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On the sample paths of the symmetric stable processes in spaces

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§ 1. Introduction and Summary.

Let $\{X(t); t \geqq 0\}$ be the symmetric stable process in R^N of index α with $0 < \alpha \leqq 2$; that is, a stochastic process with stationary independent increments such that continuous transition density, $f(t, x-y)$, relative to the Lebesgue measure in R^N is uniquely determined by its Fourier transform

$$e^{-t|\xi|^{\alpha}} = \int_{R^N} e^{i(x,\xi)} f(t, x) dx .$$

Here ξ and x are points in R^N, dx is the N-dimensional Lebesgue measure. The notation (x, ξ) means the usual inner product in R^N and $|x| = (x, x)^{1/2}$ is the usual Euclidean norm. When $\alpha = 2$, $X(t/2)$ is the standard Brownian motion process. It is well-known that we may assume that the sample functions of X have certain regularity properties. To be precise we may assume that almost all sample functions have right continuity and left-hand limits everywhere and are bounded on each bounded parameter set. We will write P_x and E_x for the conditional probability and expectation under the condition $X(0) = x$. Unless otherwise stated we assume $X(0) = 0$ with probability one, and we use the abbreviations $P_0 = P$ and $E_0 = E$. Our process is defined over some basic probability space $(W, \mathfrak{B}, P)$. We will often suppress the w's in our notation.

The main purpose of the present paper is to discuss some properties of the path functions in which the dimension number N plays an essential rôle. From this point of view, the pioneers were G. Pólya and S. Kakutani. In case of the stable process, such properties vary also according to the index α and were investigated by H. P. McKean [**12**] and others. For example, our process is recurrent if $N \leqq \alpha$, while if $N > \alpha$ then it is non-recurrent.

The first problem is concerned with the speed of wandering off to infinity in the transient case, that is, the problem of giving a characterization of upper or lower classes of monotone decreasing functions which limit the speed. It is an extension of the result by Dvoretzky-Erdös [**6**]. Our method is based on the calculation of the capacity for a ball and the fact that the hitting probability for a fixed compact set is expressible as an equilibrium Riesz potential. It turns out that the speed of escape is the quicker, the less the index α is or the larger the dimension N is.

The second topic is whether double points of the paths exist or not. For the Brownian motion case it is settled by Dvoretzky-Erdös-Kakutani [7]. Our assertion is that the symmetric stable processes have double points with probability one, if and only if $N/2<\alpha\leqq 2$. This is derived mainly from the research concerning the Hausdorff measure of the stable path due to Blumenthal-Getoor [2].

The third problem is to examine the Lebesgue measure of the range of the sample function. Although almost all Cauchy paths in R^1 constitute everywhere dense sets, their one-dimensional measures are equal to zero.

In a monograph [16] written in Japanese, the detailed proofs of the second and the third problems were published in July, 1962. Afterwards the author was informed by R. M. Blumenthal that S. J. Taylor also had discovered the theorem on double points in 1961. However Taylor's paper has not yet been printed. So the present author will give his own method in English.

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§ 2. The criterion of upper or lower classes.

In this section we discuss symmetric stable processes $X(t, w)$ in R^N with index $\alpha<N$, namely only non-recurrent case. In this case

$$P(\lim_{t\to\infty}|X(t, w)|=+\infty)=1$$

holds and this fact is shown by H. P. McKean [12] in one-dimensional case. We begin with the definition of upper class and lower class with respect to a stable process with index α. For any positive monotone non-increasing function $g(t)$, we put

$$F(w)=\{t\,;\,|X(t, w)|\leqq g(t)t^{1/\alpha}\}$$

and, if the set $F(w)$ of t is unbounded (bounded) for almost all w, then we say that $g(t)$ belongs to the upper class $\boldsymbol{U}$ (lower class $\boldsymbol{L}$) with respect to $X(t)$. Obviously from this definition, we obtain that if $g(t)\in\boldsymbol{U}$ then

$$P\left(\varliminf_{t\to\infty}\frac{|X(t, w)|}{g(t)t^{1/\alpha}}\leqq 1\right)=1,$$

and if $g(t)\in\boldsymbol{L}$ then

$$P\left(\varliminf_{t\to\infty}\frac{|X(t, w)|}{g(t)t^{1/\alpha}}\geqq 1\right)=1.$$

In other words, our problem is to study the order of the limes inferiores at time point ∞ for the oscillation of path functions. The similar matter on the limes superiores was already solved by A. Khintchine [**10**].

We shall give a criterion of lower and upper classes.

THEOREM 1. *Let $N>\alpha$. A positive monotone function $g(t)$ belongs to the lower class **L** or to the upper class **U** according as the integral*

$$\int^{\infty}\frac{1}{t}\{g(t)\}^{N-\alpha}dt \tag{1}$$

converges or diverges.

Applying Theorem 1, we easily obtain the following

COROLLARY 1. *The function*

$$g(t)=\frac{1}{(\log t)^{\frac{1+\delta}{N-\alpha}}}$$

*belongs to **U** for $\delta\leqq 0$ and belongs to **L** for $\delta>0$.*

COROLLARY 2. *For any $\delta>0$*

$$P\left(|X(t)|\geqq\frac{t^{1/\alpha}}{(\log t)^{\frac{1+\delta}{N-\alpha}}}\ \textit{for } t \textit{ large enough}\right)=1\,.$$

Hence the less the index α is or the larger the dimension N is, the quicker is the speed of wandering out to infinity, and if $N>\alpha$ almost every path approaches ∞ when $t\to\infty$.

First we give some lemmas.

LEMMA 1. *The capacity of the order $(N-\alpha)$ for a ball E with radius r is in proportion to $r^{N-\alpha}$.*

PROOF. M. Riesz [**13**] has computed that the Riesz equilibrium distribution of the exponent α in R^N for a ball E is

$$\frac{\sin\frac{\pi\alpha}{2}\Gamma\left(\frac{N}{2}\right)}{\pi^{N/2+1}}\cdot\frac{dy}{(r^2-|y|^2)^{\alpha/2}}\,,\qquad 0<\alpha<2\,.$$

The capacity being the total mass of this distribution, we obtain, by

$$\begin{aligned}\int_{|y|\leqq r}\frac{dy}{(r^2-|y|^2)^{\alpha/2}}&=\omega_N\int_0^r\frac{\rho^{N-1}d\rho}{(r^2-\rho^2)^{\alpha/2}}\\&=\omega_N\cdot r^{N-\alpha}\int_0^{\pi/2}\cos^{N-1}\theta\,\sin^{-\alpha+1}\theta\,d\theta\end{aligned}$$

with $\omega_N=2\pi^{N/2}\left[\Gamma\left(\frac{N}{2}\right)\right]^{-1}$, the value of the capacity as follows:

$$C^{N-\alpha}(E)=\frac{\sin\frac{\pi\alpha}{2}\Gamma\left(\frac{N}{2}\right)}{\pi^{N/2+1}}\int_{|y|\leqq r}\frac{dy}{(r^2-|y|^2)^{\alpha/2}}=\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{N-\alpha+2}{2}\right)}\cdot r^{N-\alpha}\,.$$

