpha\forall\beta(D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)\vdash(\beta=\psi_2^2(\alpha)\mapsto\exists\gamma(\alpha^2(\gamma,\beta)=0\wedge\gamma\neq\beta)$

$\wedge\forall\gamma(\alpha^2(\gamma,\beta)=0\wedge\gamma\neq\beta\vdash\alpha^2(\psi_0^2(\gamma),\beta)=0\wedge\psi_0^2(\gamma)\neq\beta)$

$\wedge\forall\gamma(\exists\delta(\alpha^2(\delta,\gamma)=0\wedge\delta\neq\gamma)$

$\wedge\forall\delta(\alpha^2(\delta,\gamma)=0\wedge\delta\neq\gamma\vdash\alpha^2(\psi_0^2(\delta),\gamma)=0\wedge\psi_0^2(\delta)\neq\gamma)$

$\vdash\alpha^2(\beta,\gamma)=0)))$

$\wedge\forall\alpha(D^2(\alpha^2,\alpha)\vdash D^2(\alpha^2,\psi_2^2(\alpha)))$

(abbr. $M_2^2(\alpha^2;\psi_0^2,\psi_2^2)$), which is used only under the assumption that $M_0^2(\alpha^2;\psi_0^2)$. $\psi_2^2(\alpha)$ corresponds to ω.

(ǏI$'$) $\forall\alpha\forall\beta(D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)$

$\vdash(\beta=\psi_{2'}^2(\alpha)\mapsto N(\alpha^2\restriction\beta)\wedge\forall\gamma(D^2(\alpha^2,\gamma)\wedge N(\alpha^2\restriction\gamma)\vdash\alpha^2(\beta,\gamma)=0)))$

$\wedge\forall\alpha(D^2(\alpha^2,\alpha)\vdash D^2(\alpha^2,\psi_{2'}^2(\alpha)))$

(abbr. $M_2'^2(\alpha^2,\psi_{2'}^2)$), $\psi_{2'}^2(\alpha)$ corresponds to ω_1.

(IǏI) $\forall\alpha\forall\beta(D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)\vdash(\beta=\psi_3^2(\alpha)\mapsto\alpha=\beta))$

$\wedge\forall\alpha(D^2(\alpha^2,\alpha)\vdash D^2(\alpha^2,\psi_3^2(\alpha)))$

(abbr. $M_3^2(\alpha^2;\psi_3^2)$), where ψ_3^2 corresponds to the identity function.

(IV̌) $\forall\alpha\forall\beta\forall\gamma(D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)\wedge D^2(\alpha^2,\gamma)$

$\vdash(\gamma=\psi_4^2(\alpha,\beta)\mapsto((\alpha^2(\alpha,\beta)=0\wedge\alpha\neq\beta\wedge\gamma=\psi_1^2(\alpha))$

$\vee(\alpha^2(\beta,\alpha)=0\wedge\gamma=\psi_0^2(\psi_1^2(\alpha))))))$

$\wedge\forall\alpha\forall\beta(D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)\vdash D^2(\alpha^2,\psi_4^2(\alpha,\beta)))$

(abbr. $M_4^2(\alpha^2;\psi_0^2,\psi_1^2,\psi_4^2)$), which is used only under the assumption that $M_0^2(\alpha^2;\psi_0^2)$ and $M_1^2(\alpha^2;\psi_1^2)$. ψ_4^2 corresponds to Iq.

