J. Math. Soc. Japan
Vol. 16, No. 1, 1964

On the partial recursive functions of ordinal numbers

Dedicated to Professor Y. Akizuki for his 60th birthday

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(Received May 15, 1963)

On the basis of a series of Takeuti's papers [5], [6] and [7], G. Takeuti and A. Kino [8] developed the theory of recursive functions of ordinals, defined by proposing schemata, and that of hierarchy of predicates of ordinals, built on it, and they obtained various remarkable results. We are interested in giving a formalism for those functions and in applying it, via arithmetization, to the investigation of that hierarchy which contributes not only to the theory of ordinal numbers but also to the effective and classical descriptive set theory (cf., especially, §§ 7-9 of [8]).

In the meanwhile, M. Machover [4] presented a formal system of recursive functions of ordinal numbers with infinitely many variables. His concept of a 'general recursive function' is a natural extension of that in the case of natural numbers in a certain sense; however, it is rather what we want to call 'classical' and it differs from ours, even if the number of the variables is restricted to be finite.

In this paper, we shall introduce partial recursive functions as an extension of general recursive functions in the sense of Takeuti-Kino and give a formal system for them. In much of the symbolism, the notations and terminology, we follow S. C. Kleene [2] or Machover [4]. Let ω_γ be an arbitrary, but fixed, regular initial ordinal. Throughout this paper, by a function we shall always mean a function (or a functional) with a finite number of arguments ranging over ordinals $<\omega_\gamma$ (and with a finite number of function arguments) whose values are also ordinals $<\omega_\gamma$.

In § 1 we define formally calculable functions by establishing a formalism of function calculation. Roughly speaking, our system is obtained by adapting Machover's system (with infinitistic rules of formation and transformation)

* The author wishes to express his thanks to Professors G. Takeuti and V. Devidé for their kind help in the preparation of this manuscript. This work was reported at the Symposium on the Foundations of Mathematics held at Katada, Japan, on October 17, 1962. Recently, the author knew by Amer. Math. Soc. Not., 10, No. 3 (1963) that the results similar to this paper were independently obtained by A. Lévy, Transfinite computability.

to make the formation rules finitary.—Accordingly the transformation rules are also slightly modified (in particular, a new rule is added). In our system, the transfinite list of particular symbols is assumed to be given in such a way that each member of the list biuniquely corresponds to each of ordinals $<\omega_\Upsilon$, in contrast to Machover's in which the terms, called *numerals*, are formed from the symbols 0 and $'$ as the formal expressions for the ordinals.

In § 2 we introduce partial recursive functions and show that a partial recursive function is formally calculable.

In §§ 3,4 we assume the axiom of constructibility (' $V=L$ ', see K. Gödel [**1**]) and use the results of [**6**] and [**8**, § 3]. We arithmetize, in § 3, our system in the theory of ordinal numbers, and hence we obtain the predicates (of ordinals), corresponding to the metamathematical concepts such as being a term, a system of equations, or a deduction from a system of equations, etc., as primitive recursive ones. After the arithmetization, we have, in § 4, the normal form theorem for the formally calculable functions, from which it follows that a formally calculable function is partial recursive.

The advantage of the present treatment of recursive functions is that we have the same predicates $T_n(z, x_1, \cdots, x_n, y)$ (or $T_n^{n_1\cdots n_l}(w_1, \cdots, w_l, z, x_1, \cdots, x_n, y)$) as Kleene has (cf., e. g., [**2**]) for the case of number-theoretic functions, via arithmetization of the formal system, and hence we can develop the theory of the partial recursive functions and of the hierarchy of predicates analogously as Kleene did (cf. [**2**, §§ 57, 58, 65, 66] and [**3**]), i. e. for example we have the recursion theorem, the complete form theorem, etc.

§ 1. Formal calculation of functions of ordinal numbers.

1.1. First of all, we introduce a system for formal calculation of functions.

The *primitive symbols* of the system are as follows: $=$(equals), $'$ (successor), sup (the supremum operator), $v_0, v_1, \cdots, v_n, \cdots$ (variables for ordinals $<\omega_\Upsilon$), $f_0, f_1, \cdots, f_n, \cdots$ (function letters), 0_0 (or simply 0), $0_1, \cdots, 0_\alpha, \cdots$ for each $\alpha<\omega_\Upsilon$ (specified symbols for ordinals).

The *terms* are defined by induction as follows:

1. *For each* α, *the symbol* 0_α *is a term.*
2. *A variable is a term.*
3. *If* r *is a term, then* r' *is a term.*
4. *If* $\mathrm{r}_1, \cdots, \mathrm{r}_n$ *are terms and* f *is a function letter, then* $\mathrm{f}(\mathrm{r}_1, \cdots, \mathrm{r}_n)$ *is a term.*
5. *If* $\mathrm{r}_1, \mathrm{r}_2$ *are terms and* x *is a variable, then* $\sup(\mathrm{x}, \mathrm{r}_1, \mathrm{r}_2)$ (we write this as $\sup_{\mathrm{x}<\mathrm{r}_1}\mathrm{r}_2$) *is a term.*
6. *The only terms are those given by* 1–5.

The *terms in the strict sense* are the expressions defined by restricting the basic clause 1 in the definition of the terms to

1*. 0 *is a term in the strict sense.*

An *equation* (*in the strict sense*) and a *system of equations* (*in the strict sense*) are defined analogously as by Kleene[1]. In particular, we say an equation to be *prime*, if it is of the form $\mathrm{f}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \boldsymbol{x}$ where f is a function letter and $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{x}$ are symbols for ordinals. As to the definition of *bound occurrences of variables* in a term (in the strict sense) or an equation (in the strict sense), we refer to [4].

The definitions of '*ascent*', '*supremum of an ascent*' in [4] are adapted to our case by substituting '*symbol for ordinal*', '*prime equation*' for '*numeral*', '*numerical equation*', respectively. Thus, an *ascent of length* $\alpha(0<\alpha<\omega_\gamma)$ is a transfinite sequence of prime equations of the forms:

$$
\begin{array}{ll}
(1) & \begin{array}{l}
\mathrm{f}(0_0, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \boldsymbol{x}^{(0)}, \\
\mathrm{f}(0_1, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \boldsymbol{x}^{(1)}, \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathrm{f}(0_\xi, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \boldsymbol{x}^{(\xi)}, \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}
\end{array}
$$

where $\xi<\alpha$. The *supremum of the ascent* (1) is the symbol for the least ordinal which is not smaller than any ordinal for which the symbol is a right-hand side of a member of the ascent (1). Hereafter we shall write an ascent (1) as $\{\mathrm{f}(0_\xi, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \boldsymbol{x}^{(\xi)}\}_{\xi<\alpha}$, briefly.

We have four rules of inferences:

R_1. *To pass from an equation* d *to the equation* e *which results from* d *by substituting a symbol for an ordinal for a free variable.*

R_2. *To pass from an equation* d *without free variables to the equation* e *which results from* d *by replacing an occurrence of* $0'_\alpha$ *by the symbol* $0_{\alpha+1}$.

R_3. *To pass from an equation* r = s *without free variables* (the major premise) *and a prime equation* $\mathrm{h}(\boldsymbol{z}_1, \cdots, \boldsymbol{z}_p) = \boldsymbol{z}$ (the minor premise) *to the equation which results from* r = s *by replacing an occurrence of* $\mathrm{h}(\boldsymbol{z}_1, \cdots, \boldsymbol{z}_p)$ *in* s *by* $\boldsymbol{z}$.

R_4. *To pass from an equation* r = s (the major premise) *and an ascent* $\{\mathrm{h}(0_\xi, \boldsymbol{z}_1, \cdots, \boldsymbol{z}_p) = \boldsymbol{z}^{(\xi)}\}_{\xi<\alpha}$ (whose members are the minor premises) *to the equation which results from* r = s *by replacing an occurrence of the term of the form* $\sup_{\mathrm{x}<0_\alpha}\mathrm{h}(\mathrm{x}, \boldsymbol{z}_1, \cdots, \boldsymbol{z}_p)$, *where* x *is a variable, by the supremum of the ascent.*

Now, we can define a *deduction of an equation* e *from a system* E *of equations* (*in the sirict sense*) in analogy to [2][2]. Here we must remark that our

1) We also use the auxiliary terminology, such as 'principal function letter', 'given function letter' or 'auxiliary function letter' of the system E of equations, taken from [2, § 54].

2) We also use the auxiliary terminology, such as 'principal equation', 'principal branch' or 'contributory deduction' in the deduction, taken from [2, § 54].

transformation is not finitary; in fact the rule R_4 is infinitistic. Thus, similarly to [4], we also have deductions (in the tree forms) with infinite branching (but each branch is finite in height).

$\psi_1, \cdots, \psi_l$ are partial functions of $n_1, \cdots, n_l$ variables, respectively. Let $E^{\psi_1 \cdots \psi_l}_{\mathrm{g}_1 \cdots \mathrm{g}_l}$ be the set of the prime equations $\mathrm{g}_i(\boldsymbol{z}_1, \cdots, \boldsymbol{z}_{n_i}) = \boldsymbol{z}$ where $\psi_i(z_1, \cdots, z_{n_i}) = z$ for $i = 1, 2, \cdots, l$ and all n_i-tuples $z_1, \cdots, z_{n_i}$ of ordinals $< \omega_\gamma$ for which ψ_i is defined and $\boldsymbol{z}_1, \cdots, \boldsymbol{z}_{n_i}, \boldsymbol{z}$ are the symbols for $z_1, \cdots, z_{n_i}, z$, respectively. If E is a system of equations, in E there may occur symbols, say $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m$, for particular ordinals. In this case, such a system E may be written as $E(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m)$ exhibiting the occurrences of the constant ordinals.

We say that a partial function φ is *formally calculable in* $\psi_1, \cdots, \psi_l$ $(l \geqq 0)$ with $n_1, \cdots, n_l$ variables, respectively, if we can find a system E of equations in the strict sense, with f as the principal function letter and $\mathrm{g}_1, \cdots, \mathrm{g}_l$ as the given function letters which are in order of their occurrence in the preassigned list of function letters, such that

$$(2) \qquad E^{\psi_1 \cdots \psi_l}_{\mathrm{g}_1 \cdots \mathrm{g}_l}, E \vdash \mathrm{f}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \boldsymbol{x}, \text{ if and only if } \mathrm{f}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \boldsymbol{x} \in E^{\varphi}_{\mathrm{f}},$$

where $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{x}$ are symbols for ordinals. In the above, we say following Kleene's terminology that φ is formally calculable *uniformly* in $\psi_1, \cdots, \psi_l$ if we can find a system E of equations in the strict sense, independently of the choice of functions $\psi_1, \cdots, \psi_l$ except for the number $n_1, \cdots, n_l$ of the variables, such that (2) holds for any choice of $\psi_1, \cdots, \psi_l$. If φ is completely defined and formally calculable (in the case $l = 0$), then φ may be called *effectively calculable*.

Furthermore, we say that a partial function φ is *formally calculable in* $\alpha_1, \cdots, \alpha_m$ (the constant ordinals) *and* (*uniformly*) *in* $\psi_1, \cdots, \psi_l$ $(l \geqq 0)$ if we can find a system $E(0_{\alpha_1}, \cdots 0_{\alpha_m})$ of equations such that we have (2) reading '$E(0_{\alpha_1}, \cdots, 0_{\alpha_m})$' in place of '$E$'.

1.2. For the present system, the counterparts of Lemmata IIb-IIe[3] of Kleene [**2**, § 54] hold good, we see, for example, the counterpart of Lemma IIb as follows.

LEMMA 1. *Let D be a set of equations (finite or infinite), F be a system of equations whose left members contain no function letters which occur in D, and* g *be a function letter occurring in D. Then $D, F \vdash \mathrm{g}(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_p) = \boldsymbol{y}$ where $\boldsymbol{y}_1, \cdots, \boldsymbol{y}_p, \boldsymbol{y}$ are symbols for ordinals, only if $D \vdash \mathrm{g}(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_p) = \boldsymbol{y}$.*

The proof is similar to that given by Kleene, and at the induction step there does not occur any trouble in the case of application of the new rule R_4.

We have also the counterpart of Lemma VI of [**2**, p. 344]:

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3) For the counterpart of Lemma IIe of Kleene, we refer to Theorem 1 of Machover [**4**].

LEMMA 2. *If* $\lambda x_1 \cdots x_n \varphi(\lambda s_1 \cdots s_q \theta(s_1, \cdots, s_q), \psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$ *is formally calculable uniformly in functions* $\theta, \psi_1, \cdots, \psi_l$, *then* $\lambda x_1 \cdots x_n c_1 \cdots c_p \varphi(\lambda s_1 \cdots s_q \theta^*(s_1, \cdots, s_q, c_1, \cdots, c_p), \psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$ *is formally calculable uniformly in* $\theta^*, \psi_1, \cdots, \psi_l$.

In fact, if $\lambda x_1 \cdots x_n \varphi(\lambda s_1 \cdots s_q \theta(s_1, \cdots, s_q), \psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$ is formally calculable uniformly in $\theta, \psi_1, \cdots, \psi_l$, then by the definition there is a system $E(\mathrm{f} : \mathrm{g}, \mathrm{g}_1, \cdots, \mathrm{g}_l, \mathrm{a}_1, \cdots, \mathrm{a}_n)$ of equations such that, for any fixed choice of $\theta, \psi_1, \cdots, \psi_l$, $E^{\theta \psi_1 \cdots \psi_l}_{\mathrm{g} \mathrm{g}_1 \cdots \mathrm{g}_l}, E(\mathrm{f} : \mathrm{g}, \mathrm{g}_1, \cdots, \mathrm{g}_l, \mathrm{a}_1, \cdots, \mathrm{a}_n) \vdash \mathrm{f}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \boldsymbol{x}$ where $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{x}$ are symbols for ordinals, if and only if $\mathrm{f}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \boldsymbol{x} \in E^{\lambda x_1 \cdots x_n \varphi(\theta, \psi_1, \cdots \psi_l, x_1, \cdots, x_n)}_{\mathrm{f}}$. Choose p variables, say $\mathrm{c}_1, \cdots, \mathrm{c}_p$, not occurring in E, and denote by $E^{\dagger}(\mathrm{f} : \mathrm{g}, \mathrm{g}_1, \cdots, \mathrm{g}_l, \mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{c}_1, \cdots, \mathrm{c}_p)$ the system resulting from E by changing simultaneously each part $\mathrm{h}(\mathrm{r}_1, \cdots, \mathrm{r}_s)$ where h is a function letter and $\mathrm{r}_1, \cdots, \mathrm{r}_s$ are terms to $\mathrm{h}(\mathrm{r}_1, \cdots, \mathrm{r}_s, \mathrm{c}_1, \cdots, \mathrm{c}_p)$. Let $\hat{\mathrm{g}}_1, \cdots, \hat{\mathrm{g}}_l$ be distinct function letters not occurring in E. Let $E^*(\mathrm{f} : \mathrm{g}, \hat{\mathrm{g}}_1, \cdots, \hat{\mathrm{g}}_l, \mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{c}_1, \cdots, \mathrm{c}_p)$ be the system consisting of the equations

$$\mathrm{g}_1(\mathrm{a}_1, \cdots, \mathrm{a}_{n_1}, \mathrm{c}_1, \cdots, \mathrm{c}_p) = \hat{\mathrm{g}}_1(\mathrm{a}_1, \cdots, \mathrm{a}_n),$$

$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

$$\mathrm{g}_l(\mathrm{a}_1, \cdots, \mathrm{a}_{n_l}, \mathrm{c}_1, \cdots, \mathrm{c}_p) = \hat{\mathrm{g}}_l(\mathrm{a}_1, \cdots, \mathrm{a}_{n_l})$$

and of the equations of $E^{\dagger}$.

