J. Math. Soc. Japan
Vol. 15, No. 2, 1963

Univalent functions and non-convex domains

By Noriyuki SONE

(Received Sept. 14, 1962)
(Revised Nov. 16, 1962)

§ 1. Introduction.

The following theorem due to Noshiro [1] and Wolff [2] is well known.

THEOREM A. *If $f(z)$ is regular in a convex domain D and if $\Re f'(z)>0$ in D, then $f(z)$ is univalent in D.*

This theorem has been generalized by several authors from various points of view. Ozaki [3] proved the following:

THEOREM B. *Suppose that $g(z)$ is a convex univalent function in a domain D. If $f(z)$ is regular and $\Re\{e^{i\alpha}f'(z)/g'(z)\}>0$ (α real) in D, then $f(z)$ is univalent in D.*

Subsequently Kaplan [4] introduced a class of univalent mappings which he called 'close-to-convex'. These functions $f(z)$ defined in the unit disc, are characterized by an inequality of the form $\Re\{f'(z)/g'(z)\}>0$ where $g(z)$ is a convex univalent mapping of the unit disc. Obviously this characterization is a special case of the assumption of Theorem B. He further gave a characterization of these functions without reference to a convex function $g(z)$. It is essentially equivalent to the following Umezawa's criterion for univalence [5] i.e. the condition (i) of the following theorem, as Reade [8] points out.

THEOREM B′. *Let $w=f(z)$ be regular in a simply-connected closed domain D_z whose boundary Γ_z consists of a regular curve and suppose $f'(z)\neq 0$ on Γ_z. If there holds one of the following conditions:*

(i) *For arbitrary arcs C_z on Γ_z*

$$\int_{C_z} d\arg df(z) > -\pi \quad \textit{and} \quad \int_{\Gamma_z} d\arg df(z) = 2\pi,$$

(ii) *For arbitrary arcs C_z on Γ_z* $\displaystyle\int_{C_z} d\arg df(z) < 3\pi,$

then $f(z)$ is univalent in D_z.

Recently Reade [9] proved Theorem C stated below using the following:

DEFINITION 1. Let φ be fixed, $0\leqq\varphi<\pi$. Then a domain D is said to be 'almost convex' if any distinct two points z_1, z_2 in D can be joined by a pair of straight line segments $\overline{z_1z_3}$, $\overline{z_3z_2}$ lying in D such that

(1) $$|\arg\{(z_3-z_1)/(z_2-z_3)\}| \leqq \varphi .$$

THEOREM C. *If $f(z)$ is analytic in an 'almost convex' domain D, and if the relation $\left|\int_C d\arg f'(z)\right| < \pi-\varphi$ holds for all arcs C in D, then $f(z)$ is univalent in D.*

Quite recently Cowling and Royster [11] introduced the following Definition 2, and proved Theorem D mentioned below.

DEFINITION 2. A domain D is said to have property U with a constant $\theta=\theta(D)$ if when z_1 and z_2 in D are given there exists a constant θ (independent of z_1, z_2), $\theta<\pi$, and a sequence of points $z_1=\zeta_0, \zeta_1, \zeta_2, \cdots, \zeta_n=z_2$ with the property that the segments $\overline{\zeta_0\zeta_1}, \overline{\zeta_1\zeta_2}, \cdots, \overline{\zeta_{n-1}\zeta_n}$ all belong to D and

(2) $$|\arg\{(\zeta_{k+1}-\zeta_k)/(z_2-z_1)\}| \leqq \theta, \quad k=0,1,\cdots,n-1 .$$

THEOREM D. *Let D be a domain having the property* U. *Then if for some $\theta \leqq \alpha \leqq 2\pi$, $\arg f'(z)$ satisfies $\alpha+\theta<\arg f'(z)<\pi+\alpha-\theta$, $\theta<\pi/2$, for z in D; $f(z)$ is univalent in D.*

The first purpose of this note is to generalize the above four theorems and others in the form of Theorem 2 below, and the second one is to derive another criterion for univalence of the same functions $\int_a^b f(z,\theta)d\psi(\theta)$ treated in Theorem 2 by restricting $f_z(z,\theta)/g'(z)$ (g univalent) in a circular domain as is indicated in Theorem 5. For these purposes, in §2 we introduce a new class which consists of the sets not necessarily convex and which we call 'at most φ-concave', and in §5 we employ a certain functional for which we use the symbol $\mu(D)$. Furthermore, concerning the functions in our main theorems, a coefficient theorem and the others are inserted in §4.