(V̌) $\forall\alpha\forall\beta\forall\gamma(D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)\wedge D^2(\alpha^2,\gamma)$

$\vdash(\gamma=\psi_5^2(\alpha,\beta)\mapsto(\alpha^2(\alpha,\beta)=0\wedge\gamma=\beta)\vee(\alpha^2(\beta,\alpha)=0\wedge\gamma=\alpha)))$

$\wedge\forall\alpha\forall\beta(D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)\vdash D^2(\alpha^2,\psi_5^2(\alpha,\beta)))$

(abbr. $M_5^2(\alpha^2; \psi_5^2)$). ψ_5^2 corresponds to max.

$$(\check{\mathrm{VI}}) \quad \forall\alpha\forall\beta\forall\gamma(D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)\wedge D^2(\alpha^2,\gamma)$$
$$\vdash(\gamma=\psi_6^2(\alpha,\beta)\dashv\vdash\underline{\forall}\varphi^2(\forall\alpha_1\forall\beta_1(\check{R}(\alpha^2;\alpha_1,\beta_1,\alpha,\beta)\vee(\alpha_1=\alpha\wedge\beta_1=\beta)$$
$$\vdash\exists\gamma_1(\alpha^2(\gamma_1,\gamma)=0\wedge\varphi^2(\gamma_1,\alpha_1,\beta_1)=0))$$
$$\wedge\forall\gamma_1(\alpha^2(\gamma_1,\gamma)=0\vdash\exists\alpha_1\exists\beta_1((\check{R}(\alpha^2;\alpha_1,\beta_1,\alpha,\beta)\vee(\alpha_1=\alpha\wedge\beta_1=\beta))$$
$$\wedge\varphi^2(\gamma_1,\alpha_1,\beta_1)=0))$$
$$\wedge\forall\alpha_1\forall\beta_1\forall\gamma_1\forall\alpha_2\forall\beta_2\forall\gamma_2(D^2(\alpha^2,\alpha_1)\wedge D^2(\alpha^2,\beta_1)\wedge D^2(\alpha^2,\gamma_1)$$
$$\wedge D^2(\alpha^2,\alpha_2)\wedge D^2(\alpha^2,\beta_2)\wedge D^2(\alpha^2,\gamma_2)$$
$$\wedge\varphi^2(\gamma_1,\alpha_1,\beta_1)=0\wedge\varphi^2(\gamma_2,\alpha_2,\beta_2)=0$$
$$\vdash(\alpha^2(\gamma_1,\gamma_2)=0\wedge\gamma_1\neq\gamma_2\dashv\vdash\check{R}(\alpha^2;\alpha_1,\beta_1,\alpha_2,\beta_2)))$$
$$\wedge\forall\delta(D^2(\alpha^2,\delta)\vdash\varphi^2(\psi_1^2(\delta),\psi_1^2(\delta),\psi_1^2(\delta))=0)\underline{\vdash}\varphi^2(\gamma,\alpha,\beta)=0)))$$
$$\wedge\forall\alpha\forall\beta(D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)\vdash D^2(\alpha^2,\psi_6^2(\alpha,\beta))),$$

(abbr. $M_6^2(\alpha^2; \psi_0^2, \psi_1^2, \psi_4^2, \psi_5^2, \psi_6^2)$), where $\check{R}(\alpha^2; \alpha, \beta, \alpha_1, \beta_1)$ is the abbreviation of

$$D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)\wedge D^2(\alpha^2,\alpha_1)\wedge D^2(\alpha^2,\beta_1)$$
$$\wedge(\forall\gamma(D^2(\alpha^2,\gamma)\vdash\psi_4^2(\psi_5^2(\alpha,\beta),\psi_5^2(\alpha_1,\beta_1))=\psi_1^2(\gamma))$$
$$\vee(\psi_5^2(\alpha,\beta)=\psi_5^2(\alpha_1,\beta_1)$$
$$\wedge\forall\gamma(D^2(\alpha^2,\gamma)\vdash(\psi_4^2(\beta,\beta_1)=\psi_1^2(\gamma)\vee(\beta=\beta_1\wedge\psi_4^2(\alpha,\alpha_1)=\psi_1^2(\gamma))))))$$

