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Differential forms of the first and second kind on modular algebraic varieties

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Let V be a complete non-singular variety. We denote the universal domain by K and its characteristic by p. All vector spaces and their dimensions will mean those with respect to K. Differential forms of the first kind of degree r on V are, as is well known, elements of $H^0(V, \Omega^r)$ where Ω^r is the sheaf of germs of holomorphic differential forms of degree r. The dimension of this space is denoted by $h^{r,0}$. We define the differential forms of the second kind on V after Picard and Rosenlicht [**9**] as follows. Let ω be a differential form of degree $r \geqq 1$ on V; we call ω to be *of the second kind at a point* P of V if there exists a differential form θ_P of degree $r-1$ on V such that $\omega - d\theta_P$ is holomorphic at P; if ω is of the second kind at every point of V, we call ω to be *of the second kind on* V. We denote by $\mathcal{D}_2^{(r)}(V)$ the space of all closed differential forms of the second kind of degree r on V, and by $\mathcal{D}_e^{(r)}(V)$ that of all exact differential forms among them. The dimension of the factor space $\mathcal{D}_2^{(1)}(V)/\mathcal{D}_e^{(1)}(V)$ is known to be: (1) $2h^{1,0}$ or $h^{1,0}$ respectively, in case $\dim V = 1$, according as $p = 0$ or > 1 (Rosenlicht [**9**]), (2) $2h^{1,0}$ in case $K = \boldsymbol{C}$, $\dim V$ being arbitrary (Hodge-Atiyah [**5**]).

Our purpose in §1 is to show that the dimension of the factor space $\mathcal{D}_2^{(1)}(V)/\mathcal{D}_e^{(1)}(V)$ is $h^{1,0}$, whenever $p > 1$, $\dim V$ being arbitrary. We shall prove this in making use of the operator C of Cartier; a proof of this fact in case $\dim V = 1$, making also use of C, has been given by Cartier (Tamagawa's lecture in Tokyo University 1960) and Barsotti [**2**, p. 63], but even in this case our proof is based on other property of C than that used by them. Theorem 2 generalizes Theorem 2 of [**9**]. In §2, we shall give some results on closed semi-invariant differential forms on modular abelian varieties; in case of characteristic zero, the corresponding results are found in Barsotti [**1**]. In §3 we shall prove the following result. It is known that the dimension q of the Albanese variety of a variety V is $\leqq h^{1,0}$ (Igusa [**6**]); there is a famous example of V due to Igusa [**7**] for which the strict inequality $q < h^{1,0}$ holds. We shall prove that if V is defined over a field of prime characteristic p and if V has p-torsion divisors, then the inequality $q < h^{1,0}$ holds.

NOTATIONS: $K(U)$ denotes the field of all rational functions on a variety U. If W is a subvariety of U, then $\mathfrak{o}_W$ and $\mathfrak{p}_W$ denote respectively the local ring of W on U over K and the ideal of non-units of $\mathfrak{o}_W$. A prime divisor on U means a simple subvariety of U of codimension 1; a prime divisor W gives rise to a discrete valuation of rank 1 of the field $K(U)$ over K which will be denoted by v_W.

§1. Differential forms of the second kind.

Let U be a variety defined over a field of prime characteristic p, and ω be a closed differential form of degree $r \geqq 1$ on U. Let $\{x_1, \cdots, x_m\}$ be a separating transcendence basis of $K(U)$ over K. Then ω is known to be written uniquely in the form

$$\omega = d\theta + \sum_{i_1 < \cdots < i_r} z^p_{i_1 \cdots i_r} x^{p-1}_{i_1} dx_{i_1} \wedge \cdots \wedge x^{p-1}_{i_r} dx_{i_r}, \qquad z_{i_1 \cdots i_r} \in K(U),$$

and the *operator* C is defined by the formula

$$C\omega = \sum_{i_1 < \cdots < i_r} z_{i_1 \cdots i_r} dx_{i_1} \wedge \cdots \wedge dx_{i_r}, \tag{1}$$

which is independent of the choice of $\{x_1, \cdots, x_m\}$. C is p^{-1}-*semi-linear* in the sense

$$C(\omega + \omega') = C\omega + C\omega', \qquad C(z^p \omega) = zC\omega \quad (z \in K(U)).$$

If ω is a closed differential form of degree 1, (1) is equivalent to the formula

$$(C\omega(D))^p = \omega(D^p) - D^{p-1}(\omega(D)), \tag{2}$$

and $C\omega = 0$ if and only if ω is exact, and $C\omega = \omega$ if and only if ω is of the form $\frac{df}{f}$ with $f \in K(U)$. (Cf. Cartier [4].)