Another proof which covers the case $\alpha=2$ is as follows. The probability that a symmetric stable process starting from x meets E for some $t>0$ is the value of the equilibrium Riesz potential of E at x. This fact is established by the general theory of G. A. Hunt. That is

$$P_x(\sigma_E<+\infty)^{1)}=\frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{2^\alpha \pi^{N/2}\Gamma\left(\frac{\alpha}{2}\right)}\int_E \frac{1}{|x-y|^{N-\alpha}}\mu_E(dy),$$

where μ_E is the equilibrium distribution of E.
This yields, putting $c_1{}^{2)}=\Gamma\left(\frac{N-\alpha}{2}\right)\cdot\left[2^\alpha\pi^{N/2}\Gamma\left(\frac{\alpha}{2}\right)\right]^{-1}$,

$$(2)\qquad \frac{c_1}{\max\limits_{y\in E}|x-y|^{N-\alpha}}C^{N-\alpha}(E)\leqq P_x(\sigma_E<+\infty)\leqq\frac{c_1}{\min\limits_{y\in E}|x-y|^{N-\alpha}}C^{N-\alpha}(E).$$

In view of the well-known scaling relationship

$$P_x(rX(t)\in E)=P_{rx}(X(r^\alpha t)\in E),$$

we get

$$\begin{aligned}C^{N-\alpha}(E)&=\lim_{|x|\to\infty}\frac{1}{c_1}|x|^{N-\alpha}P_x(\sigma_E<+\infty)\\&=\lim_{|x|\to\infty}\frac{1}{c_1}|x|^{N-\alpha}P_{x/r}(\sigma_B<+\infty)\\&=\lim_{|y|\to\infty}\frac{1}{c_1}r^{N-\alpha}|y|^{N-\alpha}P_y(\sigma_B<+\infty)\\&=r^{N-\alpha}\cdot C^{N-\alpha}(B)\end{aligned}$$

where B is a unit ball.

LEMMA 2. *Suppose* $|x|>r$,

$$c_2\left(\frac{r}{|x|+r}\right)^{N-\alpha}\leqq P_x(|X(t)|\leqq r\ \textit{for some}\ t>0)\leqq c_2\left(\frac{r}{|x|-r}\right)^{N-\alpha}.$$

PROOF. Let $B(x, r)$ denote the sphere with center x and radius r $(r<|x|)$. Then the probability in question obviously equals the probability that $X(t)$ ever enters $B(x, r)$ starting from 0. Thus

$$\begin{aligned}P_0(X(t)\in B(x,r)\ \text{for some}\ t>0)&=c_1\int_{B(x,r)}\frac{1}{|y|^{N-\alpha}}\mu_B(dy)\\&\leqq c_1\frac{1}{(|x|-r)^{N-\alpha}}C^{N-\alpha}(B)=c_2\frac{r^{N-\alpha}}{(|x|-r)^{N-\alpha}}.\end{aligned}$$

Another inequality is also obtained similarly.

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1) Throughout this paper $\sigma_A=\inf\{t>0;X(t)\in A\}$, i.e., σ_A is the first passage time for set A.

2) $c_1, c_2, \cdots, c_{11}$ appearing in the sequel are absolute positive constants.

LEMMA 3. *For any* $T \geqq 0$ *and* $r>0$, *put*

$$Q(r, T)=P(|X(t)| \leqq r \text{ for some } t>T).$$

Then if $T^{1/\alpha} \geqq r$, *we have*

$$Q(r, T) \geqq c_3\left(\frac{r}{T^{1/\alpha}}\right)^{N-\alpha}.$$

PROOF. Let $p(x, t)$ denote the probability density of $X(t)$ and let G be $B(0, r)$. Then

$$\begin{aligned}
Q(r, T) &= P_0(\sigma_G(w_T^+)<+\infty)^{3)}=\int_W P_{X(T,w)}(\sigma_G<+\infty)P_0(dw)\\
&\geqq \int_{|y|>r} c_2\left(\frac{r}{|y|+r}\right)^{N-\alpha} p(y, T)dy\\
&= c_2 r^{N-\alpha}E((|X(T)|+r)^{-N+\alpha}; |X(T)|>r)^{4)}\\
&= c_2 r^{N-\alpha}E((|T^{1/\alpha}X(1)|+r)^{-N+\alpha}; |T^{1/\alpha}X(1)|>r)\\
&= c_2\left(\frac{r}{T^{1/\alpha}}\right)^{N-\alpha} E((|X(1)|+rT^{-1/\alpha})^{-N+\alpha}; |X(1)|>rT^{-1/\alpha})\\
&= c_2\left(\frac{r}{T^{1/\alpha}}\right)^{N-\alpha}\int_{|y|>rT^{-1/\alpha}}(|y|+rT^{-1/\alpha})^{-N+\alpha}p(y, 1)dy.
\end{aligned}$$

The last expression above is larger than

$$c_2\left(\frac{r}{2T^{1/\alpha}}\right)^{N-\alpha}\int_{|y|>rT^{-1/\alpha}}|y|^{-N+\alpha}p(y, 1)dy,$$

since $|y|+rT^{-1/\alpha} \leqq 2|y|$ provided that $|y|>rT^{-1/\alpha}$. Recalling the assumption $rT^{-1/\alpha} \leqq 1$ and the fact that $X(1)$ has a bounded continuous density, we obtain

$$Q(r, T) \geqq c_2\left(\frac{r}{2T^{1/\alpha}}\right)^{N-\alpha}\int_{|y|>1}|y|^{-N+\alpha}p(y, 1)dy=c_3\left(\frac{r}{T^{1/\alpha}}\right)^{N-\alpha}.$$

LEMMA 4. $$Q(r, T) \leqq c_4\left(\frac{r}{T^{1/\alpha}}\right)^{N-\alpha}.$$

PROOF. Note first that

$$\begin{aligned}
P_y(|X(t)| \leqq r \text{ for some } t) &\leqq \min\left\{1, c_2\left(\frac{r}{|y|-r}\right)^{N-\alpha}\right\}\\
&\leqq c_2'\left(\frac{r}{||y|-r|}\right)^{N-\alpha}.
\end{aligned}$$

Similarly as Lemma 3, we have

$$Q(r, T)=\int_W P_{X(T)}(|X(t)| \leqq r \text{ for some } t)P_0(dw)$$

3) w_s^+ denotes the shifted path at time s, i.e., $X(t, w_s^+)=X(t+s, w)$.

