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On the limit of a monotonous sequence of Cousin's domains

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§ 0. Introduction.

In the previous paper [8] it is remarked that the limit of a monotonously decreasing sequence of Cousin-I domains in C^n is not necessarily a Cousin-I domain for $n \geqq 3$. In the present paper we shall prove that the limit of a monotonously increasing sequence of Cousin-I domains over a Stein manifold is a Cousin-I domain. Concerning the Cousin-II problem, however, we can prove that the limit of the monotonously increasing sequence of Cousin-II domains over a Stein manifold is a Cousin-II domain, only in case that it is simply connected. The proof is based on the theory of domains of holomorphy due to Docquier-Grauert [5] and the approximation theory due to Behnke [1].

§ 1. Increasing sequence of domains.

Let $\mathfrak{M}$ be a complex manifold. We say that (D, Φ) is a *domain over* $\mathfrak{M}$ if Φ is a holomorphic mapping of a complex manifold D into $\mathfrak{M}$ such that Φ is locally biholomorphic. A domain (D, Φ) over $\mathfrak{M}$ is called a *covering manifold over* $\mathfrak{M}$ if there exists a neighbourhood U of any point x of $\mathfrak{M}$ such that Φ maps each connected component of $\Phi^{-1}(U)$ biholomorphically onto U. Let (D_1, Φ_1) and (D_2, Φ_2) be domains over $\mathfrak{M}$. We say that (D_1, Φ_1) is a *domain over* (D_2, Φ_2) and write $(D_1, \Phi_1) < (D_2, \Phi_2)$ if there exists a holomorphic mapping τ of D_1 in D_2 such that $\Phi_1 = \Phi_2 \circ \tau$. By this relation $<$ the set of all domains over $\mathfrak{M}$ forms a partially ordered set. We consider a sequence $\{(D_n, \Phi_n); n = 1, 2, 3, \cdots\}$ of domains over $\mathfrak{M}$ such that $(D_n, \Phi_n) < (D_{n+1}, \Phi_{n+1})$ for $n \geqq 1$ and call it a *monotonously increasing sequence of domains over* $\mathfrak{M}$. Then there exists a holomorphic mapping τ_m^n of D_n in D_m such that $\Phi_n = \Phi_m \circ \tau_m^n$ for any $m \geqq n$.

Let E be the subset of the product set $\prod_{n=1}^{\infty} D_n$ consisting of all (x_n) which satisfies $x_n = \tau_n^N(x_N)$ $(n \geqq N)$ for some N. We say that (x_n) and $(y_n) \in E$ are equivalent modulo R if $x_n = y_n$ $(n \geqq N)$ for some N. The factor set E/R is denoted by D. Let x_n be a point of D_n. We put $x_m = \tau_m^n(x_n)$ for $m \geqq n$ and take x_m arbitrarily for $m<n$. If we define that $\tau_n(x_n)$ is the class which has (x_n) as a representative, then we have a mapping τ_n of D_n in D. Let x be an element of D, which is represented by $(x_n)\in E$ such that $x_n=\tau_n^N(x_N)$ $(n\geqq N)$. If we put $\Phi(x)=\Phi_N(x_N)$, we have a mapping Φ of D in $\mathfrak{M}$ such that $\Phi_n=\Phi\circ\tau_n$ for $n\geqq 1$. Let U_N be an open neighbourhood of x_N such that the restriction $\Phi_N|U_N$ of Φ_N to U_N is a biholomorphic mapping of U_N onto a local coordinate neighbourhood of $\Phi_N(x_N)$ in $\mathfrak{M}$. If we put $U_n=\tau_n^N(U_N)$ for $n\geqq N$, τ_n^N maps U_N biholomorphically onto U_n. Hence $\tau_N|U_N$ is an injective mapping of U_N onto the subset $V(x)$ of D consisting of all elements of D which have a representative $(x_n)\in E$ such that $x_n=\tau_n^N(x_N)$ $(n\geqq N)$ for $x_N\in U_N$. A subset of D containing such $V(x)$ is called a neighbourhood of x. If we define neighbourhoods of D in this way, D is a Hausdorff space. Let μ be a biholomorphic mapping of $\Phi_N(U_N)$ onto an open set Z of a complex Euclidean space. Then $\mu\circ(\Phi|V)=\mu\circ\Phi_N\circ(\tau_N|U_N)^{-1}$ is a homeomorphism of $V(x)$ onto Z. Let x' be another point of D. Then there exist, respectively, neighbourhoods $V'(x')$ and $U'_{N'}$ of x' and $x'_{N'}\in D_{N'}$ such that $\Phi_{N'}$ maps $U'_{N'}$ biholomorphically onto a local coordinate neighbourhood of $\Phi_{N'}(x'_{N'})$ in $\mathfrak{M}$ and that $\Phi_{N'}$ maps $U'_{N'}$ homeomorphically onto $V(x')$. Let μ' be a biholomorphic mapping of $\Phi_{N'}(U'_{N'})$ onto an open set Z' of a complex Euclidean space. Suppose that $V(x)\frown V'(x')\neq\phi$. Then

$(\mu\circ(\Phi|V))\circ(\mu'\circ(\Phi|V'))^{-1}=\mu\circ(\Phi_N|U_N)\circ(\tau^N_{N''}|U_N)^{-1}\circ(\tau^{N'}_{N''}|U'_{N'})\circ(\Phi_{N'}|U'_{N'})^{-1}\circ\mu'^{-1}$
is a biholomorphic mapping of $\mu'(\Phi'(V\frown V'))$ onto $\mu(\Phi(V\frown V'))$ where $N''=\max(N, N')$. Hence we can induce a complex structure in D such that τ_n is a holomorphic mapping of D_n in D $(n\geqq 1)$ and that Φ is a holomorphic mapping of D in $\mathfrak{M}$ which is locally biholomorphic. Therefore (D,Φ) is a domain over $\mathfrak{M}$.