THEOREM 1. *Let V be a complete non-singular variety which is defined over a field of prime characteristic p. Then $\omega \to C\omega$ induces a p^{-1}-semi-linear bijective homomorphism $\mathfrak{D}^{(1)}_2(V)/\mathfrak{D}^{(1)}_e(V) \to H^0(V, \Omega^1)$; especially we have* $\dim \mathfrak{D}^{(1)}_2(V)/\mathfrak{D}^{(1)}_e(V) = h^{1,0}$.

We first prove the following

LEMMA 1. *Let V be as in Theorem* 1, *and let ω be a closed differential form of degree $r \geqq 1$ on V. If $C\omega$ is holomorphic at $P \in V$, then ω is of the second kind at P. Especially if $C\omega$ is of the first kind on V, then ω is of the second kind on V.*

PROOF. Let $\{x_1, \cdots, x_m\}$ be a set of uniformizing coordinates of P on V. Then if ω is of the form $\omega = d\theta + \sum_{i_1 < \cdots < i_r} z^p_{i_1 \cdots i_r} x^{p-1}_{i_1} dx_{i_1} \wedge \cdots \wedge x^{p-1}_{i_r} dx_{i_r}$, we have $C\omega = \sum_{i_1 < \cdots < i_r} z_{i_1 \cdots i_r} dx_{i_1} \wedge \cdots \wedge dx_{i_r}$. If $C\omega$ is holomorphic at P, then we have $z_{i_1 \cdots i_r} \in \mathfrak{o}_P$ and so $z^p_{i_1 \cdots i_r} x^{p-1}_{i_1} \cdots x^{p-1}_{i_r} \in \mathfrak{o}_P$, which implies that $\omega - d\theta$ is holomorphic at P. This completes the proof.

LEMMA 2. *Let V be as in Theorem 1, and let ω be a closed differential form of degree 1 on V. If ω is of the second kind at $P \in V$, then $C\omega$ is holomorphic at P. Especially if ω is of the second kind on V, then $C\omega$ is of the first kind on V.*

PROOF. Let $\theta = \omega - df$ be holomorphic at P. Take any derivation D of $K(V)$ over K with $D\mathfrak{o}_P \subseteqq \mathfrak{o}_P$; then we have $\theta(D) \in \mathfrak{o}_P$, $D^{p-1}\theta(D) \in \mathfrak{o}_P$ and $\theta(D^p) \in \mathfrak{o}_P$, since θ is holomorphic at P. It follows from this and the formula (2) that $(C\theta(D))^p \in \mathfrak{o}_P$. Since the local ring $\mathfrak{o}_P$ of a simple point P is integrally closed in $K(V)$ and since $C\theta(D) \in K(V)$, we must have $C\theta(D) \in \mathfrak{o}_P$. Thus $C\omega = C\theta$ is holomorphic at P.

Theorem 1 is now an immediate consequence of Lemma 1, Lemma 2 and the fact that C maps the set of all closed differential forms on V onto that of all differential forms.

As a corollary to Lemma 1 and Lemma 2, we have

COROLLARY. *Let V be as in Theorem 1. If a closed differential form ω of degree 1 on V is of the second kind at every prime divisor of V, then ω is of the second kind on V.*

PROOF. It follows from the assumption and Lemma 2 that $C\omega$ is holomorphic at each prime divisor of V. $C\omega$ is then of the first kind on V (cf. Zariski [**11**, p. 26, Proposition 8.7]). ω is therefore of the second kind on V by Lemma 1.