and is of order 2 in $\psi_1^2, \psi_4^2, \psi_5^2$, which is used only under the assumption that $M_0^2(\alpha^2; \psi_0^2)$, $M_1^2(\alpha^2; \psi_1^2)$, $M_4^2(\alpha^2; \psi_0^2, \psi_1^2, \psi_4^2)$ and $M_5^2(\alpha^2; \psi_5^2)$. ψ_6^2 corresponds to j. We use also the predicate obtained from this by replacing the underlined logical symbols $\forall$ and $\vdash$ by $\exists$ and $\wedge$ respectively as $M_6^2(\alpha^2; \psi_0^2, \cdots, \psi_6^2)$ under the presupposition by which we can consider them to be equivalent.

$$(\check{\mathrm{VII}}_1) \quad \forall\alpha\forall\beta(D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)\vdash(\beta=\psi_7^2(\alpha)\dashv\vdash\exists\gamma(\alpha=\psi_6^2(\beta,\gamma))))$$
$$\wedge\forall\alpha(D^2(\alpha^2,\alpha)\vdash D^2(\alpha^2,\psi_7^2(\alpha)))$$

(abbr. $M_7^2(\alpha^2; \psi_0^2, \psi_1^2, \psi_4^2, \psi_5^2, \psi_6^2, \psi_7^2)$), which is used only under the assumption that $M_0^2(\alpha^2; \psi_0^2)$, $M_1^2(\alpha^2; \psi_1^2)$, $M_4^2(\alpha^2; \psi_0^2, \psi_1^2, \psi_4^2)$, $M_5^2(\alpha^2; \psi_5^2)$ and $M_6^2(\alpha^2; \psi_0^2, \psi_1^2, \psi_4^2, \psi_5^2, \psi_6^2)$. ψ_7^2 corresponds to g^1.

$$(\check{\mathrm{VII}}_2) \quad \forall\alpha\forall\beta(D^2(\alpha^2,\alpha)\wedge D^2(\alpha^2,\beta)\vdash(\beta=\psi_8^2(\alpha)\dashv\vdash\exists\gamma(\alpha=\psi_6^2(\gamma,\beta))))$$
$$\wedge\forall\alpha(D^2(\alpha^2,\alpha)\vdash D^2(\alpha^2,\psi_8^2(\alpha)))$$

(abbr. $M_8^2(\alpha^2; \psi_0^2, \psi_1^2, \psi_4^2, \psi_5^2, \psi_6^2, \psi_8^2)$), which is used only under the same assumption as in $(\check{\mathrm{VII}}_1)_2$. ψ_8^2 correponds to g^2.

We see easily that $M_i^2(\alpha^2; \psi_{i_1}^2, \cdots, \psi_i^2)$ is expressible in the $\Sigma_1^2 \cap \Pi_1^2$-form for each i $(0 \leqq i \leqq 8$ or $i = 2')$.

Let $C(a_1, \cdots, a_n)$ be a primitive recursive² function in the narrow sense. We define 'a system of equations $[f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]^{\check{\alpha}_0^2}$ with respect to α_0^2, in Kleene's theory corresponding to $[f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]^{a_0}$, where $W^2(\alpha_0^2)$ is presupposed and the order-type of α_0^2 is a_0'. The definition is given under the presupposition that $W^2(\alpha_0^2)$, $M_0^2(\alpha_0^2; \psi_0^2), \cdots, M_8^2(\alpha_0^2; \psi_0^2, \cdots, \psi_8^2)$. In the definition type-2 variables $\check{f}, \check{g}, \cdots$ and the type-1 variables $\check{a}_1, \cdots, \check{a}_n$ correspond to $f, g, \cdots$ and $a_1, \cdots, a_n$ (resp.). If $C(a_1, \cdots, a_n)$ is of the form $g(C_1(a_1, \cdots, a_n), \cdots, C_m(a_1, \cdots, a_n))$ where g is a function symbol, then $[f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]^{\check{\alpha}_0^2}$ is