4) $E_x(f(w); w \in B)=\int_B f(w)P_x(dw)$.

$$\leqq c_2' r^{N-\alpha}\int_{R^N} ||y|-r|^{-N+\alpha} p(y, T)dy$$

$$= c_2'\left(\frac{r}{T^{1/\alpha}}\right)^{N-\alpha}\int_{R^N} ||y|-rT^{-1/\alpha}|^{-N+\alpha} p(y, 1)dy .$$

Putting $|x|=||y|-rT^{-1/\alpha}|$, we obtain

$$c_2'\left(\frac{r}{T^{1/\alpha}}\right)^{N-\alpha}\left[\int_{|x|\geqq 1}+\int_{|x|<1}|x|^{-N+\alpha}p(y, 1)dy\right]$$

$$\leqq c_2'\left(\frac{r}{T^{1/\alpha}}\right)^{N-\alpha}\left(1+c_5\int_{|x|<1}|x|^{-N+\alpha}dx\right)=c_4\left(\frac{r}{T^{1/\alpha}}\right)^{N-\alpha}.$$

LEMMA 5. *For arbitrary $T\geqq 0$, $r>0$ and $K>1$, set*

$$R(r, T, K)=P(|X(t)|\leqq r \text{ for some } T\leqq t\leqq KT).$$

Then there exist a positive integer c and a positive constant c_6 such that

$$R(r, T, K)>c_6\left(\frac{r}{T^{1/\alpha}}\right)^{N-\alpha}$$

holds for every $T^{1/\alpha}\geqq r$ and $K\geqq c$.

PROOF. We obviously have

$$R(r, T, K)\geqq Q(r, T)-Q(r, TK).$$

Because of Lemmas 3 and 4, we see

$$R(r, T, K)\geqq\left(\frac{r}{T^{1/\alpha}}\right)^{N-\alpha}\left\{c_3-c_4\left(\frac{1}{K}\right)^{\frac{N-\alpha}{\alpha}}\right\}.$$

Let c be greater than $\left(\frac{c_4}{c_3}\right)^{\frac{\alpha}{N-\alpha}}$. Then the expression within the braces has a positive value. This implies the truth of the lemma.

We are now in a position to prove Theorem 1.

First we assume the convergence of the integral (1) and show that $g\in \boldsymbol{L}$. Lemma 4 gives, because of the monotonicity of $g(t)$,

$$\sum_{j=1}^{\infty} P(|X(t)|\leqq g(t)t^{1/\alpha} \text{ for some } 2^j<t\leqq 2^{j+1})$$

$$\leqq \sum_{j=1}^{\infty} P(|X(t)|\leqq g(2^j)(2^{j+1})^{1/\alpha} \text{ for some } t>2^j)$$

$$\leqq c_4\sum_{j=1}^{\infty}(2^{1/\alpha}g(2^j))^{N-\alpha}\leqq c_4 2^{N/\alpha}\int_1^{\infty}\frac{1}{t}g^{N-\alpha}(t)dt<+\infty .$$

Thus it follows from the Borel-Cantelli lemma that $g(t)$ belongs to the lower class.

Next let us consider the case in which the integral (1) for $g(t)$ is divergent. To prove the remaining case, we use the following extension of the Borel-Cantelli lemma due to Chung-Erdös [4].

LEMMA 6. *Let $\{E_i\}$ be an infinite sequence of events satisfying the following conditions:*

(i) $$\sum_{i=1}^{\infty} P(E_i) = +\infty\,;$$

(ii) *For every pair of positive integers h and n satisfying $n \geqq h$, there exist $C(h) > 0$ and $H(n, h) > n$ such that for every $m \geqq H(n, h)$ holds the inequality*

$$P(E_m / E'_h \cap E'_{h+1} \cap \cdots \cap E'_n) > C(h) P(E_m)$$

where $P(F/E)$ denotes the conditional probability of F under the hypothesis E and E' denotes the complement of E;

(iii) *There exist two absolute constants c_7 and c_8 with the following property: to each E_j there corresponds a set of events $E_{j_1}, \cdots, E_{j_s}$ belonging to $\{E_j\}$ such that*

$$\sum_{k=1}^{s} P(E_j \cap E_{j_k}) < c_7 P(E_j)$$

and that for any other E_i than E_{j_k} $(1 \leqq k \leqq s)$ which stands after E_j in the sequence (viz. $i > j$)

$$P(E_j \cap E_i) < c_8 P(E_j) P(E_i)\,.$$

Then infinitely many events E_i occur with probability 1.

Further we shall make use of the following

LEMMA 7. *Without loss of generality, we may assume that*

(3) $$\frac{1}{(\log t)^2} \leqq \{g(t)\}^{N-\alpha} \leqq 1\,.$$

PROOF. We assume that Theorem 1 holds for $g(t)$ satisfying (3) and put

(4) $$\hat{g}(t) = \max\{\min(g(t), 1), g_1(t)\}\,,$$

where $g_1(t) = (\log t)^{-\frac{2}{N-\alpha}}$. Clearly $\hat{g}(t)$ satisfies the condition (3).

For our purpose it suffices to prove the case in which the integral (1) for $g(t)$ is divergent. If there exists an increasing sequence $\{t_n\}$ such that $g(t_n) > 1$ and t_n tends to infinity with n, we have

$$\int_{t_1}^{\infty} \frac{1}{t} \{\hat{g}(t)\}^{N-\alpha} dt \geqq c_9 (\log t_n) \cdot \{\hat{g}(t_n)\}^{N-\alpha} \geqq c_9 \log t_n\,.$$

On the contrary, if $g(t) \leqq 1$ for sufficiently large t, it follows that $g(t) \leqq \hat{g}(t)$ for large t. Therefore we obtain $\hat{g}(t) \in \boldsymbol{U}$ in either case. Namely for almost all paths we can choose a sequence $\{t_n\}$ having the properties:

(5) $$|X(t_n)| \leqq t_n^{1/\alpha} \hat{g}(t_n)\,, \qquad t_n \to \infty \text{ as } n \to \infty\,.$$

On the other hand, $g_1(t) \in \boldsymbol{L}$ and so with probability 1

$$|X(t)| > t^{1/\alpha} g_1(t)$$

holds for large t. Hence $g_1(t_n) < \hat{g}(t_n)$ and also $\hat{g}(t_n) \leqq g(t_n)$. The last inequality and (5) show that $g(t) \in \boldsymbol{U}$ and the lemma follows.