§2. The 'at most φ-concave' sets.

We introduce the following two definitions:

DEFINITION 3. Let $\Gamma_z=[z_1\Gamma_z z_2]$ be a simple piece-wise analytic curve (in z-plane) with the end points z_1, z_2 and with a finite number of corners, and let Γ_z have well-defined one-side tangent vectors at the end points and those corners. Then Γ_z is said to be '(an) at most φ-concave (curve)' joining the two points z_1, z_2, if the relation

(3) $$\left|\int_{C_z} d\arg dz\right| \leqq \varphi$$

holds for all sub-curves C_z of Γ_z and for a non-negative constant φ.

REMARK 1. Throughout this paper we use the symbols $[z_1\Gamma_z z_2]$, $[z_1z_3z_2]$, etc. for the curves having the end points z_1 and z_2.

DEFINITION 4. A plane set E is said to be '(an) at most φ-concave (set)' if when z_1, z_2 in E are given there exists an at most φ-concave curve $[z_1\Gamma_z z_2]$ lying in E (where $\varphi=\varphi(E)$ is independent of z_1 and z_2).

From the above definitions we have the following:

THEOREM 1. (a) *Every (plane) convex set is at most φ-concave for every $\varphi\geqq 0$, but the converse is not always true. But every at most 0-concave set is convex.*

(b) *Every 'almost convex' domain is at most φ-concave, but the converse is not always true.*

(c) *Every domain D having the property* U *with constant $\theta=\theta(D)=\varphi/2$ is at most φ-concave, but the converse is not always true.*

(d) *Every at most φ-concave set is not necessarily simply-connected, even if $0<\varphi<\pi$.*

PROOF. (a) This is obtained at once from the definitions.

(b) Let D be an 'almost convex domain', and let z_1, z_2 be two points in D. Then there exists a broken line $[z_1 z_3 z_2]$ lying in D and satisfying (1). Hence (3) is satisfied for $\Gamma_z=[z_1 z_3 z_2]$. Therefore D is at most φ-concave. We next consider the domain D_1 in the first quadrant bounded by two concentric circles. Obviously D_1 is at most $(\pi/2)$-concave, but if the difference between the radii of the two circles is sufficiently small then the domain does not satisfy the criterion of Definition 1. Thus (b) is proved.

(c) Suppose that D has the property U with a constant $\theta=\theta(D)=\varphi/2$, and let z_1, z_2 be two points in D. Then there is a broken line $[z_1\zeta_1\zeta_2\cdots\zeta_{n-1}z_2]$ lying in D and satisfying (2) with $\theta=\varphi/2$. Hence (3) is satisfied for $\Gamma_z=[z_1\zeta_1\zeta_2\cdots\zeta_{n-1}z_2]$ and $\varphi=2\theta$. Thus the first part of (c) follows. Consider the domain D_2: the full z-plane cut from 0 to $+\infty$ along the positive real axis. Clearly D_2 is at most π-concave, but has not the property U with $\theta=\pi/2$, since the two points $z_1=1-\varepsilon i, z_2=2+\varepsilon i$ (where $\varepsilon>0$ is sufficiently small) can not be joined by any broken line lying in D and satisfying (2) with $\theta=\pi/2$. Thus (c) follows.

(d) We consider the domain D_3: the finite z-plane $|z|<+\infty$ with a hole which consists of a regular triangle and its interior. D_3 is at most $(2/3)\pi$-concave and a doubly-connected domain (but every at most 0-concave set i.e. every convex set is simply-connected). This proves (d).

§ 3. The first main theorem.