THEOREM 2. *Let U be a variety defined over a field of prime characteristic p. Let ω be a closed differential form of degree 1 on U. If ω can be approximated arbitrarily closely at a prime divisor W on U (i.e. for each natural number n, there exists $f_n \in K(U)$ such that $v_W(\omega - df_n) \geqq n$), then ω is exact.*

PROOF. Let $x = x_1$ be a uniformizing parameter of W on U i.e. $\mathfrak{p}_W = \mathfrak{o}_W x$ (which we shall denote by $\mathfrak{p}$ below), and $\{x_1, x_2, \cdots, x_m\}$ be a set of uniformizing coordinates of W on U. Put $\partial_i = \dfrac{\partial}{\partial x_i}$ $(1 \leqq i \leqq m)$ and $\theta_n = \omega - df_n = z_1 dx_1 + \cdots + z_m dx_m$, $z_i \in K(U)$. Denoting $v = v_W$, we have $v(\theta_n) = \underset{1 \leqq i \leqq m}{\mathrm{Min}} \{v(z_i)\} \geqq n$. Since $\partial_i^p = 0$, we have $(C\theta_n(\partial_i))^p = -\partial_i^{p-1}(\theta_n(\partial_i))$ by the formula (2). Since $\theta_n(\partial_i) = z_i \in \mathfrak{p}^{v(z_i)} \subseteqq \mathfrak{p}^n = \mathfrak{o}_W x^n$, z_i is of the form $z_i = x^n u_i$ with $u_i \in \mathfrak{o}_W$, so that we have $\partial_i(\theta_n(\partial_i)) = \partial_i(x^n u_i) = n x^{n-1} \cdot \partial_i x \cdot u_i + x^n \cdot \partial_i u_i \in \mathfrak{p}^{n-1}$, $\partial_i^2(\theta_n(\partial_i)) \in \mathfrak{p}^{n-2}, \cdots, \partial_i^{p-1}(\theta_n(\partial_i)) \in \mathfrak{p}^{n-(p-1)}$ $(n \geqq p-1)$. We have therefore $(C\theta_n(\partial_i))^p \in \mathfrak{p}^{n-(p-1)}$. Putting $a_i = C\omega(\partial_i) = C\theta_n(\partial_i) \in K(U)$ (which is independent of n), we have $a_i^p \in \bigcap_{n=p-1}^{\infty} \mathfrak{p}^{n-(p-1)} = \{0\}$, $a_i^p = 0$ and $a_i = 0$ $(1 \leqq i \leqq m)$. Thus we have $C\omega = a_1 dx_1 + \cdots + a_m dx_m = 0$, and that $\omega = df$ with some $f \in K(U)$, which completes the proof.

§ 2. Invariant and semi-invariant differential forms on abelian varieties.

Let A be an abelian variety. A differential form ω of degree $r \geqq 1$ on A is said to be *invariant* if $\omega \circ T_a - \omega = 0$ for any $a \in A$, where T_a denotes the translation by a; ω is said to be *semi-invariant* if for any $a \in A$ $\omega \circ T_a - \omega$ is exact. (Cf. Barsotti [**1**].) The space of all closed semi-invariant differential forms of degree r on A will be denoted by $\mathfrak{D}_s^{(r)}(A)$. Clearly we have $\mathfrak{D}_e^{(r)}(A) \subseteqq \mathfrak{D}_s^{(r)}(A)$.

PROPOSITION 1. *Let ω be a closed differential form of degree $\geqq 1$ on an abelian variety A which is defined over a field of prime characteristic p. Then ω is semi-invariant if and only if $C\omega$ is invariant.*

PROOF. For a given $a \in A$, $\omega \circ T_a - \omega$ is exact if and only if $C(\omega \circ T_a) = C\omega$; and this is so if and only if $(C\omega) \circ T_a = C\omega$, since $(C\omega) \circ T_a = C(\omega \circ T_a)$. Proposition 1 is an immediate consequence of this and the definition of "semi-invariant".

Proposition 1 may be stated also as follows.

COROLLARY 1. *Let A be as in Proposition 1. $\omega \to C\omega$ induces a p^{-1}-semi-linear bijective homomorphism $\mathfrak{D}_s^{(r)}(A)/\mathfrak{D}_e^{(r)}(A) \to H^0(A, \Omega^r)$; especially we have* $\dim \mathfrak{D}_s^{(r)}(A)/\mathfrak{D}_e^{(r)}(A) = h^{r,0}$ $(1 \leqq r \leqq \dim A)$.

COROLLARY 2. *Let A and ω be as in Proposition 1.* (1): *If ω is semi-invariant, then ω is of the second kind on A.* (2): *If ω is of degree 1 then ω is semi-invariant if and only if ω is of the second kind on A.*

PROOF. Note that a differential form ω of degree $r \geqq 1$ on A is invariant if and only if it is of the first kind on A. (1) follows immediately from Proposition 1 and Lemma 1. (2) follows from Lemma 2 and Proposition 1.

