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Supplement to "Projective modules over weakly noetherian rings"

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In this note we shall use the same notations and terminologies as in [**2**]. In [**2**] we proved that, if R is a weakly noetherian ring for which dim(m-spec(R)) is finite, any nonfinitely generated projective R-module is a direct sum of finitely generated projective modules (Theorem 7.1 of [**2**]). Recently H. Bass proved in [**1**] that, if $R/J(R)$ is left noetherian, any uniformly big projective module is free. This result suggests the following

THEOREM. *Let R be an indecomposable weakly noetherian ring. Then any nonfinitely generated projective R-module is free.*

Our objective in this note is to prove this theorem. The writer wishes to thank S. Endo for helpful suggestions concerning the proof of the following Lemma 2.

LEMMA 1 (Kaplansky). *Any projective module over a ring is a direct sum of countably generated projective modules.*

By virtue of this lemma, it suffices to prove the main theorem under the condition that the projective module is countably generated.

Now we state the key result:

LEMMA 2 (cf. Lemma 6.3 of [**2**]). *Let R be an indecomposable weakly noetherian ring, P a countably generated projective module which is not finitely generated, u any element of P and M a submodule of P such that $Ru+M=P$. Then there exists an element $m \in M$ such that $R(u+m)$ is a free direct summand of P.*

PROOF. Let F be a countably generated free module, $\{f_i; i=1, 2, \cdots\}$ a free basis of F, and $F=P\oplus Q$. If p is an element of P, we write $p=\sum_j r(j, p)f_j$, $r(j, p)\in R$, $n(p)=\max(\{j \mid r(j, p)\neq 0\})$. First we notice that the set $\{n(p) \mid p\in M\}$ is not bounded since P is not finitely generated. Let p_1 be an element of M such that $n_1=n(p_1)>n(u)$. Put $V_1=V(\{r(n_1, m) \mid m\in M\})$. Since R is weakly noetherian there is a finite set of elements $m_1, \cdots, m_\alpha$ of M such that $V_1=V(\{r(n_1, m_i), i=1, 2, \cdots, \alpha\})$ (see Lemma 3.2 of [**2**]). Put $n_1'=\max(n(m_1), \cdots, n(m_\alpha))$. If $V_1\neq\phi$, we take an element $\mathfrak{x}_1\in V_1$. Since $\dim(M+\mathfrak{x}P/\mathfrak{x}P: R/\mathfrak{x})=\infty$ for any $\mathfrak{x}\in X$ ($=m$-spec(R)) (cf. Cor. 5.2 of [**2**]), there is an element p_2 of M such that $n(p_2)>n_1'$, $(r(n_1'+1, p_2), \cdots, r(n(p_2), p_2)) \not\subset \mathfrak{x}_1$. Let $r(n_2, p_2) \notin \mathfrak{x}_1$, $n(p_2) \geqq n_2 > n_1'$. Since $r(n_1, p_2) \in \mathfrak{x}_1$, we have $r(n_1, p_2)+r(n_2, p_2) \notin \mathfrak{x}_1$. Put $V_2 = V(\{r(n_1, m)+r(n_2, m) | m \in M\})$. Since $r(n_2, m_1)=r(n_2, m_2) = \cdots = r(n_2, m_\alpha)=0$ and $\mathfrak{x}_1 \notin V_2$, we have $V_1 \supsetneqq V_2$. We find a finite set of elements $m_{\alpha+1}, \cdots, m_\beta$ of M such that $V_2 = V(\{r(n_1, m_i)+r(n_2, m_i); i=1, 2, \cdots, \beta\})$. If $V_2 \neq \phi$, we take an element $\mathfrak{x}_2 \in V_2$ and find an element p_3 of M such that $n(p_3) > n_2' = \max(n(m_1), \cdots, n(m_\beta))$, $(r(n_2'+1, p_3), \cdots, r(n(p_3), p_3)) \not\subset \mathfrak{x}_2$. Let $r(n_3, p_3) \notin \mathfrak{x}_2$, $n(p_3) \geqq n_3 > n_2'$ $(\geqq n_2)$. Put $V_3 = V(\{r(n_1, m)+r(n_2, m)+r(n_3, m) | m \in M\})$ and we have $V_1 \supsetneqq V_2 \supsetneqq V_3$. In this way, we can construct a strictly decreasing sequence of closed sets of X. Since R is weakly noetherian, we can find an integer c such that $V_c = V(\{r_i; i=1, 2, \cdots, \gamma\}) = \phi$ where $r_i = r(n_1, m_i) + \cdots + r(n_c, m_i)$, hence we have $(r_1, \cdots, r_\gamma) = R$. Thus there are elements s_i of R such that $\sum_{i=1}^{\gamma} s_i r_i = 1$. Put $g_{n_i} = f_{n_i} - f_{n_c}$ for $i=1, 2, \cdots, c-1$; $g_i = f_i$ otherwise. Then $\{g_i\}$ is a free basis of F and we have that the coefficient of g_{n_c} in m_i is r_i. Now put $m = \sum s_i m_i$, then the coefficient of g_{n_c} in $u+m$ is 1 since $n(u) < n_1 < n_2 < \cdots < n_c$. Hence $R(u+m)$ is a free direct summand of P by Lemma 1.3 of [**2**].

LEMMA 3 (cf. Theorem 7.1 of [**2**]). *Over a weakly noetherian ring, any countably generated projective module is a direct sum of finitely generated projective modules.*

This lemma follows from Lemma 2 in the same way as in the proof of Theorem 7.1 of [**2**].

Now the following lemma suffices to complete the proof of the main theorem.

LEMMA 4 (cf. Proposition 2.4 of [**1**]). *Let R be an indecomposable weakly noetherian ring and P_i $(\neq(0))$ $(i=1, 2, \cdots)$ countably generated projective modules. Then the countable direct sum $P = \sum \oplus P_i$ is free.*

PROOF. Put $P_j^* = \sum' \oplus P_i$ where i ranges over all i such that $2^{j-1} | i, 2^j \nmid i$. Then P_j^* is not finitely generated and countably generated. Thus P_j^* has a free direct summand Ru_j, by Lemma 2. Hence P $(=\sum \oplus P_j^*)$ has a free direct summand $F = \sum \oplus Ru_j$. Put $P = F \oplus Q$. Since Q is a countably generated projective module, there exists a countably generated free module F' such that $Q \oplus F' = F''$ is free, by Eilenberg's lemma. Now we have $F \cong F'$, hence $P \cong F''$.

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References

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