Let (D',Φ') be a domain over $\mathfrak{M}$ such that $(D_n,\Phi_n)<(D',\Phi')$ $(n\geqq 1)$ with a holomorphic mapping τ'_n of D_n in D' satisfying $\tau'_n=\tau'_m\circ\tau^n_m$ for $m\geqq n$. In this case $\{(D_n,\Phi_n)\,;\,n=1,2,3,\cdots\}$ is called a *monotonously increasing sequence over* (D',Φ'). Let $(x_n)\in E$ be a representative of $x\in D$ such that $x_n=\tau^N_n(x_N)$ $n\geqq N$ for $x_N\in D_N$. If we put $\tau'(x)=\tau'_N(x_N)$, τ' is well-defined and a holomorphic mapping of D in D' such that $\Phi=\Phi'\circ\tau'$. Hence (D,Φ) is a domain over $\mathfrak{M}$ such that $\{(D_n,\Phi_n)\,;\,n=1,2,3,\cdots\}$ is a monotonously increasing sequence over (D,Φ) and that $(D,\Phi)<(D',\Phi')$ for all (D',Φ') over which $\{(D_n,\Phi_n)\,;\,n=1,2,3,\cdots\}$ is a monotonously increasing sequence. We call this domain (D,Φ) over $\mathfrak{M}$ the *limit of a monotonously increasing sequence* $\{(D_n,\Phi_n)\,;\,n=1,2,3,\cdots\}$ *of domains over* $\mathfrak{M}$ and denote it by $\lim(D_n,\Phi_n)$. If D_n is a domain in $\mathfrak{M}$ for each n, then D coincides with the usual $\lim D_n=\bigcup_{n=1}^{\infty}D_n$.

LEMMA 1. *Let G be a relatively compact subdomain of the limit (D,Φ) of a monotonously increasing sequence $\{(D_n,\Phi_n)\,;\,n=1,2,3,\cdots\}$ of domains over* *a complex manifold* $\mathfrak{M}$. *Then there exist an integer* m *and a relatively compact subdomain* G_m *of* D_m *such that* τ_m *maps* G_m *biholomorphically onto* G.

PROOF. Let $E_n \subset D_n$ be a relatively compact subdomain of D_n such that $\tau_m^n(E_n) \subset E_m$ and $\bar{G} \subset \bigcup_{n=1}^{\infty} \tau_n(E_n)$ for $m \geqq n \geqq 1$. Since $\{\tau_n(E_n); n=1, 2, 3, \cdots\}$ is an open covering of a compact set $\bar{G}$, there exists an integer n such that $\bar{G} \subset \tau_n(\bar{E}_n)$. We shall prove that τ_m maps $K_m = \tau_m^n(\bar{E}_n)$ injectively into D for sufficiently large m. If this is not true, there exist sequences $\{y_\nu; \nu=n, n+1, n+2, \cdots\}$, $\{x_\nu'; \nu=n, n+1, n+2, \cdots\}$ and $\{x_\nu''; \nu=n, n+1, n+2, \cdots\}$ of points y_ν in D and x_ν', x_ν'' in $\bar{E}_n$ such that $\tau_\nu^n(x_\nu') \neq \tau_\nu^n(x_\nu'')$ and $y_\nu = \tau_n(x_\nu') = \tau_n(x_\nu'')$. Since $\bar{E}_n$ and $\tau_n(\bar{E}_n)$ are compact, there exists a subsequence $\{p_\nu\}$ of $\{n, n+1, n+2, \cdots\}$ such that $x_{p_\nu}' \to x' \in \bar{E}_n$, $x_{p_\nu}'' \to x'' \in \bar{E}_n$ and $y_{p_\nu} \to y \in \tau_n(\bar{E}_n)$ as $\nu \to \infty$. Since $y_\nu = \tau_n(x_\nu') = \tau_n(x_\nu'')$, we have $y = \tau_n(x') = \tau_n(x'')$. Hence there exists an integer l such that $\tau_l^n(x') = \tau_l^n(x'')$. Therefore there exist neighbourhoods U' and U'' of x' and x'' such that τ_l^n maps U' and U'' biholomorphically onto $U_l = \tau_l^n(U') = \tau_l^n(U'')$ and that τ_l maps U_l biholomorphically onto $\tau_l(U)$. Since $x_{p_\nu}' \to x'$ and $x_{p_\nu}'' \to x''$ as $\nu \to \infty$, there exists an integer μ such that $x_\mu' \in U'$, $x_\mu'' \in U''$ and $\mu > l$. Therefore we have $\tau_l^n(x_\mu') = \tau_l^n(x_\mu'')$. Hence we have $\tau_\mu^n(x_\mu') = \tau_\mu^n(x_\mu'')$. But this is a contradiction. If we put $G_m = \tau_m^{-1}(G) \cap K_m$, we have our lemma.

§ 2. Domain of holomorphy.

Let $\{(D_i, \Phi_i); i \in I\}$ be a set of domains over $\mathfrak{M}$. We denote by D the set of all (x_i) such that a neighbourhood U of a point x in $\mathfrak{M}$ and a neighbourhood U_i of x_i in D_i for each i satisfy $x = \Phi_i(x_i)$ and $U = \Phi_i(U_i)$. We can naturally induce a complex structure in D such that the canonical mapping λ_i of D in D_i is holomorphic for each i and the mapping Φ defined by $\Phi = \Phi_i \circ \lambda_i$ is a mapping of D in $\mathfrak{M}$ which is locally biholomorphic. Hence (D, Φ) is a domain over $\mathfrak{M}$. (D, Φ) is called the *intersection of domains* (D_i, Φ_i) $(i \in I)$ and denoted by $\bigcap_{i \in I} (D_i, \Phi_i)$. If each D_i is a subdomain of $\mathfrak{M}$, then D coincides with the open kernel of the usual intersection $\bigcap_{i \in I} D_i$.