Then we have

$$E^{\lambda s_1 \cdots s_q \theta^*(s_1, \cdots, s_q, c_1, \cdots, c_p) \psi_1 \cdots \psi_l}_{\mathrm{g} \qquad\qquad\qquad\qquad \mathrm{g}_1 \cdots \mathrm{g}_l}, E \vdash \mathrm{f}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \boldsymbol{x}$$

where $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n$ and $\boldsymbol{x}$ are symbols for ordinals, if and only if

$$E^{\theta^* \psi_1 \cdots \psi_l}_{\mathrm{g}\, \hat{\mathrm{g}}_1 \cdots \hat{\mathrm{g}}_l}, E^* \vdash \mathrm{f}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, 0_{c_1}, \cdots, 0_{c_p}) = \boldsymbol{x}\,,$$

for each choice of $\theta^*, \psi_1, \cdots, \psi_l, c_1, \cdots, c_p$. The proof of this is parallel to that in [**2**, p. 345], and the argument given there can be also applied to our case with rules R_2, R_4 added.

In particular, we see in course of the proof that the following holds.

(3)

$$E^{\lambda s_1 \cdots s_q \theta^*(s_1, \cdots, s_q, c_1, \cdots, c_p) \psi_1 \cdots \psi_l}_{\mathrm{g} \qquad\qquad\qquad\qquad \mathrm{g}_1 \cdots \mathrm{g}_l}, E \vdash \mathrm{f}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \boldsymbol{x}\,,$$

where $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n$ and $\boldsymbol{x}$ are symbols for ordinals, if and only if

$$E^{\lambda s_1 \cdots s_q \theta^*(s_1, \cdots, s_q, c_1, \cdots, c_p)}_{\{\mathrm{a}_1 \cdots \mathrm{a}_q\} \mathrm{g}(\mathrm{a}_1, \cdots, \mathrm{a}_q, 0_{c_1}, \cdots, 0_{c_p})}, \; E^{\psi_1 \cdots \psi_l}_{\hat{\mathrm{g}}_1 \cdots \hat{\mathrm{g}}_l}, E^* \vdash \mathrm{f}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, 0_{c_1}, \cdots, 0_{c_p}) = \boldsymbol{x},$$

for each choice of $\theta^*, \psi_1, \cdots, \psi_l, c_1, \cdots, c_p$.

In the above, $E^{\lambda s_1 \cdots s_q \theta^*(s_1, \cdots, s_q, c_1, \cdots, c_p)}_{\{\mathrm{a}_1 \cdots \mathrm{a}_q\} \mathrm{g}(\mathrm{a}_1, \cdots, \mathrm{a}_q, 0_{c_1}, \cdots, 0_{c_p})}$ is the set of the equations $\mathrm{g}(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_q, 0_{c_1}, \cdots, 0_{c_p}) = \boldsymbol{y}$ which are in $E^{\theta^*}_{\mathrm{g}}$, for each fixed ordinals $c_1, \cdots, c_p$.

§2. The partial recursive functions and their formal calculability.

2.1. In this section, we use various notations and terminology, which are given in [8] (in particular, in §1 of [8]), without further notice. We slightly modify the *primitive recursive schemata* of Takeuti-Kino, but our modification is not essential.

By the definition given by Takeuti-Kino, a function $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$, where $l \geqq 0$ and $n > 0$, is primitive recursive, if it is introduced by a finite series of applications of the following schemata (I)-(XIII), where $\psi, \psi_1, \cdots, \psi_l$ are function variables, $\chi, \chi_1, \cdots, \chi_m$ are previously introduced functions and $m > 0$:

(I) $\varphi(x) = x'$.

(IIa) $\varphi(x) = 0$. (IIb) $\varphi(x) = \omega$.

(III) $\varphi(x) = x$.

(IV) $\varphi(x, y) = I_q(x, y)$.

(V) $\varphi(x, y) = \max(x, y)$.

(VI) $\varphi(x, y) = j(x, y)$.

(VIIa) $\varphi(x) = g^1(x)$. (VIIb) $\varphi(x) = g^2(x)$.

(VIII) $\varphi(\psi, x_1, \cdots, x_n) = \psi(x_1, \cdots, x_n)$.

(IX) $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n) = \chi(\psi_1, \cdots, \psi_l, \chi_1(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n), \cdots, \chi_m(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n))$.

(Xa) $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n, x) = \chi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$.

(Xb) $\varphi(\psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n) = \chi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$.

(XIa) $\varphi(\psi_1, \cdots, \psi_l, \psi, x_1, \cdots, x_n) = \chi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$.

(XIb) $\varphi(\psi, \psi_1, \cdots, \psi_l, x_1, \cdots, x_n) = \chi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$.

(XII) $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n, x) = \mu z_{z<x}[\chi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n, z) = 0]$,

where $\mu z_{z<x} R(\cdots, z)$ is the least ordinal $z < x$ such that $R(\cdots, z)$ if $(Ez)_{z<x} R(\cdots, z)$ and x otherwise.

(XIII) $\varphi(\psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n) = C(\lambda z \varphi^x(\psi_1, \cdots, \psi_l, z, x_1, \cdots, x_n), \psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n)$,

where $C(\psi, \psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n)$ is a function combination and ψ is a function variable of one argument.

A function combination $C(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$, an expression for the ambiguous value of a function, can be defined by induction so that it is constructed syntactically from some of the ordinal number variables $x_1, \cdots, x_n$, some of the function variables $\psi_1, \cdots, \psi_l$, the constants: $0, \omega, I_q$, max, j, g^1, g^2, the operators: $'$, $\mu z_{z<x}$, and functions introduced by the application of Schema (XIII) to previously constructed function combinations, without the use of λ-notation (cf. [2, §44]).

Now, we take Schema (XIII′) in place of (XIII), as follows:

$$\text{(XIII}'\text{)} \qquad \varphi(\psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n) = \chi(\lambda z \varphi^x(\psi_1, \cdots, \psi_l, z, x_1, \cdots, x_n), \psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n),$$

where $\chi(\psi, \psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n)$ is a previously introduced function and ψ is a function variable with one argument. Then we can define the concept of a primitive recursive description of a function $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$ in terms of Schemata (I)-(XII) and (XIII′), and we have:

A function $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$ *is primitive recursive in the preceding sense, if and only if there is a primitive recursive description of* φ *in terms of Schemata* (I)-(XII) *and* (XIII′).

PROOF. Given a primitive recursive function φ, we show by induction on the number of applications of Schemata (I)-(XIII) that there exists a primitive recursive description of φ. For this, it will suffice to treat only the case for the schema (XIII) of transfinite recursion. Let $\varphi(\psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n) = C(\lambda z \varphi^x(\psi_1, \cdots, \psi_l, z, x_1, \cdots, x_n), \psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n)$ where $C(\psi, \psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n)$ is a function combination, and assume that the previously by (XIII) introduced functions, which are used to construct C, are exactly $\xi_1, \cdots, \xi_j$. Put $\chi(\psi, \psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n) = C(\psi, \psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n)$. Then similarly to $\sharp A$ of [2, p. 224] (also cf. [2, §44, Example 1, p. 221]), we obtain a primitive recursive derivation of χ from $\psi_1, \cdots, \psi_l$, say in order of $\xi_1, \cdots \xi_j$:

$$(4) \qquad \varphi_1, \cdots, \varphi_{n_1}, \xi_1, \cdots, \varphi_{n_2}, \xi_2, \cdots, \varphi_{n_j}, \xi_j, \cdots, \chi$$

with a fixed analysis[4] where Schemata (I)-(XII) are applied.

Now by the hypothesis of the induction, there is a primitive recursive description $\varphi_{i1}, \cdots, \varphi_{ik_i} (= \xi_i)$, for each $i = 1, \cdots, j$. In (4) we replace each function ξ_i by the sequence $\varphi_{i1}, \cdots, \varphi_{ik_i}$, and supply φ as the last. The resulting sequence is a (probably redundant) primitive recursive description of φ in terms of (I)-(XII) and (XIII′).

Conversely, given a primitive recursive description $\varphi_1, \cdots, \varphi_k (= \varphi)$, we show by course-of-values induction on the length k of the description that φ is primitive recursive (i.e. definable by a series of applications of Schemata (I)-(XIII)); at the same time we shall see that φ is expressible as a function combination.

Cases 1-8: φ is introduced by one of Schemata (I)-(VIII). These cases

4) Cf. [2, p. 234]

are trivial. In fact, say $\varphi(x,y)=\max(x,y)$ by (V), where $\max(x,y)$ is a function combination by itself.

Case 9: $\varphi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n)=\chi(\psi_1,\cdots,\psi_l,\chi_1(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n),\cdots,\chi_m(\psi_1,\cdots,\psi_l,x,\cdots,x_n))$ by Schema (IX), where $\chi,\chi_1,\cdots,\chi_m$ precede $\varphi(=\varphi_k)$ in the description. Then by the hypothesis of the induction φ is primitive recursive. Also assume as the hypothesis of the induction that there are function combinations expressing $\chi,\chi_1,\cdots,\chi_m$, denoted by $C(\psi_1,\cdots,\psi_l,y_1,\cdots,y_m), C_1(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n),\cdots,C_m(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n)$, respectively. Then φ is expressible as the function combination $C(\psi_1,\cdots,\psi_l,C_1(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n),\cdots,C_m(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n))$.

Case 10: $\varphi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,x)=\chi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n)$ or $\varphi(\psi_1,\cdots,\psi_l,x,x_1,\cdots,x_n)=\chi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n)$ by Schema (X), where χ precedes φ in the description. Assume as the hypothesis of the induction that χ is expressible as a function combination $C(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n)$. Then $C(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n)$ is a function combination expressing $\varphi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,x)$ (or $\varphi(\psi_1,\cdots,\psi_l,x,x_1,\cdots,x_n)$) by itself, and is also denoted by $C(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,x)$ (or $C(\psi_1,\cdots,\psi_l,x,x_1,\cdots,x_n)$).

Case 11: Similar to the case 10.

Case 12: $\varphi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,x)=\mu z_{z<x}[\chi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,z)=0]$ by Schema (XII), where χ precedes φ in the description. Assume as the hypothesis of the induction that χ is primitive recursive and is expressible as a function combination, denoted by $C(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,z)$. Then by the definition, φ is primitive recursive. The function combination $\mu z_{z<x}C(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,z)$ expresses the ambiguous value of φ.

Case 13: $\varphi(\psi_1,\cdots,\psi_l,x,x_1,\cdots,x_n)=\chi(\lambda z\varphi^x(\psi_1,\cdots,\psi_l,z,x_1,\cdots,x_n),\psi_1,\cdots,\psi_l,x,x_1,\cdots,x_n)$ by Schema (XIII$'$), where $\chi(\psi,\psi_1,\cdots,\psi_l,x,x_1,\cdots,x_n)$ precedes φ in $\varphi_1,\cdots,\varphi_k(=\varphi)$. Assume as the hypothesis of the induction that χ is expressible as a function combination, denoted by $C(\psi,\psi_1,\cdots,\psi_l,x,x_1,\cdots,x_n)$. Then, for each fixed choice of $\psi_1,\cdots,\psi_l$ and $x_1,\cdots,x_n$, we see by transfinite induction on x:

$$\varphi(\psi_1,\cdots,\psi_l,x,x_1,\cdots,x_n)=C(\lambda z\varphi^x(\psi_1,\cdots,\psi_l,z,x_1,\cdots,x_n),\psi_1,\cdots,\psi_l,x,x_1,\cdots,x_n)$$

Thus, φ can be introduced by Schema (XIII), and $\varphi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n)$ is a function combination itself. Q. E. D.

Now we define the partial recursive functions. For this, we rewrite Schemata (I)-(XII), (XIII$'$) with '$\simeq$'[5] in place of '$=$', except for '$=$' in the bracket [ ] on the right-hand side of Schema (XII), and let $\psi,\psi_1,\cdots,\psi_l$ range over the partial functions. Here we use the interpretation that $\mu z_{z<x}[\chi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,z)=0]$ is defined if and only if $(z)_{z<x}[\chi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,z)$ is defined] (likewise for the partial predicates $(Ez)_{z<x}[\chi(\psi_1,\cdots,\psi_l,x_1,\cdots x_n,z)=0]$,

5) See, e. g. [2, § 63].

$(z)_{z<x}[\chi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n, z)=0])$ but in (XIII′) $\varphi^x(\psi_1, \cdots, \psi_l, z, x_1, \cdots, x_n)$ is undefined exactly if $z<x \wedge [\varphi(\psi_1, \cdots, \psi_l, z, x_1, \cdots, x_n)$ is undefined] where

$$\varphi^x(\psi_1, \cdots, \psi_l, z, x, \cdots, x) \simeq \begin{cases} \varphi(\psi_1, \cdots, \psi_l, z, x_1, \cdots, x_n) & \text{if } z<x, \\ 0 & \text{otherwise}. \end{cases}$$

To those schemata we add further

(XIV) $\qquad \varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n) \simeq \mu z[\chi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n, z)=0]$

under the interpretation that $\mu z[\chi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n, z)=0]$ is defined if and only if $(Ez)[\chi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n, z)=0 \wedge (t)_{t \leqq z}[\chi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n, t)$ is defined]].

A function $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$ is *partial recursive*, if there is a partial recursive description of it in terms of Schemata (I)-(XII), (XIII′) and (XIV). A partial recursive function $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$, where $\psi_1, \cdots, \psi_l$ range over the completely defined functions, is *general recursive*, if it is defined for all the values of arguments.

We also call a function $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$, where $l \geqq 0$ and $n>0$, *primitive, partial (general) recursive in the classical sense*, if it can be introduced by adding a schema:

(0) $$\varphi(x)=\alpha,$$

where α is a constant ordinal $<\omega_\Upsilon$, to the primitive, partial recursive schemata, respectively. Then, if a function $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$ is primitive (or partial, general) recursive in the classical sense, a finite number of constant ordinals, say $\alpha_1, \cdots, \alpha_m$, are used in the definition of φ. We say for such a function to be *primitive* (resp. *partial, general*) *recursive in* $\alpha_1, \cdots, \alpha_m$.

2.2. To argue the formal calculability of the partial recursive functions, we begin by showing the effective calculability of some particular primitive recursive functions.

