J. Math. Soc. Japan
Vol. 17, No. 2, 1965

On the equivalence problems associated with a certain class of homogeneous spaces

By Noboru TANAKA

(Received Sept. 19, 1964)

Introduction

The present paper, first of all, introduces the notion of an l-system on which our theory is based. An l-system L is defined to be a system of a real semi-simple Lie algebra $\mathfrak{g}$ and three subalgebras of $\mathfrak{g}$ satisfying certain conditions (Definition 1.1). To every l-system L we associate a homogeneous space $M_L = G/G'$ of a Lie group G over a closed subgroup G' of G (see § 1). It is remarkable that the homogeneous space $M_L = G/G'$ is a prolongation of a compact Riemannian symmetric space in the following sense (cf. Proposition 3.2): A maximal compact subgroup K of G acts transitively on M_L, and the homogeneous space $M_L = K/K \cap G'$ is a compact Riemannian symmetric space. A recent work of T. Nagano [7] proves that, roughly speaking, any prolongation of a compact Riemannian symmetric space is locally isomorphic with a homogeneous space of the form $M_L = G/G'$.

Now let L be an l-system and let $M_L = G/G'$ be the corresponding homogeneous space. G being considered as a transformation group on M_L, the linear isotropy group $\tilde{G}$ of G at the origin o of M_L is a subgroup of the general linear group $GL(\mathfrak{m})$ of the tangent vector space $\mathfrak{m}$ to M_L at o. In this way, to every l-system L there corresponds a representation $(\tilde{G}, \mathfrak{m})$. Therefore there can be defined the notion of a $\tilde{G}$-structure: A $\tilde{G}$-structure on a manifold M is a principal fiber bundle $\tilde{P}$ over the base space M with structure group $\tilde{G}$ which is a subbundle of the bundle of frames of M (Definition 5.1).

The main purpose of the present paper is, for a given l-system L, to study conditions for the equivalence of two $\tilde{G}$-structures. Our main results (Theorems 9.3, 9.4, 10.1 and 10.2) may be stated as follows: Under general hypotheses on L, to every $\tilde{G}$-structure $\tilde{P}$ there is associated a system, called the normal connection of type (L), in such a way that the equivalence of two $\tilde{G}$-structures can be characterized. The normal connection of type (L) is a Cartan connection corresponding to the homogeneous space $M_L = G/G'$ and is found to be a generalization of the normal conformal connection. It should be here noted that there also exists the notion of the normal connection of type (L) under a weaker hypothesis on L (Theorems 9.1 and 9.2), which just generalizes the notion of the normal projective connection.

In § 1, we construct the homogeneous space $M_L = G/G'$ and study the fundamental properties of it. In § 2, we define the notion of an irreducible l-system (Definition 2.1). It is shown that an arbitrary l-system L is decomposed into a product of irreducible l-systems (Propositions 2.1 and 2.2) and that the set of all irreducible l-systems is devided into two classes called of type (R) and of type (C) (Definition 2.2). § 3 is devoted to the study of the duality which lies among l-systems. This leads to the investigation of the maximal compact subgroups of the group G. In § 4, it is shown that the set of all isomorphism classes of irreducible l-systems of type (C) is in a one to one correspondence with the set of all isomorphism classes of compact irreducible hermitian symmetric spaces. § 5, § 6 and § 7 are preliminary to the subsequent two sections. In § 8, it is proved that to every $\tilde{G}$-structure $\tilde{P}$ on a manifold M there is associated a tensor field T of type $\binom{1}{2}$ on M called the torsion tensor field of $\tilde{P}$. This notion turns out to be a generalization of the notion of the Nijenhuis tensor field of an almost complex structure.

§ 9 is concerned with the construction of the normal connection of type (L), which will be carried out following the construction of the normal projective connection given in [**9**]. Hypotheses on L for the existence of the normal connection of type (L) will be stated in terms of an endomorphism Φ_L and an integer $\delta(L)$ (as for the cohomological interpretations of these hypotheses, see Appendix). The endomorphism Φ_L has an intimate relationship to the quadratic form which appears in the study of discrete subgroups of a Lie group, for example, in [**11**]. The integer $\delta(L)$ is the dimension of the first derived space associated with the representation $(\tilde{G}, \mathfrak{m})$. Finally in § 10, we study the endomorphism Φ_L and the integer $\delta(L)$ by using the results in § 2, § 3 and § 4 and show that hypotheses on L for the existence of the normal connection of type (L) are generally satisfied. Owing to the results in § 2, the problems will be reduced to the case where L is an irreducible l-system of type (C).

Preliminary remark

Throughout this paper, we always assume the differentiability of class C^∞, and by a manifold we shall mean a manifold satisfying the second countability axiom. $\boldsymbol{R}$ (resp. $\boldsymbol{C}$) will denote the field of real numbers (resp. of complex numbers).

§ 1. The groups G, G' and $\tilde{G}$

DEFINITION 1.1. Let $\mathfrak{g}$ be a real semi-simple Lie algebra and let $\mathfrak{m}$, $\mathfrak{m}^*$ and $\tilde{\mathfrak{g}}$ be three subalgebras of $\mathfrak{g}$. The (ordered) system $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ is called an l-system if it satisfies the following conditions:

(l. 1) $\mathfrak{g}=\mathfrak{m}+\mathfrak{m}^*+\tilde{\mathfrak{g}}$ (direct sum of vector spaces);

(l. 2) Both $\mathfrak{m}$ and $\mathfrak{m}^*$ are abelian;

(l. 3) $[\tilde{\mathfrak{g}}, \mathfrak{m}]\subset\mathfrak{m}$ and $[\tilde{\mathfrak{g}}, \mathfrak{m}^*]\subset\mathfrak{m}^*$;

(l. 4) $[\mathfrak{m}, \mathfrak{m}^*]=\tilde{\mathfrak{g}}$.