Let (X, Φ) be a domain over $\mathfrak{M}$ and f be a holomorphic function in X. A domain (X', Φ') over $\mathfrak{M}$ is called a *domain of holomorphic prolongation of* f if there exist a holomorphic function f' in X' and a holomorphic mapping τ of X in X' such that $\Phi = \Phi' \circ \tau$ and $f = f' \circ \tau$. In this case it holds that $(X, \Phi) < (X', \Phi')$. f' is called a *holomorphic prolongation of* f *over* (X', Φ'). Consider a fixed domain (X, Φ) over $\mathfrak{M}$ and a holomorphic function f in X. A domain $(\tilde{X}_f, \tilde{\Phi}_f)$ is called the *domain of maximal holomorphic prolongation of* f if the following conditions are satisfied:

(i) There exists a holomorphic function $\tilde{f}$ in $\tilde{X}_f$ which is a holomorphic prolongation of f over $(\tilde{X}_f, \tilde{\Phi}_f)$.

(ii) If f' is a holomorphic prolongation of f over a domain (X', Φ') over $\mathfrak{M}$, then $\tilde{f}$ is a holomorphic prolongation of f' over $(\tilde{X}_f, \tilde{\Phi}_f)$.

A domain over $\mathfrak{M}$ is called a *domain of holomorphy* if it is the domain of maximal holomorphic prolongation of a holomorphic function in a domain over $\mathfrak{M}$. Due to Cartan [3] (exposé 7) there exists such domain $(\tilde{X}_f, \tilde{\Phi}_f)$ for any holomorphic function f in a domain (X, Φ) over $\mathfrak{M}$. If $\mathfrak{M}$ is a Stein manifold, a domain of holomorphy over $\mathfrak{M}$ is holomorphically convex from Docquier-Grauert [5].

Conversely, suppose that (X, Φ) is a holomorphically convex domain over a Stein manifold $\mathfrak{M}$. We can construct a holomorphic function f in X by using Bochner-Martin's method [2] such that f is unbounded at each boundary point of (X, Φ). Since the holomorphically convex domain (X, Φ) over $\mathfrak{M}$ is a Stein manifold (see Grauert [7]), X is holomorphically separable, that is, there exists a holomorphic function in X which takes different values at two given different points in X. Let $\Delta=\{x_i\,;\, i=1, 2, 3, \cdots\}$ be a dense subset of X such that $\Phi^{-1}\{x_i\}\subset\Delta$ for any i. There exists a holomorphic function $f_{ij}=f_{ji}$ in X such that $f_{ij}(x_i)\neq f_{ij}(x_j)$ for $i\neq j$. If we take a suitable double sequence $\{a_{ij}\,;\, i, j=1, 2, 3, \cdots\}$ of complex numbers, $g=\sum_{i\neq j} a_{ij}f_{ij}$ converges absolutely and uniformly in any compact subset of X and $g(x_i)\neq g(x_j)$ for any $i\neq j$. Then, for suitable complex numbers a and b, $h=af+bg$ is a holomorphic function in X which is unbounded at each boundary point of (X, Φ) and satisfies $h(x_i)\neq h(x_j)$ for any $i\neq j$. (X, Φ) is the domain of maximal holomorphic prolongation of h and is a domain of holomorphy.

Hence we obtained

LEMMA 2. *A domain (X, Φ) over a Stein manifold $\mathfrak{M}$ is holomorphically convex, if and only if (X, Φ) is a domain of holomorphy.*

Hereafter we shall denote a Stein manifold by $\mathfrak{M}$. Let (X, Φ) be a domain over $\mathfrak{M}$ and O_X be the set of all holomorphic functions in X. For any $f\in O_X$ we denote by $(\tilde{X}_f, \tilde{\Phi}_f)$ the domain of maximal holomorphic prolongation of f. We denote by $(\tilde{X}, \tilde{\Phi})$ the intersection of $(\tilde{X}_f, \tilde{\Phi}_f)$ for all $f\in O_X$. $(\tilde{X}, \tilde{\Phi})$ can be characterized by the following properties:

(i) For any $f\in O_X$, there exists a holomorphic prolongation of f over $(\tilde{X}, \tilde{\Phi})$.

(ii) If a domain (X', Φ') has the above property, then $(X', \Phi')<(\tilde{X}, \tilde{\Phi})$.

$(\tilde{X}, \tilde{\Phi})$ is called the *envelope of holomorphy of a domain* (X, Φ).

LEMMA 3. *The envelope of holomorphy $(\tilde{X}, \tilde{\Phi})$ of a domain (X, Φ) over a Stein manifold $\mathfrak{M}$ is a domain of holomorphy.*

PROOF. Let O_X be the set of all holomorphic functions in X and $(\tilde{X}_f, \tilde{\Phi}_f)$ be the domain of maximal holomorphic prolongation of $f\in O_X$. We consider an open covering $\{V_i; i\in I\}$ of $\mathfrak{M}$ with the following properties:

There exists a holomorphic mapping μ_i of V_i onto a domain of holomorphy W_i of a complex Euclidean space. We put $U_i=\tilde{\Phi}^{-1}(V_i)$ for any $i\in I$. Then $(U_i, \mu_i\circ(\tilde{\Phi}|U_i))$ is the intersection of holomorphically convex open sets $(\tilde{\Phi}_f^{-1}(V_i), \mu_i\circ\tilde{\Phi}_f|\tilde{\Phi}_f^{-1}(V_i))$ over the complex Euclidean space for all $f\in O_X$. Therefore $(U_i, \mu_i\circ(\tilde{\Phi}|U_i))$ is holomorphically convex by Cartan-Thullen's results [**4**]. Hence $(\tilde{X}, \tilde{\Phi})$ is p_4-convex in the sense of Docquier-Grauert [**5**]. Therefore $(\tilde{X}, \tilde{\Phi})$ is holomorphically convex and is a domain of holomorphy from Lemma 2.