Now we prove the following principal:

THEOREM 2. *Suppose that $f(z,\theta)$ is continuous in (z,θ) for $z\in D$, $\theta\in[a,b]$ and regular in z for each fixed θ, where D is a domain (not necessarily simply-connected if f is single-valued), and $[a,b]$ is a finite closed interval. Suppose* *furthermore that $\zeta=g(z)$ is regular and univalent in D and maps D onto at most φ-concave domain D_ζ, φ being $<\pi$, and that for some $0\leqq\alpha\leqq 2\pi$, the relation*

$$\left|\arg\left\{e^{i\alpha}\frac{f_z(z,\theta)}{g'(z)}\right\}\right|<\frac{\pi-\varphi}{2} \qquad (\textit{where } f_z \textit{ means } \frac{\partial}{\partial z}f) \tag{4}$$

holds for all $z\in D$ and $\theta\in[a,b]$. Then

$$F(z)=\int_a^b f(z,\theta)d\psi(\theta)$$

is regular and univalent in D, where $\psi(\theta)$ ($\not\equiv$ constant) is any bounded and monotone function defined for $[a,b]$.

PROOF. We first note that $F(z)$ is regular and $F'(z)=\int_a^b f_z(z,\theta)d\psi(\theta)$ in D. The proof of this fact is analogous to the case $\psi(\theta)\equiv\theta$, and may be omitted.

Let z_1, z_2 be two distinct points in D, and let $\zeta=g(z)$, $\zeta_i=g(z_i)$, $i=1,2$. Since D_ζ is at most φ-concave, there exists an at most φ-concave curve $[\zeta_1\Gamma_\zeta\zeta_2]$ lying in D_ζ, and hence we may construct a narrow band B_ζ (cf. [**9**] for a special case) satisfying the following conditions:

1° B_ζ bounds a simply-connected domain D_ζ' whose closure is in D_ζ and whose interior contains $[\zeta_1\Gamma_\zeta\zeta_2]$.

2° If we denote the oriented boundary curve of D_ζ' by $\Gamma_\zeta', \Gamma_\zeta'$ is a piecewise analytic curve having well-defined one-side tangent vectors at its corners.

3° Further we have

$$\int_{C'_\zeta} d\arg d\zeta \geqq -\varphi \text{ for all arcs } C_\zeta'\subset\Gamma_\zeta', \text{ and } \int_{\Gamma_\zeta'} d\arg d\zeta=2\pi. \tag{5}$$

Let g^{-1} be the inverse function of g, and let $B_z, D_z', \Gamma_z', C_z'$ and $[z_1\Gamma_z z_2]$ be the images of $B_\zeta, D_\zeta', \Gamma_\zeta', C_\zeta'$ and $[\zeta_1\Gamma_\zeta\zeta_2]$ by g^{-1} respectively. Then, since $g^{-1}(\zeta)$ is regular and univalent in D_ζ, these images also satisfy the conditions 1° and 2° with z instead of ζ.

Now we write for all $C_z'\subset\Gamma_z'$

$$\begin{aligned}\int_{C'_z} d\arg dF(z) &= \int_{C'_z} d\arg\left[\left\{\int_a^b f_z(z,\theta)d\psi(\theta)\right\}dz\right] \\ &= \int_{C'_z} d\arg\left\{\int_a^b \frac{f_z(z,\theta)}{g'(z)}d\psi(\theta)\right\}+\int_{C'_\zeta} d\arg d\zeta.\end{aligned} \tag{6}$$

Without loss of generality let $\psi(\theta)\uparrow$. Then

$$\frac{1}{\psi(b)-\psi(a)}\int_a^b \frac{f_z(z,\theta)}{g'(z)}d\psi(\theta)\equiv G(z)$$

is the centroid of a non-negative 'mass distribution of total mass one' over the points $f_z(z,\theta)/g'(z)$ ($\theta\in[a,b]$) for each fixed $z\in C_z'$. Hence $G(z)$ lies in the convex hull of the set of these points. On the other hand from (4) the convex hull lies in an angular domain $|\arg\{e^{i\alpha}\xi\}|<(\pi-\varphi)/2$, and hence so does $G(z)$. Hence we have the fact that

(7) $\int_a^b \frac{f_z(z,\theta)}{g'(z)}d\psi(\theta)$ also lies in the same domain for $z\in C_z'$.