Let $L_i=(\mathfrak{g}_i, \mathfrak{m}_i, \mathfrak{m}_i^*, \tilde{\mathfrak{g}}_i)$ $(i=1, 2)$ be an l-system. An isomorphism f of the Lie algebra $\mathfrak{g}_1$ onto the Lie algebra $\mathfrak{g}_2$ is called an isomorphism of L_1 onto L_2 if $f(\mathfrak{m}_1)=\mathfrak{m}_2$, $f(\mathfrak{m}_1^*)=\mathfrak{m}_2^*$ (and hence $f(\tilde{\mathfrak{g}}_1)=\tilde{\mathfrak{g}}_2$).

Let $\mathfrak{g}$ be a real Lie algebra, let $\mathfrak{n}$ be a subspace of $\mathfrak{g}$ and let $\tilde{\mathfrak{g}}$ be a subalgebra of $\mathfrak{g}$. The system $S=(\mathfrak{g}, \mathfrak{n}, \tilde{\mathfrak{g}})$ will be called an *infinitesimal (affine) symmetric space* or briefly an *s-system* if it satisfies the following conditions:

(s. 1) $\mathfrak{g}=\mathfrak{n}+\tilde{\mathfrak{g}}$ (direct sum);

(s. 2) $[\tilde{\mathfrak{g}}, \mathfrak{n}]\subset\mathfrak{n}$;

(s. 3) $[\mathfrak{n}, \mathfrak{n}]\subset\tilde{\mathfrak{g}}$.

We see from Def. 1.1 that[1], for any l-system $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$, the system $S=(\mathfrak{g}, \mathfrak{m}+\mathfrak{m}^*, \tilde{\mathfrak{g}})$ forms an s-system.

From now on, we shall consider a fixed l-system $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ and denote by φ the Killing form of $\mathfrak{g}$.

We have easily

LEMMA 1.1.

$$\varphi(\mathfrak{m}, \mathfrak{m})=\varphi(\mathfrak{m}^*, \mathfrak{m}^*)=\varphi(\mathfrak{m}+\mathfrak{m}^*, \tilde{\mathfrak{g}})=\{0\}$$

We put

$$\langle\xi, \omega\rangle=\varphi(\xi, \omega)$$

for all $\xi\in\mathfrak{m}$ *and* $\omega\in\mathfrak{m}^*$.

PROPOSITION 1.1. (1) *The bilinear mapping* $m\times m^*\ni(\xi, \omega)\to\langle\xi, \omega\rangle\in\boldsymbol{R}$ *gives a duality between the two vector spaces* $\mathfrak{m}$ *and* $\mathfrak{m}^*$, *i.e., if* $\xi\in\mathfrak{m}$ *(resp.* $\omega\in\mathfrak{m}^*$*) and if* $\langle\xi, \mathfrak{m}^*\rangle=\{0\}$ *(resp.* $\langle\mathfrak{m}, \omega\rangle=\{0\}$*), then* $\xi=0$ *(resp.* $\omega=0$*). In particular,* $\dim\mathfrak{m}=\dim\mathfrak{m}^*$. (2) *The restriction of* φ *to* $\tilde{\mathfrak{g}}$ *is non-degenerate.*

This follows immediately from Lemma 1.1 and the fact that φ is non-degenerate.

LEMMA 1.2. (1) *If* $\xi\in\mathfrak{m}$ *(resp.* $\omega\in\mathfrak{m}^*$*) and if* $[\xi, \mathfrak{m}^*]=\{0\}$ *(resp.* $[\mathfrak{m}, \omega]=\{0\}$*), then* $\xi=0$ *(resp.* $\omega=0$*).* (2) *If* $A\in\tilde{\mathfrak{g}}$ *and if* $[A, \mathfrak{m}]=\{0\}$ *or* $[A, \mathfrak{m}^*]$

1) Therefore, some of our results (in § 1–§ 4) essentially follow from Berger [**1**].

$=\{0\}$, *then* $A=0$.

PROOF. (1) Suppose that a $\xi\in\mathfrak{m}$ satisfies the condition $[\xi, \mathfrak{m}^*]=\{0\}$. We have $\langle[\xi, \tilde{\mathfrak{g}}], \mathfrak{m}^*\rangle=\varphi(\tilde{\mathfrak{g}}, [\xi, \mathfrak{m}^*])=\{0\}$, whence $[\xi, \tilde{\mathfrak{g}}]=\{0\}$ (Prop. 1.1, (1)). It follows that ξ is in the center of $\mathfrak{g}$ (conditions (l. 1) and (l. 2)). Since $\mathfrak{g}$ is semi-simple, we have $\xi=0$. The second assertion can be similarly proved. (2) Suppose that an $A\in\tilde{\mathfrak{g}}$ satisfies the condition $[A, \mathfrak{m}]=\{0\}$. We have $\langle\mathfrak{m}, [A, \mathfrak{m}^*]\rangle=\varphi([A, \mathfrak{m}], \mathfrak{m}^*)=\{0\}$, whence $[A, \mathfrak{m}^*]=\{0\}$. Since $\tilde{\mathfrak{g}}=[\mathfrak{m}, \mathfrak{m}^*]$ (condition (l. 4)), we get $[A, \tilde{\mathfrak{g}}]=\{0\}$. If follows that A is in the center of $\mathfrak{g}$ and hence $A=0$. The second assertion can be similarly proved.