$$(D^2(\alpha_0^2, \check{a}_1) \wedge \cdots \wedge D^2(\alpha_0^2, \check{a}_n) \vdash \check{f}(\check{a}_1, \cdots, \check{a}_n) = \check{g}(\check{h}_1(\check{a}_1, \cdots, \check{a}_n), \cdots, \check{h}_m(\check{a}_1, \cdots, \check{a}_n)))$$
$$\wedge \forall \check{x}_1 \cdots \forall \check{x}_n([h_1(x_1, \cdots, x_n) = C_1(x_1, \cdots, x_n)]^{\check{\alpha}_0^2})$$
$$\wedge \cdots$$
$$\wedge \forall \check{x}_1 \cdots \forall \check{x}_n([h_m(x_1, \cdots, x_n) = C_m(x_1, \cdots, x_n)]^{\check{\alpha}_0^2})$$

where $\check{g}$ is ψ_i $(0 \leqq i \leqq 8)$ or k_l $(1 \leqq l \leqq j)$. If $C(a_1, \cdots, a_n)$ is of the form $\mu x_{x < C_0(a_1, \cdots, a_n)} C_1(a_1, \cdots, a_n, x)$, then $[f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]^{\check{\alpha}_0^2}$ is

$$(D^2(\alpha_0^2, \check{a}_1) \wedge \cdots \wedge D^2(\alpha_0^2; \check{a}_n) \vdash ((\alpha_0^2(\check{f}(\check{a}_1, \cdots, \check{a}_n), \check{h}_0(\check{a}_1, \cdots, \check{a}_n)) = 0$$
$$\wedge \check{f}(\check{a}_1, \cdots, \check{a}_n) \neq \check{h}_0(\check{a}_1, \cdots, \check{a}_n) \wedge \check{h}_1(\check{a}_1, \cdots, \check{a}_n, \check{f}(\check{a}_1, \cdots, \check{a}_n)) = \psi_1^2(\check{a}_1)$$
$$\wedge \forall \check{x}(\alpha_0^2(\check{x}, \check{f}(\check{a}_1, \cdots, \check{a}_n)) = 0 \wedge \check{x} \neq \check{f}(\check{a}_1, \cdots, \check{a}_n)$$
$$\vdash \check{h}_1(\check{a}_1, \cdots, \check{a}_n, \check{x}) \neq \psi_1^2(\check{a})))$$
$$\vee (\check{f}(\check{a}_1, \cdots, \check{a}_n) = \psi_1^2(\check{a}_1)$$
$$\wedge \forall \check{x}(\alpha_0^2(\check{x}, \check{h}_0(\check{a}_1, \cdots, \check{a}_n)) = 0 \wedge \check{x} \neq \check{h}_0(\check{a}_1, \cdots, \check{a}_n)$$
$$\vdash \check{h}_1(\check{a}_1, \cdots, \check{a}_n, \check{x}) \neq \psi_1^2(\check{a}_1)))))$$
$$\wedge \forall \check{x}_1 \cdots \forall \check{x}_n([h_0(x_1, \cdots, x_n) = C_0(x_1, \cdots, x_n)]^{\check{\alpha}_0^2})$$
$$\wedge \forall \check{x}_1 \cdots \forall \check{x}_n \forall \check{x}([h_1(x_1, \cdots, x_n, x) = C_1(x_1, \cdots, x_n, x)]^{\check{\alpha}_0^2}).$$

$[f(a_1, \cdots, a_n) = g(a_1, \cdots, a_n)]^{\check{\alpha}_0^2}$ for a function symbol g is $D^2(\alpha_0^2, \check{a}_1) \wedge \cdots \wedge D^2(\alpha_0^2, \check{a}_n) \vdash \check{f}(\check{a}_1, \cdots, \check{a}_n) = \check{g}(\check{a}_1, \cdots, \check{a}_n)$, where $\check{g}$ is ψ_i^2 or k_l according as g is introduced by one of (I)-(VII), (II') or (XII).