§ 3. p-torsions and the inequality $q < h^{1,0}$[1].

Let V be a complete non-singular variety defined over a field of prime characteristic p, and $V \xrightarrow{\varphi} A$ be the Albanese variety of V. Pic(V) and Pic(A) denote the group of all divisor classes (with respect to the linear equivalence) on V and A respectively; Pic$(V)_p$ and Pic$(A)_p$ denote respectively the subgroup of the elements in Pic(V) and Pic(A) whose orders divide p. We shall denote the algebraic and linear equivalence relations between divisors by $\equiv$ and $\sim$ respectively. If $\mathfrak{a}$ is a divisor on A with $\mathfrak{a} \sim 0$, then $\varphi^{-1}(\mathfrak{a}) \sim 0$ on V (if it is defined); $\mathfrak{a} \to \varphi^{-1}(\mathfrak{a})$ induces a homomorphism φ^*: Pic$(A) \to$ Pic(V) and Pic$(A)_p \to$ Pic$(V)_p$ (Lang [**8**, p. 236, p. 65]). A divisor X on V is said to be *p-torsion* if $X \not\equiv 0$ but $pX \equiv 0$. $\omega \to \omega \circ \varphi$ induces the homomorphism $\delta\varphi$ of the space $H^0(A, \Omega^1)$ into the subspace of closed differential forms in $H^0(V, \Omega^1)$, and which is known to be injective (Igusa [**6**]). $\mathfrak{L}(V)$ denotes the additive

1) We owe to D. Mumford the formulation which follows.

group of differential forms in $H^0(V, \Omega^1)$ which are invariant by C; $\mathcal{L}(A)$ denotes the similar group on A. $\delta\varphi$ induces an injective homomorphism of $\mathcal{L}(A)$ into $\mathcal{L}(V)$, since $C(\omega\circ\varphi)=(C\omega)\circ\varphi$ for $\omega\in\mathcal{L}(A)$.

We use the following result of Cartier [**3**, Theorem 5].

(3) *Notations being as above, the group* $\mathrm{Pic}(V)_p$ *is canonically isomorphic to* $\mathcal{L}(V)$.

The proof of this fact is not published. However, a proof of this fact in case $\dim V=1$ is given in Serre [**10**, p. 28, Proposition 10], and it can be applied to the case of $\dim V\geqq 1$. In fact, note that a differential form on V is of the first kind on V if and only if it is holomorphic at every prime divisor of V (cf. [**11**, p. 26, Proposition 8.7]). Then, as in the proof of [**10** Proposition 10], it can be seen that $Cl(X)$[2] $\rightarrow \frac{df}{f}$ with $pX=(f)$ gives an injective homomorphism θ: $\mathrm{Pic}(V)_p\rightarrow\mathcal{L}(V)$. In order to see that θ is surjective, we have only to show that, if a differential form $\frac{df}{f}(f\in K(V))$ is holomorphic at a prime divisor W on V, then $v_W(f)\equiv 0 \pmod p$. Let t be a uniformizing parameter of W on V, and let $f=t^e u$ with a unit u in $\mathfrak{o}_W$ and $e=v_W(f)$. We have $\frac{df}{f}=e\frac{dt}{t}+\frac{du}{u}$. The differential form $\frac{dt}{t}$ is not holomorphic at W, since the derivation $D=\frac{\partial}{\partial t}$ of $K(V)/K$ is holomorphic at W and $\frac{dt}{t}(D)=\frac{1}{t}\notin\mathfrak{o}_W$; this implies $e\equiv 0 \pmod p$ since $e\frac{dt}{t}=\frac{df}{f}-\frac{du}{u}$ is holomorphic at W. Thus we see that θ is bijective.

PROPOSITION 2. *Notations being as above,* φ^*: $\mathrm{Pic}(A)_p\rightarrow\mathrm{Pic}(V)_p$ *is surjective if and only if* V *has no* p*-torsion divisor.*

PROOF. $\hat{V}$ denotes the subgroup of $\mathrm{Pic}(V)$ of classes which are represented by divisors algebraically equivalent to zero (i.e. the Picard variety of V); $\hat{A}$ denotes the similar group on A. If D is a Poincaré divisor for A, then $D\circ\varphi$ is a Poincaré divisor for V; φ^* gives therefore a canonical bijective homomorphism $\hat{A}\rightarrow\hat{V}$. (Cf. Lang [8, p. 148, Proposition 1 and Theorem 1].)