NOTATION. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. $N(\mathfrak{h}, \mathfrak{g})$ (resp. $C(\mathfrak{h}, \mathfrak{g})$) will denote the normalizer (resp. the centralizer) of $\mathfrak{h}$ in $\mathfrak{g}$.

By Lemma 1.2, we have easily

PROPOSITION 1.2. (1) $N(\mathfrak{m}, \mathfrak{g})=\mathfrak{m}+\tilde{\mathfrak{g}}$ *and* $N(\mathfrak{m}^*, \mathfrak{g})=\mathfrak{m}^*+\tilde{\mathfrak{g}}$. (2) $C(\mathfrak{m}, \mathfrak{g})=\mathfrak{m}$ *and* $C(\mathfrak{m}^*, \mathfrak{g})=\mathfrak{m}^*$, *in other words, both* $\mathfrak{m}$ *and* $\mathfrak{m}^*$ *are maximal abelian subalgebras of* $\mathfrak{g}$.

We shall denote by $A(\mathfrak{g})$ the group of all automorphisms of the Lie algebra $\mathfrak{g}$. $\mathfrak{g}$ being semi-simple, the Lie algebra of $A(\mathfrak{g})$ may be identified with the Lie algebra $\mathfrak{g}$ in such a way that $adaX=aX$ for all $a\in A(\mathfrak{g})$ and $X\in\mathfrak{g}$. We prefer the notation $adaX$ to the one aX.

NOTATION. Let G be a Lie group and let $\mathfrak{g}$ be the Lie algebra of G. Given a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $N(\mathfrak{h}, G)$ (resp. $C(\mathfrak{h}, G)$) will denote the normalizer (resp. the centralizer) of $\mathfrak{h}$ in G, which is, by definition, the subgroup of G consisting of all the elements a such that $ada\mathfrak{h}=\mathfrak{h}$ (resp. $adaX=X$ for all $X\in\mathfrak{h}$). The Lie algebra of $N(\mathfrak{h}, G)$ (resp. $C(\mathfrak{h}, G)$) coincides with $N(\mathfrak{h}, \mathfrak{g})$ (resp. $C(\mathfrak{h}, \mathfrak{g})$).

We shall denote by $\tilde{G}$ the intersection of the normalizer of $\mathfrak{m}$ in $A(\mathfrak{g})$ and that of $\mathfrak{m}^*$ in $A(\mathfrak{g})$, i. e., $\tilde{G}=N(\mathfrak{m}, A(\mathfrak{g}))\cap N(\mathfrak{m}^*, A(\mathfrak{g}))$. It follows from Prop. 1.2, (1) that the Lie algebra of $\tilde{G}$ is identical with the Lie algebra $\tilde{\mathfrak{g}}$. Note that the group $\tilde{G}$ may be characterized as the group of all automorphisms of L. We now define a representation ρ of $\tilde{G}$ on $\mathfrak{m}$ by $\rho(a)\xi=ada\xi$ for all $a\in\tilde{G}$ and $\xi\in\mathfrak{m}$ and denote by the same letter ρ the corresponding representation of $\tilde{\mathfrak{g}}$ on $\mathfrak{m}$; we have $\rho(A)\xi=[A, \xi]$ for all $A\in\tilde{\mathfrak{g}}$ and $\xi\in\mathfrak{m}$.

In the same manner as in Lemma 1.2, (2), we can prove

LEMMA 1.3. *The representation* ρ *of* $\tilde{G}$ *on* $\mathfrak{m}$ *is faithful.*

By Prop. 1.1, (1), the vector space $\mathfrak{m}^*$ may be identified with the dual space of $\mathfrak{m}$ and, by Lemma 1.3, the group $\tilde{G}$ (resp. the Lie algebra $\tilde{\mathfrak{g}}$) may be identified with a subgroup (resp. a subalgebra) of the general linear group $GL(\mathfrak{m})$ of $\mathfrak{m}$ (resp. the Lie algebra $\mathfrak{gl}(\mathfrak{m})$ of all endomorphisms of $\mathfrak{m}$). Under these identifications, we have the equalities:

$$(1.1)\qquad ada\xi = a\xi,\quad ada\omega = {}^t a^{-1}\omega,\quad adaB = aBa^{-1},$$
$$[A, \xi] = A\xi,\quad [A, \omega] = -{}^t A\omega,\quad [A, B] = AB - BA$$

for all $a \in \tilde{G}$, $\xi \in \mathfrak{m}$, $\omega \in \mathfrak{m}^*$ and $A, B \in \tilde{\mathfrak{g}}$, where ${}^t A$ stands for the transpose of an endomorphism A of $\mathfrak{m}$ with respect to the duality between $\mathfrak{m}$ and $\mathfrak{m}^*$.

Let $\tilde{\varphi}$ be the Killing form of $\tilde{\mathfrak{g}}$. Then a direct calculation gives

LEMMA 1.4.

$$\varphi(A, B) = \tilde{\varphi}(A, B) + 2TrAB$$

for all $A, B \in \tilde{\mathfrak{g}}$, *where* TrC *is the trace of an endomorphism* C *of* $\mathfrak{m}$.

