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On torsion classes of Abelian groups

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§1. Introduction.

Recently, Serre classes of Abelian groups have been classified by Balcerzyk [1], which answers a question posed by Hu [4]. Closely related are *torsion classes*, or *T-classes*, which are classes of groups closed under homomorphic images, formation of infinite direct sums, and under group extensions, i.e., if the sequence $0\rightarrow A\rightarrow B\rightarrow C\rightarrow 0$ is exact with A and C in the class, then B is also in the class. One does not require, however, that the class be closed under subgroups, as in the case of Serre classes.

It is easily verified that if $\mathcal{T}$ is a torsion class, then each group G has a maximal $\mathcal{T}$-subgroup T, and that the quotient G/T has no $\mathcal{T}$-subgroups except (0). If $\mathcal{S}$ is any class of groups, the minimal T-class containing $\mathcal{S}$ is denoted by $T(\mathcal{S})$. For further results concerning torsion classes in general Abelian categories, the reader is referred to [2] or [3].

In this paper we shall discuss the problem of finding all torsion classes of Abelian groups. Our main results are the complete classification of T-classes contained in the class $\mathcal{T}_0$ of usual torsion groups and the result that for any T-class $\mathcal{T}$ of groups, the equality $\mathcal{T}=T[(\mathcal{T}\cap\mathcal{T}_0)\cup(\mathcal{T}\cap\mathcal{F}_0)]$, which then reduces the classification of all torsion classes for groups to the classification of those torsion classes generated by subclasses of the class $\mathcal{F}_0$ of usual torsion-free groups. Also, we show that if $\mathcal{T}$ is any non-zero T-class of groups, $\mathcal{T}\cap\mathcal{T}_0\neq(0)$, and that the smallest T-class having non-zero intersection with any non-zero T-class in the class of divisible torsion groups.

We shall use the following notation: $C(n)$ denotes the cyclic group of order n, $Z(p^{\infty})$ denotes the p-primary quasicyclic group, Q denotes the additive group of rational numbers, Q_p denotes the subgroup of Q of fractions with denominator a power of p, and finally, Z denotes the integers.

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§ 2. *T*-classes of torsion groups.

In this section we classify all T-classes consisting of ordinary torsion groups.

LEMMA 2.1. *The class $T(C(p))$ is the class of all p-groups.*

PROOF. First, since the p-groups form a T-class, we have $T(C(p))$ contained in the p-groups, by minimality of the class $T(C(p))$. On the other hand, given $T(C(p))$, we note that $C(p^n) \in T(C(p))$, as can be easily proved by induction on n and the fact that $T(C(p))$ is closed under extensions. Now for any p-group H, the map $\varphi: \sum_{x \in H} \langle x \rangle \to H$ given by $\varphi(\{n_x x\}_{x \in H}) = \sum_{x \in H} n_x x$ (where $n_x = 0$ for all but finitely many $x \in H$) is an epimorphism, and each $\langle x \rangle$ is isomorphic to $C(p^n)$ for some n. This shows that any p-group is in $T(C(p))$. The following lemma is obvious.

LEMMA 2.2. *The class $T(Z(p^\infty))$ is the class of all divisible p-groups.*

DEFINITION 2.3. A family $\{\mathcal{T}_a\}_{a \in I}$ of torsion classes will be called *independent* if for every group G, the family $\{G_a\}_{a \in I}$ of subgroups of G is independent, where G_a is the maximal $\mathcal{T}_a$-subgroup of G.

The proof of the following lemma is elementary, and is omitted.

LEMMA 2.4. *If the torsion group X is an extension of the group A by the group B, then X_p is an extension of A_p by B_p, where A_p, B_p, and X_p denote the p-primary parts of A, B, and X, respectively.*

THEOREM 2.5. *Let $\mathcal{A}$ be a non-empty subset of the set $\mathcal{P}$ of primes. For each $p \in \mathcal{A}$, let $\mathcal{T}_p$ be either all p-groups, or all divisible p-groups. Then the T-classes $\mathcal{T}_p$ ($p \in \mathcal{A}$) are independent. Moreover, if $\mathcal{T} = T[\cup \{\mathcal{T}_p | p \in \mathcal{A}\}]$, then for any group G, $G_{\mathcal{T}} = \sum G_{\mathcal{T}_p}$ ($p \in \mathcal{A}$), where the sum is direct.*