Therefore we get

(8) $$\int_{C'_z} d\arg\left\{\int_a^b \frac{f_z(z,\theta)}{g'(z)}d\psi(\theta)\right\}>\varphi-\pi \text{ for all } C_z'\subset\Gamma_z',$$

and especially

(9) $$\int_{\Gamma'_z} d\arg\left\{\int_a^b \frac{f_z(z,\theta)}{g'(z)}d\psi(\theta)\right\}=0.$$

Thus we have from (6), (8) and the first part of (5)

$$\int_{C'_z} d\arg dF(z)>-\pi \text{ for all } C_z'\subset\Gamma_z'.$$

We also have from (6), (9) and the second part of (5)

$$\int_{\Gamma'_z} d\arg dF(z)=2\pi.$$

On the other hand from (4), in the same way as for (7), we get

$$F'(z)=g'(z)\int_a^b \frac{f_z(z,\theta)}{g'(z)}d\psi(\theta)\neq 0$$

for all $z\in D_z'\cup\Gamma_z'$. Hence we can use Theorem B′, in a slightly generalized form [**9**], to conclude that $F(z)$ is univalent in D_z'. Therefore $F(z_1)\neq F(z_2)$ for $z_1\neq z_2$. This completes the proof.

The special cases of the theorem are listed below.

COROLLARY 1: Theorem A: *the case where* $f(z,\theta)\equiv f(z)$, $g(z)\equiv z$, $(\alpha=0)$ *and* $\varphi=0$.

COROLLARY 2: Theorem B: *the case where* $f(z,\theta)\equiv f(z)$ *and* $\varphi=0$.

COROLLARY 3: Theorem C: *the case where* $f(z,\theta)\equiv f(z), g(z)\equiv z$ *and D is an 'almost convex' domain.*

COROLLARY 4: Theorem D (with $(-\alpha-\pi/2)$ instead of α): *the case where* $f(z,\theta)\equiv f(z), g(z)\equiv z$, $(\varphi=2\theta)$ *and D is a domain having the property* U *with* $\theta=\varphi/2$.

A necessary and sufficient condition that $f(z)$ should be close-to-convex for $|z|<1$ is that the hypothesis of Corollary 2 (with $\alpha=0$) should be satisfied for $|z|<1$ instead of D. Hence we have

COROLLARY 5. (Ozaki-Kaplan's theorem [**4**]). *Every close-to-convex function is univalent.*

Furthermore we have the following interesting corollaries by specializing the choice of at most φ-concave domains in such a case of Theorem 2 that $f(z,\theta)\equiv f(z)$ and $g(z)\equiv z$.

COROLLARY 6. (A generalization of Theorem 2.2 of [11].) *Let D be a domain any two points z_1, z_2 of which may be joined by a convex (or concave) curve $z=z(t)$ $(t_1\leqq t\leqq t_2, z_i=z(t_i), i=1,2)$ lying in D and satisfying that*

$$|\arg dz_1-\arg dz_2|\leqq\varphi<\pi \qquad (\varphi \text{ constant}),$$

where dz_i are the one-side tangent vectors to the curve at the points z_i. Then if $f(z)$ is regular in D and if for some $0\leqq\alpha\leqq 2\pi$

$$(10) \qquad |\arg\{e^{i\alpha}f'(z)\}|<(\pi-\varphi)/2$$

for all $z\in D, f(z)$ is univalent in D.

COROLLARY 7. (A generalization of Theorem 2.4 of [11].) *Let D be a domain any two points of which may be joined by a finite number of line segments lying in D and parallel to the oblique coordinate axis i. e. the set $\{z; \Im z=0$ or $\Im(e^{-i\varphi}z)=0, 0<\varphi<\pi\}$ such that the x and y coordinates of the end points form either a non-decreasing or non-increasing sequence. Then if $f(z)$ is regular in D and if for some $0\leqq\alpha\leqq 2\pi$ and all $z\in D$ the relation* (10) *holds, $f(z)$ is univalent in D.*

§ 4. A coefficient theorem and the others.