Now let $C(a_1, \cdots, a_n)$ be a primitive recursive² function in the narrow sense. We define the result of the translation of $b = C(a_1, \cdots, a_n)$ in Kleene hierarchy which is denoted $(b = C(a_1, \cdots, a_n))^{\vee}(\varphi^2, \varphi_1^2, \cdots, \varphi_n^2)$ (where $\varphi^2, \varphi_1^2, \cdots, \varphi_n^2$ correspond to $b, a_1, \cdots, a_n$). This has two equivalent forms $(b = C(a_1, \cdots, a_n))^{\check{\forall}}(\varphi^2, \varphi_1^2, \cdots, \varphi_n^2)$ and $(b = C(a_1, \cdots, a_n))^{\check{\exists}}(\varphi^2, \varphi_1^2, \cdots, \varphi_n^2)$. $(b = C(a_1, \cdots, a_n))^{\check{\forall}}(\varphi^2, \varphi_1^2, \cdots, \varphi_n^2)$ is

$$\underline{\forall}\alpha_0^2 \underline{\forall}\check{a}_1 \cdots \underline{\forall}\check{a}_n \underline{\forall}\check{b} \underline{\forall}\psi_0^2 \underline{\forall}\psi_1^2 \cdots \underline{\forall}\psi_8^2 \underline{\forall}\check{h}_1 \cdots \underline{\forall}\check{h}_m \underline{\forall}\check{k}_1 \cdots \underline{\forall}\check{k}_j \underline{\forall}\check{f}$$
$$(W^2(\alpha_0^2) \wedge D^2(\alpha_0^2, \check{a}_1) \wedge \cdots \wedge D^2(\alpha_0^2, \check{a}_n) \wedge D^2(\alpha_0^2, \check{b})$$

$$\wedge = (\varphi_1^2, \alpha_0^2 \restriction \check{a}_1) \wedge \cdots \wedge = (\varphi_n^2, \alpha_0^2 \restriction \check{a}_n) \wedge = (\varphi^2, \alpha_0^2 \restriction \check{b})$$
$$\wedge M_0^2(\alpha_0^2, \psi_0^2) \wedge \cdots \wedge M_8^2(\alpha_0^2, \psi_0^2, \cdots, \psi_8^2)$$
$$\wedge Cl^2(\alpha_0^2 ; \psi_6^2, \check{k}_1, \cdots, \check{k}_j) \wedge [f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]^{\check{\alpha}_0^2}$$
$$\underline{\vdash} \check{b} = \check{f}(\check{a}_1, \cdots, \check{a})),$$

where $h_1, \cdots, h_m$ are auxiliary functions of $[f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]$ and $k_1, \cdots, k_j$ are functions introduced by the primitive recursion occurring in the construction of C. $(b = C(a_1, \cdots, a_n))^{\check{\exists}}(\varphi^2, \varphi_1^2, \cdots, \varphi_n^2)$ is obtained from $(b = C(a_1, \cdots, a_n))^{\check{\forall}}(\varphi^2, \varphi_1^2, \cdots, \varphi_n^2)$ by underlined $\forall$'s and $\vdash$ by $\exists$'s and $\wedge$ respectively.
From the definition we have

THEOREM 1. *Every primitive recursive² predicate in the narrow sense is expressible by a $\Sigma_1^2 \cap \Pi_1^2$-predicate in Kleene hierarchy.*

COROLLARY 1. *Every predicate containing no function variable and expressible in the $\Sigma_k^{2,pr}$-or $\Pi_k^{2,pr}$-form $(k \geqq 1)$ is expressible by the Σ_k^2-or Π_k^2-predicate (resp.).*

PROOF. Using Theorem 1, it can be proved similary as in the proof of Theorem 8 of [8].