If V has no p-torsion divisor, then $\mathrm{Pic}(V)_p$ is a subgroup of $\hat{V}$; φ^* maps therefore $\mathrm{Pic}(A)_p$ *onto* $\mathrm{Pic}(V)_p$ *isomorphically*, since the abelian variety A has no p-torsion divisor.

Conversely, if V has a divisor X such that $X\not\equiv 0$ but $pX\equiv 0$, then we can find a divisor Y on V such that $Y\equiv X$ and $pY\sim 0$ (Lang [8, p. 101, Corollary 4 to Theorem 4]). Then $y=Cl(Y)\in\mathrm{Pic}(V)_p$ can not be an image from $\mathrm{Pic}(A)_p$, which shows that φ^*: $\mathrm{Pic}(A)_p\rightarrow\mathrm{Pic}(V)_p$ is not surjective. Proposition 2 is thereby proved.

2) $Cl(X)$ denotes the linear equivalence class containing the divisor X.

θ denotes the canonical isomorphism $\mathrm{Pic}(V)_p \to \mathcal{L}(V)$ in (3).

PROPOSITION 3. *Notations being as above, the following diagram is commutative.*

$$\begin{array}{ccc} \mathrm{Pic}(V)_p & \xleftarrow{\varphi^*} & \mathrm{Pic}(A)_p \\ \wr\Vert\,\theta & & \wr\Vert\,\theta \\ \mathcal{L}(V) & \xleftarrow{\delta\varphi} & \mathcal{L}(A) \end{array}$$

PROOF. Let $a = Cl(\mathfrak{a}) \in \mathrm{Pic}(A)_p$ with $p\mathfrak{a} = (\alpha)$, $\alpha \in K(A)$. Then we have $\theta(a) = \dfrac{d\alpha}{\alpha} \in \mathcal{L}(A)$ and $\varphi^*(a) = Cl(\varphi^{-1}(\mathfrak{a})) \in \mathrm{Pic}(V)_p$. It will not be difficult to see here that $\alpha \circ \varphi$ is defined and is not constant 0. This being so, we have $p\varphi^*(a) = Cl(p\varphi^{-1}(\mathfrak{a})) = Cl(\varphi^{-1}(\alpha)) = Cl((\alpha \circ \varphi))$. It follows from this that $\theta(\varphi^*(a)) = \dfrac{d(\alpha \circ \varphi)}{\alpha \circ \varphi} = \delta\varphi(\theta(a))$, which implies the commutativity of the diagram.

In view of the commutative diagram in Proposition 3, Proposition 2 is equivalent to the following.

THEOREM 3. *Let V be a complete non-singular variety which is defined over a field of prime characteristic p, and $V \xrightarrow{\varphi} A$ be its Albanese variety. Denote by $\mathcal{L}(V)$ the additive group of differential forms $\omega \in H^0(V, \Omega^1)$ such that $C\omega = \omega$, and by $\mathcal{L}(A)$ the similar group on A. Then $\delta\varphi : \mathcal{L}(A) \to \mathcal{L}(V)$ is surjective if and only if V has no p-torsion divisor.*

COROLLARY. *Let V, p and A be as in Theorem 3, and q the dimension of A. If V has a p-torsion divisor, then we have $q < h^{1,0}$.*

PROOF. If V has a p-torsion divisor, then we may find by Theorem 3 a differential form $\dfrac{df}{f} \in \mathcal{L}(V)$ which is not an image from $\mathcal{L}(A)$. *Assume* for a moment that $\dfrac{df}{f} = \omega \circ \varphi$ for some differential form ω of the first kind on A. We have $C(\omega \circ \varphi) = (C\omega) \circ \varphi$, since ω is closed. (Cf. [**1**, p. 93, 2.1].) Since $(C\omega - \omega) \circ \varphi = C(\omega \circ \varphi) - \omega \circ \varphi = C\left(\dfrac{df}{f}\right) - \dfrac{df}{f} = 0$, and since $C\omega - \omega$ is of the first kind by Lemma 2, we would have $C\omega - \omega = 0$ because of the injectivity of $\delta\varphi$. ω would be therefore of the form $\omega = \dfrac{d\alpha}{\alpha}$ with $\alpha \in K(A)$. This would imply that $\dfrac{df}{f} = \omega \circ \varphi$ is an image from $\mathcal{L}(A)$, which is a contradiction.

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