Let S denote a system of equations of the forms $\mathrm{s}(\mathrm{a}, 0)=\mathrm{a}$, $\mathrm{h}_1(\mathrm{c}, \mathrm{a})=\mathrm{s}(\mathrm{a}, \mathrm{c})'$ and $\mathrm{s}(\mathrm{a}, \mathrm{b})=\sup_{\mathrm{z}<\mathrm{b}} \mathrm{h}_1(\mathrm{z}, \mathrm{a})$, M a system consisting of the equations of S and of equations of the forms $\mathrm{m}(\mathrm{a}, 0)=0$, $\mathrm{h}_2(\mathrm{c}, \mathrm{a})=\mathrm{s}(\mathrm{m}(\mathrm{a}, \mathrm{c}), \mathrm{a})$ and $\mathrm{m}(\mathrm{a}, \mathrm{b})=\sup_{\mathrm{z}<\mathrm{b}} \mathrm{h}_2(\mathrm{z}, \mathrm{a})$, where s is the principal function letter of S, and R a system consisting of the equations of M and of equations of the forms $\mathrm{r}(\mathrm{a}, 0)=0'$, $\mathrm{h}_3(\mathrm{c}, \mathrm{a})=\mathrm{m}(\mathrm{r}(\mathrm{a}, \mathrm{c}), \mathrm{a})$ and $\mathrm{r}(\mathrm{a}, \mathrm{b})=\sup_{\mathrm{z}<\mathrm{b}} \mathrm{h}_3(\mathrm{z}, \mathrm{a})$, where m is the principal function letter of M.

Then by transfinite induction on y (for (6), (7), using Lemma 1, successively) we see easily:

(5) $\quad S \vdash \mathrm{s}(0_x, 0_y)=0_z$ if and only if $x+y=z$ for any ordinals x, y and z.

(6) $\quad M \vdash \mathrm{m}(0_x, 0_y)=0_z$ if and only if $x \cdot y=z$ for any ordinals x, y and z.

(7) $\quad R \vdash \mathrm{r}(0_x, 0_y)=0_z$ if and only if $x^y=z$ for any ordinals x, y and z.

Now we choose function letters f, g and h which do not occur in R, and consider E consisting of the equations of R, whose principal function letter is r, and of equations of the forms:

$$\mathrm{h}(\mathrm{z}, \mathrm{a}_1, \cdots, \mathrm{a}_n) = \mathrm{r}(0, \mathrm{g}(\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{z})), \ \mathrm{f}(\mathrm{a}_1, \cdots, \mathrm{a}_n, 0) = 0'.$$

$$\mathrm{f}(\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{a}) = \mathrm{r}(0, \sup_{z<\mathrm{a}} \mathrm{h}(\mathrm{z}, \mathrm{a}_1, \cdots, \mathrm{a}_n)).$$

Let $\psi(x_1, \cdots, x_n, z)$ be a given function. By (7) we see easily that for any ordinals $x_1, \cdots, x_n, x$

$$E_{\mathrm{g}}^{\psi}, E \vdash \mathrm{f}(0_{x_1}, \cdots, 0_{x_n}, 0_x) = 0_y$$

only for $y=0$ or 1; because to deduce $\mathrm{f}(0_{x_1}, \cdots, 0_{x_n}, 0_x) = 0_y$, for any symbol 0_y, we must use $\mathrm{f}(\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{a}) = \mathrm{r}(0, \sup_{z<\mathrm{a}} \mathrm{h}(\mathrm{z}, \mathrm{a}_1, \cdots, \mathrm{a}_n))$ as the principal equation, and we have by Lemma 1 that, for any ordinals v and w, $\mathrm{r}(0, 0_v) = 0_w$ is deducible from the equations of R only, hence $0^v = w$. Moreover, we see the following

LEMMA 3. *For any ordinals* $x_1, \cdots, x_n$ *and* x,

$$E_{\mathrm{g}}^{\psi}, E \vdash \mathrm{f}(0_{x_1}, \cdots, 0_{x_n}, 0_x) = 0$$

if and only if $(Ez)_{z<x}[\psi(x_1, \cdots, x_n, z) = 0]$.

PROOF. Consider any $n+1$ ordinals $x_1, \cdots, x_n, x$ and assume $(Ex)_{z<x}[\psi(x_1, \cdots, x_n, z) = 0]$. Then for every $z < x$, $\psi(x_1, \cdots, x_n, z)$ is defined and there is an ordinal $b < x$ such that $\psi(x_1, \cdots, x_n, b) = 0$. Hence, $\mathrm{g}(0_{x_1}, \cdots, 0_{x_n}, 0_z) = 0_y \in E_{\mathrm{g}}^{\psi}$ for every $z < x$ and some y, and in particular $\mathrm{g}(0_{x_1}, \cdots, 0_{x_n}, 0_b) = 0 \in E_{\mathrm{g}}^{\psi}$ for that ordinal b. Now we choose the equation $\mathrm{f}(\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{a}) = \mathrm{r}(0, \sup_{z<\mathrm{a}} \mathrm{h}(\mathrm{z}, \mathrm{a}_1, \cdots, \mathrm{a}_n))$ and apply the rule R_1 successively to obtain

$$\mathrm{f}(0_{x_1}, \cdots, 0_{x_n}, 0_x) = \mathrm{r}(0, \sup_{z<0_x} \mathrm{h}(\mathrm{z}, 0_{x_1}, \cdots, 0_{x_n})). \tag{8}$$

At this position, we establish the contributory deductions of the minor premises for the application of R_4. Using the equation $\mathrm{h}(\mathrm{z}, \mathrm{a}_1, \cdots, \mathrm{a}_n) = \mathrm{r}(0, \mathrm{g}(\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{z}))$, and applying R_1 to substitute the respective symbols corresponding to the ordinals $x_1, \cdots, x_n$ and each ordinal $z < x$ for the variables $\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{z}$, respectively, we obtain the equations $\mathrm{h}(0_z, 0_{x_1}, \cdots, 0_{x_n}) = \mathrm{r}(0, \mathrm{g}(0_{x_1}, \cdots, 0_{x_n}, 0_z))$. From each of the latters it results $\mathrm{h}(0_z, 0_{x_1}, \cdots, 0_{x_n}) = \mathrm{r}(0, 0_y)$ by the rule R_3 with $\mathrm{g}(0_{x_1}, \cdots, 0_{x_n}, 0_z) = 0_y$ in E^{ψ} as a minor premise. In particular, $\mathrm{h}(0_b, 0_{x_1}, \cdots, 0_{x_n}) = \mathrm{r}(0, 0)$ is deducible. On the other hand, by (7) there is a deduction of $\mathrm{r}(0, 0_y) = 0_{0^y}$ from the equations of R for any ordinal y, where 0_{0^y} is the symbol corresponding to 1 or 0 according to $y=0$ or $y>0$. By R_3 we then have an ascent

$$\{\mathrm{h}(0_z, 0_{x_1}, \cdots, 0_{x_n}) = 0_{0^y}\}_{z<x}. \tag{9}$$

This includes a prime equation $\mathrm{h}(0_b, 0_{x_1}, \cdots, 0_{x_n}) = 0_1$, so the supremum of it is 0_1.

Now, we can use R_4 with (8) as the major premise and (9) as the minor premises to deduce $\mathrm{f}(0_{x_1}, \cdots, 0_{x_n}, 0_x) = \mathrm{r}(0, 0_1)$, which implies $\mathrm{f}(0_{x_1}, \cdots, 0_{x_n}, 0_x) = 0$ by R_3, since by (7) $R \vdash \mathrm{r}(0, 0_1) = 0$.

Conversely, to deduce $\mathrm{f}(0_{x_1}, \cdots, 0_{x_n}, 0_x) = 0$ from E we must use the same principal equation and the same substitutions as above. Hence $\mathrm{r}(0, \sup_{z<0_x}\mathrm{h}(z, 0_{x_1}, \cdots, 0_{x_n}))$ must be replaced by 0 in consequence of a series of applying rules and of using further equations of E. By (7) and Lemma 1, we see easily that for all $z < x$ $\mathrm{h}(0_z, 0_{x_1}, \cdots, 0_{x_n}) = 0_y$ must be infered, including at least one of prime equation with 0_y for $y > 0$ as the right member. Here, to deduce $\mathrm{h}(0_z, 0_{x_1}, \cdots, 0_{x_n}) = 0_y$ for any $z < x$ and y we must use the equation $\mathrm{h}(\mathrm{z}, \mathrm{a}_1, \cdots, \mathrm{a}_n) = \mathrm{r}(0, \mathrm{g}(\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{z}))$ and the equations $\mathrm{g}(0_{x_1}, \cdots, 0_{x_n}, 0_z) = 0_y$, whence $\mathrm{g}(0_{x_1}, \cdots, 0_{x_n}, 0_z) = 0_y \in E_{\mathrm{g}}^{\psi}$; in particular, as is easily seen from (7) and Lemma 1, $\mathrm{g}(0_{x_1}, \cdots, 0_{x_n}, 0_z) = 0$ for some ordinal z must belong to E^{ψ}. The latters imply that $\psi(x_1, \cdots, x_n, z)$ are all defined for $z < x$ and $\psi(x_1, \cdots, x_n, z) = 0$ for some ordinal $z < x$. Thus we have $(Ez)_{z<x}[\psi(x_1, \cdots, x_n, z) = 0]$.

In the above, we saw that the system E of equations defines the representing function of $(Ez)_{z<x}[\psi(x_1, \cdots, x_n, z) = 0]$ as a function formally calculable in ψ. Then we may write such a system as

$$E(\mathrm{f}, \exists \mathrm{z} < \mathrm{a}\ \mathrm{g}(\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{z})), \tag{10}$$

where f, g denote the principal, given function letters, respectively.

Dually (or more easily), we have a system of equations which may be written for the sake of brevity as

$$E(\mathrm{f}, \forall \mathrm{z} < \mathrm{a}\ \mathrm{g}(\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{z})), \tag{11}$$

where f, g denote the principal, given function letters, respectively, and which defines the representing function of the partial predicate $(z)_{z<x}[\psi(x_1, \cdots, x_n, z) = 0]$ as a function formally calculable from ψ.

LEMMA 4. *$\mu z_{z<x}[\psi(x_1, \cdots, x_n, z) = 0]$ is formally calculable in ψ.*

Indeed, $\mathrm{f}, \mathrm{g}, \mathrm{h}_1, \mathrm{h}_2, \mathrm{s}$ being distinct function letters, we consider the system $E(\mathrm{h}_1, \exists \mathrm{x} < \mathrm{z}\ \mathrm{g}(\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{x}))$ of equations taken in such a way that s is the principal function letter of its subsystem S and f, h_2 do not occur in it, followed by the equations of the forms: $\mathrm{f}(\mathrm{a}_1, \cdots, \mathrm{a}_n, 0) = 0$, $\mathrm{h}_2(\mathrm{z}, \mathrm{a}_1, \cdots, \mathrm{a}_n) = \mathrm{s}(\mathrm{f}(\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{z}), \mathrm{h}_1(\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{z}'))$ and $\mathrm{f}(\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{a}) = \sup_{\mathrm{z}<_3}\mathrm{h}_2(\mathrm{z}, \mathrm{a}_1, \cdots, \mathrm{a}_n)$. We shall denote it by $E(\mathrm{f}, \mu\mathrm{z} < \mathrm{a}\ \mathrm{g}(\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{z}))$. Then by (5), Lemma 1 and 3, we have that, for any ordinals $x_1, \cdots, x_n, x$ and y,

$$E_{\mathrm{g}}^{\psi}, E(\mathrm{f}, \mu\mathrm{z} < \mathrm{a}\ \mathrm{g}(\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{z})) \vdash \mathrm{f}(0_{x_1}, \cdots, 0_{x_n}, 0_x) = 0_y,$$

if and only if $\mu y_{y<z}[\psi(x_1, \cdots, x_n, z) = 0]$ is defined and its value is y (the informal reasoning for the case of number-theoretic functions in [**2**, 228–229] suggests

the present proof).

Next, let f, g, h_1, h_2 be distinct function letters. We take the system consisting of the equations of $E(h_1, \exists x < z\, g(a_1, \cdots, a_n, x))$, chosen in such a way that it does not contain f, h_2, and of the equations of the forms:

$$h_2(0', 0, z) = z\,,$$

$$f(a_1, \cdots, a_n) = h_2(h_1(a_1, \cdots, a_n, z), h(a_1, \cdots, a_n, z'), z)\,,$$

and denote it by $E(f, \mu z g(a_1, \cdots, a_n, z))$. Then, we have the following lemma, the proof of which is obtained similarly to [**2**, 279–281] by using Lemmata 1, 3:

LEMMA 5. *$E_g^{\psi}, E(f, \mu zg(a_1, \cdots, a_n, z)) \vdash f(0_{x_1}, \cdots, 0_{x_n}) = 0_y$ if and only if $(Ez)[(t)_{t\leqq z}(\psi(x_1, \cdots, x_n, t)$ is defined$) \wedge \psi(x_1, \cdots, x_n, z) \simeq 0]$, where $y \simeq \mu z[\psi(x_1, \cdots, x_n, z) = 0]$.*

This shows that the function $\mu z[\psi(x_1, \cdots, x_n, z) = 0]$ is formally calculable in ψ.

2.3. LEMMA 6. *The functions $\lambda x \cdot x'$, $\lambda x \cdot 0$, $\lambda x \cdot \omega$, $\lambda x \cdot x$, $\lambda xyIq(x, y)$, $\lambda xy \max(x, y)$, $\lambda xy\, j(x, y)$, $\lambda xg^1(x)$ and $\lambda xg^2(x)$ are all effectively calculable.*

PROOF. For the functions $\lambda x \cdot x'$, $\lambda x \cdot 0$, $\lambda x \cdot x$, we have immediately the systems $E_{\mathrm{I}}, E_{\mathrm{IIa}}, E_{\mathrm{III}}$ of equations (each of them consists of one single equation), by translating Schemata (I), (IIa), (III), respectively, into the formalism, so that each of them defines the corresponding one as a effectively calculable function.

Let f, h_1, h_2, r be distinct function letters. We choose R, $E(h_2, \forall x < a\, h_1(y, x))$, $E(f, \exists y < b\, h_2(a, y))$ in such a way that the auxiliary function letters of each of them be distinct from those of the others and from f, h_1, h_2, r and that the principal function letter of R be r. Denote by $E(f, <)$ the system consisting of the equations of R, $f(0, b) = r(0, b)$, $h_1(y, x) = f(x, y)$, the equations of $E(h_2, \forall x < a\, h_1(y, x))$ and $E(f, \exists y < b\, h_2(a, y))$, in the order exhibited. As it is easily seen, the following recursion holds:

$$Iq(0, b) = 0^b\,,$$

$$Iq(a, b) = \begin{cases} 0 & \text{if } (Ey)_{y<b}(x)_{x<a}[Iq(x, y) = 0]\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

By (7), Lemma 3 and its dual and using Lemma 1, this informal argument suggests the proof that

$$E(f, <) \vdash f(0_x, 0_y) = 0_z \text{ if and only if } Iq(x, y) = z$$

and hence $\lambda xyIq(x, y)$ is effectively calculable.

Using this, the dual of Lemma 3, Lemma 5 and Lemma 1 (or using the counterpart of Lemma IIe of [**2**, §54]), we see that $\lambda x \cdot \omega$ is also effectively calculable. In fact, E_{Ib}: the equations of $E(\mathrm{h}_1, <), \mathrm{h}_2(\mathrm{a, x, y}) = \mathrm{h}_1(\mathrm{y}', \mathrm{x})$, the equations of $E(\mathrm{h}_3, \forall \mathrm{y} < \mathrm{x}\, \mathrm{h}_2(\mathrm{a, x, y}))$, deleted the equation $\mathrm{h}_3(\mathrm{a}, 0) = 0$, and the equations of $E(\mathrm{f}, \mu\mathrm{x}\, \mathrm{h}_3(\mathrm{a, x}))$ (with the same conditions as those mentioned above for the definition of $E(\mathrm{f}, <)$) defines $\lambda x \cdot \omega$, i.e. $E_{\mathrm{IIb}} \vdash \mathrm{f}(0_x) = 0_\omega$ for any ordinal x; for, we have $\omega = \mu x\{Iq(0, x) = 0 \wedge (y)_{y<x}[Iq(y', x) = 0]\}$, where $(Ex)\{Iq(0, x) = 0 \wedge (y)_{y<x}[Iq(y', x) = 0]\}$.