LEMMA 4. *Let $(D_1, \Phi_1)<(D_2, \Phi_2)$ be domains over a Stein manifold $\mathfrak{M}$ and τ be a holomorphic mapping of D_1 in D_2 with $\Phi_1=\Phi_2\circ\tau$. Let $(\tilde{D}_1, \tilde{\Phi}_1)$ and $(\tilde{D}_2, \tilde{\Phi}_2)$ be, respectively, the envelopes of holomorphy of (D_1, Φ_1) and (D_2, Φ_2). Let λ_1 and λ_2 be, respectively, the canonical mapping of D_1 in $\tilde{D}_1$ and that of D_2 in $\tilde{D}_2$. Then there exists a holomorphic mapping $\tilde{\tau}$ of $\tilde{D}_1$ in $\tilde{D}_2$ such that $\lambda_2\circ\tau=\tilde{\tau}\circ\lambda_1$.*

PROOF. By Remmert's result [**9**] there exists a biholomorphic mapping μ of $\tilde{D}_2$ onto a regular analytic set A in C^α as $\tilde{D}_2$ is a Stein manifold. Then $\mu\circ\lambda_2\circ\tau$ is a holomorphic mapping of D_1 in C^α. Since $(\tilde{D}_1, \tilde{\Phi}_1)$ is the envelope of holomorphy of (D_1, Φ_1), there exists a holomorphic mapping ϕ of $\tilde{D}_1$ in C^α such that $\mu\circ\lambda_2\circ\tau=\phi\circ\lambda_1$. We shall prove that $\phi(\tilde{D}_1)\subset A$. Suppose that $\phi(x_1)\notin A$ for $x_1\in\tilde{D}_1$. x_1 can be joined by a smooth Jordan curve $C=\{x(t); 0\leqq t\leqq 1\}$ in $\tilde{D}_1$ with a point $x_0\in\lambda_1(D_1)\subset\tilde{D}_1$ such that $x_0=x(0)$ and $x_1=x(1)$. Let t_0 be the supremum of t' such that $\{x(t); 0\leqq t\leqq t'\}\subset A$. Since A is closed, we have $0<t_0<1$. Since $z_0=\phi(x(t_0))\in A$, there exist a neighbourhood V of z_0 and holomorphic functions $f_1, f_2, \cdots$ and f_s in C^α such that $V\cap A=\{z; f_1(z)=f_2(z)=\cdots=f_s(z)=0, z\in V\}$. From the theorem of identity $f_i\circ\mu\circ\lambda_2\circ\tau$ is identically zero in D_1 for any i. There exists t_1 such that $t_0<t_1<1$ and $\phi(x(t_1))\in V-A$. Then f_i satisfies $a=f_i(\phi(x(t_1)))\neq 0$ for some $1\leqq i\leqq s$. $1/(f_i\circ\phi-a)$ is a meromorphic function in $\tilde{D}_1$ which is identically $-1/a$ in the open subset $\lambda_1(D_1)$ of $\tilde{D}_1$ and has a pole at a point $x(t_1)$ of the envelope of holomorphy $(\tilde{D}_1, \tilde{\Phi}_1)$ of (D_1, Φ_1). But this is a contradiction. Hence we have $\phi(\tilde{D}_1)\subset A$. Therefore the mapping $\tilde{\tau}=\mu^{-1}\circ\phi$ is a holomorphic mapping of $\tilde{D}_1$ in $\tilde{D}_2$ such that $\lambda_2\circ\tau=\tilde{\tau}\circ\lambda_1$. Since $\tilde{\Phi}_1\circ\lambda_1=\tilde{\Phi}_2\circ\tilde{\tau}\circ\lambda_1$, we have $\tilde{\Phi}_1=\tilde{\Phi}_2\circ\tilde{\tau}$.

LEMMA 5. *Let $\{(D_n, \Phi_n); n=1, 2, 3, \cdots\}$ be a monotonously increasing sequence of domains of holomorphy over a Stein manifold $\mathfrak{M}$. Then its limit (D, Φ) is also a domain of holomorphy.*

PROOF. It suffices to prove that D is p_6-convex in the sense of Docquier-Grauert [**5**]. We put $B(a)=\{z; |z_1|\leqq 1, |z_2|<a, \cdots, |z_\alpha|<a\}$ and $\delta B(a)=\{z; |z_1|=1, |z_2|<a, \cdots, |z_\alpha|<a\}$ where α is the dimension of $\mathfrak{M}$. Let φ be a biholomorphic mapping of $B=B(1)$ in D such that $\varphi(\delta B)\Subset D$. Let W be a relatively compact open neighbourhood of $\varphi(\delta B\cup\overline{B(1/2)})$. From Lemma 1

there exist an integer m_0 and a relatively compact open set W_0 in D_{m_0} such that τ_{m_0} maps W_0 biholomorphically onto W. For any $1/2<a<1$ there exists $\varepsilon>0$ such that $\varphi(\overline{G(a)})\subset W_0$ for $G(a)=\{z;\ 1-\varepsilon<|z_1|<1+\varepsilon,\ |z_2|<a,\cdots,|z_\alpha|<a\}\cup\{z;\ |z_1|<1+\varepsilon,\ |z_2|<1/2,\cdots,|z_\alpha|<1/2\}$. $\eta=(\tau_{m_0}|W_0)^{-1}\circ\varphi$ maps $G(a)$ biholomorphically in D_{m_0}. As in the proof of Lemma 4 there exists a holomorphic mapping $\tilde{\eta}_a$ of $\widetilde{G(a)}=\{z;|z_1|<1+\varepsilon,|z_2|<a,\cdots,|z_\alpha|<a\}$, which is the envelope of holomorphy of $G(a)$ (see e.g. Bochner-Martin [2], Chap. IV), in D_{m_0} such that $\tilde{\eta}_a=\eta_a$ in $G(a)$. From the theorem of identity we have $\varphi=\tau_{m_0}\circ\tilde{\eta}_a$ in $\widetilde{G(a)}$. Since φ is biholomorphic, $\tilde{\eta}_a$ is also biholomorphic. Thus we have proved that there exists a biholomorphic mapping $\tilde{\eta}$ of B in D_{m_0} such that $\varphi=\tau_{m_0}\circ\tilde{\eta}$ and $\tilde{\eta}(\delta B)\Subset D_{m_0}$. Since D_{m_0} is p_6-convex, we have $\tilde{\eta}(B)\Subset D_{m_0}$. Therefore we have $\varphi(B)=\tau_{m_0}(\eta(B))\Subset D$. Hence D is p_6-convex.

§ 3. Cohomology of an increasing sequence of domains.