Define the events

$$E_i = \{w\,;\, |X(t)| \leqq g(c^{i+1})c^{i/\alpha} \text{ for some } c^i \leqq t \leqq c^{i+1}\} \qquad i = 1, 2, \cdots\cdots,$$

where c is the integer that appeared in Lemma 5. It is seen by Lemmas 5 and 7 and choosing a certain constant c_{10}

$$\sum_{i=1}^{\infty} P(E_i) > c_6 \sum_{i=1}^{\infty} \{g(c^{i+1})\}^{N-\alpha} \geqq c_{10} \int_1^{\infty} \frac{1}{t} g^{N-\alpha}(t)dt = \infty\,.$$

This implies condition (i) of Lemma 6.

To deduce condition (ii), denote by F_n the event

$$\{w\,;\, |X(c^{n+1})| \leqq a_{n,h}\}\,,$$

where $a_{n,h}$ is a constant such that for given $h < n$ we have

$$P(E_h' \cap \cdots\cdots \cap E_n' \cap F_n) > \frac{1}{2} P(E_h' \cap \cdots\cdots \cap E_n')\,. \tag{6}$$

Suppose $h < n < m$. Then we find by (6)

$$\begin{aligned} P(E_m/E_h' \cap \cdots\cdots \cap E_n') &= \frac{P(E_m \cap E_h' \cap \cdots\cdots \cap E_n')}{P(E_h' \cap \cdots\cdots \cap E_n')} \\ &> \frac{P(E_m \cap E_h' \cap \cdots\cdots \cap E_n' \cap F_n)}{2P(E_h' \cap \cdots\cdots \cap E_n' \cap F_n)} = \frac{1}{2} P(E_m/E_h' \cap \cdots\cdots \cap E_n' \cap F_n)\,. \end{aligned}$$

Now it follows from the Markov property of the given process that

$$\begin{aligned} &P(E_m \cap E_h' \cap \cdots\cdots \cap E_n' \cap F_n) \\ &\geqq P(|X(t - c^{n+1})| \leqq g(c^{m+1})c^{m/\alpha} - a_{n,h} \text{ for some } c^m \leqq t \leqq c^{m+1}) \\ &\qquad \times P(E_h' \cap \cdots\cdots \cap E_n' \cap F_n)\,. \end{aligned}$$

Therefore

$$\begin{aligned} &P(E_m/E_h' \cap \cdots\cdots \cap E_n' \cap F_n) \\ &\geqq P(|X(t - c^{n+1})| \leqq g(c^{m+1})c^{m/\alpha} - a_{n,h} \text{ for some } c^m \leqq t \leqq c^{m+1})\,. \end{aligned}$$

Applying Lemmas 5 and 4 and letting m be sufficiently large, we see that the last expression above is larger than

$$c_6\{g(c^{m+1}) - a_{n,h}c^{-m/\alpha}\}^{N-\alpha} > \frac{1}{2} c_6 \{g(c^{m+1})\}^{N-\alpha} > \frac{c_6}{2} \frac{1}{c_4} P(E_m)\,.$$

Hence we conclude

$$P(E_m/E_h' \cap \cdots\cdots \cap E_n') > \frac{1}{4} \frac{c_6}{c_4} P(E_m)\,.$$

Finally the verification of condition (iii) may be carried out as follows. Let us define

$$\sigma_j(w)=\begin{cases} \inf\{t;\,|X(t)|\leqq g(c^{j+1})c^{j/\alpha},\ c^j\leqq t\leqq c^{j+1}\} & \text{if there is such } t \\ c^{j+1}+1 & \text{otherwise.}\end{cases}$$

For any $j<i$, it follows from the strong Markov property of the process that

$$P(E_j\cap E_i)=P(\sigma_j\leqq c^{j+1},\ \sigma_i\leqq c^{i+1})$$

$$(7)\qquad =\int_{c^j}^{c^{j+1}}P(|X(t)|\leqq g(c^{i+1})c^{i/\alpha}\ \text{for some}\ c^i\leqq t\leqq c^{i+1}/\sigma_j=s)P(\sigma_j\in ds)$$

$$\leqq P(|X(t)|\leqq g(c^{i+1})c^{i/\alpha}+g(c^{j+1})c^{j/\alpha}\ \text{for some}\ c^i-c^{j+1}\leqq t)P(\sigma_j\leqq c^{j+1}).$$

When $i-j$ is sufficiently large, we can show from Lemma 7 that

$$(8)\qquad \frac{g(c^{j+1})c^{j/\alpha}}{(c^i-c^{j+1})^{1/\alpha}}\leqq\Big(\frac{c^j}{c^i-c^{j+1}}\Big)^{1/\alpha}<\{(i+1)\log c\}^{-2/(N-\alpha)}\leqq g(c^{i+1}).$$

Combining Lemma 4 with (7) and (8), we have

$$P(E_j\cap E_i)\leqq c_4\Big\{\frac{g(c^{i+1})c^{i/\alpha}+g(c^{j+1})c^{j/\alpha}}{(c^i-c^{j+1})^{1/\alpha}}\Big\}^{N-\alpha}\cdot P(E_j)$$

$$\leqq c_4\{g(c^{i+1})\}^{N-\alpha}\Big[\Big(1-\frac{1}{c}\Big)^{-\frac{N-\alpha}{\alpha}}+1\Big]P(E_j)=c_{11}\cdot P(E_j)P(E_i).$$

This is the second half of (iii) in Lemma 6. When i is not large enough compared with j, we obviously get

$$P(E_j\cap E_i)\leqq P(E_j).$$

The first half of (iii) may be proved by observing that such i's form a finite set. Thus we have proved that the events E_i occur infinitely often with probability 1 and so $g(t)$ belongs to the upper class.

REMARK. For the stable process with $N=\alpha$, there is a problem similar to Theorem 1, that is, the exact estimates of the frequency of small values of $|X(t)|$. This problem for $N=\alpha=2$ is worked out by F. Spitzer [**15**] and that for $N=\alpha=1$ is still open[5]. Because of the existence of zero points, this problem has no counterpart in the case $N=1<\alpha\leqq 2$.