As E_V, the following system of equations may be chosen:

$$\mathrm{h}(0, \mathrm{a, b}) = \mathrm{a},\ \mathrm{h}(0', \mathrm{a, b}) = \mathrm{b} \ \text{ and } \ \mathrm{f}(\mathrm{a, b}) = \sup_{\mathrm{x}<0''} \mathrm{h}(\mathrm{x, a, b}),$$

from which $\mathrm{f}(0_x, 0_y) = 0_z$ is deducible if and only if $\max(x, y) = z$.

Now we consider the function

$$\varphi(a, b) = \chi(\max(a, b)) + b + a \cdot Iq(b, \max(a, b))$$

where $\chi(c)$ is defined by recursion as follows:

$$\begin{cases} \chi(0) = 0, \\ \chi(c) = \mu z((y)_{y<c}(\chi(y) + (y \cdot 2)' \leqq z)). \end{cases}$$

Here, $\chi(c)$ is effectively calculable; for, translating this definition of χ into the forms:

$$\mathrm{f}(0) = 0,\ \mathrm{h}(\mathrm{y}) = \mathrm{s}(\mathrm{f}(\mathrm{y}), (\mathrm{m}(\mathrm{y}, 0''))'),\ \mathrm{f}(\mathrm{a}) = \sup_{\mathrm{y}<\mathrm{a}} \mathrm{h}(\mathrm{y}),$$

we see easily by (5), (6) and Lemma 1 that, for any ordinals x and y, $\mathrm{f}(0_x) = 0_y$ is deducible from M and these equations if and only if $\chi(x) = y$, when M is chosen in such a way that it does not contain f nor h, its principal function letter is m and the principal function letter of its subsystem S is s. Hence, by the effective calculability of the functions $\lambda xy \max(x, y)$, $\lambda xy \cdot x + y$, $\lambda xy \cdot xy$ and $\lambda xy Iq(x, y)$ and using Lemma 1 (or the counterpart of Lemma IIe of [**2**, § 54]), we have the effective calculability of $\lambda xy\, \varphi(x, y)$. On the other hand, we see that

$$\varphi(a, b) < \varphi(c, d) \rightleftarrows \max(a, b) < \max(c, d)$$
$$\vee (\max(a, b) = \max(c, d) \wedge (b < d \vee (b = d \wedge a < c))),$$

and therefore $\varphi(a, b) = j(a, b)$ for any ordinals a and b (see, [**8**, § 1] and cf. Gödel [**1**, Chapter III, 7.81, p. 28]). This shows that $\lambda xy\ j(x, y)$ is effectively calculable.

Finally, as to the function g^1, g^2, by Lemmata 3, 4 and 1 (or using the counterpart of Lemma IIe of [**2**, § 54]) it will suffice to remark that

$$g^1(a) = \mu x_{x<a}(Ey)_{y<a}[j(x, y) = a]$$
$$g^2(a) = \mu x_{x<a}(j(g^1(a), x) = a)$$

and

$$x=y \rightleftarrows Iq(x,y)=1 \wedge Iq(y,x)=1$$
$$\rightleftarrows Iq(0, Iq(x,y)\cdot Iq(y,x))=0\,.$$

2.4. Now, we are going to prove the main theorem of this section:

THEOREM 1. *For each* $l \geqq 0, n>0$: *If* $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$ *is partial recursive, then it is formally calculable uniformly in* $\psi_1, \cdots, \psi_l$.

PROOF. Given a partial recursive function $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$ with any given numbers $l \geqq 0, n>0$, we construct a system $E(\mathrm{f}: \mathrm{g}_1, \cdots, \mathrm{g}_l, \mathrm{a}_1, \cdots, \mathrm{a}_n)$ which defines φ as a formally calculable function uniformly in $\psi_1, \cdots, \psi_l$, by course-of-values induction on the length k of a given partial recursive description $\varphi_1, \cdots, \varphi_k$ of φ. The cases (I)-(XII), (XIII′), (XIV) correspond to Schemata (I)-(XII), (XIII′), (XIV) by which $\varphi_k(=\varphi)$ may occur in the description.

Case (I)-(VII): These were already established in Lemma 6.

Case (VIII): $\varphi(\psi, x_1, \cdots, x_n) \simeq \psi(x_1, \cdots, x_n)$. Let $E(\mathrm{f}: \mathrm{g}, \mathrm{a}_1, \cdots, \mathrm{a}_n)$ be the system of an equation of the form $\mathrm{f}(\mathrm{a}_1, \cdots, \mathrm{a}_n)=\mathrm{g}(\mathrm{a}_1, \cdots, \mathrm{a}_n)$.

Case (IX): $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n) \simeq \chi(\psi_1, \cdots, \psi_l, \chi_1(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n), \cdots, \chi_m(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n))$. By the hypothesis of the induction, there exist systems $H_1(\mathrm{h}_1: \mathrm{g}_1, \cdots, \mathrm{g}_l, \mathrm{a}_1, \cdots, \mathrm{a}_n), \cdots, H_m(\mathrm{h}_m: \mathrm{g}_1, \cdots, \mathrm{g}_l, \mathrm{a}_1, \cdots, \mathrm{a}_n)$ and $H(\mathrm{h}: \mathrm{g}_1, \cdots, \mathrm{g}_l, \mathrm{a}_1, \cdots, \mathrm{a}_m)$ of equations such that, for each choice of $\psi_1, \cdots, \psi_l$,

$E^{\psi_1 \cdots \psi_l}_{\mathrm{g}_1 \cdots \mathrm{g}_l}, H_j \vdash h_j(0_{x_1}, \cdots, 0_{x_n})=0_{y_j}$

if and only if $\chi_j(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n) \simeq y_j$ for $j=1, \cdots, m$,

and

$E^{\psi_1 \cdots \psi_l}_{\mathrm{g}_1 \cdots \mathrm{g}_l}, H \vdash \mathrm{h}(0_{y_1}, \cdots, 0_{y_m})=0_y$ if and only if $\chi(\psi_1, \cdots, \psi_l, y_1, \cdots, y_m) \simeq y$,

for any ordinals $x_1, \cdots, x_n, y_1, \cdots, y_m$ and y. Here we can choose these systems so that the function letters occurring in each one of $H_1, \cdots, H_m, H$ are distinct from those occurring in the others excepting the given function letters $\mathrm{g}_1, \cdots, \mathrm{g}_l$. Let f be a function letter not occurring in $H_1, \cdots, H_m$ nor in H. Denote by $E(\mathrm{f}: \mathrm{g}_1, \cdots, \mathrm{g}_l, \mathrm{a}_1, \cdots, \mathrm{a}_n)$ the system consisting of the equations of $H_1, \cdots, H_m, H$ and of the equation

$$\mathrm{f}(\mathrm{a}_1, \cdots, \mathrm{a}_n)=\mathrm{h}(\mathrm{h}_1(\mathrm{a}_1, \cdots, \mathrm{a}_n), \cdots, \mathrm{h}_m(\mathrm{a}_1, \cdots, \mathrm{a}_n))\,.$$

Then, we get easily that for each choice of $\psi_1, \cdots, \psi_l$,

$E^{\psi_1 \cdots \psi_l}_{\mathrm{g}_1 \cdots \mathrm{g}_l}, E(\mathrm{f}: \mathrm{g}_1, \cdots, \mathrm{g}_l, \mathrm{a}_1, \cdots, \mathrm{a}_n) \vdash \mathrm{f}(0_{x_1}, \cdots, 0_{x_n})=0_y\,,$

if and only if $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n) \simeq y$, for any ordinals $x_1, \cdots, x_n$ and y,

In the above, to establish the consistency property 'only if', we shall use the counterpart of Lemma IIc of [2, §54].

Case (Xa): $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n, x) \simeq \chi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$. By the hypothesis of the induction, there is a system $H(\mathrm{h}: \mathrm{g}_1, \cdots, \mathrm{g}_l, \mathrm{a}_1, \cdots, \mathrm{a}_n)$ of equations such that, for each choice of $\psi_1, \cdots, \psi_l$,

$$(12)\qquad E^{\psi_1\cdots\psi_l}_{g_1\cdots g_l}, H\vdash \mathrm{h}(0_{x_1},\cdots,0_{x_n})=0_y, \text{ if and only if } \chi(\psi_1,\cdots,\psi_l, x_1,\cdots,x_n)\simeq y, \text{ for any ordinals } x_1,\cdots,x_n \text{ and } y.$$

Then let $E(\mathrm{f}:\mathrm{g}_1,\cdots,\mathrm{g}_l,\mathrm{a}_1,\cdots,\mathrm{a}_n,\mathrm{a})$ be the system consisting of the equations of H and of the equation

$$\mathrm{f}(\mathrm{a}_1,\cdots,\mathrm{a}_n,\mathrm{a})=\mathrm{h}(\mathrm{a}_1,\cdots,\mathrm{a}_n)$$

where f is a function letter not occurring in H.

Case (Xb): $\varphi(\psi_1,\cdots,\psi_l,x,x_1,\cdots,x_n)\simeq\chi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n)$. Similarly to the above, we have a system $E(\mathrm{f}:\mathrm{g}_1,\cdots,\mathrm{g}_l,\mathrm{a},\mathrm{a}_1,\cdots,\mathrm{a}_n)$ of equations with the desired property.

Case (XI): $\varphi(\psi_1,\cdots,\psi_l,\psi,x_1,\cdots,x_n)\simeq\chi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n)$ or $\varphi(\psi,\psi_1,\cdots,\psi_l,x_1,\cdots,x_n)\simeq\chi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n)$. By the hypothesis of the induction, for χ there is a system $H(\mathrm{h}:\mathrm{g}_1,\cdots,\mathrm{g}_l,\mathrm{a}_1,\cdots,\mathrm{a}_n)$ of equations such that, for each choice of $\psi_1,\cdots,\psi_l$, we have (12). Let f, g be function letters not occurring in H. Then the system $E(\mathrm{f}:\mathrm{g}_1,\cdots,\mathrm{g}_l,\mathrm{g},\mathrm{a}_1,\cdots,\mathrm{a}_n)$, consisting of the equation $\mathrm{g}(\mathrm{a}_1,\cdots,\mathrm{a}_p)=\mathrm{g}(\mathrm{a}_1,\cdots,\mathrm{a}_p)$ where p is the number of arguments of ψ and of those of H followed by $\mathrm{f}(\mathrm{a}_1,\cdots,\mathrm{a}_n)=\mathrm{h}(\mathrm{a}_1,\cdots,\mathrm{a}_n)$, is the desired one for φ.

Case (XII): $\varphi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,x)\simeq\mu z_{z<x}[\chi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,z)=0]$. By the hypothesis of the induction, there is a system $H(\mathrm{h}:\mathrm{g}_1,\cdots,\mathrm{g}_l,\mathrm{a}_1,\cdots,\mathrm{a}_n,\mathrm{z})$ of equations such that, for each choice of $\psi_1,\cdots,\psi_l$,

$$(13)\qquad \begin{aligned}&E^{\psi_1\cdots\psi_l}_{g_1\cdots g_l}, H\vdash \mathrm{h}(0_{x_1},\cdots,0_{x_n},0_z)=0_w \text{ if and only if}\\ &\chi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,z)\simeq w \text{ (i.e. } \mathrm{h}(0_{x_1},\cdots,0_{x_n},0_z)=0_w\\ &\in E_{\mathrm{h}}^{\lambda x_1\cdots x_n z\chi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,z)}), \text{ for any ordinals } x_1,\cdots,x_n,z \text{ and } w.\end{aligned}$$

Here we choose the system $E(\mathrm{f}:\mu\mathrm{z}<\mathrm{a}\,\mathrm{h}(\mathrm{a}_1,\cdots,\mathrm{a}_n,\mathrm{z}))$ defined in Lemma 4 so that the function letters except h do not occur in H. Let $E(\mathrm{f}:\mathrm{g}_1,\cdots,\mathrm{g}_l,\mathrm{a}_1,\cdots,\mathrm{a}_n,\mathrm{a})$ be the system:

$$H(\mathrm{h}:\mathrm{g}_1,\cdots,\mathrm{g}_l,\mathrm{a}_1,\cdots,\mathrm{a}_n,\mathrm{z}),\ E(\mathrm{f},\mu\mathrm{z}<\mathrm{a}\,\mathrm{h}(\mathrm{a}_1,\cdots,\mathrm{a}_n,\mathrm{z})).$$

Then we can see, for any fixed choice of $\psi_1,\cdots,\psi_l$, the following:

$$\begin{aligned}&\varphi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,x)\simeq y\\ &\rightleftarrows \mu z_{z<x}[\chi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,z)=0]\simeq y\\ &\rightleftarrows E_{\mathrm{h}}^{\lambda x_1\cdots x_n z\chi(\psi_1,\cdots,\psi_l,x_1,\cdots,x_n,z)}, E(\mathrm{f},\mu\mathrm{z}<\mathrm{a}\,\mathrm{h}(\mathrm{a}_1,\cdots,\mathrm{a}_n,\mathrm{z}))\\ &\quad\vdash \mathrm{f}(0_{x_1},\cdots,0_{x_n},0_x)=0_y\end{aligned}$$

(by Lemma 4)

$$\rightleftarrows E^{\psi_1\cdots\psi_l}_{g_1\cdots g_l}, H\,E(\mathrm{f},\mu\mathrm{z}<\mathrm{a}\,\mathrm{h}(\mathrm{a}_1,\cdots,\mathrm{a}_n,\mathrm{z}))\vdash \mathrm{f}(0_{x_1},\cdots,0_{x_n},0_x)=0_y$$

using (13), and also—for the proof of '$\leftarrow$' — by the counterpart of Lemma

IIc of [2, § 54]), for any ordinals $x_1, \cdots, x_n, x$ and y.

Therefore, we have that, for each choice of $\psi_1, \cdots, \psi_l$,

$$E^{\psi_1\cdots\psi_l}_{g_1\cdots g_l}, E(f : g_1, \cdots, g_l, a_1, \cdots, a_n, a) \vdash f(0_{x_1}, \cdots, 0_{x_n}, 0_x) = 0_y,$$

if and only if $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n, x) \simeq y$, for any ordinals $x_1, \cdots, x_n, x$ and y.

Case (XIII): $\varphi(\psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n) \simeq \chi(\lambda z\varphi^x(\psi_1, \cdots, \psi_l, z, x_1, \cdots, x_n), \psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n)$. By the hypothesis of the induction $\lambda xx_1 \cdots x_n\chi(\psi, \psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n)$, where ψ is a function variable with one argument, is formally calculable uniformly from $\psi, \psi_1, \cdots, \psi_l$; i.e. there exists a system of equations $H(h : g, g_1, \cdots, g_l, a, a_1, \cdots, a_n)$ with h as the principal function letter and with $g, g_1, \cdots, g_l$ as the given function letters such that, for each choice of $\psi, \psi_1, \cdots, \psi_l$,

$$E^{\psi\psi_1\cdots\psi_l}_{gg_1\cdots g_l}, H(h : g, g_1, \cdots, g_l, a, a_1, \cdots, a_n) \vdash h(0_x, 0_{x_1}, \cdots, 0_{x_n}) = 0_y,$$

(14) if and only if $\chi(\psi, \psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n) \simeq y$,

for any ordinals $x, x_1, \cdots, x_n$ and y.