Let $\{(D_n,\Phi_n);n=1,2,3,\cdots\}$ be a monotonously increasing sequence of domains over $\mathfrak{M}$, (D,Φ) be its limit and τ_m^n and τ_n be, respectively, the canonical mapping of D_n in D_m $(m\geqq n)$ and that of D_n in D $(n\geqq 1)$. Then there exists a canonical homomorphism π_n^m of $H^1(D_m,\mathfrak{O})$ in $H^1(D_n,\mathfrak{O})$ for $m\geqq n$ such that $\pi_n^l=\pi_n^m\circ\pi_m^l$ for $l\geqq m\geqq n$ where $\mathfrak{O}$ is the sheaf of all germs of holomorphic functions. Hence $\{H^1(D_n,\mathfrak{O});\pi_n^m\}$ is an inverse system of C-module over a directed set $\{1,2,3,\cdots\}$. We consider its inverse limit and denote it by $\lim H^1(D_n,\mathfrak{O})$. The canonical homomorphisms of $H^1(D,\mathfrak{O})$ in $H^1(D_n,\mathfrak{O})$ induce the canonical homomorphism of $H^1(D,\mathfrak{O})$ in $\lim H^1(D_n,\mathfrak{O})$. Under these assumptions we have

LEMMA 6. *The canonical homomorphism of $H^1(D,\mathfrak{O})$ in $\lim H^1(D_n,\mathfrak{O})$ is injective.*

PROOF. Since the canonical homomorphism of $H^1(\mathfrak{U},\mathfrak{F})$ in $H^1(X,\mathfrak{F})$ is injective for any sheaf $\mathfrak{F}$ of abelian groups in a topological space X and for any open covering $\mathfrak{U}$ of X, it suffices to prove the following statement:

If $\{f_{ij}\}$ is an element of $Z^1(\mathfrak{V},\mathfrak{O})$ (cocycle) for any open covering $\mathfrak{V}=\{V_i;\ i\in I\}$ of D such that $\{f_{ij}\circ\tau_n\}\in B^1(\tau_n^{-1}(\mathfrak{V});\mathfrak{O})$ (coboundary) for any $n\geqq 1$ where $\tau_n^{-1}(\mathfrak{V})=\{\tau_n^{-1}(V_i);i\in I\}$, then $\{f_{ij}\}\in B^1(\mathfrak{V},\mathfrak{O})$.

Let $(\tilde{D}_n,\tilde{\Phi}_n)$ and $(\tilde{D},\tilde{\Phi})$ be, respectively, the envelope of holomorphy of (D_n,Φ_n) $(n=1,2,3,\cdots)$ and (D,Φ). From Lemma 4 $\{(\tilde{D}_n,\tilde{\Phi}_n);n=1,2,3,\cdots\}$ is a monotonously increasing sequence of domains over (D,Φ). Hence from Lemma 5 $\lim(\tilde{D}_n,\tilde{\Phi}_n)$ is a domain of holomorphy satisfying $(D,\Phi)<\lim(\tilde{D}_n,\tilde{\Phi}_n)<(\tilde{D},\tilde{\Phi})$. Since $(\tilde{D},\tilde{\Phi})$ is the envelope of holomorphy of (D,Φ), we have $(\tilde{D},\tilde{\Phi})=\lim(\tilde{D}_n,\tilde{\Phi}_n)$. We denote by τ_n, τ_m^n, $\tilde{\tau}_n$, $\tilde{\tau}_m^n$, λ_n and λ the canonical mapping of D_n in D, that of D_n in D_m, that of $\tilde{D}_n$ in $\tilde{D}$, that of $\tilde{D}_n$ in $\tilde{D}_m$, that of D_n in $\tilde{D}_n$ and that of D in $\tilde{D}$, respectively. Then the commutativity holds in the following diagram:

$$\begin{array}{ccccc} D_n & \xrightarrow{\tau_m^n} & D_m & \xrightarrow{\tau_m} & D \\ \downarrow{\scriptstyle \lambda_n} & & \downarrow{\scriptstyle \lambda_m} & & \downarrow{\scriptstyle \lambda} \\ \tilde{D}_n & \xrightarrow[\tilde{\tau}_m^n]{} & \tilde{D}_m & \xrightarrow[\tilde{\tau}_m]{} & \tilde{D} \end{array}$$

Let $\{Q_n\,;\, n=1, 2, 3, \cdots\}$ be a sequence of relatively compact open subsets of D such that $Q_n \Subset Q_{n+1}$ $(n \geqq 1)$ and $D=\bigcup_{n=1}^{\infty} Q_n$. From Lemmas 2 and 3 $\tilde{D}$ is holomorphically convex. There exists a sequence of analytic polycylinders P_n defined by holomorphic functions in $\tilde{D}$ such that $P_n \Subset P_{n+1}$, $\lambda(Q_n) \subset P_n$ for $n \geqq 1$ and $\tilde{D}=\bigcup_{n=1}^{\infty} P_n$. Since $(D, \Phi)=\lim (D_n, \Phi_n)$ and $(\tilde{D}, \tilde{\Phi})=\lim (\tilde{D}_n, \tilde{\Phi}_n)$, there exists a monotonously increasing sequence $\{\nu_n\,;\, n=1, 2, 3, \cdots\}$ of integers such that τ_{ν_n} and $\tilde{\tau}_{\nu_n}$ map, respectively, relatively compact open subsets $'Q_{\nu_n}$ of D_{ν_n} and $'P_{\nu_n}$ of $\tilde{D}_{\nu_n}$ biholomorphically onto Q_{ν_n} of D and P_{ν_n} of $\tilde{D}$ for $n \geqq 1$ and further that

$$\tau_{\nu_{n+1}}^{\nu_n}('Q_{\nu_n}) \subset 'Q_{\nu_{n+1}}, \quad \tau_{\nu_{n+1}}^{\nu_n}('P_{\nu_n}) \subset 'P_{\nu_{n+1}}, \quad \lambda_{\nu_n}('Q_{\nu_n}) \subset 'P_{\nu_n} \qquad (n \geqq 1).$$

Without losing generality, we may suppose that $\nu_n = n$.