PROPOSITION 1.3.

$$\tilde{G} = N(\tilde{\mathfrak{g}}, GL(\mathfrak{m}))\,.$$

PROOF. We have clearly $\tilde{G} \subset N(\tilde{\mathfrak{g}}, GL(\mathfrak{m}))$. Take any $a \in N(\tilde{\mathfrak{g}}, GL(\mathfrak{m}))$ and define an automorphism $\hat{a}$ of $\mathfrak{g}$ (as a vector space) as follows: $\hat{a}\xi = a\xi$, $\hat{a}\omega = {}^t a^{-1}\omega$ and $\hat{a}A = aAa^{-1}$ for all $\xi \in \mathfrak{m}$, $\omega \in \mathfrak{m}^*$ and $A \in \tilde{\mathfrak{g}}$. We show that $\hat{a}$, thus obtained, is an automorphism of the Lie algebra $\mathfrak{g}$. By (1.1), it suffices to verify the equality: $\hat{a}[\xi, \omega] = [\hat{a}\xi, \hat{a}\omega]$ for all $\xi \in \mathfrak{m}$ and $\omega \in \mathfrak{m}^*$. By Lemma 1.4, we have $\varphi(\hat{a}A, \hat{a}B) = \varphi(A, B)$ for all $A, B \in \tilde{\mathfrak{g}}$. Hence, we have $\varphi(\hat{a}[\xi, \omega], \hat{a}A) = \varphi([\xi, \omega], A) = \langle [A, \xi], \omega \rangle = \langle [\hat{a}A, \hat{a}\xi], \hat{a}\omega \rangle = \varphi([\hat{a}\xi, \hat{a}\omega], \hat{a}A)$ for all $A \in \tilde{\mathfrak{g}}$. It follows from Prop. 1.1, (2) that $\hat{a}[\xi, \omega] = [\hat{a}\xi, \hat{a}\omega]$, which proves our assertion. We have clearly $a = \hat{a} \in \tilde{G}$.

COROLLARY. *The identity transformation* E_L *of* $\mathfrak{m}$ *is in the center of* $\tilde{\mathfrak{g}}$.

PROPOSITION 1.4.

$$\langle \xi, \omega \rangle = 2\,Tr[\xi, \omega]$$

for all $\xi \in \mathfrak{m}$ *and* $\omega \in \mathfrak{m}^*$.

PROOF. By Lemma 1.4 and Cor. to Prop. 1.3, we have $\varphi([\xi, \omega], E_L) = 2Tr[\xi, \omega]$. We have $\varphi([\xi, \omega], E_L) = \langle \xi, [\omega, E_L] \rangle = \langle \xi, \omega \rangle$.

We shall denote by G' the normalizer of $\mathfrak{m}^*$ in $A(\mathfrak{g})$, i.e., $G' = N(\mathfrak{m}^*, A(\mathfrak{g}))$. By Prop. 1.2, (1), the Lie algebra of G' is given by $\mathfrak{g}' = \mathfrak{m}^* + \tilde{\mathfrak{g}}$. We have clearly $\tilde{G}$, $\exp \mathfrak{m}^* \subset G'$, where $\exp \mathfrak{m}^*$ denotes the abelian subgroup of $A(\mathfrak{g})$ generated by $\mathfrak{m}^*$.

LEMMA 1.5.

$$G' = N(\mathfrak{g}', A(\mathfrak{g}))\,.$$

PROOF. We have clearly $G' \subset N(\mathfrak{g}', A(\mathfrak{g}))$. Take any a in $N(\mathfrak{g}', A(\mathfrak{g}))$. We have $\varphi(ada\mathfrak{m}^*, \mathfrak{g}') = \varphi(\mathfrak{m}^*, \mathfrak{g}') = \{0\}$ (Lemma 1.1). Hence, it follows from Lemma 1.1 and Prop. 1.1 that $ada\mathfrak{m}^* \subset \mathfrak{m}^*$, i.e., $a \in G'$.

We have

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{g}' \quad \text{(direct sum)}\,.$$

Taking account of Lemma 1.5, we now define a homomorphism l of G' into $GL(\mathfrak{m})$ by

(1.2) $$ada\xi \equiv l(a)\xi \bmod \mathfrak{g}'$$

for all $a \in G'$ and $\xi \in \mathfrak{m}$. We shall denote by the same letter l the corresponding homomorphism of $\mathfrak{g}'$ into $\mathfrak{gl}(\mathfrak{m})$, i. e.,

$$[X, \xi] \equiv l(X)\xi \bmod \mathfrak{g}'$$

for all $X \in \mathfrak{g}'$ and $\xi \in \mathfrak{m}$. We see easily that $l(X)$ coincides with the $\tilde{\mathfrak{g}}$-component of X in the decomposition $\mathfrak{g}' = \mathfrak{m}^* + \tilde{\mathfrak{g}}$. Now it is clear that the restriction of l $(: G' \to GL(\mathfrak{m}))$ to $\tilde{G}$ is the identity transformation of $\tilde{G}$. We have

$$ad(\exp \omega)\xi = \xi + [\omega, \xi] + \frac{1}{2}[\omega, [\omega, \xi]]$$

for all $\omega \in \mathfrak{m}^*$ and $\xi \in \mathfrak{m}$, which implies that the group $\exp \mathfrak{m}^*$ is contained in the kernel of l.

PROPOSITION 1.5.

$$l(G') = \tilde{G}.$$

PROOF. As we have seen above, the restriction of l to $\tilde{G}$ is the identity transformation of $\tilde{G}$. Hence $l(\tilde{G}) = \tilde{G}$. Therefore we have only to prove that $l(G') \subset \tilde{G}$. Take any a in G'. We have $[adaA, \xi] = ada[A, ada^{-1}\xi] \equiv ada(Al(a)^{-1}\xi) \equiv l(a)Al(a)^{-1}\xi \bmod \mathfrak{g}'$ for all $\xi \in \mathfrak{m}$ and $A \in \tilde{\mathfrak{g}}$. This implies that $l(a)Al(a)^{-1}$ coincides with the $\tilde{\mathfrak{g}}$-component of $adaA$ (in the decomposition $\mathfrak{g}' = \mathfrak{m}^* + \tilde{\mathfrak{g}}$) for all $A \in \tilde{\mathfrak{g}}$. Therefore, $l(a) \in \tilde{G}$ by Prop. 1.3.