Now, for each choice of $\psi_1, \cdots, \psi_l$, by the definition of φ and (14) we see immediately that

$$E^{\psi_1\cdots\psi_l}_{g_1\cdots g_l}, E^{\lambda z\varphi^x(\psi_1, \cdots, \psi_l, z, x_1, \cdots x_n)}_{g}, H(h : g, g_1, \cdots, g_l, a, a_1, \cdots, a_n)$$

(15) $\vdash h(0_x, 0_{x_1}, \cdots, 0_{x_n}) = 0_y$, if and only if $\varphi(\psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n)$

$\simeq y$, for any ordinals $x, x_1, \cdots, x_n$ and y.

Using Lemma 2, we can take the system $H^*(h : g, \hat{g}_1, \cdots, \hat{g}_l, a, a_1, \cdots, a_n, c, c_1, \cdots, c_n)$ to be such that H^* defines $\lambda xx_1 \cdots x_n cc_1 \cdots c_n \chi(\lambda z\psi^*(z, c, c_1, \cdots, c_n), \psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n)$ as a formally calculable function uniformly in $\psi^*, \psi_1, \cdots, \psi_l$.

Let f, f_1, f_2, f_3, be function letters not occurring in H^*, and consider the equations:

$$f_2(0, b, a_1, \cdots, a_n) = f(b, a_1, \cdots, a_n),$$
$$f_3(0, b, a_1, \cdots, a_n) = 0$$
$$g(b, a, a_1, \cdots, a_n) = f_2(f_1(b, a), b, a_1, \cdots, a_n),$$
$$g(b, a, a_1, \cdots, a_n) = f_3(f_1(0, f_1(b, a)), b, a_1, \cdots, a_n).$$

We shall denote the system of these equations by $G(g, f^a(b, a_1, \cdots, a_n))$. Let $\psi_1, \cdots, \psi_l$ be given functions with the specified number of arguments, respectively. For each $n+1$-tuple of ordinals $x, x_1, \cdots, x_n$, we define $F^x_{x_1\cdots x_n}$ to be the set of equations $f(0_z, 0_{x_1}, \cdots, 0_{x_n}) = 0_y$, where $\varphi(\psi_1, \cdots, \psi_l, z, x_1, \cdots, x_n) \simeq y$, for all $z < x$ and y. In particular $F^x_{x_1\cdots x_n}$ is empty, if $x = 0$. Then we see immediately that

$$F^x_{x_1\cdots x_n}, E^{Iq}_{f_1}, G(g, f^a(b, a_1, \cdots, a_n)) \vdash g(0_z, 0_x, 0_{x_1}, \cdots, 0_{x_n}) = 0_y,$$

if and only if

$$g(0_z, 0_x, 0_{x_1}, \cdots, 0_{x_n}) = 0_y \in E^{\lambda z\varphi^x(\psi_1,\cdots,\psi_l,z,x_1,\cdots,x_n)}_{\{b\}g(b,0_x,0_{x_1},\cdots,0_{x_n})}$$

Therefore, when we choose $E(f_1, <)$ so that the auxiliary function letters do not occur in $G(g, f^a(b, a_1, \cdots, a_n))$, by the counterpart of Lemma IIc of [2, § 54], we have

$$(16)\quad \begin{aligned} &F^x_{x_1\cdots x_n}, E(f_1, <)\, G(g, f^a(b, a_1, \cdots, a_n)) \vdash g(0_z, 0_x, 0_{x_1}, \cdots, 0_{x_n}) = 0_y, \\ &\text{if any only if } g(0_z, 0_x, 0_{x_1}, \cdots, 0_{x_n}) = 0_y \in E^{\lambda z\varphi^x(\psi_1,\cdots,\psi_l,z,x_1,\cdots,x_n)}_{\{b\}g(b,0_x,0_{x_1},\cdots,0_{x_n})}. \end{aligned}$$

Now let $E(f : \hat{g}_1, \cdots, \hat{g}_l, a, a_1, \cdots, a_n)$ be the system consisting of the equations of $E(f_1, <)\, G(g, f^a(b, a_1, \cdots, a_n))$ (which may be also abbreviated by E_1, where $E(f_1, <)$ is chosen so that the auxiliary function letters occur neither in $G(g, f^a(b, a_1, \cdots, a_n))$ nor in H^*), those of H^* and of the equation

$$f(a, a_1, \cdots, a_n) = h(a, a_1, \cdots, a_n, a, a_1, \cdots, a_n).$$

Let $\psi_1, \cdots, \psi_l$ be given functions with the specified number of arguments, respectively. For each $n+1$-tuple of ordinals $x, x_1, \cdots, x_n$, we denote by $G^x_{x_1\cdots x_n}$ be the set of equations $f(0_z, 0_{x_1}, \cdots, 0_{x_n}) = 0_w$ for any $z < x$ deducible from $E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}, E(f : \hat{g}_1, \cdots, \hat{g}_l, a, a_1, \cdots, a_n)$. Then we see:

$$(17)\quad \begin{aligned} &\text{If } h(0_x, 0_{x_1}, \cdots, 0_{x_n}, 0_x, 0_{x_1}, \cdots, 0_{x_n}) \text{ is deducible from } E^{\psi_1\cdots\psi}_{\hat{g}_1\cdots\hat{g}_l}, E, \\ &\text{then it is deducible from } E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}, G^x_{x_1\cdots x_n}, E_1 H^* \end{aligned}$$

For, the letter f does not occur in H^*, so the last equation of E is used only to deduce equations of the form $g(\boldsymbol{z}, 0_x, 0_{x_1}, \cdots, 0_{x_n}) = \boldsymbol{w}$ in the given deduction; but by the definition of E_1, only the equations of $G^x_{x_1\cdots x_n}$ are used in those subdeductions.

For any fixed choice of $\psi_1, \cdots, \psi_l$ (with the specified number of arguments, respectively), we shall show by transfinite induction on x that

$$(18)\quad \begin{aligned} &E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}, E(f : \hat{g}_1, \cdots, \hat{g}_l, a, a_1, \cdots, a_n) \vdash f(0_x, 0_{x_1}, \cdots, 0_{x_n}) = 0_y, \\ &\text{if and only if } \varphi(\psi_1, \cdots, \psi_l, a, a_1, \cdots, a_n) \simeq y \text{ (i.e. } \varphi \text{ is defined} \\ &\text{and its value is } y\text{), for any ordinals } x, x_1, \cdots, x_n \text{ and } y. \end{aligned}$$

BASIS: Let $x=0$. For the proof of 'if', suppose $\varphi(\psi_1, \cdots, \psi_l, 0, x_1, \cdots, x_n) \simeq y$. Then by the definition of φ, $\chi(\lambda z\varphi^0(\psi_1, \cdots, \psi_l, z, x_1, \cdots, x_n), \psi_1, \cdots, \psi_l, 0, x_1, \cdots, x_n) \simeq y$ (is true), where $\lambda z\varphi^0(\psi_1, \cdots, \psi_l, z, x_1, \cdots, x_n) = \lambda z \cdot 0$; it follows by the completeness property of (15) that

$$(19)\quad \begin{aligned} &E^{\lambda z\varphi^0(\psi_1,\cdots,\psi_l,z,x_1,\cdots,x_n)\psi_1\cdots\psi_l}_{g\ \ g_1\cdots g_l}, H(h : g, g_1, \cdots, g_l, a, a_1, \cdots, a_n) \\ &\vdash h(0, 0_{x_1}, \cdots, 0_{x_n}) = 0_y. \end{aligned}$$

Using (3) as given in the proof of Lemma 2 (for the choice $0, x_1, \cdots, x_n$ of $c, c_1, \cdots, c_n$), we see that (19) implies

$$E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}, E^{\lambda z\varphi^0(\psi_1,\cdots,\psi_l,z,x_1,\cdots,x_n)}_{\{b\}g(b,0,0x_1,\cdots,0x_n)}, H^* \tag{20}$$
$$\vdash \mathrm{h}(0, 0_{x_1}, \cdots, 0_{x_n}, 0, 0_{x_1}, \cdots, 0_{x_n}) = 0_y\,.$$

Then by the completeness property of (16) and the general property of deducibility, $\mathrm{h}(0, 0_{x_1}, \cdots, 0_{x_n}, 0, 0_{x_1}, \cdots, 0_{x_n}) = 0_y$ is deducible from $E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}, E_1 H^*$, since $F^0_{x_1\cdots x_n} = \phi$; hence $\mathrm{f}(0, 0_{x_1}, \cdots, 0_{x_n}) = 0_y$ is deducible from $E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}, E(\mathrm{f} : \hat{\mathrm{g}}_1, \cdots, \hat{\mathrm{g}}_l, \mathrm{a}, \mathrm{a}_1, \cdots, \mathrm{a}_n)$ by using the last equation of E as the principal equation.

Conversely, suppose that $E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}, E(\mathrm{f} : \hat{\mathrm{g}}_1, \cdots, \hat{\mathrm{g}}_l, \mathrm{a}, \mathrm{a}_1, \cdots, \mathrm{a}_n) \vdash \mathrm{f}(0, 0_{x_1}, \cdots, 0_{x_n}) = 0_y$. In $E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}, E$ only the equation $\mathrm{f}(\mathrm{a}, \mathrm{a}_1, \cdots, \mathrm{a}_n) = \mathrm{h}(\mathrm{a}, \mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{a}, \mathrm{a}_1, \cdots, \mathrm{a}_n)$ contains the function letter f in the left member. Then to deduce $\mathrm{f}(0, 0_{x_1}, \cdots, 0_{x_n}) = 0_y$ we must take it as the principal equation, to which the rule R_1 is to be applied successively to obtain $\mathrm{f}(0, 0_{x_1}, \cdots, 0_{x_n}) = \mathrm{h}(0, 0_{x_1}, \cdots, 0_{x_n}, 0, 0_{x_1}, \cdots, 0_{x_n})$ followed by the application of R_3 to eliminate the letter h, since no other applications of rules along the principal branch yield $\mathrm{f}(0, 0_{x_1}, \cdots, 0_{x_n}) = 0_y$ as the end equation. Therefore we must take the deduction of $\mathrm{h}(0, 0_{x_1}, \cdots, 0_{x_n}, 0, 0_{x_1}, \cdots, 0_{x_n}) = 0_y$ from $E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}, E$ as only the contributory deduction.

Now we can use (17). In consequence it must be deducible also from $E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}, G^0_{x_1\cdots x_n}, E_1 H^*$, where $G^0_{x_1\cdots x_n}$ is empty by definition. We note that the left members of equations of H^* contain no function letters which occur in $E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}$ or in E_1 by the definitions of them, h is the principal function letter of H^* (hence it occurs neither in $E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}$ nor in E_1), and that only the function letter g occurs in both E_1 and H^*. Then by the counterpart of Lemma IIc of [**2**, §54] and using (16) for the case when $x=0$ (in this case $F^x_{x_1\cdots x_n} = \phi$) we necessarily have (20). From this, it follows (19), hence $\varphi(\psi_1, \cdots, \psi_l, 0, x_1, \cdots, x_n) \simeq y$ by using (3) as given in the proof of Lemma 2 (for the choice $0, x_1, \cdots, x_n$ of $c, c_1, \cdots, c_n$), the consistency property of (15) successively.

Thus, (18) is proved in case $x=0$.

INDUCTION STEP: Let $x > 0$. Assume as the hypothesis of transfinite induction that (18) holds for every ordinal $z < x$. Then by the definitions,

$$F^x_{x_1\cdots x_n} = G^x_{x_1\cdots x_n}\,.$$

Then we can prove (18) for x similarly to the case for $x=0$. We give the outline of the proof:

$$\varphi(\psi_1, \cdots, \psi_l, x, x_1, \cdots, x_n) \simeq y$$
$$\rightleftarrows E^{\lambda z\varphi^x(\psi_1,\cdots,\psi_l,z,x_1,\cdots,x_n)\psi_1\cdots\psi_l}_{g\ \hat{g}_1\cdots\hat{g}_l}, H \vdash \mathrm{h}(0_x, 0_{x_1}, \cdots, 0_{x_n}) = 0_y$$

(by (15))

$$\rightleftarrows E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}, E^{\lambda z\varphi^x(\psi_1,\cdots,\psi_l,z,x_1,\cdots,x_n)}_{\{b\}g(b,0x,0x_1,\cdots,0x_n)}, H^* \vdash \mathrm{h}(0_x, 0_{x_1}, \cdots, 0_{x_n}, 0_x, 0_{x_1}, \cdots, 0_{x_n}) = 0_y$$

(by (3) as given in the proof of Lemma 2 (for the choice $x, x_1, \cdots, x_n$ of $c, c_1,$

$\cdots, c_n))$

$$\rightleftarrows E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}, F^x_{x_1\cdots x_n}, E_1 H^* \vdash \mathrm{h}(0_x; 0_{x_1}, \cdots, 0_{x_n}, 0_x, 0_{x_1}, \cdots, 0_{x_n}) = 0_y$$

(by (16), and for '$\leftarrow$', using the counterpart of Lemma IIc of [**2**, § 54] and by the definitions of E_1 and H^*)

$$\rightleftarrows E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}, G^x_{x_1\cdots x_n}, E_1 H^* \vdash \mathrm{h}(0_x, 0_{x_1}, \cdots, 0_{x_n}, 0_x, 0_{x_1}, \cdots, 0_{x_n}) = 0_y$$

(by the hypothesis of transfinite induction)

$$\rightleftarrows E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}, E \vdash \mathrm{h}(0_x, 0_{x_1}, \cdots, 0_{x_n}, 0_x, 0_{x_1}, \cdots, 0_{x_n}) = 0_y$$

(for '$\rightarrow$', by the definition of $G^x_{x_1\cdots x_n}$ and using the general property of deducibility; for '$\leftarrow$', by (17))

$$\rightleftarrows E^{\psi_1\cdots\psi_l}_{\hat{g}_1\cdots\hat{g}_l}, E \vdash \mathrm{f}(0_x, 0_{x_1}, \cdots, 0_{x_n}) = y$$

(for '$\leftarrow$', by analysing the given deduction.)

Thus, (18) is proved for any fixed functions $\psi_1, \cdots, \psi_l$; hence, we know that $E(\mathrm{f} : \hat{\mathrm{g}}_1, \cdots, \hat{\mathrm{g}}_l, \mathrm{a}, \mathrm{a}_1, \cdots, \mathrm{a}_n)$ is a desired system for the present case.

Case (XIV): $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n) \simeq \mu x[\chi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n, x) = 0]$. Using Lemma 5, we can define a system $E(\mathrm{f} : \mathrm{g}_1, \cdots, \mathrm{g}_l, \mathrm{a}_1, \cdots, \mathrm{a}_n)$ of equations similarly to the case (XII) such that, for each choice of $\psi_1, \cdots, \psi_l$,

$E^{\psi_1\cdots\psi_l}_{g_1\cdots g_l}, E \vdash \mathrm{f}(0_{x_1}, \cdots, 0_{x_n}) = 0_y$, if and only if $\varphi(\psi_1, \cdots, \psi_l,$ $x_1, \cdots, x_n) \simeq y$, for any ordinals $x_1, \cdots, x_n$ and y.