Since $\{f_{ij} \circ \tau_n\} \in B^1(\tau_n^{-1}(\mathfrak{B}), \mathfrak{O})$, there exists $\{f_i^n\} \in C^0(\tau_n^{-1}(\mathfrak{B}), \mathfrak{O})$ such that $f_{ij} \circ \tau_n = f_i^n - f_j^n$ in $\tau_n^{-1}(V_i) \cap \tau_n^{-1}(V_j) \neq \phi$. If we put $f^n = f_i^n - f_i^{n+1} \circ \tau_{n+1}^n$ in $\tau_n^{-1}(V_i)$, then, f^n is well-defined and holomorphic in D_n. Since $(\tilde{D}_n, \tilde{\Phi}_n)$ is the envelope of holomorphy of (D_n, Φ_n), there exists a holomorphic prolongation $\tilde{f}^n$ of f^n over $(\tilde{D}_n, \tilde{\Phi}_n)$. There holds $f^n = \tilde{f}^n \circ \lambda_n$ for $n \geqq 1$. Since $\tilde{f}^n \circ (\tilde{\tau}_n | 'P_n)^{-1}$ is a holomorphic function in P_n, there exists a holomorphic function $\tilde{h}^n$ in $\tilde{D}$ $(n \geqq 1)$ which satisfies $|\tilde{f}^n \circ (\tilde{\tau}_n | 'P_n)^{-1} - \tilde{h}^n| < 2^{-n}$ in P_{n-1} for $n \geqq 2$ from Behnke's approximation theory [**1**]. If we put $h^n = \tilde{h}^n \circ \lambda$, the holomorphic function h^n in D satisfies $|f^n \circ (\tau_n | 'Q_n)^{-1} - h^n| < 2^{-n}$ in Q_{n-1} for $n \geqq 2$. We consider holomorphic functions in D defined by $g^1 = 0$, $g^n = h^1 + h^2 + \cdots + h^{n-1}$ for $n \geqq 2$. Then the coboundary of $\{f_i^n \circ (\tau_n | 'Q_n)^{-1} + g^n\} \in C^0(\mathfrak{B} \cap Q_n, \mathfrak{O})$, where $\mathfrak{B} \cap Q_n = \{V_i \cap Q_n\,;\, i \in I\}$, is $\{f_{ij} | Q_n\} \in Z^1(\mathfrak{B} \cap Q_n, \mathfrak{O})$. There holds

$$(f_i^n \circ (\tau_n | 'Q_n)^{-1} + g^n) - (f_i^{n+1} \circ (\tau_{n+1} | 'Q_{n+1})^{-1} + g^{n+1}) = f^n \circ (\tau_n | 'Q_n)^{-1} - h^n$$

in any $V_i \cap Q_n$. Hence $f_i^n \circ (\tau_n | 'Q_n)^{-1} + g^n$ converges uniformly in any compact subset of V_i to a holomorphic function f_i in V_i. Since

$$f_{ij} = (f_i^n \circ (\tau_n | 'Q_n)^{-1} + g^n) - (f_j^n \circ (\tau_n | 'Q_n)^{-1} + g^n)$$

in $V_i \cap V_j \cap Q_n$, the coboundary of $\{f_i\} \in C^0(\mathfrak{V}, \mathfrak{O})$ is just the original cocycle $\{f_{ij}\}$.

§ 4. Cousin domains.

A collection $\mathfrak{C} = \{(m_i, V_i) ; i \in I\}$ of pairs of an open subset V_i of a complex manifold X and a meromorphic function m_i in V_i is called a *Cousin-I* (or *Cousin-II*) *distribution in* X if $m_i - m_j \in H^0(V_i \cap V_j, \mathfrak{O})$ (or $m_i/m_j \in H^0(V_i \cap V_j, \mathfrak{O}^*)$) for any $V_i \cap V_j \neq \phi$ and $\{V_i ; i \in I\}$ is an open covering of X where $\mathfrak{O}^*$ is the sheaf of all germs of holomorphic mapping in $\dot{C} = GL(1, C)$. A meromorphic function m in X is called a *solution of the Cousin-I* (or *Cousin-II*) *distribution* $\mathfrak{C}$ if $m - m_i \in H^0(V_i, \mathfrak{O})$ (or $m/m_i \in H^0(V_i, \mathfrak{O}^*)$) for any $i \in I$. A meromorphic function M in the universal covering manifold $(X^\#, \lambda^\#)$ of X is called a *multiform solution of* $\mathfrak{C}$ if M is the solution of the Cousin distribution $\{(m_i \circ \lambda^\#, \lambda^{\#-1}(V_i)) ; i \in I\}$. If any Cousin-I (or Cousin-II) distribution in X has a solution, X is called a *Cousin-I* (or *Cousin-II*) *manifold*. If any Cousin-I (or Cousin-II) distribution in X has a multiform solution, X is called a *multiform Cousin-I* (or *Cousin-II*) *manifold*. A complex manifold X with the vanishing fundamental group $\pi_1(X)$ is called *simply connected*.

PROPOSITION 1. *The limit* (D, Φ) *of a monotonously increasing sequence* $\{(D_n, \Phi_n) ; n = 1, 2, 3, \cdots\}$ *of Cousin-I domains over a Stein manifold* $\mathfrak{M}$ *is a Cousin-I domain. However, for any* $\alpha \geqq 3$ *there exists an example of the limit of a monotonously decreasing sequence of Cousin-I domains in* C^α *which is not even a multiform Cousin-I domain.*

PROOF. Let $\mathfrak{C} = \{(m_i, V_i) ; i \in I\}$ be a Cousin-I distribution in D. Then $\{(m_i \circ \tau_n, \tau_n^{-1}(V_i)) ; i \in I\}$ is a Cousin-I distribution in D_n. If we put $f_{ij} = m_i - m_j \in H^0(V_i \cap V_j, \mathfrak{O})$ and $f_{ij}^n = m_i \circ \tau_n - m_j \circ \tau_n \in H^0(\tau_n^{-1}(V_i \cap V_j), \mathfrak{O})$, then $\{f_{ij}^n\} \in Z^1(\tau_n^{-1}(\mathfrak{V}), \mathfrak{O})$ is the canonical image of $\{f_{ij}\} \in Z^1(\mathfrak{V}, \mathfrak{O})$ where $\mathfrak{V} = \{V_i ; i \in I\}$ and $\tau_n^{-1}(\mathfrak{V}) = \{\tau_n^{-1}(V_i) ; i \in I\}$. Since D_n is a Cousin-I domain for any n, $\{f_{ij}^n\} \in B^1(\tau_n^{-1}(\mathfrak{V}), \mathfrak{O})$ for any n. Hence there exists a holomorphic function f_i in V_i for any $i \in I$ such that $f_{ij} = f_i - f_j$ in $V_i \cap V_j$ from Lemma 6. If we put $m = m_i - f_i$ in V_i, m is well-defined and a solution of $\mathfrak{C}$.