COROLLARY 2. *Every general recursive² predicate containing no function variable is expressible by a $\Sigma_1^2 \cap \Pi_1^2$-predicate.*

PROOF. This follows from Corollary 1 and Theorems 1² and 2².

§ 8. Expression of predicates of order 3 in our hierarchy.

In this section we shall consider Kleene's predicates of order 3 containing only variables of type $\leqq 2$ and show that every predicate expressible in the Σ_k^2- or Π_k^2-form is expressible in the $\Sigma_k^{2,ord}$- or $\Pi_k^{2,ord}$-form (resp.); especially every predicate expressible in the $\Sigma_1^2 \cap \Pi_1^2$-form is expressible as a general recursive² predicate.

Let $F^2(f)$ be

$$\forall x(x < f \wedge x \in f \vdash \exists y \exists z(<y, z> = x \wedge y < \omega \wedge y \in \omega \wedge z < \omega_1 \wedge F(z))$$
$$\wedge \forall x \forall y \forall z(x < \omega_1 \wedge y < \omega \wedge z < \omega \wedge <y, x> \in f \wedge <z, x> \in f \vdash y \equiv z)$$
$$\wedge \forall x(F(x) \wedge x < \omega_1 \vdash \exists y(y < \omega \wedge <\mathrm{m}(y), x> \in f)) \wedge \forall x(x < f \vdash \neg x \equiv f)$$

(cf. § 8 of [8] for $F(x)$ and $\mathrm{m}(y)$) and $f^{\#} g$ be

$$\mu y_{y<\omega}(<\mathrm{m}(y), g> \in f \wedge \mathrm{m}(y) \in \omega).$$

We shall define the $\#$-operation from predicates of order 3 to predicates of ordinal numbers, by which a type-0 variable turns a variable of ordinal numbers $<\omega$, a type-1 variable turns a variable f of ordinal numbers such that $F(f)$ and a type-2 variable turns a variable f of ordinal numbers such that $F^2(f)$. By means of XXXV and XXXVII b of [4], we define the $\#$-operation as follows:

$(j=\alpha(i))^{\#}$ is $F(f)\wedge j=f^{\#}i\wedge j<\omega\wedge i<\omega$;
$(j=\alpha^2(\alpha))^{\#}$ is $F^2(f)\wedge F(g)\wedge j=f^{\#}g\wedge j<\omega\wedge g<\omega_1$;
$(k=i+j)^{\#}$ is $k=i+j\wedge i<\omega\wedge j<\omega\wedge k<\omega$;
$(k=i\cdot j)^{\#}$ is $k=J(j,i)\wedge i<\omega\wedge j<\omega\wedge k<\omega$.
$(\neg A)^{\#}$ is $\neg A^{\#}$; $(A\wedge B)^{\#}$ is $A^{\#}\wedge B^{\#}$; $(A\vee B)^{\#}$ is $A^{\#}\vee B^{\#}$.

Let $A(\alpha_1^2,\cdots,\alpha_l^2,\alpha_1,\cdots,\alpha_m,a_1,\cdots,a_n,a)$ be a predicate of order 3 and $(A(\alpha_1^2,\cdots,\alpha_l^2,\alpha_1,\cdots,\alpha_m,a_1,\cdots,a_n,a))^{\#}$ be defined. Then

$$(\exists x A(\alpha_1^2,\cdots,\alpha_l^2,\alpha_1,\cdots,\alpha_m,a_1,\cdots,a_n,x))^{\#}$$

is $\exists x(x<\omega\wedge(A(\alpha_1^2,\cdots,\alpha_l^2,\alpha_1,\cdots,\alpha_m,a_1,\cdots,a_n,x))^{\#})$;