In [**6**] Dvoretzky-Erdös who concerned with the Brownian motion case, derived their theorem from the explicit evaluation of the probability that a Brownian motion starting at x $(r<|x|=R)$ ever meets the sphere with radius

5) After the manuscript was written, this problem was resolved. See, S. Watanabe and J. Takeuchi, On the Spitzer test for Cauchy process on a line, to appear in Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete. Moreover the author gave a similar criterion concerning the local asymptotic behaviours of the transient process. Cf. Proc. Japan Acad., **40** (1964), 141-144.

r and center at the origin. It is known that this probability is equal to $\left(\frac{r}{R}\right)^{N-2}$. Concerning such a probability for a stable process, we show, in passing, the following

THEOREM 2. *Let Γ_1 and Γ_2 be two balls with centers at the origin and radii r_1 and r_2 ($r_1 < r_2 \leqq \infty$). If σ_{Γ_1}, σ_{Γ_2} are the first passage time for Γ_1, Γ_2, then for $a \in \Gamma_2 - \Gamma_1$ with $|a| = \rho$,*

$$P_a(\sigma_{\Gamma_1} < \sigma_{\Gamma_2}) < \begin{cases} \dfrac{\rho^{-N+\alpha} - r_2^{-N+\alpha}}{r_1^{-N+\alpha} - r_2^{-N+\alpha}} & \text{if } N > \alpha, \quad 0 < \alpha < 2, \\ \log \dfrac{r_2}{\rho} \Big/ \log \dfrac{r_2}{r_1} & \text{if } N = \alpha = 1, \end{cases}$$

$$P_a(\sigma_{\Gamma_1} < +\infty) < \left(\frac{r_1}{\rho}\right)^{N-\alpha} \qquad \text{if } N > \alpha, \quad 0 < \alpha < 2.$$

PROOF. Let us write $X(t)$ and $Y(t)$ for the symmetric stable process with $N > \alpha$ and the Cauchy process on the line respectively. H. P. McKean [**12**] proved that the process $-\frac{1}{|X(t)|^{N-\alpha}}$ and $\log|Y(t)|$ are semi-martingales. His paper discusses only the one-dimensional case but the result may be extended to processes in R^N. We now follow a reasoning of J. L. Doob [**5**]. Let $X^*(t)$ be the process stopped when $X(t)$ enters the interior of Γ_1 or the exterior of Γ_2. It follows that $|X^*(t)|^{-N+\alpha}$ is also a lower semi-martingale. Hence

$$\rho^{-N+\alpha} = E(|X^*(0)|^{-N+\alpha}) \geqq E(|X^*(t)|^{-N+\alpha}) \qquad 0 \leqq t < \infty, \tag{9}$$

and the right side of this inequality is monotone non-increasing in the variable t. Since

$$\lim_{t\to\infty} |X^*(t)|^{-N+\alpha} = |X^*(\infty)|^{-N+\alpha}$$

with probability one and these random variables are uniformly bounded, we have

$$\rho^{-N+\alpha} \geqq E(|X^*(\infty)|^{-N+\alpha}). \tag{10}$$

Setting $p = P_a(\sigma_{\Gamma_1} < \sigma_{\Gamma_2})$, (10) can be written in the form

$$\rho^{-N+\alpha} \geqq p r_1^{-N+\alpha} + (1-p) r_2^{-N+\alpha},$$

therefore

$$p \leqq \frac{\rho^{-N+\alpha} - r_2^{-N+\alpha}}{r_1^{-N+\alpha} - r_2^{-N+\alpha}}. \tag{11}$$

On the contrary, suppose $u(t)$ is a non-negative and Borel measurable function and that $u(X(t))$ is a martingale, then $u(t)$ is essentially constant. H. P. McKean [**12**] has shown this important property to which the case $\alpha = 2$ is an exception. According to McKean's theorem, $|X(t)|^{-N+\alpha}$ process is not a martingale and so there is no equality (9), (10) and (11). These observations yield the first part of the theorem. The method used above is also applicable to the second inequality. Setting $r_2=+\infty$ and letting $X^*(t)$ be the process stopped if $X(t)$ enters the interior of Γ_1, we get the last assertion. For the Cauchy process in R^1, it is no longer valid to let $r_2=\infty$ and obtain $P_a(\sigma_{\Gamma_1}<\infty)<1$. In fact the recurrent character of this process yields $P_a(\sigma_{\Gamma_1}<\infty)=1$. The reason why it is false is that $\log|x|$ is unbounded as $x\to\infty$ and therefore the formal invariance of the semi-martingale property of a stochastic process under optional stopping is not applicable.

§3. Double points of the paths.

We will begin with a short summary of the relevant definitions and known facts concerning the notions of Hausdorff measure and generalized capacity.

Let α be a positive real number and E a subset of R^N. For each $\varepsilon>0$, put

$$\Lambda_\varepsilon^\alpha(E)=\inf\sum_{i=1}^\infty(\operatorname{diam}E_i)^\alpha$$

where $\{E_i\,;\,i\geqq 1\}$ is a covering of E by subsets of R^N all of diameters less than ε, and the infimum is taken over all such coverings. Let

$$\Lambda^\alpha(E)=\lim_{\varepsilon\to0}\Lambda_\varepsilon^\alpha(E)\,.$$

Then Λ^α is called the Hausdorff α-dimensional outer measure. It is a metric outer measure and so the Borel sets are always measurable. We will need the following fact, which was proved by A. S. Besicovitch [**1**]: If E is a Borel set with $\Lambda^\alpha(E)=M\leqq\infty$ and if $0<h<M$, then there exists a closed set F contained in E such that $\Lambda^\alpha(F)=h$. It is also true that

$$\sup\{\alpha:\Lambda^\alpha(E)=\infty\}=\inf\{\alpha\,;\,\Lambda^\alpha(E)=0\}\ .$$

This common value is called the Hausdorff dimension of E.

We also need to define the α-capacity. Suppose A is any bounded Borel set in R^N. Let $\mathfrak{M}(A)$ be the family of all countably additive measures m defined for all subsets of A with $m(A)=1$. We define

$$\mathrm{C}_\alpha(A)=\Big[\inf_{m\in\mathfrak{M}(A)}\int_A\int_A\frac{1}{|x-y|^\alpha}m(dx)m(dy)\Big]^{-1/\alpha}$$

as the α-capacity of A. Thus $\mathrm{C}_\alpha(A)>0$ if and only if there exists a measure of total mass 1 for which the double integral above is convergent. This definition of the α-capacity is essentially the same as Frostman's and is different from that of Vallée Poussin who has defined the capacity to be the total mass of the measure inducing an equilibrium potential. We denote by C_α the former and by C^α the latter to discriminate between them. The numerical values of both capacities do not necessarily equal but one has a positive value for a given compact set if and only if the other is also positive. This fact is well known in potential theory and its proof is given in [**16**].

Frostman and Kametani have proved that if $\Lambda^\alpha(A)>0$, then $C_\beta(A)>0$ for all $\beta<\alpha$. On the other hand, if $\Lambda^\alpha(A)<+\infty$ then $C_\alpha(A)=0$. This is due to Kametani who used the method of Ugaheri. The proofs of these results and a complete bibliography may be found in S. Kametani [**9**] and in S. J. Taylor [**17**].