LEMMA 1.6. *Let $a \in G'$ be in the kernel of l.* (1) *$ada\omega = \omega$ for all $\omega \in \mathfrak{m}^*$.* (2) *If we put $\omega = adaE_L - E_L$, then we have $\omega \in \mathfrak{m}^*$ and*

$$ada\xi - \xi \equiv [\omega, \xi] \bmod \mathfrak{m}^*$$

for all $\xi \in \mathfrak{m}$.

PROOF. (1) By Lemma 1.1, we have $\langle \xi, ada\omega \rangle = \varphi(ada^{-1}\xi, \omega) = \langle \xi, \omega \rangle$ for all $\xi \in \mathfrak{m}$, whence $ada\omega = \omega$. (2) It follows from the proof of Prop. 1.5 that $adaA \equiv A \bmod \mathfrak{m}^*$ for all $A \in \tilde{\mathfrak{g}}$. Hence $\omega \in \mathfrak{m}^*$. Since E_L is in the center of $\tilde{\mathfrak{g}}$, we get $[E_L, ada^{-1}\xi] \equiv \xi \bmod \mathfrak{m}^*$. Therefore we have $[\omega, \xi] = ada[E_L, ada^{-1}\xi] - \xi \equiv ada\xi - \xi \bmod \mathfrak{m}^*$.

PROPOSITION 1.6. *The kernel of l is equal to* $\exp \mathfrak{m}^*$.

PROOF. As we have seen above, the group $\exp \mathfrak{m}^*$ is contained in the kernel of l. Take any $a \in G'$ in the kernel of l. By Lemma 1.6, we can find an $\omega \in \mathfrak{m}^*$ such that $ada\xi - \xi \equiv [\omega, \xi] \bmod \mathfrak{m}^*$ for all $\xi \in \mathfrak{m}$. If we put $b = (\exp(-\omega))a$, then we have easily $adb\xi \equiv \xi \bmod \mathfrak{m}^*$ for all $\xi \in \mathfrak{m}$. Let us prove that $adb\xi = \xi$ for all $\xi \in \mathfrak{m}$. We set $\gamma(\xi) = adb\xi - \xi$, which is in $\mathfrak{m}^*$. We have $adb[\xi, \xi'] = [\xi, \gamma(\xi')] - [\xi', \gamma(\xi)] = 0$. Hence we get $\langle \xi, \gamma(\xi') \rangle = \langle \xi', \gamma(\xi) \rangle$ by Prop. 1.4. On the other hand, we have $adb^{-1}\xi' = \xi' - \gamma(\xi')$ by Lemma 1.6, (1). Hence we have $\langle \xi', \gamma(\xi) \rangle = \varphi(\xi', adb\xi - \xi) = \varphi(\xi', adb\xi) = \varphi(adb^{-1}\xi', \xi) = \varphi(\xi' - \gamma(\xi'), \xi) = -\langle \xi, \gamma(\xi') \rangle$. Consequently, we have $\gamma = 0$, i. e.,

$ad b\xi=\xi$ for all $\xi\in\mathfrak{m}$. By Lemma 1.6 (1), we have $ad b\omega=\omega$ for all $\omega\in\mathfrak{m}^*$. It follows that $ad bX=X$ for all $X\in\mathfrak{g}$, which means $b=e$, i.e., $a=\exp\omega$, where e is the identity element of $A(\mathfrak{g})$.

PROPOSITION 1.7. *Every element of G' is uniquely expressed in the form $a\exp\omega$, where $a\in\tilde{G}$ and $\omega\in\mathfrak{m}^*$.*

PROOF. Take any $a\in G'$. Since $l(G')=\tilde{G}\subset G'$ and since the restriction of l to $\tilde{G}$ is the identity transformation of $\tilde{G}$, we see that $l(a)^{-1}a$ is in the kernel of l, i.e., of the form $\exp\omega$ with an $\omega\in\mathfrak{m}^*$ (Prop. 1.6). Hence $a=l(a)\exp\omega$. It remains to prove the uniqueness. Suppose that an $a\in G'$ is expressed as $b\exp\omega$ with a $b\in\tilde{G}$ and an $\omega\in\mathfrak{m}^*$. We have $l(a)=l(b)=b$ and hence $c=l(a)^{-1}a=\exp\omega$. But we have $ad c\xi\equiv\xi+[\omega,\xi] \bmod \mathfrak{m}^*$ for all $\xi\in\mathfrak{m}$. Since $\mathfrak{m}$ is a maximal abelian subalgebra of $\mathfrak{g}$ (Prop. 1.2, (2)), this equality means that ω is uniquely determined by c and hence by a.

Finally, let $A(\mathfrak{g})^\circ$ be the connected component of the identity of $A(\mathfrak{g})$ that is just the adjoint group of $\mathfrak{g}$. We set $G=A(\mathfrak{g})^\circ\cdot\tilde{G}$, which is an open subgroup of $A(\mathfrak{g})$. The homogeneous space $M_L=G/G'$ will be called *associated* with the l-system L. Since $G/G'=A(\mathfrak{g})^\circ/G'\cap A(\mathfrak{g})^\circ$, the space M_L is connected. By (1.2) and Prop. 1.5, we see that the group $\tilde{G}$ may be identified with the linear isotropy group of G/G' and that the homomorphism l may be identified with the homomorphism of the isotropy group G' of G/G' onto the linear isotropy group $\tilde{G}$. Later on, we shall see that M_L is compact and the action of G on M_L is effective.