PROOF. Let $\sum \mathcal{T}_p$ denote the class of groups of the form $\sum M_p$, $M_p \in \mathcal{T}_p$, ($p \in \mathcal{A}$). It is immediate that $\sum \mathcal{T}_p$ is closed under direct sums. Now let $M = \sum M_p$ be an element of $\sum \mathcal{T}_p$, and let X be any group. Then if $f: \sum M_p \to X$, we have that $\sum (\operatorname{Im} f)_{\mathcal{T}_p}$ is a direct sum by the independence of the classes $\mathcal{T}_p$. Suppose now that $y \in \operatorname{Im} f$. Then $y = f(x_{p_1} + \cdots + x_{p_n}) = \sum_{i=1}^{n} f(x_{p_i}) \in \sum (\operatorname{Im} f)_{\mathcal{T}_p}$. Hence $\operatorname{Im} f = \sum (\operatorname{Im} f)_{\mathcal{T}_p}$ so that $\operatorname{Im} f \in \sum \mathcal{T}_p$. Now we show that $\sum \mathcal{T}_p$ is closed under extensions. Suppose the sequence

$$0 \to T_p^1 \to X \to T_p^2 \to 0$$

is exact. Then X is torsion and hence $X = \sum X_p$ ($p \in \mathcal{P}$) by the primary decomposition for torsion groups. We note also that by Lemma 2.4, $X_p = 0$ for $p \notin \mathcal{A}$. Hence we need only show that X_p is divisible for those $p \in \mathcal{A}$ for which $\mathcal{T}_p$ consists of the divisible p-groups. But by Lemma 2.4, X_p is an extension of the p-component of $\sum T_p^1$ by the p-component of $\sum T_p^2$, each of which is divisible, so that X_p is divisible. This completes the proof of the theorem.

We now introduce some notation. Let $\pi p^{e(p)}$ be a formal symbol, called a *restricted Steinitz number*, where $e(p)$ is 0, 1, or ∞. The formal product is taken over all the primes p. We now can prove the following fundamental result.

THEOREM 2.6. *There is a one to one correspondence between the torsion subclasses of the class* $\mathcal{T}_0$ *of usual torsion groups and the set of all restricted Steinitz numbers. In this correspondence, if* $\pi p^{e(p)}$ *is a Steinitz number, the class corresponding to it is* $\mathcal{T}=T(\bigcup_p \mathcal{T}_p)$ *where*

$$\mathcal{T}_p=\begin{cases} 0, \text{ if } e(p)=0 \\ \text{all } p\text{-groups, if } e(p)=1 \\ \text{all divisible } p\text{-groups, if } e(p)=\infty . \end{cases}$$

PROOF. First of all, if $\mathcal{T}\subseteqq\mathcal{T}_0$, let p be any prime. If there is a non-zero p-group $G\in\mathcal{T}$, let $G=D\oplus R$, where D is the maximum divisible subgroup of G. Then if $R\neq 0$, R reduced implies $pR\neq R$. (Note that $qR=R$ for $q\neq p$, so that if $pR=R$, T would be divisible and hence zero.) But then $R\to R/pR$ is a non-zero map, and there is a summand of R/pR isomorphic to $C(p)$. Hence we see that if $R\neq 0$, $T(G)$ is the class of all p-groups by Lemma 2.1. Now if G is divisible, then $T(G)$ is simply the class of all divisible p-groups by Lemma 2.2, since $Z(p^\infty)\in T(G)$. Thus if there is a non-zero reduced p-group in $\mathcal{T}$, we set $e(p)=1$, and if there is no such reduced p-group, but there is a non-zero divisible p-group in $\mathcal{T}$, we set $e(p)=\infty$. Set $e(p)=0$ if there are no non-zero p-groups in $\mathcal{T}$. Conversely, suppose $\pi p^{e(p)}$ is a Steinitz number of the above type. Now define $\mathcal{G}=\{G_p\mid G_p=0, C(p)$, or $Z(p^\infty)$, according as $e(p)=0, 1$, or $\infty\}$. We take $\mathcal{T}=T(\mathcal{G})$.

In order to show that the correspondence is one-one, let $\mathcal{T}$ be a torsion class contained in $\mathcal{T}_0$, and $\pi p^{e(p)}$ be its Steinitz number. We claim $\mathcal{T}=T(\mathcal{G})$ where $\mathcal{G}$ is constructed as above from the Steinitz number of $\mathcal{T}$. Clearly, $T(\mathcal{G})\subseteqq\mathcal{T}$. On the other hand, if $T\in\mathcal{T}$, T is an ordinary torsion group, so that $T=\sum_p T_p$ by the primary decomposition theorem. For each p, T_p is a p-group, and hence T_p is a member of $T(C(p))$ or $T(Z(p^\infty))$ for each p. Also, if $e(p)=0$, then also $T_p=0$ holds. Therefore since $T(\mathcal{G})=T(\bigcup_p\{T(G_p)\mid G_p\in\mathcal{G}\})$, and $T(\mathcal{G})$ is closed under direct sums, we have $\mathcal{T}\subseteqq T(\mathcal{G})$, and thus the equality $\mathcal{T}=T(\mathcal{G})$.