Q. E. D.

Similarly we have

THEOREM 2. *If $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$ is partial recursive in the ordinals $\alpha_1, \cdots, \alpha_m$, then it is formally calculable in $\alpha_1, \cdots, \alpha_m$ and uniformly in $\psi_1, \cdots, \psi_l$.*

§ 3. Arithmetization of the formalism in the theory of ordinals.

3.1. We introduce a Gödel numbering of objects of the formalism by ordinals in such a way that the Gödel number of a system of equations in the strict sense is a natural number (i.e. an ordinal $< \omega$). To the symbol 0_α for the ordinal α we correlate the Gödel number $j(3, \alpha)$, to the $i+1$-st variable v_i the Gödel number $j(4, i)$, and to the $i+1$-st function letter f_i the Gödel number $j(5, i)$. Suppose that the Gödel numbers $r, r_1, \cdots, r_n$ have already been correlated to the terms $\mathrm{r}, \mathrm{r}_1, \cdots, \mathrm{r}_n$, respectively. Then, to r' we correlate the Gödel number $j(6, r)$, to $\mathrm{f}(\mathrm{r}_1, \cdots, \mathrm{r}_n)$ the Gödel number $j(f, j(r_1, j(r_2, \cdots, j(r_{n-1}, r_n) \cdots)))$ where f is the Gödel number of the function letter f, and to $\sup_{\mathrm{x}<\mathrm{r}_1}\mathrm{r}_2$ the Gödel number $j(7, j(x, j(r_1, r_2)))$ where x is the Gödel number of the variable x. Next, to $\mathrm{r}_1 = \mathrm{r}_2$ we correlate the Gödel number $j(8, j(r_1, r_2))$. To a system of equations $\mathrm{e}_1, \cdots, \mathrm{e}_k$ we correlate the Gödel number $j(1, j(e_1, \cdots, j(e_{k-1}, e_k) \cdots))$

where $e_1, \cdots, e_k$ are the Gödel numbers of $\mathrm{e}_1, \cdots, \mathrm{e}_k$, respectively.

In order to define the Gödel number of a transfinite sequence of equations (say, the Gödel number of an ascent) or of a transfinite sequence of deductions, we introduce some auxiliary notions by using the model of set theory constructed in the theory of primitive recursive functions, given in [8] (for this model, cf. also [5], [6]). Let $S(a, b)$ be

$$(u)_{u<a}(v)_{v<a}(w)_{w<a}(\langle v, u\rangle \in a \wedge \langle w, u\rangle \in a \rightarrow v \equiv w)$$
$$\wedge (x)_{x<a}(x \in a \rightarrow (Ey)_{y<a}(Ez)_{z<b}(O(y) \wedge x \equiv \langle y, j(0, z, 0)\rangle))$$
$$\wedge (x)_{x<b}(Ey)_{y<a}(\langle y, j(0, x, 0)\rangle \in a).$$

Then $S(a, b)$ is a primitive recursive predicate by [8, § 3][6].

Now, we assume the axiom of constructibility '$V=L$' (Gödel [1]). Let $S^*(a ; \psi, b)$ $(S^*(a ; \{a_x\}_{x<b}))$ denote the predicate

$$S(a, b) \wedge (x)_{x<b}(\psi(x) = u(a \dagger j(0, x, 0))))^{7)}$$
$$(S(a, b) \wedge (x)_{x<b}(a_x = u(a \dagger j(0, x, 0)))),$$

and $S(a ; \psi, b)$ $(S(a ; \{a_x\}_{x<b}))$ be

$$S^*(a ; \psi, b) \wedge (x)_{x<a} \neg S^*(x ; \psi, b) \ (S^*(a ; \{a_x\}_{x<b}) \wedge (y)_{y<a} \neg S^*(y ; \{a_x\}_{x<b})).$$

Then for any given ordinal b and function ψ with one argument (or a transfinite sequence $\{a_x\}_{x<b}$ of ordinals), there exists one and only one ordinal a such that we have $S(a ; \psi, b)$ (or $S(a ; \{a_x\}_{x<b})$). Using this, we correlate to an ascent $\{\mathrm{e}_\xi\}_{\xi<\alpha}$ the Gödel number $j(\omega, a)$ where $S(a ; \{a_\xi\}_{\xi<\alpha})$ and a_ξ is the Gödel number of the prime equation e_ξ of the ascent, under the assumption of the axiom of constructibility.

By the definition, a deduction is in the tree form with infinitely many, but finite in length, branches in general. Hence, by induction corresponding to the definition of 'deduction', we can assign the Gödel number d to a deduction D, whose end equation is e with the Gödel number e, from a set F of equations as follows:

0. If D consists of only one equation in F, then $d = j(2, e)$.

1, 2. If D_1 is a deduction with the Gödel number d_1 from F, and D is

6) For the definitions of the functions or predicates: $j(x, y, z)$, $x \in y$, $x \equiv y$, $\langle x, y\rangle$, $O(y)$, see [8, p. 204, p. 205 and p. 208]. These are all primitive recursive by their definitions.

7) $u(x)$ is a primitive recursive function such that $u(a)$=the ordinal to which a corresponds if a is an ordinal in the model, $u(a)=0$ otherwise; cf. [8, p. 206]. $a \dagger b$ is defined as follows:

$$a \dagger b = \begin{cases} \mu y_{y<a}(\langle y, b\rangle \in a) & \text{if there exists a } y \text{ such that } y<a \text{ and } \langle y, b\rangle \in a, \\ 0 & \text{otherwise.} \end{cases}$$

The function $x \dagger y$ is evidently primitive recursive; cf. [8, p. 207].

$\frac{D_1}{e}$ by R_1 or R_2, then $d = j(2, j(e, g^2(d_1)))$.

3. If D_1, D_2 are deductions with the Gödel numbers d_1, d_2, respectively, from F and D is $\frac{D_1 \; D_2}{e}$ by R_3, then $d = j(2, j(e, j(g^2(d_1), g^2(d_2))))$.

4. If D_1 is a deduction with the Gödel number d_1 from F, $\{D_{2,\xi}\}_{\xi<\alpha}$ is a sequence of deductions of the members of an ascent $\{e_\xi\}_{\xi<\alpha}$ with the Gödel number a from F and D is $\frac{D_1 \{D_{2,\xi}\}_{\xi<\alpha}}{e}$ by R_4, then $d = j(2, j(e, j(g^2(d_1), j(a, d_2))))$ where $S(d_2, \{g^2(d_{2,\xi})\}_{\xi<\alpha})$ and $d_{2,\xi}$ is the Gödel number of $D_{2,\xi}$.

Thus, the Gödel number can be uniquely correlated to each object of the system, under the assumption of the axiom of constructibility if necessary. As is easily seen from the property of the function j, distinct ordinals are, of course, assigned as Gödel numbers to distinct objects.

3.2. Let $\nu(x, y)$ be defined by

$$\begin{aligned} \nu(x, y) &= y && \text{if } x = 0 \\ &= g^2(\nu(\delta(x), y))^{8)} && \text{if } x > 0 \wedge x < \omega, \\ &= 0 && \text{otherwise}. \end{aligned}$$

Using [8, Proposition 3], we see easily that this function is primitive recursive. We use below the abbreviation $[a]_n$ for the function $g^1(\nu(n, a))$, where $n < \omega$.

Now, we define primitive recursive predicates and functions corresponding to the respective metamathematical predicates and functions for our formal system, via the Gödel numbering defined above for the objects, similarly to [2, §56].

$Or(a)$, $V(a)$ and $Fl(a)$ are the predicates expressing that '*a is the Gödel number of a symbol for ordinal*', '*a is the Gödel number of a variable*' and '*a is the Gödel number of a function letter*', respectively. Then we have

$$Or(a) \rightleftarrows g^1(a) = 3,$$

$$V(a) \rightleftarrows g^1(a) = 4 \wedge a < \omega,$$

$$Fl(a) \rightleftarrows g^1(a) = 5 \wedge a < \omega.$$

Let $Or^{-1}(a)$ be defined by

$$Or^{-1}(a) = \mu x_{x<a} \, (a = j(3, x)).$$

If a is the Gödel number of the symbol for an ordinal b, then $Or^{-1}(a) = b$; $Or^{-1}(a) = a$, otherwise.

$Tm^*(a)$ is '*a is the Gödel number of a term in the strict sense*', that is

8) $\delta(x) = z$ if $x = z'$; otherwise, $\delta(x) = x$. See [8, p. 201].

$$Tm^*(a) \rightleftarrows a=9 \vee V(a) \vee [g^1(a)=6 \wedge Tm^*(g^2(a))]$$
$$\vee [Fl([a]_0) \wedge (En)_{0<n<\omega}(Tm^*(\nu(n, a)) \wedge (i)_{0<i<n} Tm^*([a]_i))]$$
$$\vee [g^1(a)=7 \wedge V([a]_1) \wedge Tm^*([a]_2) \wedge Tm^*(\nu(3, a))] .$$

When $a=0$, all disjunctive members on the right-hand side are evidently false; hence we have $\neg Tm^*(0)$. When $a>0$, we see that $g^2(a)<a$ ($g^1(a) \leqq a$), by the property of j and the definition of g^2 (g^1), hence we have also $[a]_i<a$ and $\nu(n, a)<a$, for $0<i<n$ and $n<\omega$. Let $\tau(x)$ be the representing function of $Tm^*(x)$. Then, putting

$$R(\psi, a) \rightleftarrows a=9 \vee V(a) \vee [g^1(a)=6 \wedge \psi(g^2(a))=0]$$
$$\vee [Fl([a]_0) \wedge (En)_{0<n<\omega}(\psi(\nu(n, a))=0 \wedge (i)_{0<i<n}(\psi([a]_i)=0))]$$
$$\vee [g^1(a)=7 \wedge V([a]_1) \wedge \psi([a]_2)=0 \wedge \psi(\nu(3, a))=0] ,$$

we get $\tau(a)=\chi(\lambda z \tau^a(z), a)$ where $\chi(\psi, a)$ is the representing function of the primitive recursive predicate $R(\psi, a)$. Therefore, the predicate $Tm^*(a)$ is primitive recursive (cf. [**2**, § 43, Example 3]). This argument can be applied also to predicates given below.

$$Tm(a) \rightleftarrows Or(a) \vee V(a) \vee [g^1(a)=6 \wedge Tm(g^2(a))]$$
$$\vee [Fl([a]_0) \wedge (En)_{0<n<\omega}(Tm(\nu(n, a)) \wedge (i)_{0<i<n} Tm([a]_i))]$$
$$\vee [g^1(a)=7 \wedge V([a]_1) \wedge Tm([a]_2) \wedge Tm(\nu(3, a))] .$$

Then the predicate $Tm(a)$ is true if and only if *a is the Gödel number of a term.*

$Eq^*(e)$, $Eq(e)$, $SE^*(z)$ and $SE(z)$ are the predicates expressing that '*e is the Gödel number of an equation in the strict sense*', '*e is the Gödel number of an equation*', '*z is the Gödel number of a system of equations in the strict sense*' and '*z is the Gödel number of a system of equations*', respectively. Then we have

$$Eq^*(e) \rightleftarrows g^1(e)=8 \wedge Tm^*(g^1(g^2(e))) \wedge Tm^*(g^2(g^2(e))) ,$$
$$Eq(e) \rightleftarrows g^1(e)=8 \wedge Tm(g^1(g^2(e))) \wedge Tm(g^2(g^2(e))) ,$$
$$SE^*(z) \rightleftarrows [z]_0=1 \wedge (En)_{0<n<\omega}(Eq^*(\nu(n, z)) \wedge (i)_{0<i<n} Eq^*([z]_i)) ,$$

and

$$SE(z) \rightleftarrows [z]_0=1 \wedge (En)_{0<n<\omega}(Eq(\nu(n, z)) \wedge (i)_{0<i<n} Eq([z]_i)) .$$

$Sb(d, e, t, x)$ is '*t, x and e are the Gödel numbers of a term* t, *a variable* x *and a term or an equation* e, *respectively, and d is the Gödel number of the term or the equations* d *which results from* e *by substituting* t *for* x', that is

$$Sb(d, e, t, x) \rightleftarrows Tm(t) \wedge V(x) \wedge (Tm(e) \vee Eq(e))$$
$$\wedge [(e = x \wedge d = t) \vee (V(e) \wedge e \neq x \wedge d = e) \vee (Or(e) \wedge d = e)$$
$$\vee ([d]_0 = [e]_0 \neq 7 \wedge (En)_{0<n<\omega}(Sb(\nu(n, d), \nu(n, e), t, x)$$
$$\wedge (i)_{0<i<n} Sb([d]_i, [e]_i, t, x)))$$
$$\vee ([d]_0 = [e]_0 = 7 \wedge [d]_1 = [e]_1 \neq x \wedge Sb([d]_2, [e]_2, t, x)$$
$$\wedge Sb(\nu(3, d), \nu(3, e), t, x))] .$$

To see that $Sb(d, e, t, x)$ is primitive recursive, it will suffice to consider in the same way as in [**2**, p. 257] the predicate $Sb(z, t, x)$ such that

$$Sb(z, t, x) \rightleftarrows Sb(g^1(z), g^2(z), t, x) ;$$

because $j([g^1(z)]_i, [g^2(z)]_i) < z$ and $j(\nu(n, g^1(z)), \nu(n, g^2(z))) < z$ for $0 < i < n$ and $n < \omega$, when $z > 0$ and $g^1(z) > 0$.

Let $Ct(e, x)$ be the predicate

$$(Tm(e) \vee Eq(e)) \wedge V(x) \wedge \neg Sb(e, e, 9, x) .$$

Then $Ct(e, x)$ is true, if and only if *e is the Gödel number of a term or an equation* e *and x is the Gödel number of a variable* x *such that* e *contains* x *free.*

$$Cn_1(c, d) \rightleftarrows Eq(d) \wedge (Ex)_{x<d}(Ea)_{a<c}(Or(a) \wedge Ct(d, x) \wedge Sb(c, d, a, x)) .$$
$$Cn_2(c, d) \rightleftarrows Eq(d) \wedge (x)_{x<d} \neg Ct(d, x) \wedge (Et)_{t<d}(Ea)_{a<d}[Tm(t) \wedge Ct(t, 16)$$
$$\wedge Or(a) \wedge Sb(d, t, j(6, a), 16) \wedge Sb(c, t, j(3, Or^{-1}(a)+1), 16)] .$$
$$Cn_3(c, d, e) \rightleftarrows Eq(e) \wedge Fl(g^1([e]_1)) \wedge (En)_{0<n<\omega}[Or(\nu(n, [e]_1))$$
$$\wedge (i)_{0<i<n} Or([e]_{1,i})] \wedge Or(\nu(2, e)) \wedge Eq(d)$$
$$\wedge (x)_{x<d} \neg Ct(d, x) \wedge c = j(8, j([d]_1, \nu(2, c)))$$
$$\wedge (Et)_{t<d}[Tm(t) \wedge Ct(t, 16) \wedge Sb(\nu(2, d), t, [e]_1, 16)$$
$$\wedge Sb(\nu(2, c), t, \nu(2, e), 16)]$$

where $[d]_{i,j}$ abbreviates $[[d]_i]_j$. Then $Cn_1(c, d)$ $(Cn_2(c, d)), Cn_3(c, d, e)$ express

'*c is the Gödel number of an equation* c *which is an immediate consequence of an equation* d *with the Gödel number d by R_1 (R_2)*',

'*c, d and e are the Gödel numbers of equations* c, d *and* e, *respectively, such that* c *is an immediate consequence of* d *and* e *by R_3*',

respectively.