THEOREM 3. *Suppose that for each fixed $\theta\in[a,b]$*

$$f(z,\theta)=f_1(\theta)z+f_2(\theta)z^2+\cdots+f_n(\theta)z^n+\cdots$$

is regular in the unit circle $|z|<1$, and that f is continuous in (z,θ) for $|z|<1$ and $\theta\in[a,b]$. Let

$$g(z)=z+g_2z^2+\cdots+g_nz^n+\cdots$$

be regular, univalent and convex for $|z|<1$. Then if the relations

$$(11) \qquad \Re\{e^{i\alpha}f_z(z,\theta)/g'(z)\}>0, \quad |f_1(\theta)|\leqq 1$$

hold for $|z|<1$ and $\theta\in[a,b]$ and for some $0\leqq\alpha\leqq 2\pi$;

$$F(z)=\int_a^b f(z,\theta)d\psi(\theta)=c_1z+c_2z^2+\cdots+c_nz^n+\cdots$$
$$(\psi(\theta)\uparrow, 0<\psi(b)-\psi(a)<+\infty)$$

is regular and univalent for $|z|<1$, and we have

$$(12) \qquad |c_n|\leqq n(\psi(b)-\psi(a)),\ n=1,2,\cdots.$$

These inequalities are sharp for every n.

PROOF. The first part of the assertion is a special case of Theorem 2. Now by the hypothesis of $g(z)$ we get

$$g(z) \ll z/(1-z) \quad \text{and so} \quad g'(z) \ll 1/(1-z)^2.$$

On the other hand, applying Carathéodory's theorom, from (11) we see that

$$e^{i\alpha}\frac{f_z(z,\theta)}{g'(z)} = e^{i\alpha}f_1(\theta)+\cdots \ll 1+2\Re\{e^{i\alpha}f_1(\theta)\}\sum_{n=1}^{\infty} z^n \ll \frac{1+z}{1-z}.$$

Therefore we get

$$f(z,\theta) = \int_0^z \frac{f_z(z,\theta)}{g'(z)} g'(z)dz \ll \int_0^z \frac{1+z}{1-z}\frac{1}{(1-z)^2}dz = \frac{z}{(1-z)^2},$$

and so

$$|f_n(\theta)| \leqq n, \quad n=1,2,\cdots.$$

Hence, for each fixed z in $|z|<1$, the series $f(z,\theta)=f_1(\theta)z+f_2(\theta)z^2+\cdots$ is uniformly convergent for $a\leqq\theta\leqq b$, and so we may write

$$F(z)=\sum_{n=1}^{\infty}\left[z^n\int_a^b f_n(\theta)d\psi(\theta)\right] \quad \text{for} \quad |z|<1.$$

Thus we have for each $n=1,2,\cdots$

$$|c_n| = \left|\int_a^b f_n(\theta)d\psi(\theta)\right| \leqq \int_a^b |f_n(\theta)|\, d\psi(\theta) \leqq n(\psi(b)-\psi(a)).$$

For the special case such that $f(z,\theta)=f_0(z,\theta)\equiv z/(1-z)^2, g(z)=g_0(z)\equiv z/(1-z)$ and $\alpha=0$, the hypotheses of the theorem are satisfied, and then we have the equality in (12) for the function

$$F_0(z)=\int_a^b f_0(z,\theta)d\psi(\theta) = \frac{z}{(1-z)^2}(\psi(b)-\psi(a)).$$

This completes the proof.

COROLLARY 8. *Let $f(z)=z+c_2z^2+\cdots+c_nz^n+\cdots$ be close-to-convex for $|z|<1$, then $|c_n|\leqq n, n=2,3,\cdots$. These inequalities are sharp for every n* [10].

PROOF. In Theorem 3 if we put $f(z,\theta)\equiv f(z), f_1(\theta)\equiv 1, \alpha=0$ and $\psi(b)-\psi(a)=1$, then we have this corollary.

REMARK 2. Concerning this corollary, suppose that $w=f(z)=z+c_2z^2+\cdots+c_nz^n+\cdots$ is close-to-convex and not starlike with respect to $w=($ for $|z|<1$, then at least one of c_n is not member of the complex quadratic field K i.e. the set $\{a+ibN^{1/2}\}$, where a and b are rational numbers, and $N\geqq 1$ s a rational integer having no square factor. This follows from the more ge neral Corollary 10 stated below and from Corollary 5, in addition to the f ıct that the star mappings are included in the close-to-convex functions.