As for the latter half, we put

$$D = \{z ; |z_1| < 1, |z_2| < 1, \cdots, |z_\alpha| < 1\} - \{z ; z_1 = z_2 = 0\}$$

and

$$D_p = \{z ; |z_1| < (p+1)/p, |z_2| < (p+1)/p, \cdots, |z_\alpha| < (p+1)/p\} \\ - \bar{D} \cap \{z ; z_1 = z_2 = 0\}$$

for $p = 1, 2, 3, \cdots$. As shown in the previous paper [8], D is the limit (precisely the open kernel of $\bigcap_{p=1}^{\infty} D_p$) of the monotonously decreasing sequence of Cousin-I domains D_p in C^α but is not a Cousin-I domain. Since D is simply connected, D is not even a multiform Cousin-I domain.

Let (D, Φ) be a domain over a Stein manifold $\mathfrak{M}$ and $(D^\#, \lambda^\#)$ be the universal covering manifold of D. If we put $\Phi^\# = \Phi \circ \lambda^\#$, $(D^\#, \Phi^\#)$ is a domain over $\mathfrak{M}$. If (D, Φ) is a domain of holomorphy, it is p_7-convex in the sense of Docquier-Grauert [5]. Hence $(D^\#, \Phi^\#)$ is p_7-convex and is a domain of holomorphy. This follows also from the result of Stein [11] that a covering manifold over a Stein manifold is a Stein manifold.

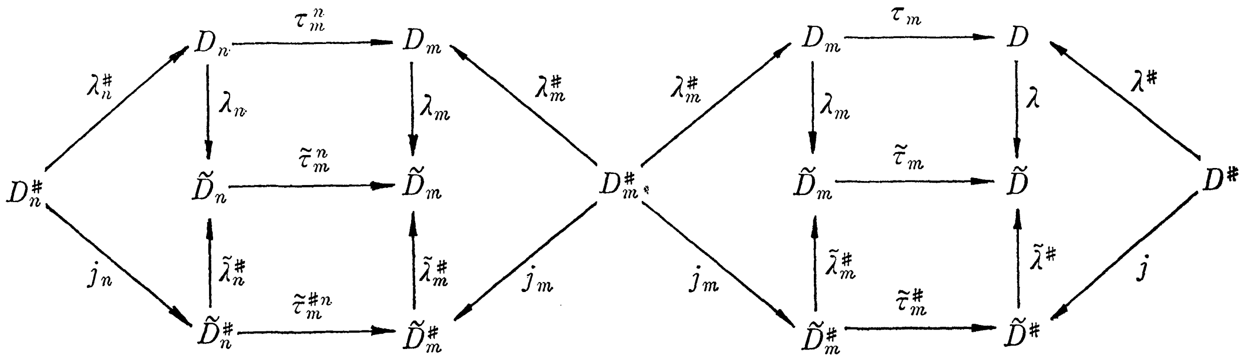
Let $\{(D_n, \Phi_n)\,;\, n=1, 2, 3, \cdots\}$ be a monotonously increasing sequence of domains over $\mathfrak{M}$, (D, Φ) be its limit, $(\tilde{D}_n, \tilde{\Phi}_n)$ and $(\tilde{D}, \tilde{\Phi})$ be, respectively, the envelope of holomorphy of (D_n, Φ_n) and (D, Φ). Let $(D_n^\#, \lambda_n^\#)$, $(D^\#, \lambda^\#)$, $(\tilde{D}_n^\#, \tilde{\lambda}_n^\#)$ and $(\tilde{D}^\#, \tilde{\lambda}^\#)$ be, respectively, the universal covering manifolds of D_n, D, $\tilde{D}_n$ and $\tilde{D}$ for any n. If we put

$$\Phi_n^\# = \Phi_n \circ \lambda_n^\#\,, \quad \Phi^\# = \Phi \circ \lambda^\#\,, \quad \tilde{\Phi}_n^\# = \tilde{\Phi}_n \circ \lambda_n^\#\,, \quad \tilde{\Phi}^\# = \tilde{\Phi} \circ \lambda^\#\,,$$

then the commutativity holds in the three-dimensional diagram obtained by adding holomorphic mappings

$$D_n^\# \xrightarrow{\tau_m^{\#n}} D_m^\# \xrightarrow{\tau_m^\#} D^\#$$

to the following diagram and identifying the same symbols in it where each holomorphic mapping is the canonical one $(m \geqq n)$:



From Lemma 1 $(D^\#, \Phi^\#)$ and $(\tilde{D}^\#, \tilde{\Phi}^\#)$ are, respectively, the limits of monotonously increasing sequences $\{(D_n^\#, \Phi_n^\#)\,;\, n=1, 2, 3, \cdots\}$ and $\{(\tilde{D}_n^\#, \tilde{\Phi}_n^\#)\,;\, n=1, 2, 3, \cdots\}$. Let $\{P_n\,;\, n=1, 2, 3, \cdots\}$, $\{Q_n\,;\, n=1, 2, 3, \cdots\}$, $\{R_n\,;\, n=1, 2, 3, \cdots\}$ and $\{S_n\,;\, n=1, 2, 3, \cdots\}$ be, respectively, sequences of relatively compact subdomains of $\tilde{D}$, D, $\tilde{D}^\#$ and $D^\#$ with the following properties:

$$P_n \subset P_{n+1}\,, \quad Q_n \subset Q_{n+1}\,, \quad R_n \subset R_{n+1}\,, \quad S_n \subset S_{n+1}\,,$$

$$\bigcup_{n=1}^{\infty} P_n = \tilde{D}\,, \quad \bigcup_{n=1}^{\infty} Q_n = D\,, \quad \bigcup_{n=1}^{\infty} R_n = \tilde{D}^\#\,, \quad \bigcup_{n=1}^{\infty} S_n = D^\#\,,$$

$$\lambda(Q_n) \subset P_n\,, \quad \tilde{\lambda}^\#(R_n) \subset P_n\,, \quad \lambda^\#(S_n) \subset Q_n\,, \quad j(S_n) \subset R_n\,.$$

P_n and Q_n are, respectively, analytic polycylinders defined by holomorphic functions in $\tilde{D}$ and $\tilde{D}^{\#}$.