Blumenthal-Getoor [**2**] proved that the Hausdorff dimension of the range of the sample paths of general stable process in R^N with $N\geqq\alpha$ is equal to α with probability 1. We shall further need the following

LEMMA 8. *Let* $L(a,b\,;w)=L(a,b)=\{x\in R^N;\ x=X(t,w)$ *for some* $t\in[a,b]\}$. *Then*

$$P(\Lambda^\alpha(\overline{L(0,1)})<\infty)=1\,^{6)}\,.$$

PROOF. The argument in Blumenthal-Getoor [**2**; 373-374] is available; so we only sketch it, describing in some detail the point how to adapt the argument. For each $\varepsilon>0$ and all $k\geqq 1$ we define

$$T_{1\varepsilon}=\inf\{t>0\,;\,|X(t)|>\varepsilon^{1/\alpha}\}\,,$$

$$T_{k+1,\varepsilon}=\inf\{t>0\,;\,|X(t+T_{1\varepsilon}+\cdots+T_{k\varepsilon})-X(T_{1\varepsilon}+\cdots+T_{k\varepsilon})|>\varepsilon^{1/\alpha}\}\,,$$

and put

$$N_\varepsilon=\min\{n\geqq 1\,;\,T_{1\varepsilon}+\cdots+T_{n\varepsilon}>1\}\,.$$

If $S(0,\varepsilon)$ denotes the solid closed sphere with center at the origin 0 and radius $\varepsilon^{1/\alpha}$, and if $S(k,\varepsilon)$ denotes a similar sphere with center at $X(T_{1\varepsilon}+\cdots+T_{k\varepsilon})$, then $\overline{L(0,1)}\subset\bigcup_{k=0}^{N_\varepsilon-1}S(k,\varepsilon)$ and

$$\sum_{k=0}^{N_\varepsilon-1}(\operatorname{diam}S(k,\varepsilon))^\alpha=2^\alpha\varepsilon N_\varepsilon\,.$$

$T_{11}=\inf\{t>0\,;\,|X(t)|>1\}$ being positive with probability 1 because of the right continuity of the paths, it follows that $E(T_{11})>0$. Let x satisfy $[E(T_{11})]^{-1}<x<\infty$. Putting $\varepsilon=x/n$, we get

$$P(\varepsilon N_\varepsilon\leqq x)=P\Big(x\frac{T_{11}+\cdots+T_{n1}}{n}>1\Big)\,. \tag{12}$$

Then, by the law of large numbers, (12) approaches to 1 as $n\to\infty$ $(\varepsilon\to 0)$. Hence $P(\Lambda^\alpha(\overline{L(0,1)})\leqq 2^\alpha x<\infty)=1$.

LEMMA 9. *Almost all symmetric stable paths of index* α *in* R^N *have the*

6) The notation $\bar{A}$ means closure of A.

following properties.

If $0<\alpha\leqq\beta$, *then* $C_\beta(\overline{L(0,t)})=0$.

If $0<\beta<\alpha$, *there exists a compact set* F *such that* $C_\beta(F)>0$ *and* $F\subset L(0,t)$.

PROOF. By the Blumenthal-Getoor theorem and the previous lemma, we have $\Lambda^\beta(L(0,1))<\infty$ if $\alpha\leqq\beta$. By obvious modifications of the proof it follows that, for given $t>0$, $\Lambda^\beta(L(0,t))<\infty$ with probability 1. In general we would get the same value of the Hausdorff measure if we restrict all members of the covering to be closed sets. Thus we obtain $C_\beta(\overline{L(0,t)})=0$ by the theorem of Ugaheri-Kametani [**9**].

Now using the result of Blumenthal-Getoor and noting the definition of Hausdorff dimension, we get $\Lambda^\gamma(L(0,t))=\infty$ if $\beta<\gamma<\alpha$. According to the theorem of Besicovitch, there exists a compact set F contained in $L(0,t)$ such that $0<\Lambda^\gamma(F)<\infty$. Hence we obtain $C_\beta(F)>0$ by the theorem of Frostman-Kametani and so the lemma is proved.

LEMMA 10. *Let* $X(t)$ *be the symmetric stable process in* R^N *of index* α *with* $N>\alpha$. *For any compact set* F, *put*

$$\Phi(x;F)=P_x(X(t)\in F \text{ for some } t>0).$$

Then for any point x, $\Phi(x;F)$ *is positive or zero according as* $C_{N-\alpha}(F)$ *is positive or zero.*

PROOF. Since (2) in Lemma 1 holds with E replaced by F, we find immediately that $\Phi(x;F)>0$ if and only if $C_{N-\alpha}(F)>0$. In view of what we have remarked above, whether $\Phi(x;F)$ is positive or not is equivalent to $C_{N-\alpha}(F)$ is positive or zero.

We are now ready to prove the theorem on double points. A path w is said to have a double point if there exist a point ζ and a pair of real numbers s,t with $s<t$ such that $X(s,w)=X(t,w)=\zeta$.

THEOREM 3. *For a symmetric stable process, there exist infinitely many values of the time parameter which correspond to the double points of the path with probability* 1 *if* $N\leqq 3$ *and* $N/2<\alpha\leqq 2$. *However if* $N\geqq 4$, *or if* $0<\alpha\leqq N/2$ *even in the case* $N\leqq 3$, *almost every path is free from the double points.*

PROOF. We proceed in several steps.

(i) The case: $N\leqq 3$, $N/2<\alpha\leqq 2$ and $N>\alpha$.