§ 3. *T*-classes closed under subgroups.

The results of the previous section can be applied to classify all those T-classes of groups which are closed under the operation of taking subgroups.

We say a T-class is *proper*, if it does not contain all groups. We then have the following already much-proved result (see [1], [5] and [6]).

THEOREM 3.1. *The proper T-classes of groups which are closed under taking subgroups are in one-one correspondence with the Steinitz numbers $\pi p^{e(p)}$, where $e(p)=0$ or 1 for each prime p.*

PROOF. Let $\mathcal{T}$ be such a torsion class. If $\mathcal{T}$ contains any mixed group, $\mathcal{T}$ consists of all groups. For then $\mathcal{T}$ contains the group Z and since any group is a homomorphic image of a direct sum of copies of Z, $\mathcal{T}$ contains all groups, and is not proper. Thus it suffices to consider the class $\mathcal{T}$ contained in $\mathcal{T}_0$. But then by Theorem 2.6, we need only consider the case where $e(p)=\infty$ for some p. But then $Z(p^\infty)\in\mathcal{T}$ for this p, and we consider a subgroup of $Z(p^\infty)$ isomorphic to $C(p)$. This subgroup is in $\mathcal{T}$, but this implies that $e(p)=1$, a contradiction. Hence $e(p)=1$ or 0, as desired.

Conversely, if a class $\mathcal{T}\subseteqq\mathcal{T}_0$ has its Steinitz number of the form $\pi p^{e(p)}$ with $e(p)=1$ or 0 for all p, then $\mathcal{T}$ is closed under subgroups.

§4. Examples of T-classes of groups.

It will be of interest to consider several specific examples of torsion and torsion-free classes. We have already discussed the usual torsion groups and all the torsion subclasses of this class. We have noted that for any prime p, the p-groups and the divisible p-groups are torsion classes of groups. Also, the class $\mathcal{D}$ of all divisible Abelian groups is a T-class. In fact $\mathcal{D}=T(Q)$. We have already seen that $\mathcal{T}_0=T(\sum_p C(p))$ where the summation ranges over all the primes (cf. 2.6).

One easily verifies that the class $\mathcal{D}_p$ of p-divisible groups (i.e., groups G such that $pG=G$) is a torsion class, and we have the following result.

PROPOSITION 4.1. *$\mathcal{D}_p=T(Q_p)$, where Q_p is as in §1.*

PROOF. Note first that Q_p is p-divisible, so that $\mathcal{D}_p\supseteqq T(Q_p)$. Now let D be any p-divisible torsion-free group. Then D is a module over the ring Q_p and hence is a Q_p-homomorphic image of a direct sum of copies of Q_p. But then D is a Z-homomorphic image of that direct sum, so that $D\in T(Q_p)$. Hence we see that $Q\in T(Q_p)$, and moreover, $Z(q^\infty)\in T(Q_p)$ for all primes q. Also, since Q_p is q-reduced for $q\neq p$, we have $Q_p\to Q_p/qQ_p$ a non-zero map for $q\neq p$, so that $C(q)\in T(Q_p)$ for $q\neq p$. This shows that the divisible p-groups and all q-groups for $q\neq p$ are in $T(Q_p)$. So now let G be any p-divisible group. If G_t denotes the usual torsion subgroup of G, $G_t=\sum_q (G_t)_q$. Now G_t pure in G implies that G_t is p-divisible. In particular, $(G_t)_p$ is p-divisible and hence divisible. This shows that $G_t\in T(Q_p)$. Also, G/G_t is torsion-free and p-divisible, therefore is in $T(Q_p)$. Finally $G\in T(Q_p)$, since $T(Q_p)$ is closed under extensions.

and the sequence

$$0 \to G_t \to G \to G/G_t \to 0$$

is exact with each end group in $T(Q_p)$.

§ 5. General T-classes of Abelian groups.

In this section we reduce the problem of classifying all T-classes of groups to that of classifying those T-classes generated by classes of ordinary torsion-free groups.

LEMMA 5.1. *Let $\mathcal{T}$ be a torsion class of groups. Let $\mathcal{A}=\{p \mid C(p) \in \mathcal{T}\}$, $\mathcal{B}=\{p \mid C(p) \notin \mathcal{T}\}$. Then if $\mathcal{S}=\{C(p) \mid p \in \mathcal{A}\}$, we have the inclusions*

$$T(\mathcal{S}) \subseteqq \mathcal{T} \subseteqq \cap \{\mathcal{D}_p \mid p \in \mathcal{B}\}\ .$$

Hence each member of $\mathcal{T}$ is p-divisible for each $p \in \mathcal{B}$.