COROLLARY 9. *Let $f(z)=z+c_2z^2+\cdots+c_nz^n+\cdots$ be regular for $|z|\leqq 1$, and let C be the part of $|z|=1$ on which*

$$1+\Re\{zf''(z)/f'(z)\}>0.$$

Then if

$$\int_C\left(1+\Re\frac{zf''(z)}{f'(z)}\right)d\theta<3\pi \quad (z=e^{i\theta}),$$

we have that $|c_n|\leqq n, n=2, 3, \cdots$ [**6**].

PROOF. In this case we can appeal to Theorem B′ to conclude that $f(z)$ is close-to-convex for $|z|<1$. Hence the corollary holds at once from Corollary 8.

REMARK 3. As Umezawa [**6**] points out in this case $f(z)$ is not merely close-to-convex but also convex in one direction [**12**], [**6**], [**7**], etc..

Concerning Remark 2 we prove the following:

THEOREM 4. *Suppose that* $w=f(z)=z^k(1+a_1z+a_2z^2+\cdots+a_nz^n+\cdots)^{-\lambda}$ $=z^k(1+b_1z+b_2z^2+\cdots+b_nz^n+\cdots)$ *is single-valued* ($f(z)\neq 0$), *regular and* $|k|$-*valent for* $0<|z|<1$, *where* $k\ (\neq 0)$ *is a rational integer,* λ *is a real number and* $\lambda k>0$. *Suppose further that* $f(z)$ *is not starlike with respect to* $w=0$ *for* $|z|<1$. *Then at least one of* a_n *is not a member of* K *as in Remark* 2.

PROOF. Let $a_n\in K$ for all $n\geqq 1$. Then by 'Theorem 4 in our previous paper [**13**]' we see that $f(z)$ has the form $z^k\{(-1)^m\prod_{j=1}^{m}(ze^{i\theta_j}-1)\}^{-\lambda}$ (θ_j real) unless z^k. Hence $f(z)$ is starlike with respect to $w=0$ for $|z|<1$ by 'Theorem 1 in [**13**]'. This contradicts the hypothesis, and so the theorem follows.

REMARK 4. Let $\lambda=k=1$ in Theorem 4. Then, if b_n ($n\geqq 1$) are integers in a given quadratic field, then a_n ($n\geqq 1$) are also integers in the same field, and vice versa [**14**]. Hence we have the following:

COROLLARY 10. *Suppose that for* $|z|<1$ $w=f(z)=z+c_2z^2+\cdots+c_nz^n+\cdots$ *is regular, univalent and not starlike with respect to* $w=0$, *then at least one of* c_n *is not a member of* K *as above.*

§ 5. The second main theorem.

We prove the following theorem using the method suggested by S. Ozaki.

THEOREM 5. *Suppose that* f *is continuous in* (z,θ) *for* $z\in D$ *and* $\theta\in[a,b]$ *and regular in* z *for each fixed* θ, *where* D *is a domain and* $[a,b]$ *is a finite closed interval. Suppose that* $\zeta=g(z)$ *is regular and univalent in* D *and maps* D *onto a domain* D_ζ, *and that for some complex constant* $A\neq 0$, *the relation*

$$\left|A\frac{f_z(z,\theta)}{g'(z)}-1\right|<\mu(D_\zeta)\equiv\inf_{\zeta_1,\zeta_2\in D_\zeta}\frac{|\zeta_1-\zeta_2|}{l(\zeta_1,\zeta_2)} \quad \left(\textit{where } f_z \textit{ means } \frac{\partial f}{\partial z}\right) \tag{13}$$

holds for all $z\in D$ *and* $\theta\in[a,b]$. *Then*

$$F(z)=\int_a^b f(z,\theta)d\psi(\theta)$$

is regular and univalent in D, *where* $l(\zeta_1,\zeta_2)\equiv\inf_{[\zeta_1\Gamma_\zeta\zeta_2]\subset D_\zeta}L[\zeta_1\Gamma_\zeta\zeta_2]$, *and* $L[\zeta_1\Gamma_\zeta\zeta_2]$

is the length of the rectifiable curve $[\zeta_1\Gamma_\zeta\zeta_2]$ *lying in D, and* $\psi(\theta)$ ($\not\equiv$ *constant*) *is a bounded and monotone function defined for* $[a, b]$.