Hereafter we assume the axiom of constructibility.

Let $As(a)$ be '*a is the Gödel number of an ascent*', and $\mathrm{Sup}(a, b)$ be '*b is the Gödel number of the supremum of an ascent with the Gödel number a*' Then these are

$$As(a) \rightleftarrows g^1(a) = \omega \wedge (Ey)_{y<a}(Et)_{t<a}[S(g^2(a), y) \wedge Fl(g^1(t))$$
$$\wedge \{g^2(t) = 16 \vee (En)_{1<n<\omega}([t]_1 = 16 \wedge Or(\nu(n, t)) \wedge (i)_{1<i<n} Or([t]_i))\}$$
$$\wedge (x)_{x<y}(g^1(a^{(x)}) = 8 \wedge Sb([a^{(x)}]_1, t, j(3, x), 16) \wedge Or(\nu(2, a^{(x)})))]$$

where $a^{(x)} = u(g^2(a)\dagger j(0, x, 0))$,

$$\mathrm{Sup}(a, b) \rightleftarrows As(a) \wedge (x)_{x<l(a)}(\nu(2, a^{(x)}) \leqq b) \wedge Or(b)$$
$$\wedge (y)_{y<b}[Or(y) \rightarrow (Ex)_{x<l(a)}(y < \nu(2, a^{(x)}))]$$

where $l(a) = \mu z_{z<a} S(g^2(a), z)$.

We can see easily that $\mathrm{Sup}(a, b) \rightarrow b < a'$. Hence, put

$$\mathrm{Sup}(a) = \mu z_{z<a'} \mathrm{Sup}(a, z).$$

Then we have a primitive recursive predicate $Cn_4(c, d, a)$ expressing that '*c, d and a are the Gödel numbers of equations* c, d *and an ascent A, respectively, such that* c *is an immediate consequence of* d *and A by R_4*'. In fact,

$$Cn_4(c, d, a) \rightleftarrows As(a) \wedge Eq(d) \wedge (x)_{x<d} \neg Ct(d, x)$$
$$\wedge c = j(8, j([d]_1, \nu(2, c))) \wedge (Eu)_{u<d}(Ex)_{x<d}(Et)_{t<d}[Tm(u)$$
$$\wedge Ct(u, 16) \wedge V(x) \wedge Tm(t) \wedge Ct(t, x) \wedge Sb([a^{(0)}]_1, t, 9, x)$$
$$\wedge Sb(\nu(2, d), u, j(7, j(x, j(j(3, l(a)), t))), 16) \wedge Sb(\nu(2, c), u, \mathrm{Sup}(a), 16)].$$

Let $D(z, y)$ denote the predicate '*z is the Gödel number of a system of equations Z, and y is the Gödel number of a deduction from Z*', and $D(\psi_1, \cdots, \psi_l, z, y)$ the predicate '*z is the Gödel number of a system of equations Z* (whose given function letters are $g_1, \cdots, g_l$), *and y is the Gödel number of a deduction from $E^{\psi_1 \cdots \psi_l}_{g_1 \cdots g_l} Z$*' for completely defined functions $\psi_1, \cdots, \psi_l$ with $n_1, \cdots, n_l$ variables, respectively. Let $[x]_* = x$ if $g^1(x) = 8$; otherwise, $= g^1(x)$. Then by the definition of the Gödel number of a deduction, we have

$$D(z, y) \rightleftarrows SE(z) \wedge g^1(y) = 2$$
$$\wedge \{[Eq(g^2(y)) \wedge (Ei)_{0<i<\omega}(g^2(y) = [z]_i \vee (g^2(y) = \nu(i, z)))]$$
$$\vee [Cn_1([y]_1, [\nu(2, y)]_*) \wedge D(z, j(2, \nu(2, y)))]$$
$$\vee [Cn_2([y]_1, [\nu(2, y)]_*) \wedge D(z, j(2, \nu(2, y)))]$$
$$\vee [Cn_3([y]_1, [y]_{2,*}, [\nu(3, y)]_*) \wedge D(z, j(2, [y]_2)) \wedge D(z, j(2, \nu(3, y)))]$$
$$\vee [Cn_4([y]_1, [y]_{2,*}, [y]_3) \wedge S(\nu(4, y), l([y]_3)) \wedge (x)_{x<l([y]_3)}([y]_3^{(x)}$$
$$= [u(\nu(4, y)\dagger j(0, x, 0))]_* \wedge D(z, j(2, u(\nu(4, y)\dagger j(0, x, 0)))))]\},$$

and that $D(\psi_1, \cdots, \psi_l, z, y)$ is expressible by the predicate obtainable by inserting

(21) $$[[y]_1 = 8 \wedge [y]_{2,0} = g_1 \wedge (Eu_1)_{u_1<y} \cdots (Eu_{n_1})_{u_{n_1}<y}([y]_{2,1} = j(3, u_1) \wedge \cdots \wedge [y]_{2,n_1-1} = j(3, u_{n_1-1}) \wedge \nu(n_1, [y]_2) = j(3, u_{n_1})) \wedge \nu(3, y) = j(3, \psi_1(u_1, \cdots, u_{n_1}))]$$
$$\vee \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad \vee$$
$$[[y]_1 = 8 \wedge [y]_{2,0} = g_l \wedge (Eu_1)_{u_1<y} \cdots (Eu_{n_l})_{u_{n_l}<y}([y]_{2,1} = j(3, u_1) \wedge \cdots \wedge [y]_{2,n_l-1} = j(3, u_{n_l-1}) \wedge \nu(n_l, [y]_2) = j(3, u_{n_l})) \wedge \nu(3, y) = j(3, \psi_l(u_1, \cdots, u_{n_l}))],$$

where $g_1, \cdots, g_l$ are the Gödel numbers of $\mathrm{g}_1, \cdots, \mathrm{g}_l$, as the first disjunctive member in the braces { } in the right member of the above equivalence, using $D(\psi_1, \cdots, \psi_l, z, *)$ in place of $D(z, *)$.

Now, we consider the predicate '*z is the Gödel number of a system of equations Z, and y is the Gödel number of a deduction from Z of a prime equation* $\mathrm{f}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \boldsymbol{x}$, *where* f *is the principal function letter of* $Z, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n$ *are the symbols corresponding to the ordinals* $x_1, \cdots, x_n$, *respectively, and* $\boldsymbol{x}$ *is a symbol for ordinal*', and denote it by $S_n(z, x_1, \cdots, x_n, y)$. We can also consider the predicate denoted by $S_n(\psi_1, \cdots, \psi_l, z, x_1, \cdots, x_n, y)$, reading 'from $E^{\psi_1 \cdots \psi_l}_{\mathrm{g}_1 \cdots \mathrm{g}_l}, Z$' in place of 'from Z' in the above.

Then by the definition, we see that

$$S_n(z, x_1, \cdots, x_n, y) \rightleftarrows D(z, y) \wedge (Ei)_{0<i<\omega}\{[\nu(i, z)]_0 = 8 \wedge [Fl([g^2(y)]_{*,1,0}) \wedge [g^2(y)]_{*,1,0} = [\nu(i, z)]_{1,0} \wedge [g^2(y)]_{*,1,1} = j(3, x_1) \wedge \cdots \wedge [g^2(y)]_{*,1,n-1} = j(3, x_{n-1}) \wedge \nu(n, [g^2(y)]_{*,1}) = j(3, x_n) \wedge Or(\nu(2, [g^2(y)]_*))]\}.$$

Therefore, $S_n(z, x_1, \cdots, x_n, y)$ is primitive recursive. For $S_n(\psi_1, \cdots, \psi_l, z, x_1, \cdots, x_n, y)$, we have an equivalence like $S_n(z, x_1, \cdots, x_n, y)$, except reading '$D(\psi_1, \cdots, \psi_l, z, y)$' in place of '$D(z, y)$'; hence it is also primitive recursive.

Let $U(y)$ be defined by

$$U(y) = Or^{-1}(\nu(2, [g^2(y)]_*)).$$

Then $U(y)$ is a primitive recursive function such that $U(y) = x$, *whenever* y *is the Gödel number of a deduction of an equation of the form* $\mathrm{r} = 0_x$.

Thus, we have established the following results, under the assumption of the axiom of constructibility:

A function $\varphi(x_1, \cdots, x_n)$ *is formally calculable (uniformly) in* $\psi_1, \cdots, \psi_l$ $(l \geqq 0)$ *where* $\psi_1, \cdots, \psi_l$ *are any completely defined functions of* $n_1, \cdots, n_l$ *variables, respectively, if and only if there exists an ordinal* e *such that*

(22) $$e < \omega$$

$$(23) \qquad (x_1)\cdots(x_n)[\varphi(x_1,\cdots,x_n) \text{ is defined} \rightleftarrows (Ey)S_n(\psi_1,\cdots,\psi_l,e,x_1,\cdots,x_n,y)],$$

and

$$(24) \qquad (x_1)\cdots(x_n)(y)[S_n(\psi_1,\cdots,\psi_l,e,x_1,\cdots,x_n,y)\rightarrow U(y)\simeq\varphi(x_1,\cdots,x_n)].$$

In general, *a function $\varphi(x_1,\cdots,x_n)$ is formally calculable in $\alpha_1,\cdots,\alpha_m$, and (uniformly) in $\psi_1,\cdots,\psi_l$, where $\alpha_1,\cdots,\alpha_m$ are any constant ordinals and $\psi_1,\cdots,\psi_l$ are any completely defined functions of $n_1,\cdots,n_l$ variables, respectively, if and only if there exists an ordinal e (the Gödel number of a system $E(0_{\alpha 1},\cdots,0_{\alpha m})$ of equations) such that* (23) *and* (24) *hold, where generally $e>\omega$ but e is constructed from $\alpha_1,\cdots,\alpha_m$ by primitive recursive functions.*

§ 4. Theorems on recursive functions and the hierarchy $\{\sum_k^{ord}, \prod_k^{ord}\}_{k=1,2,\cdots}$

4.1. In this section, we assume the axiom of constructibility, and suppose $\psi_1,\cdots,\psi_l$ range over the completely defined functions of $n_1,\cdots,n_l$ variables, respectively.

For each $l\geqq 0$, let

$$T_n(\psi_1,\cdots,\psi_l,z,x_1,\cdots,x_n,y)\rightleftarrows S_n(\psi_1,\cdots,\psi_l,z,x_1,\cdots,x_n,y) \wedge (t)_{t<y}\neg S_n(\psi_1,\cdots,\psi_l,z,x_1,\cdots,x_n,t),$$

following Kleene's famous notation. Then, we have

$$(25) \qquad (Ey)T_n(\psi_1,\cdots,\psi_l,z,x_1,\cdots,x_n,y)\rightleftarrows(Ey)S_n(\psi_1,\cdots,\psi_l,z,x_1,\cdots,x_n,y).$$

Now, using these primitive recursive predicates we can proceed as in the Kleene's theory of recursive functions of natural numbers. First of all, we have the normal from theorem (cf. [**2**, p. 288, p. 292 and p. 330]):

THEOREM 3. *For each $l\geqq 0$ and $n>0$: Given any function $\varphi(x_1,\cdots,x_n)$ formally calculable (uniformly) in $\psi_1,\cdots,\psi_l$, a natural number e can be found such that*

$$(26) \qquad \varphi(x_1,\cdots,x_n) \textit{ is defined} \rightleftarrows (Ey)T_n(\psi_1,\cdots,\psi_l,e,x_1,\cdots,x_n,y),$$

$$(27) \qquad \varphi(x_1,\cdots,x_n)\simeq U(\mu yT_n(\psi_1,\cdots,\psi_l,e,x_1,\cdots,x_n,y)),$$

and

$$(28) \qquad (x_1)\cdots(x_n)(y)[T_n(\psi_1,\cdots,\psi_l,e,x_1,\cdots,x_n,y)\rightarrow U(y)\simeq\varphi(x_1,\cdots,x_n)].$$

Furthermore, we have the following theorem.

THEOREM 4. *For each $l\geqq 0$, $m>0$ and $n>0$: Given any function $\varphi(x_1,\cdots,x_n)$ formally calculable in constant ordinals $\alpha_1,\cdots,\alpha_m$, and (uniformly) in functions $\psi_1,\cdots,\psi_l$, an ordinal e can be found such that*

$$(29) \qquad e=\nu(\alpha_1,\cdots,\alpha_m).$$

where ν is a primitive recursive function known from the syntactical form of the system $E(0_{\alpha_1}, \cdots, 0_{\alpha_m})$ of equations which defines $\varphi(x_1, \cdots, x_n)$ as a formally calculable function from $\alpha_1, \cdots, \alpha_m, \psi_1, \cdots, \psi_l$, (26), (27), and (28) hold. Thus, it holds that

$$\varphi(x_1, \cdots, x_n) \simeq U(\mu T_n(\psi_1, \cdots, \psi_l, \nu(\alpha_1, \cdots, \alpha_m), x_1, \cdots, x_n, y)). \tag{30}$$

When a function $\varphi(x_1, \cdots, x_n)$ is formally calculable uniformly in $\psi_1, \cdots, \psi_l$ (in $\alpha_1, \cdots, \alpha_m$, and uniformly in $\psi_1, \cdots, \psi_l$) we write this as $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$, and say $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$ to be formally calculable (in $\alpha_1, \cdots, \alpha_m$). Then, from the above theorems, if follows that *a formally calculable* (in $\alpha_1, \cdots, \alpha_m$) *function $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$ is partial recursive* (in $\alpha_1, \cdots, \alpha_m$). Unifying this with Theorem 1 (2), we have

THEOREM 5. *For each $l \geqq 0$ and $n > 0$: A function $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$ is partial recursive (in $\alpha_1, \cdots, \alpha_m$), if and only if it is formally calculable (in $\alpha_1, \cdots, \alpha_m$), when $\psi_1, \cdots, \psi_l$ range over the completely defined functions.*

To obtain the predicate $T_n^{n_1 \cdots n_l}(w_1, \cdots, w_l, z, x_1, \cdots, x_n, y)$ similar to Kleene's (cf. [**2**, 290-291]), we proceed as follows.

When ψ is a function variable with n arguments, let $\sigma_n(\psi, y)$ be the function defined by

$$\begin{aligned} \sigma_n(\psi, y) &= \mu w S(w, \{\psi(x)\}_{x<y}) && \text{for } n = 1, \\ &= \mu w S(w, \{\psi(x_1, \cdots, x_n)\}_{j(x_1, \cdots, j(x_{n-1}, x_n) \cdots) < y}) && \text{for each } n > 1. \end{aligned}$$

By the definition of the predicate S, we see immediately that, for each n:

$$\psi(x) = u(\sigma_1(\psi, y) \dagger j(0, x, 0)) \qquad \text{when} \quad x < y, \tag{31}$$

$$\psi(x_1, \cdots, x_n) = u(\sigma_n(\psi, y) \dagger j(0, j(x_1, \cdots, j(x_{n-1}, x_n) \cdots), 0)) \tag{32}$$

$$\text{when } j(x_1, \cdots, j(x_{n-1}, x_n) \cdots) < y,$$

and that $\sigma_n(\psi, y)$ is general recursive. We shall abbreviate functions $u(\sigma_1(\psi, y) \dagger j(0, x, 0))$, $u(\sigma_n(\psi, y) \dagger j(0, j(x_1, \cdots, j(x_{n-1}, x_n) \cdots), 0))$ $(n > 1)$ by $(\sigma_1(\psi, y))_x$, $(\sigma_n(\psi, y))_{x_1 \cdots x_n}$, respectively.