Let $A(\alpha_1^2,\cdots,\alpha_l^2,\alpha_1,\cdots,\alpha_m,\alpha,a_1,\cdots,a_n)$ be a predicate of order 3 and $(A(\alpha_1^2,\cdots,\alpha_l^2,\alpha_1,\cdots,\alpha_m,\alpha,a_1,\cdots,a_n))^{\#}$ be defined as $A^{\#}(g_1,\cdots,g_l,f_1,\cdots,f_m,f,a_1,\cdots,a_n)$. Then

$$(\exists\alpha A(\alpha_1^2,\cdots,\alpha_l^2,\alpha_1,\cdots,\alpha_m,\alpha,a_1,\cdots,a_n))^{\#}$$

is $\exists f(f<\omega_1\wedge F(f)\wedge A^{\#}(g_1,\cdots,g_l,f_1,\cdots,f_m,f,a_1,\cdots,a_n))$.

Let $A(\alpha_1^2,\cdots,\alpha_l^2,\alpha^2,\alpha_1,\cdots,\alpha_m,a_1,\cdots,a_n)$ be a predicate of order 3 and $(A(\alpha_1^2,\cdots,\alpha_l^2,\alpha^2,\alpha_1,\cdots,\alpha_m,a_1,\cdots,a_n))^{\#}$ is defined as $A^{\#}(g_1,\cdots,g_l,g,f_1,\cdots,f_m,a_1,\cdots,a_n)$. Then

$$(\exists\alpha^2 A(\alpha_1^2,\cdots,\alpha_l^2,\alpha^2,\alpha_1,\cdots,\alpha_m,a_1,\cdots,a_n))^{\#}$$

is $\exists g(F^2(g)\wedge A^{\#}(g_1,\cdots,g_l,g,f_1,\cdots,f_m,a_1,\cdots,a_n))$.

Similarly for dual forms. Then from the definition of the $\#$-operation and Proposition 7 of [8], Proposition 7^2, we have

THEOREM. *Every predicate of order 2 is expressible by a primitive recursive² predicate and every predicate expressible in the Σ_k^2- or Π_k^2-form is expressible in the $\Sigma_k^{2,ord}$- or $\Pi_k^{2,ord}$-form (resp.); especially every predicate expressible in the $\Sigma_1^2\cap\Pi_1^2$-form is expressible by a general recursive² predicate.* (Cf. Theorem 1^2.)

§ 9. Remarks for predicates of ordinal numbers in higher number classes and predicates of finite types in Kleene hierarchy.

In the previous sections we considered predicates of ordinal numbers in the third class and type-2 objects in Kleene hierarchy. Let us consider, in general, the $n+2$ nd number class ($n\geqq 0$). We shall define primitive and general recursive functions of ordinal numbers in the $n+2$ nd class in the same way as in [8] or in §§ 1, 2 of this paper using $\omega,\omega_1,\cdots,\omega_n$ as initial functions of primitive recursive functions and call them primitive and general recursive^{n+1} functions, respectively. Then our arguments about predicates of ordinal numbers in the third class can straightly be extended to predicates of ordinal numbers in the $n+2$ nd class for $n\geqq 2$. We shall denote the classes corresponding to $\Sigma_k^{2,ord}$ and $\Pi_k^{2,ord}$ as $\Sigma_k^{n+1,ord}$ and $\Pi_k^{n+1,ord}$, respectively.

To express our predicates of the k-quantifier form in Kleene hierarchy of finite types, we shall use objects of higher finite types stated in § 7. We shall make correspond an ordinal number a (in the $n+2$ nd class) to a type-$n+1$ object from 2-places type-n objects to $\{0,1\}$ which gives a well-ordering of type-n objects by means of the axiom of constructibility. Then we can continue our arguments in the same way as in § 7 and obtain

THEOREM. *For $n \geqq 1$; every general recursive^{n+1} predicate containing no function variable is expressible in the $\Sigma_1^{n+1} \cap \Pi_1^{n+1}$-form. Every predicate expressible in the $\Sigma_k^{n+1,ord}$- or $\Pi_k^{n+1,ord}$-form is expressible in the Σ_k^{n+1}- or Π_k^{n+1}-form (resp.) in Kleene hierarchy.*