PROOF. The first inclusion is clear. Now let $p \in \mathcal{B}$, and $A \in \mathcal{T}$ be given. Consider the group A/pA. Since $A/pA \in T(C(p)) \cap \mathcal{T}$, $A=pA$, or A is p-divisible.

THEOREM 5.2. *Let $\mathcal{T}$ be any torsion class of Abelian groups. Then if $G \in \mathcal{T}$ and G_t denotes the ordinary torsion subgroup of G, then $G \in \mathcal{T}$ if and only if G_t and $G/G_t \in \mathcal{T}$.*

PROOF. Given G_t and $G/G_t \in \mathcal{T}$, $G \in \mathcal{T}$ follows since $\mathcal{T}$ is closed under extensions. Conversely, if $G \in \mathcal{T}$, $G/G_t \in \mathcal{T}$ follows. To see that $G_t \in \mathcal{T}$, note $G_t=\sum_p (G_t)_p = \sum_{p \in \mathcal{A}} (G_t)_p \oplus \sum_{p \in \mathcal{B}} (G_t)_p$, where $\mathcal{A}$ and $\mathcal{B}$ are as in Lemma 5.1. Now $\sum_{p \in \mathcal{A}} (G_t)_p \in \mathcal{T}$, and G_t is p-divisible for $p \in \mathcal{B}$ since G_t is pure in G and G has this property by 5.1. But $\sum_{p \in \mathcal{B}} (G_t)_p$ is p-divisible for each $p \in \mathcal{B}$, since this group is a homomorphic image of G_t. But then each summand $(G_t)_p$ for $p \in \mathcal{B}$ is p-divisible, and is therefore divisible. It follows that $\sum_{p \in \mathcal{B}} (G_t)_p$ is a summand of G and therefore is a member of $\mathcal{T}$. But then the entire subgroup G_t is a member of $\mathcal{T}$ as desired.

COROLLARY 5.3. *If $\mathcal{T}$ is any torsion class of groups, we have the equality $\mathcal{T}=T[(\mathcal{T} \cap \mathcal{T}_0) \cup (\mathcal{T} \cap \mathcal{F}_0)]$, where $\mathcal{T}_0$ and $\mathcal{F}_0$ are the usual torsion and torsion-free groups, respectively.*

PROOF. Let $G \in \mathcal{T}$. Then $G_t \in \mathcal{T} \cap \mathcal{T}_0$, $G/G_t \in \mathcal{T} \cap \mathcal{F}_0$, so that one inclusion holds. The other inclusion is obvious.

REMARK 5.4. The above results reduce the problem of classifying all torsion classes of groups to that of classifying the torsion classes generated by oridinary torsion-free groups. The latter problem is still unsolved as of this writing.

PROPOSITION 5.5. *If $\mathcal{T}$ is any non-zero torsion class of Abelian groups, then $\mathcal{T} \cap \mathcal{T}_0 \neq (0)$.*

PROOF. The proposition follows immediately from the observation that any group whatsoever has a homomorphic image in $\mathcal{T}_0$.

Our final result characterizes the class $\mathcal{T}_0 \cap \mathcal{D}$ of divisible torsion groups.

THEOREM 5.6. *The class* $\mathcal{T}_0 \cap \mathcal{D} = T(Q/Z)$ *is the unique smallest torsion class having a non-zero intersection with any other non-zero torsion class.*

PROOF. Let $\mathcal{T}$ be any class satisfying the hypothesis of the theorem. Certainly such classes exist, since the class of all Abelian groups has this property. Consider the class $T(Z(p^\infty))$, which consists of all divisible p-groups. If $\mathcal{T} \cap T(Z(p^\infty)) \neq (0)$, clearly $Z(p^\infty) \in \mathcal{T}$. This holds for any prime p, so that $Q/Z \in \mathcal{T}$. On the other hand, if $\mathcal{T} = T(Q/Z)$, $\mathcal{T}$ has the required property. For if $\mathcal{T}_1$ is any torsion class with $G \in \mathcal{T}_1$, where G has non-zero divisible part, we are through. If G is reduced, $pG \neq G$ for some p so that a summand $C(p)$ of G/pG is in $\mathcal{T}_1$, from which it follows that $Z(p^\infty) \in \mathcal{T}_1$. But then $\mathcal{T}_1 \cap T(Q/Z) \neq (0)$.

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