Using (31) or (32), we can write $\psi_i(u_1, \cdots, u_n)$ $(i = 1, \cdots, l)$ in (21) as $(\sigma_{n_i}(\psi_i, y))_{u_1 \cdots u_{n_i}}$; for, even if $n_i > 1$, we have not only $u_j < y$ $(j = 1, \cdots, n_i)$ but also $j(u_1, \cdots, j(u_{n_i-1}, u_{n_i}) \cdots) < y$, as is easily seen from the definitions. Hence, we obtain the predicate $D^{n_1 \cdots n_l}(w_1, \cdots, w_l, z, y)$ instead of $D(\psi_1, \cdots, \psi_l, z, y)$ by replacing $\psi_i(u_1, \cdots, u_{n_i})$ in (21) by $(w_i)_{u_1 \cdots u_{n_i}}$ for each $i = 1, \cdots, l$. Then, using $D^{n_1 \cdots n_l}(w_1, \cdots, w_l, z, y)$ in place of $D(\psi_1, \cdots, \psi_l, z, y)$, we have a primitive recursive predicate $S_n^{n_1 \cdots n_l}(w_1, \cdots, w_l, z, x_1, \cdots, x_n, y)$ for each $l > 0$, $n > 0$ such that

$$\begin{aligned} &S_n^{n_1 \cdots n_l}(\sigma_{n_1}(\psi_1, y), \cdots, \sigma_{n_l}(\psi_l, y), z, x_1, \cdots, x_n, y) \\ &\qquad \leftrightarrows S_n(\psi_1, \cdots, \psi_l, z, x_1, \cdots, x_n, y). \end{aligned}$$

Let $T_n^{n_1\cdots n_l}(w_1, \cdots, w_l, z, x_1, \cdots, x_n, y)$ be the predicate

$$S_n^{n_1\cdots n_l}(w_1, \cdots, w_l, z, x_1, \cdots, x_n, y) \wedge (t)_{t<y} \neg S_n^{n_1\cdots n_l}(w_1, \cdots, w_l, z, x_1, \cdots, x_n, t).$$

Using this, we have by the normal form theorem that, *given a partial recursive function* $\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n)$, *a natural number* e *can be found such that*

$$\varphi(\psi_1, \cdots, \psi_l, x_1, \cdots, x_n) \simeq U(\mu y T_n^{n_1\cdots n_l}(\sigma_{n_1}(\psi_1, y), \cdots, \sigma_{n_l}(\psi_l, y), e, x_1, \cdots, x_n, y)).$$

This shows the following: For example, let $\omega_\gamma = \omega_1$, $l=1$ and $n_1 = 1$. Given a partial recursive function $\varphi(\psi, x_1, \cdots, x_n)$, for each $\psi, x_1, \cdots, x_n$, if $\varphi(\psi, x_1, \cdots, x_n)$ is defined, its value *depends only on at most countably many values* $\psi(z)$ for the function argument ψ.

REMARK. $T_n(\psi_1, \cdots, \psi_l, z, x_1, \cdots, x_n, y)$, $T_n^{n_1\cdots n_l}(w_1, \cdots, w_l, z, x_1, \cdots, x_n, y)$ are primitive recursive, but, unfortunately, $T_n^{n_1\cdots n_l}(\sigma_{n_1}(\psi_1, y), \cdots, \sigma_{n_l}(\psi_l, y), z, x_1, \cdots, x_n, y)$ is general (not primitive, at present) recursive, in contrast to the Kleene's case (cf. [**2**, 290-291]).

4.2. On account of Theorem 5, we say, following to Kleene's terminology, that any natural (ordinal) number e such that (27) holds *defines* φ *recursively* (*in* $\alpha_1, \cdots, \alpha_m$) or is a *Gödel number of* φ (*from* $\alpha_1, \cdots, \alpha_m$). Now we have a counterpart of [**2**, Theorem XXIII] for our case.

THEOREM 6. *For each* $m, n > 0$, *there is a primitive recursive function* $Sb_n^m(z, y_1, \cdots, y_m)$ *such that, if* e *defines recursively* $\lambda y_1 \cdots y_m x_1 \cdots x_n \varphi(y_1, \cdots, y_m, x_1, \cdots, x_n)$, *then, for each fixed m-tuple* $y_1, \cdots, y_m$ *of ordinal numbers,* $Sb_n^m(e, y_1, \cdots, y_m)$ *defines recursively* $\lambda x_1 \cdots x_n \varphi(y_1, \cdots, y_m, x_1, \cdots, x_n)$ *in* $y_1, \cdots, y_m$, *and, when* $z, y_1, \cdots, y_m$ *are natural numbers, so is* $Sb_n^m(z, y_1, \cdots, y_m)$.

Similarly for the case $l > 0$.

PROOF. Similar to the proof of [**2**, Theorem XXIII]. By the definition, $\varphi(y_1, \cdots, y_m, x_1, \cdots, x_n) \simeq U(\mu y T_{m+n}(e, y_1, \cdots, y_m, x_1, \cdots, x_n, y))$. Since $\lambda z y_1 \cdots y_m x_1 \cdots x_n U(\mu y T_{m+n}(z, y_1, \cdots, y_m, x_1, \cdots, x_n, y))$ is partial recursive, it is formally calculable by Theorem 1. Then, we can find a system of equations defining that function as a formally calculable function. Let D be such a system, and g denote its principal function letter. Now, we choose a function letter f which does not occur in D. For any choice of ordinals $z, y_1, \cdots, y_m$, let C consist of the equations of D followed by the equation

$$\mathrm{f}(\mathrm{a}_1, \cdots, \mathrm{a}_n) = \mathrm{g}(0_z, 0_{y_1}, \cdots, 0_{y_m}, \mathrm{a}_1, \cdots, \mathrm{a}_n).$$

Let $d, f, g, a_1, \cdots, a_n$ be the Gödel numbers of $D, \mathrm{f}, \mathrm{g}, \mathrm{a}_1, \cdots, \mathrm{a}_n$, respectively. Then, the Gödel number of C is the ordinal

$$j(1, j([d]_1, \cdots, j([d]_{s-1}, j(\nu(s, d), j(8, j(j(f, j(a_1, \cdots, j(a_{n-1}, a_n) \cdots)),$$
$$j(g, j(j(3, z), j(j(3, y_1), \cdots, j(j(3, y_m), j(a_1, \cdots, j(a_{n-1}, a_n) \cdots)) \cdots))))))) \cdots))$$

where $s=\mu x_{x<d}Eq(\nu(x,d))$. We write this as $Sb_n^m(z, y_1, \cdots, y_m)$. $Sb_n^m(z, y_1, \cdots, y_m)$ is a desired function. Indeed, $Sb_n^m(z, y_1, \cdots, y_m)$ is primitive recursive and, for each fixed m-tuple $y_1, \cdots, y_m$ of ordinals, $Sb_n^m(e, y_1, \cdots, y_m)$ defines $\lambda x_1 \cdots x_n \varphi(y_1, \cdots, y_m, x_1, \cdots, x_n)$ recursively in $y_1, \cdots, y_m$.

COROLLARY. *For each natural number $n>0$: If a function $\varphi(x_1, \cdots, x_n)$ is partial recursive in $\alpha_1, \cdots, \alpha_m$, then a natural number e can be found such that $Sb_n^m(e, \alpha_1, \cdots, \alpha_m)$ is a Gödel number of φ from $\alpha_1, \cdots, \alpha_m$.*

THE OUTLINE OF THE PROOF. Given a function $\varphi(x_1, \cdots, x_n)$ partial recursive in $\alpha_1, \cdots, \alpha_m$, we can find, by Theorem 2, a system $E(0_{\alpha_1}, \cdots, 0_{\alpha_m})$ of equations which defines φ as a function formally calculable from $\alpha_1, \cdots, \alpha_m$. Choose variables $\mathrm{y}_1, \cdots, \mathrm{y}_m$ and a function letter f not occurring in $E(0_{\alpha_1}, \cdots, 0_{\alpha_m})$. In each equation of $E(0_{\alpha_1}, \cdots, 0_{\alpha_m})$, change simultaneously each part $\mathrm{h}(\mathrm{r}_1, \cdots, \mathrm{r}_p)$ to $\mathrm{h}(\mathrm{r}_1, \cdots, \mathrm{r}_p, \mathrm{y}_1, \cdots, \mathrm{y}_m)$ and then each symbol 0_{α_i} $(i=1, \cdots, m)$ to the variable y_i, respectively. To the system of equations thus obtained, add the equation

$$\mathrm{f}(\mathrm{y}_1, \cdots, \mathrm{y}_m, \mathrm{a}_1, \cdots, \mathrm{a}_n)=\mathrm{g}(\mathrm{a}_1, \cdots, \mathrm{a}_n, \mathrm{y}_1, \cdots, \mathrm{y}_m),$$

where g is the principal function letter of $E(0_{\alpha_1}, \cdots, 0_{\alpha_m})$, as the last equation. Let E be the resulting system. The Gödel number e is a desired one; i.e. $e<\omega$ and e defines $\lambda y_1 \cdots y_m x_1 \cdots x_n \varphi(y_1, \cdots, y_m, x_1, \cdots, x_n)$ recursively, where

$$\varphi(x_1, \cdots, x_n) \simeq \varphi(\alpha_1, \cdots, \alpha_m, x_1, \cdots, x_n).$$

As the other corollary to Theorem 6, we have the following recursion theorem (cf., e.g., [**2**, Theorem XXVII]).

THEOREM 7. *For each natural number $n>0$: Given any partial recursive function $\psi(z, x_1, \cdots, x_n)$, a natural number e can be found which defines $\varphi(x_1, \cdots, x_n)$ recursively, where*

$$\varphi(x_1, \cdots, x_n) \simeq \psi(e, x_1, \cdots, x_n).$$

The proof is similar to that of Theorem XXVII in [**2**], and $e=Sb_n^1(f,f)<\omega$, where f is a natural number which defines $\lambda y x_1 \cdots x_n \psi(Sb_n^1(y,y), x_1, \cdots, x_n)$ recursively.

4.3. We can apply usefully the predicates $T_n(z, x_1, \cdots, x_n, y)$ $(n=1, 2, \cdots)$ to develop the theory of hierarchy $\{\sum_k^{ord}, \prod_k^{ord}\}_{k=1,2,\cdots}$ (cf. [8, § 5]), built on the quantified forms of the predicates of ordinals which are expressible syntactically by starting with general recursive predicates and using the symbolism of the first order predicate calculus. Hence, we have a version or another easy proof for each of the enumeration ([8, Corollary to Theorem 4]), hierarchy ([8, Theorem 5]), Post's Theorem ([8, Theorem 3]), etc. of [8, §§ 5, 6].

For example, given any general recursive predicate $R(a,b)$ of two variables, consider the partial recursive function $\mu x R(a,x)$. Then, by the definition,

$$\mu x R(a, x) \text{ is defined} \rightleftarrows (Ex)R(a, x).$$

By Theorems 1, 3, a natural number e can be found such that it defines $\mu y R(x, y)$ recursively. Hence, using (26) of Theorem 3,

$$(33) \qquad (Ex)R(a, x) \rightleftarrows (Ex)T_1(e, a, x).$$

Thus, we see: *The predicate* $(Ex)[z < \omega \wedge T_1(z, a, x)]$ *enumerates*[9] *the predicates* $P(a)$ *of the form* $\sum_1^{ord}$ *with general recursive scope.* Similarly we have the enumerating predicates for the classes of predicates of the other forms in $\{\sum_k^{ord}, \prod_k^{ord}\}_{k=1,2,\cdots}$ From this, it follows immediately that *the class of the predicates of the form* $\sum_k^{ord}$ (or $\prod_k^{ord}$) *is the same whether a general recursive or only a primitive recursive predicate be allowed as the scope.* (cf. [**8**, the second part of Theorem 3]).

Now, we can give another proof of Theorem 5 (the hierarchy theorem) of [8]. Indeed, for the form $\sum_1^{ord}$ it suffices to take the predicate $(Ex)T_1(a, a, x)$. This is evidently of the form $\sum_1^{ord}$, but it can not be expressed in the dual form $\prod_1^{ord}$. Similarly for the other forms. Furthermore, we add the complete form theorem (cf., e.g., [**3**, VII]).

THEOREM 8. *The predicate* $(Ex)T_1(a, a, x)$ *is a complete predicate*[10] *of the form* $\sum_1^{ord}$. *Similarly for the other forms in* $\{\sum_k^{ord}, \prod_k^{ord}\}_{k=1,2,\cdots}$

PROOF. Given any predicate $(Ex)R(a_1, \cdots, a_n, x)$ with general recursive scope R, consider the function $\lambda a_1 \cdots a_n z \mu x R(a_1, \cdots, a_n, x)$. Since this function is partial recursive, there is a Gödel number, say e $(e < \omega)$, of it. Then, for any fixed n-tuple $a_1, \cdots, a_n$ of ordinals, $Sb_1^n(e, a_1, \cdots, a_n)$ is a Gödel number of the function $\lambda z \mu x R(a_1, \cdots, a_n, x)$, which is defined if and only if $(Ex)R(a_1, \cdots, a_n, x)$ for any z. Hence, by Theorem 4, we have

$$(Ex)R(a_1, \cdots, a_n, x) \rightleftarrows (Ex)T_1(Sb_1^n(e, a_1, \cdots, a_n), z, x)$$
$$\rightleftarrows (Ex)T_1(Sb_1^n(e, a_1, \cdots, a_n), Sb_1^n(e, a_1, \cdots, a_n), x)$$

(by substituting $Sb_1^n(e, a_1, \cdots, a_n)$ for z).

We consider the predicates obtained by using 'general recursive in the classical sense' or 'general recursive in the ordinals $\alpha_1, \cdots, \alpha_m$' in place of 'general recursive' in the definition of the (gr)-predicates (see [**8**, p. 206]), and denote the corresponding quantified form (or the class of the predicates of that form) by $\sum_k^{ord}(\omega_T)$ ($\prod_k^{ord}(\omega_T)$) or $\sum_k^{ord, \alpha_1, \cdots, \alpha_m}$ ($\prod_k^{ord, \alpha_1, \cdots, \alpha_m}$), respectively.

The following will be remarkable.

The predicate $(Ex)T_1(z, a, x)$ ($(Ex)[z < \omega \wedge T_1(Sb_1^m(z, \alpha_1, \cdots, \alpha_m), a, x)]$) *enumerates the predicates of the form* $\sum_1^{ord}(\omega_T)(\sum_1^{ord, \alpha_1, \cdots, \alpha_m})$, *and the predicate* $(Ex)T_1(a, a, x)$, *which is of the form* $\sum_1^{ord}$ *and is a complete predicate of it, is*

9) Cf. [**2**, Discussion of § 57, p. 282].

10) For the terminology 'complete predicate', e.g., see [**3**, p. 196].

a complete predicate also of the predicates in $\sum_1^{ord}(\omega_\gamma)$. *Similarly for the other forms.*

In § 4.2-3, we treated only the case where the number l of function variables $=0$. We remark that one can extend that to the case $l>0$, without any difficulty.

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