By means of Lemmas 9 and 10, there exists a compact set K such that $\Phi(x;K)>0$ and $K\subset L(0,t)$ for almost all paths, if $N-\alpha<\alpha$; in this case $N/2<\alpha\leqq 2$ and $N\leqq 3$ necessarily hold. Let $0\leqq a<b<c<\infty$. Then, on the strength of the Markov property, we obtain

$$P_0(w;L(a,b;w)\cap L(c,\infty;w)\neq\phi)$$
$$=P_0(w;L(a,b;w)\cap L(0,\infty;w_c^+)\neq\phi)$$

$$(13) \qquad = \int_W P_{X(c,w)}(w' ; L(a, b ; w) \cap L(0, \infty ; w') \neq \phi) P_0(dw)$$

$$= \int_W \Phi(X(c, w) ; L(a, b ; w)) P_0(dw)$$

$$\geqq \int_W \Phi(X(c, w) ; K(w)) P_0(dw)$$

where $K(w)$ is a compact set such that $K(w) \subset L(a, b ; w)$ and $\Phi(x ; K(w)) > 0$. When we fix a path w, $K(w)$ is also fixed. As the last integral in (13) is positive, we see $P_0(L(a, b) \cap L(c, \infty) \neq \phi) > 0$. It follows that there exists a real number d with $c < d < \infty$ such that $P(L(a, b) \cap L(c, d) \neq \phi) = \delta > 0$. Let us put $a_k = a + kd$, $b_k = b + kd$, $c_k = c + kd$, $d_k = (k+1)d$, $(k \geqq 1)$. Then $P(L(a_k, b_k) \cap L(c_k, d_k) \neq \phi) = \delta > 0$ for all $k \geqq 1$. Since the process has independent increments, we can apply the Borel-Cantelli lemma and consequently we have

$$P(L(a_k, b_k) \cap L(c_k, d_k) \neq \phi \text{ for infinitely many } k) = 1 .$$

(ii) The case: $N \geqq 4$, or $N \leqq 3$ and $0 < \alpha \leqq N/2$.

With the help of Lemmas 9 and 10, we find that for almost all paths $\Phi(x ; \overline{L(0, t)}) = 0$ if $\alpha \leqq N - \alpha$, that is, if $0 < \alpha \leqq N/2$. By the arguments similar to those in the case above, we infer that

$$P(L(a, b) \cap L(c, \infty) \neq \phi) \leqq \int_W \Phi(X(c, w) ; \overline{L(a, b ; w)}) P(dw)$$

and therefore the left side of this inequality is also equal to zero. Hence for any rational numbers $r_k (1 \leqq k \leqq 4)$ with $0 \leqq r_1 < r_2 < r_3 < r_4 < \infty$, we have $P(L(r_1, r_2) \cap L(r_3, r_4) \neq \phi) = 0$. Let a, b, c and d be any real numbers satisfying

$$0 \leqq r_1 \leqq a < b \leqq r_2 < r_3 \leqq c < d \leqq r_4 < \infty .$$

Then we can conclude

$$P(L(a, b) \cap L(c, d) = \phi) = 1 .$$

(iii) The case: $N = \alpha$.

The case of a two-dimensional Brownian motion is implied by the result in the case $N = 3$ and the mutual independence of the component processes. Thus it suffices to prove only for the Cauchy process in R^1. We need the notion of logarithmic capacity in the sequel. Let A be an arbitrary analytic set. Its logarithmic capacity is given by

$$\mathrm{C}_0(A) = \exp\Big(\inf_{m \in \mathfrak{M}(A)} \int_A \int_A \log |x - y| \, m(dx) m(dy) \Big)$$

where $\mathfrak{M}(A)$ has the same meaning as before. If the logarithmic capacity of A is positive, then for any point x, $P_x(X(t) \in A$ for some $t > 0) = 1$. This result for $N = \alpha = 2$ is well-known. The proof for general recurrent cases due to S. Watanabe may be found in [**16**].

We shall first prove that $C_0(L(0, 1))>0$ outside a null w-set. It is easy to see that

$$E(\log|X(t, w)-X(s, w)|)=E(\log|X(t-s, w)|)$$
$$=E(\log\{|t-s||X(1)|\})=\log|t-s|+\frac{2}{\pi}\int_0^\infty \frac{\log u}{1+u^2}du\,.$$

The second term of the last expression is zero, because

$$\int_0^\infty \frac{\log u}{1+u^2}du=\int_0^1 \frac{\log u}{1+u^2}du-\int_0^1 \frac{\log t}{1+t^2}dt=0\,.$$

Hence

$$E(\log|X(t, w)-X(s, w)|)=\log|t-s|\,,$$

and accordingly

$$\int_0^1\int_0^1 E(\log|X(t, w)-X(s, w)|)dtds=\int_0^1\int_0^1 \log|t-s|dtds>-\infty\,.$$

An application of Fubini's theorem gives

$$\int_0^1\int_0^1 \log|X(t, w)-X(s, w)|dtds>-\infty$$

for almost all w. Put

$$m_w(dx)=|\{t\,;\,X(t, w)\in dx\}\cap[0, 1]|\,,$$

where $|A|$ denotes the Lebesgue measure of A. Then $m_w\in\mathfrak{M}(L(0, 1))$ and

$$\int_{L(0,1)}\int_{L(0,1)} \log|x-y|m_w(dx)m_w(dy)>-\infty$$

holds with probability 1. This implies that $C_0(L(0, 1))$ is positive. The rest of the proof can be done in the same way as in the case (i).

(iv) The case: $N=1<\alpha\leqq 2$.

In this case, $P_x(X(t)=y$ for some $t>0)=1$ for any given points x, y on the line. This is indicated by S. Watanabe [**18**] and Blumenthal-Getoor [**3**]. Combining the strong Markov property with $P_y(X(t)=x$ for some $t>0)=1$, particles of our process will return to the starting point almost surely. Hence the starting point is a double point and, what is more, a multiple point of multiplicity k for any positive integer k. In other words, there exist infinitely many values of the time parameter which correspond to the double points.

§ 4. Lebesgue measure of the paths.

It is well known that almost every Brownian path in the plane describes an everywhere dense curve, but the two-dimensional Lebesgue measure of the curve is zero with probability 1. This interesting theorem was discovered by P. Lévy [**11**]. The aim of the present section is simply to note the occurrence of a similar phenomenon in the Cauchy process on the line. The trajectories of almost all symmetric stable paths with $N=1\leqq\alpha\leqq 2$ are dense on the line, while, in the case $N>\alpha$, almost every path constitutes a nowhere dense set in R^N. Actually H. P. McKean [**12**] has shown this result for processes in R^1, but the proof may be carried over to R^N without essential changes.

As for the Lebesgue measure of the paths, we can deduce the following

THEOREM 4. *For the symmetric stable paths in* R^N,

$$P(|L(0,\infty)|=0)=1 \qquad \text{if} \quad N\geqq\alpha\,,$$
$$P(|L(0,\infty)|=\infty)=1 \qquad \text{if} \quad N<\alpha\,,$$

where $L(0,\infty)=\{x\in R^N\,;\, x=X(t,w) \text{ for some } t\in[0,\infty)\}$ *and* $|A|$ *denotes the N-dimensional Lebesgue measure of A.*

PROOF. It is known that the N-dimensional Lebesgue measure is zero if and only if the value of its N-dimensional Hausdorff measure is equal to zero. (See S. Saks [**14**; p. 54] for a discussion of this fact.) Combining this property with the theorem of Blumenthal-Getoor [**2**], the conclusion of the theorem for the case $N>\alpha$ is immediate. For $N=1<\alpha$, the answer is derived from the discussion in Blumenthal-Getoor [**3**; p. 314].