EXAMPLE. Let $P^m(K)$ be the m-dimensional projective space over a field K, where $K=\boldsymbol{R}$ or $\boldsymbol{C}$. The group G of all projective transformations of $P^m(K)$ acts transitively on $P^m(K)$ and hence the space $P^m(K)$ may be represented by a homogeneous space G/G', G' being the isotropy group of G at a point o of $P^m(K)$. Now, with the projective space $P^m(K)$ there is associated an l-system L as follows: The Lie algebra $\mathfrak{g}$ of G may be identified with the Lie algebra $\mathfrak{sl}(m+1,K)$[2]. Therefore every element of $\mathfrak{g}$ is uniquely expressed as

$$M(\xi,\omega,A)=\begin{pmatrix} -\frac{1}{m+1}TrA & \omega \\ \xi & A-\frac{1}{m+1}TrA \end{pmatrix}.$$

where

$$\xi=\begin{pmatrix}\xi_1\\ \vdots\\ \xi_m\end{pmatrix},\quad \omega=(\omega_1,\cdots,\omega_m),\ A=(A_{ij})_{1\leq i,j\leq m}.$$

Setting $\tilde{\xi}=M(\xi,0,0)$, $\tilde{\omega}=M(0,\omega,0)$ and $\tilde{A}=M(0,0,A)$, we have $M(\xi,\omega,A)$

2) For the classical groups and their Lie algebras, we use the notations given in C. Chevalley, Theory of Lie groups I.

$=\tilde{\xi}+\tilde{\omega}+\tilde{A}$, $[\tilde{\xi}, \tilde{\xi}']=[\tilde{\omega}, \tilde{\omega}']=0$, $[\tilde{A}, \tilde{A}']=\widetilde{[A, A']}$, $[\tilde{A}, \tilde{\xi}]=\widetilde{A\xi}$, $[\tilde{A}, \tilde{\omega}]=-\widetilde{\omega A}$ and $[\tilde{\xi}, \tilde{\omega}]=\widetilde{\xi\omega+\omega\xi}$. Let $\mathfrak{m}$ (resp. $\mathfrak{m}^*$, resp. $\tilde{\mathfrak{g}}$) be the subalgebra of $\mathfrak{g}$ consisting of all elements $\tilde{\xi}$ (resp. $\tilde{\omega}$, resp. $\tilde{A}$). Then we see from the above consideration that the system $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ forms an l-system. An l-system which is isomorphic with the l-system L, will be called of type $P^m(K)$. One notes that the homogeneous space $P^m(K)=G/G'$ may be regarded as the homogeneous space associated with the l-system L.

§2. Decomposition of an *l*-system

LEMMA 2.1. *Let* $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ *be an l-system and let* $\mathfrak{h}$ *be an ideal of* $\mathfrak{g}$. *Then we have*

$$\mathfrak{h}=(\mathfrak{h}\cap\mathfrak{m})+(\mathfrak{h}\cap\mathfrak{m}^*)+(\mathfrak{h}\cap\tilde{\mathfrak{g}}),$$

and the system $(\mathfrak{h}, \mathfrak{h}\cap\mathfrak{m}, \mathfrak{h}\cap\mathfrak{m}^*, \mathfrak{h}\cap\tilde{\mathfrak{g}})$ *forms an l-system.*

PROOF. Take any $X\in\mathfrak{h}$ and express it as $\xi+\omega+A$, where $\xi\in\mathfrak{m}$, $\omega\in\mathfrak{m}^*$ and $A\in\tilde{\mathfrak{g}}$. We have $[E_L, X]=\xi-\omega$ and $[E_L, [E_L, X]]=\xi+\omega$. Since $[E_L, \mathfrak{h}]$, $[E_L, [E_L, \mathfrak{h}]]\subset\mathfrak{h}$, it follows that $\xi, \omega\in\mathfrak{h}$ and hence $A\in\mathfrak{h}$. The second half of Lemma 2.1 can be easily proved by considering the complementary ideal of $\mathfrak{h}$ in $\mathfrak{g}$.

Let $L_i=(\mathfrak{g}_i, \mathfrak{m}_i, \mathfrak{m}_i^*, \tilde{\mathfrak{g}}_i)$ $(1\leqq i\leqq s)$ be an l-system. If we set $\mathfrak{g}=\mathfrak{g}_1\times\cdots\times\mathfrak{g}_s$, $\mathfrak{m}=\mathfrak{m}_1\times\cdots\times\mathfrak{m}_s$, $\mathfrak{m}^*=\mathfrak{m}_1^*\times\cdots\times\mathfrak{m}_s^*$ and $\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}_1\times\cdots\times\tilde{\mathfrak{g}}_s$, then we see that the system $(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ forms an l-system, which will be called the product of $L_1, \cdots, L_s$ and denoted by $L_1\times\cdots\times L_s$.