PROOF. Let z_1, z_2 be two distinct points in D, and let $\zeta_i = g(z_i), i=1, 2$. Then $\zeta_1 \neq \zeta_2$ and $g'(z) \neq 0$ in D since $g(z)$ is univalent in D. Let D^* be the closed disc $|\zeta - \zeta_1| \leqq r$ such that $D^* \subset D_\zeta$ and $|\zeta_1 - \zeta_2| > r$, and let us set such that

$$\max_{\substack{\zeta \in D^* \\ \theta \in [a,b]}} \left| A \frac{f_z(z, \theta)}{g'(z)} - 1 \right| \equiv \mu(D_\zeta) - \delta . \tag{14}$$

Then from (13) we find that $\delta > 0$. Hence there exists the curve $\Gamma \equiv [\zeta_1 \Gamma \zeta_2]$ lying in D and satisfying that

$$0 \leqq L(\Gamma) - l(\zeta_1, \zeta_2) < r\delta / \mu(D_\zeta) , \tag{15}$$

where $L(\Gamma)$ is the length of Γ.

Now let us subdivide Γ in such a way that

$$\Gamma = [\zeta_1 \Gamma_1 \zeta_3] + [\zeta_3 \Gamma_2 \zeta_2], \quad |\zeta_1 - \zeta_3| = r \quad \text{and} \quad [\zeta_1 \Gamma_1 \zeta_3] \subset D^* .$$

Then, using the relations (14), (13) and (15), we may write for $\zeta \in \Gamma$

$$\begin{aligned}
&\left| A\{F(z_2) - F(z_1)\} - (\zeta_2 - \zeta_1) \int_a^b d\psi(\theta) \right| \\
&\leqq \left| \int_a^b d\psi(\theta) \int_{\zeta_1}^{\zeta_3} \left\{ A \frac{f_z(z, \theta)}{g'(z)} - 1 \right\} d\zeta \right| + \left| \int_a^b d\psi(\theta) \int_{\zeta_3}^{\zeta_2} \left\{ A \frac{f_z(z, \theta)}{g'(z)} - 1 \right\} d\zeta \right| \\
&\leqq \left[(\mu(D_\zeta) - \delta) \int_{\zeta_1}^{\zeta_3} |d\zeta| + \mu(D_\zeta) \int_{\zeta_3}^{\zeta_2} |d\zeta| \right] \left| \int_a^b d\psi(\theta) \right| \\
&\leqq \left[\mu(D_\zeta) L(\Gamma) - \delta r \right] \left| \int_a^b d\psi(\theta) \right| \\
&< \mu(D_\zeta) l(\zeta_1, \zeta_2) \left| \int_a^b d\psi(\theta) \right| \leqq \left| (\zeta_1 - \zeta_2) \int_a^b d\psi(\theta) \right| .
\end{aligned}$$

Thus the point $A\{F(z_2) - F(z_1)\}$ lies in the domain bounded by the circle with the centre $(\zeta_2 - \zeta_1)|\psi(b) - \psi(a)|$ and the radius $|\zeta_2 - \zeta_1||\psi(b) - \psi(a)|$. Hence $A\{F(z_2) - F(z_1)\} \neq 0$. This proves the theorem.

REMARK 5. For the case where $f(z, \theta) \equiv z, g(z) \equiv z$, and D is a convex domain, since $\mu(D) = 1$, we have Theorem A again by making $A \to +0$.

We state the following theorem which illustrates the relation between Theorems 2 and 5.

THEOREM 6. *Let* Φ *be the set of the at most* φ*-concave sets,* φ being $< \pi$. *Then for each* $E \in \Phi$ *the inequalities*

$$\cos(\varphi/2) \leqq \mu(E) \leqq 1 \tag{16}$$

hold, where $\mu(E)$ *is defined as in Theorem* 5. (*These inequalities hold not only in case where* $\varphi = 0$ *and hence* E *is convex, but also in case when* E *is not con-*

vex.) The second of these inequalities holds when E is the set obtained out of a convex set by excluding a finite number ($\neq 0$) *or an infinity of isolated points, and the first of them holds when E is the set whose boundary contains a pair* $[z_1z_3z_2]$ *of oriented straight line segments* $\overrightarrow{z_1z_3}\ \overrightarrow{z_3z_2}$ *such that*