There remains thus to prove the case $\alpha=N$ and it suffices to show the case of the Cauchy process in R^1. The proof given below is a slight modification of the arguments in K. Itô [8] and is valid also for $0<\alpha\leqq N=1$. First of all, define

$$f_X(x,w)=\begin{cases} 1 & \text{if} \quad x\in\overline{L_X(0,t)}\,,\\ 0 & \text{if} \quad x\notin\overline{L_X(0,t)}\,,\end{cases}$$

where $L_X(0,t)$ is the range of $X(s)$ as s runs over $[0,t]$. Let $\mathfrak{B}(W)$ denote the σ-algebra of subsets of W generated by all sets of the form $\{w\,;\,X(s,w)\in E\}$ where s ranges over $[0,t]$, and E over the Borel sets of R^1. Let $\mathfrak{B}(R^1)$ be the class of Borel subsets of R^1. Then we shall show that $f_X(x,w)$ is $\mathfrak{B}(R^1)\times\mathfrak{B}(W)$-measurable in the pair (x,w). It suffices to prove

$$\{(x,w)\,;\,x\notin\overline{L_X(0,t)}\}\in\mathfrak{B}(R^1)\times\mathfrak{B}(W)\,.$$

For any open set U in R^1, we have

$$\{w\,;\,\overline{L_X(0,t)}\subset U^c\}=\bigcap_{\substack{0\leqq r\leqq t\\ r:\ \text{rational}}}\{w\,;\,X(r,w)\in U^c\}$$

so that the left side is $\mathfrak{B}(W)$-measurable. Now let $\{U_n\,;\,n\geqq 1\}$ be a countable base for R^1. Then it is possible to see

$$\{(x,w)\,;\,x\notin\overline{L_X(0,t)}\}$$
$$=\bigcup_{n=1}^{\infty}[U_n\times\{w\,;\,\overline{L_X(0,t)}\subset U_n^c\}]\in\mathfrak{B}(R^1)\times\mathfrak{B}(W)\,.$$

In view of the Fubini's theorem and

$$|\overline{L_X(0, t)}| = \int_{R^1} f_X(x, w)dx,$$

it results that $|\overline{L_X(0, t)}|$ is measurable in w. From now on, we consider the following three processes which obey the same probability law as the process $\{X(s, w); 0 \leqq s \leqq t\}$;

$$\begin{aligned} Y(s, w) &= X(s+t, w) - X(t, w) && 0 \leqq s \leqq t, \\ Z(s, w) &= -(X(t, w) - X(t-s, w)) && 0 \leqq s \leqq t, \\ U(s, w) &= 2^{-1/\alpha} X(2s, w) && 0 \leqq s \leqq t. \end{aligned}$$

Let $L_Y(0, t)$, $L_Z(0, t)$ and $L_U(0, t)$ have the meanings similar to $L_X(0, t)$. Thus we get

$$E(|\overline{L_X(0, t)}|) = \int_{R^1} E(f_X(x, w))dx = \int_{R^1} E(f_Y(x, w))dx = E(|\overline{L_Y(0, t)}|).$$

The same reason yields

$$E(|\overline{L_X(0, t)}|) = E(|\overline{L_Z(0, t)}|) = E(|\overline{L_U(0, t)}|).$$

If $[A+r]$ denotes the translation of A by r and $\equiv$ denotes the congruence, then we find

$$\begin{aligned} \overline{L_X(0, 2t)} &= \overline{L_X(0, t)} \cup [\overline{L_Y(0, t)} + X(t)] \\ &\equiv [\overline{L_X(0, t)} - X(t)] \cup \overline{L_Y(0, t)} \\ &= \overline{L_Z(0, t)} \cup \overline{L_Y(0, t)}. \end{aligned}$$

Hence we have

$$|\overline{L_X(0, 2t)}| + |\overline{L_Y(0, t)} \cap \overline{L_Z(0, t)}| = |\overline{L_Y(0, t)}| + |\overline{L_Z(0, t)}|$$

and

$$\begin{aligned} & E(|\overline{L_X(0, 2t)}|) + E(|\overline{L_Y(0, t)} \cap \overline{L_Z(0, t)}|) \\ (14) \qquad & = E(|\overline{L_Y(0, t)}|) + E(|\overline{L_Z(0, t)}|) = 2E(|\overline{L_X(0, t)}|). \end{aligned}$$

It is also true that

$$(15) \qquad E(|\overline{L_X(0, 2t)}|) = E(|2^{1/\alpha}\overline{L_U(0, t)}|) = 2^{1/\alpha}E(|\overline{L_U(0, t)}|) = 2^{1/\alpha}E(|\overline{L_X(0, t)}|).$$

Clearly (14) and (15) taken together yield

$$(16) \qquad (2^{1/\alpha} - 2)E(|\overline{L_X(0, t)}|) + E(|\overline{L_Y(0, t)} \cap \overline{L_Z(0, t)}|) = 0.$$

If $0 < \alpha < 1$, both terms in (16) are non-negative and $2^{1/\alpha} - 2 > 0$. Therefore we have

$$E(|\overline{L_X(0, t)}|) = 0.$$

If $\alpha = 1$,

$$E(|\overline{L_Y(0, t)} \cap \overline{L_Z(0, t)}|) = 0 .$$

This can be written as

$$E\left(\int_{R^1} f_Y(x, w) f_Z(x, w) dx\right) = \int_{R^1} E(f_Y(x, w) f_Z(x, w)) dx = 0 .$$

$X(s)$ is a process with independent increments, so that $Y(s)$ and $Z(s)$ are independent of each other, therefore

$$E(f_Y(x, w) f_Z(x, w)) = E(f_Y(x, w)) E(f_Z(x, w)) = [E(f_X(x, w))]^2 .$$

Consequently

$$\int_{R^1} [E(f_X(x, w))]^2 dx = 0 ,$$

that is, $E(f_X(x, w)) = 0$ except possibly in a null set of x's in R^1. Again we obtain

$$\int_{R^1} E(f_X(x, w)) dx = E(|\overline{L_X(0, t)}|) = 0 .$$

Thus for $0 < \alpha \leqq 1$,

$$P(|\overline{L_X(0, t)}| = 0) = 1 .$$

Combining $|\overline{L_X(0, t)}| \geqq |L_X(0, t)|$ with the countable additivity of measure, we complete the proof.

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Added in proof. S. J. Taylor and D. Ray (private communication) determined the "exact Hausdorff measure" of the sample paths of the stable processes. Our Theorem 4 is an immediate corollary of their recent works.

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