By Lemma 2.1, we get

PROPOSITION 2.1. *Let* $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ *be an l-system and let* $\mathfrak{g}=\mathfrak{g}_1+\cdots+\mathfrak{g}_s$ *be the decomposition of* $\mathfrak{g}$ *into simple ideals. We set* $\mathfrak{m}_i=\mathfrak{g}_i\cap\mathfrak{m}$, $\mathfrak{m}_i^*=\mathfrak{g}_i\cap\mathfrak{m}^*$ *and* $\tilde{\mathfrak{g}}_i=\mathfrak{g}_i\cap\tilde{\mathfrak{g}}$. *Then the systems* $L_i=(\mathfrak{g}_i, \mathfrak{m}_i, \mathfrak{m}_i^*, \tilde{\mathfrak{g}}_i)$ *are l-systems and the given l-system L is isomorphic with the product* $L_1\times\cdots\times L_s$.

Given an l-system $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$, we shall denote by $(\tilde{\mathfrak{g}}, \mathfrak{m})$ the (identity) representation of $\tilde{\mathfrak{g}}$ on $\mathfrak{m}$.

PROPOSITION 2.2. *Let* $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ *be an l-system. A necessary and sufficient condition that the representation* $(\tilde{\mathfrak{g}}, \mathfrak{m})$ *is irreducible is that* $\mathfrak{g}$ *is simple.*

PROOF. Necessity follows from Prop. 2.1. Sufficiency is proved as follows: Take any $\tilde{\mathfrak{g}}$-stable subspace $\mathfrak{m}'$ of $\mathfrak{m}$ and set $\mathfrak{g}'=\mathfrak{m}'+[[\mathfrak{m}', \mathfrak{m}^*], \mathfrak{m}^*]+[\mathfrak{m}', \mathfrak{m}^*]$. We see easily that $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$. Therefore we have $\mathfrak{m}'=\{0\}$ or $\mathfrak{m}$ according as $\mathfrak{g}'=\{0\}$ or $\mathfrak{g}$.

DEFINITION 2.1. An l-system $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ is called irreducible if the representation $(\tilde{\mathfrak{g}}, \mathfrak{m})$ is irreducible.

DEFINITION 2.2. An irreducible l-system $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ is called of type

(R) (resp. of type (C)) if the representation $(\tilde{\mathfrak{g}}, \mathfrak{m})$ is of first class (resp. of second class).

In general, let $(\mathfrak{g}, V)$ be a real representation, i.e., V is a real vector space and $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$. V^C denoting the complexification of V, $\mathfrak{g}$ may be regarded as a (real) subalgebra of the complex Lie algebra of all complex endomorphisms of V^C. Assuming that the representation $(\mathfrak{g}, V)$ is irreducible, we say that $(\mathfrak{g}, V)$ is *of first class* (resp. *of second class*) if the complex representation $(\mathfrak{g}, V^C)$ is irreducible (resp. reducible) [**3**].

Let $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ be an l-system. We shall denote by $\tilde{C}$ (resp. $\tilde{\mathfrak{c}}$) the center of the group $\tilde{G}$ (resp. the Lie algebra $\tilde{\mathfrak{g}}$). By Prop. 1.3, we have $\tilde{\mathfrak{c}}=C(\tilde{\mathfrak{g}}, \mathfrak{gl}(\mathfrak{m}))$ and $\tilde{C}\subset C(\tilde{\mathfrak{g}}, GL(\mathfrak{m}))\subset \tilde{G}$.

This being said, we have

PROPOSITION 2.3. *The notation being as above, we assume that L is irreducible.* (1) *If L is of type* (R), *then $\tilde{\mathfrak{c}}$ consists of all the elements λE_L, where $\lambda\in \boldsymbol{R}$.* (2) *If L is of type* (C), *then there is an element I_L of $\tilde{\mathfrak{c}}$ such that $I_L^2=-E_L$ and such that $\tilde{\mathfrak{c}}$ consists of all the elements $\lambda E_L+\mu I_L$, where $\lambda, \mu\in\boldsymbol{R}$. Moreover, I_L is unique up to the factor* -1. (3) $\tilde{C}=C(\tilde{\mathfrak{g}}, GL(\mathfrak{m}))=\tilde{\mathfrak{c}}\cap GL(\mathfrak{m})$.

We shall now show that an irreducible l-system of type (C) is really "complex".

LEMMA 2.2. *Let $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ be an l-system. Then we have*

$$A\cdot[\xi, \omega]=[[A, \xi], \omega]=-[\xi, [A, \omega]]$$

for all $A\in\tilde{\mathfrak{c}}$, $\xi\in\mathfrak{m}$ and $\omega\in\mathfrak{m}^*$.

PROOF. We have $A\cdot[\xi, \omega]\cdot\xi'=[A, [[\xi, \omega], \xi']]=[A, [\xi, [\omega, \xi']]]=[[A, \xi], [\omega, \xi']]=[[A, \xi], \omega]\cdot\xi'$ for all $\xi'\in\mathfrak{m}$. Hence $A\cdot[\xi, \omega]=[[A, \xi], \omega]$. The second equality is clear.

By Lemma 2.2, we have

PROPOSITION 2.4. *Let $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ be an irreducible l-system of type* (C). (1) $I_L\cdot\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}$. (2) *$\mathfrak{g}$ is given a complex structure as follows: $\sqrt{-1}\,\xi=[I_L, \xi]$, $\sqrt{-1}\,\omega=-[I_L, \omega]$ and $\sqrt{-1}\,A=I_L\cdot A$ for all $\xi\in\mathfrak{m}$, $\omega\in\mathfrak{m}^*$ and $A\in\tilde{\mathfrak{g}}$.* (3) *$\mathfrak{g}$ is a complex Lie algebra with respect to this complex structure, so that $\mathfrak{m}$, $\mathfrak{m}^*$ and $\tilde{\mathfrak{g}}$ are complex subalgebras of $\mathfrak{g}$.*

Finally, let $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ be an l-system and let $\mathfrak{g}^C$, $\mathfrak{m}^C$, $\mathfrak{m}^{*C}$ and $\tilde{\mathfrak{g}}^C$ be the complexifications of $\mathfrak{g}$, $\mathfrak{m}$, $\mathfrak{m}^*$ and $\tilde{\mathfrak{g}}$ respectively. If we consider $\mathfrak{g}^C$, $\mathfrak{m}^C$, $\mathfrak{m}^{*C}$ and $\tilde{\mathfrak{g}}^C$ as real Lie algebras, we find that the system $(\mathfrak{g}^C, \mathfrak{m}^C, \mathfrak{m}^{*C}, \tilde{\mathfrak{g}}^C)$ forms an l-system, which will be called the *complexification* of L and denoted by L^C.