$$\int_{[z_1z_3z_2]} d\arg dz = -\varphi\,. \tag{17}$$

PROOF. Let z_1, z_2 be two distinct points. Among the at most φ-concave curves ($\varphi<\pi$) by each of which z_1, z_2 are joined, the curve which has the maximum length consists of a pair of straight line segments $\overline{z_1z_3}$, $\overline{z_3z_2}$ such that $|z_1-z_3|=|z_3-z_2|$ and $|\arg(z_3-z_1)/(z_2-z_3)|=\varphi$. This may be proved geometrically. Hence the left-side inequality in (16) holds. The rest is easy and may be omitted.

By virtue of Theorem 6 we compare Theorem 2 and Theorem 5 in the case where $g(z)\equiv z$. If D is an at most φ-concave domain, $\varphi<\pi$, whose boundary contains a broken line segment $[z_1z_3z_2]$ satisfying (17) since $\mu(D) = \cos(\varphi/2)$, if (13) holds then (4) holds for $\alpha=\arg A$, but the converse is not always true. Hence for this case Theorem 2 is better than Theorem 5. But, for example, let D be an at most φ-concave domain any two points of which may be joined by a circular arc (lying in D) $z=z_0+ae^{i\beta}(\cos t+i\sin t)$, for which $t_1\leqq t\leqq t_2$, $0\leqq t_1$, $t_2\leqq\varphi$, $0<\varphi<\pi$, and where $a>0$, β real. (a, β may depend on the two points to be connected.) Then we get

$$\cos\frac{\varphi}{2}<\left(\sin\frac{\varphi}{2}\right)\Big/\frac{\varphi}{2}\leqq\mu(D)\leqq 1\,.$$

Hence we find that for this case (13) does not imply (4) and (4) does not imply (13). Furthermore for the case where $\varphi\geqq\pi$, (4) is senseless, but there are some cases for which (13) makes sense. Thus we shall see that both of the theorems are worth stating.

I wish to express here my hearty gratitude to Prof. S. Ozaki for his kind guidance during my research.

Yamanashi University

Bibliography

[1] K. Noshiro, On the theory of schlicht functions, J. Fac. Sci., Hokkaido Imp. Univ., 2 (1934), 129–155.

[2] J. Wolff, L'intégrale d'une fonction holomorphe et à partie réelle positive dans un demi-plan est univalente, C. R. Acad. Sci. Paris, 198 (1934), 1209–1210.

[3] S. Ozaki, On the theory of multivalent functions, Sci. Rep. Tokyo Bunrika Daigaku A, 2 (1935), 167–188.

[4] W. Kaplan, Close-to-convex schlicht functions, Michigan Math. J., **1** (1952), 169-185.
[5] T. Umezawa, On the theory of univalent functions, Tohoku Math. J., **7** (1955), 212-228.
[6] T. Umezawa, Analytic functions convex in one direction, J. Math. Soc. Japan, **4** (1952), 194-202.
[7] T. Umezawa, Star-like theorems and convex-like theorems in an annulus, J. Math. Soc. Japan, **6** (1954), 68-75.
[8] M. O. Reade, On Umezawa's criteria for univalence, J. Math. Soc. Japan, **9** (1957), 234-237.
[9] M. O. Reade, On Umezawa's criteria for univalence II, J. Math. Soc. Japan, **10** (1958), 255-259.
[10] M. O. Reade, Sur une classe de fonctions univalentes, C. R. Acad. Sci. Paris, **239** (1954), 1758-1759.
[11] V. F. Cowling and W. C. Royster, Some applications of the Weierstrass mean value theorem, J. Math. Soc. Japan, **13** (1961), 104-108.
[12] M. S. Robertson, Analytic functions star-like in one direction, Amer. J. Math., **58** (1936), 465-472.
[13] N. Sone, Note on Bernardi's theorem, Mem. Fac. Liberal Arts & Education Yamanashi Univ., **8** (1957), 124-127.
[14] S. D. Bernardi, Note on an inequality of Prawitz, Duke Math. J., **23** (1956), 385-391.