Conversely, to express predicates of order $n+2$ in our hierarchy, we shall define ordinal numbers which are functions in the model of the set theory and correspond to variables of higher types; let $F(f), F^2(f), \cdots, F^{n+1}(f)$ mean that f is the ordinal number of this kind. We can consider that $f < \omega_m$ if $F^m(f)$ and $m \leqq n$. We can easily extend arguments in § 8 in this case and obtain

THEOREM. *Every predicate of order $n+1$ is expressible by a primitive recursive^{n+1} predicate and every predicate expressible in the Σ_k^{n+1}- or Π_k^{n+1}-form is expressible in the $\Sigma_k^{n+1,ord}$- or $\Pi_k^{n+1,ord}$-form (resp.); especially every predicate expressible in the $\Sigma_1^{n+1} \cap \Pi_1^{n+1}$-form is expressible by a general recursive^{n+1} predicate.*

§ 10. Classical hierarchy and classically expressible ordinal numbers.

We shall call a function f to be *c-recursive* if there exist a general recursive function g and an ordinal number $e \leqq \omega_1$ such that

$$f(a_1, \cdots, a_n) = g(a_1, \cdots, a_n, e).$$

A *c-recursive predicate* is a predicate whose representing function is c-recursive. We shall consider the predicates constructed from c-recursive predicates, propositional connectives and quantifiers and call the hierarchy which consists of these predicates the *classical hierarchy* (cf. [2] for the notion 'classical'). We use the notation Σ_k^c or Π_k^c to express the counterpart of Σ_k^{ord} or Π_k^{ord} (resp.) for the classical hierarchy. An ordinal number a is called to be *classically expressible,* if a is expressible by using c-recursive functions and any ordinal number not greater than ω_1. Let ω_1^* be the least ordinal number which is not classically expressible.

Let $\mathfrak{R}_1 = \{f(a) \mid f \text{ is } c\text{-recursive and } a < \omega_1^*\}$, φ_1 be a one to one mapping from $\mathfrak{R}_1$ onto $\{x \mid x < \mathfrak{r}_1\}$ satisfying

$$a_1 \in \mathfrak{R}_1, a_2 \in \mathfrak{R}_1, a_1 < a_2 \rightarrow \varphi_1(a_1) < \varphi_1(a_2).$$

and $\mathfrak{B}_1 = \{\{x\}f(x, b) \mid f$ is c-recursive and $b \in \mathfrak{R}_1\}$. If $f \in \mathfrak{B}_1$ and $\varphi_1(f(a)) = g(\varphi_1(a))$ for every $a \in \mathfrak{R}_1$, then we say 'g is an f^{φ_1}' (cf. [7]). Then the propositions and theorems given in § 10 of [8] remain valid by the following modification: Replace 'general recursive' (or 'recursive' which means general recursive) by 'c-recursive'; the letter $\mathfrak{R}$ by $\mathfrak{R}_1$, the letter $\mathfrak{B}$ by $\mathfrak{B}_1$, the letter φ by φ_1, 'recursively expressible' by 'classically expressible', ω^* by ω_1^*, ω by ω_1, Σ_1^{ord} by Σ_1^c, Π_1^{ord} by Π_1^c and moreover, the equivalence given in the proof of Theorem 17 by the following one;

$$a < \omega_1^* \rightleftarrows \exists e(e \leqq \omega_1 \wedge \exists f(f \wedge \omega \wedge \exists x(T(f, j_4(0, e, g^1(x), g^2(x))) = 0$$
$$\wedge a = g^1(x) \wedge \forall y(y < x \vdash T(f, j_4(0, e, g^1(y), g^2(y))) \neq 0)))),$$

and supply $\varphi_1(\omega_1) = \omega_1$ to the counterpart of Proposition 13.

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