There exists a subsequence $\{\nu_n ; n=1, 2, 3, \cdots\}$ of $\{1, 2, 3, \cdots\}$ with the following properties:

There exist, respectively, subdomains P'_n, Q'_n, R'_n and S'_n of $\tilde{D}_{\nu_n}$, D_{ν_n}, $\tilde{D}^{\#}_{\nu_n}$ and $D^{\#}_{\nu_n}$ such that $\tilde{\tau}_{\nu_n}$, τ_{ν_n}, $\tilde{\tau}^{\#}_{\nu_n}$ and $\tau^{\#}_{\nu_n}$ map biholomorphically P'_n, Q'_n, R'_n and S'_n onto P_{ν_n}, Q_{ν_n}, R_{ν_n} and S_{ν_n} and that

$$\lambda_{\nu_n}(Q'_n) \subset P'_n\,, \quad \tilde{\lambda}^{\#}_{\nu_n}(R'_n) \subset P'_n\,, \quad \lambda^{\#}_{\nu_n}(S'_n) \subset Q'_n\,, \quad j_{\nu_n}(S'_n) \subset R'_n\,,$$

$$\tilde{\tau}^{\nu_n}_{\nu_{n+1}}(P'_n) \subset P'_{n+1}\,, \quad \tau^{\nu_n}_{\nu_{n+1}}(Q'_n) \subset Q'_{n+1}\,, \quad \tilde{\tau}^{\#\nu_n}_{\nu_{n+1}}(R'_n) \subset R'_{n+1}\,, \quad \tau^{\#\nu_n}_{\nu_{n+1}}(S'_n) \subset S'_{n+1}\,.$$

We may suppose that $\nu_n = n$. Let $\mathfrak{C} = \{(m_i, V_i) ; i \in I\}$ be a Cousin-II distribution in (D, Φ). We shall suppose that the Cousin-II distribution $\{(m_i \circ \tau_n, \tau_n^{-1}(V_i)) ; i \in I\}$ has a solution in D_n for any n. There exists a meromorphic function m^n in D_n such that $m^n/m_i \circ \tau_n \in H^0(\tau_n^{-1}(V_i), \mathfrak{O}^*)$ for any $i \in I$. If we put $f^n = m^n/m^{n+1} \circ \tau^n_{n+1}$, then $f^n \in H^0(D_n, \mathfrak{O}^*)$. Since $(\tilde{D}_n, \tilde{\Phi}_n)$ is the envelope of holomorphy of (D_n, Φ_n), there exists a holomorphic prolongation $\tilde{f}^n$ of f^n which satisfies $\tilde{f}^n \in H^0(\tilde{D}_n, \mathfrak{O}^*)$ and $f^n = \tilde{f}^n \circ \lambda_n$ for any n. Then $\log(\tilde{f}^n \circ \tilde{\lambda}^{\#}_n) \in H^0(\tilde{D}^{\#}_n, \mathfrak{O})$ for $n \geqq 1$. There holds $\log(\tilde{f}^n \circ \tilde{\lambda}^{\#}_n \circ (\tilde{\tau}^{\#}_n | 'R_n)^{-1}) \in H^0(R_n, \mathfrak{O})$ for $n \geqq 1$. Since R_n is an analytic polycylinder defined by holomorphic functions in $\tilde{D}^{\#}$, from Behnke's approximation theory [1] there exists a holomorphic function $\tilde{h}^n$ in $\tilde{D}^{\#}$ such that

$$|\log(\tilde{f}^n \circ \tilde{\lambda}^{\#}_n \circ (\tilde{\tau}^{\#}_n | 'R_n)^{-1}) - \tilde{h}^n| < 2^{-n-2} \text{ in } R_{n-1} \text{ for } n \geqq 2\,.$$

We put $H^n = \exp(\tilde{h}_n \circ j) \in H^0(D^{\#}, \mathfrak{O}^*)$. There holds $|f^n \circ (\tau_n | 'Q_n)^{-1} \circ \lambda^{\#}/H^n - 1| < 2^{-n}$ in S_{n-1} for $n \geqq 2$. We put $G^1 = 1$, $G^n = H^1 H^2 \cdots H^{n-1} \in H^0(D^{\#}, \mathfrak{O}^*)$ $(n \geqq 2)$. Then $M^n = (m^n \circ (\tau_n | Q'_n)^{-1} \circ \lambda^{\#}) G^n$ is a meromorphic function in S_n. There holds $|M^n/M^{n+1} - 1| < 2^{-n}$ in S_{n-1}. Therefore $\{M^n ; n=1, 2, 3, \cdots\}$ converges uniformly to a meromorphic function M in any compact subset of $D^{\#}$. There holds $M/m_i \circ \lambda^{\#} \in H^0(\lambda^{\#-1}(V_i), \mathfrak{O}^*)$ for any $i \in I$. Hence M is the solution of the Cousin-II distribution $\{m_i \circ \lambda^{\#}, \lambda^{\#-1}(V_i) ; i \in I\}$ in $D^{\#}$ and is a multiform solution of $\mathfrak{C}$. Therefore we have

PROPOSITION 2. *The limit of a monotonously increasing sequence of Cousin-II domains over a Stein manifold is a multiform Cousin-II domain.*

COROLLARY. *If the limit of a monotonously increasing sequence of Cousin-II domains over a Stein manifold is simply connected, it is a Cousin-II domain.*

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