By Prop. 2.3, we have

PROPOSITION 2.5. *If L is an irreducible l-system of type* (R), *then the complexification L^C of L is irreducible of type* (C).

§ 3. The duality

The following lemma is a generalization of Prop. 1.3. The proof (which is omitted) is analogous to that of Prop. 1.3.

LEMMA 3.1. *Let* $L_i=(\mathfrak{g}_i, \mathfrak{m}_i, \mathfrak{m}_i^*, \tilde{\mathfrak{g}}_i)$ $(i=1, 2)$ *be an l-system. Let* f' *be an isomorphism of* $\tilde{\mathfrak{g}}_1$ *onto* $\tilde{\mathfrak{g}}_2$ *and let* f'' *be an isomorphism of* $\mathfrak{m}_1$ *onto* $\mathfrak{m}_2$. *If* $f''[A, \xi]=[f'A, f''\xi]$ *for all* $A\in\tilde{\mathfrak{g}}_1$, *and* $\xi\in\mathfrak{m}_1$, *there exists a unique isomorphism* f *of* L_1 *onto* L_2 *such that* $fA=f'A$ *and* $f\xi=f''\xi$ *for all* $A\in\mathfrak{g}_1$ *and* $\xi\in\mathfrak{m}_1$.

Hereafter we shall study a fixed l-system $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ and use the notations in the previous sections. We shall say that the l-system $L^*=(\mathfrak{g}, \mathfrak{m}^*, \mathfrak{m}, \tilde{\mathfrak{g}})$ is the *dual* of L and that an isomorphism θ of L onto L^* is *involutive* if θ is an involutive automorphism of $\mathfrak{g}$. Moreover an involutive isomorphism θ of L onto L^* will be called a *$*$-isomorphism* if θ is the involutive automorphism of $\mathfrak{g}$ associated with a certain Cartan decomposition of $\mathfrak{g}$, or equivalently if the quadratic form $\mathfrak{g}\ni X\to\varphi(X, \theta X)\in\boldsymbol{R}$ is negative definite.

PROPOSITION 3.1. *There exists at least one $*$-isomorphism θ of L onto L^*.*

PROOF. By Prop. 2.1, we may assume without loss of generality that L is irreducible. The representation $(\tilde{\mathfrak{g}}, \mathfrak{m})$ being irreducible, the Lie algebra $\tilde{\mathfrak{g}}$ is reductive: $\tilde{\mathfrak{g}}=\mathfrak{c}+[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ (direct sum) and the Lie algebra $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ is semi-simple. By Prop. 2.3 and the theorem of E. Cartan, Mostow and Iwasawa [8], we can find an involutive automorphism θ' of $\tilde{\mathfrak{g}}$ and a negative definite inner product $(\,,)$ on $\mathfrak{m}$ which satisfy the following conditions: The quadratic form $\tilde{\mathfrak{g}}\ni A\to\tilde{\varphi}(A, \theta' A)\in\boldsymbol{R}$ is negative semi-definite and

$$(A\xi, \xi')+(\xi, \theta' A\cdot\xi')=0 \tag{3.1}$$

for all $A\in\tilde{\mathfrak{g}}$ and $\xi, \xi'\in\mathfrak{m}$. It follows from Lemma 1.4 that the quadratic form $\tilde{\mathfrak{g}}\ni A\to\varphi(A, \theta' A)\in\boldsymbol{R}$ is negative definite. We now define an isomorphism θ'' of $\mathfrak{m}$ onto $\mathfrak{m}^*$ by $\langle\xi, \theta''\xi'\rangle=(\xi, \xi')$ for all $\xi, \xi'\in\mathfrak{m}$. From (3.1), we get $\theta''[A, \xi]=[\theta' A, \theta''\xi]$ for all $A\in\tilde{\mathfrak{g}}$ and $\xi\in\mathfrak{m}$. Therefore by Lemma 3.1, there is a unique isomorphism θ of L onto L^* such that $\theta A=\theta' A$ and $\theta\xi=\theta''\xi$ for all $A\in\tilde{\mathfrak{g}}$ and $\xi\in\mathfrak{m}$. Since we have $\theta^{-1}A=\theta' A$ and $\theta^{-1}\xi=\theta''\xi$ for all $A\in\tilde{\mathfrak{g}}$ and $\xi\in\mathfrak{m}$, it follows from Lemma 3.1 that θ is involutive. It remains to prove that θ is a $*$-isomorphism. Every $X\in\mathfrak{g}$ can be uniquely expressed as $\xi+\theta\xi'+A$, where $\xi, \xi'\in\mathfrak{m}$ and $A\in\tilde{\mathfrak{g}}$. We have $\varphi(X, \theta X)=(\xi, \xi)+(\xi', \xi')+\varphi(A, \theta A)$. Since both the inner product $(\,,)$ and the quadratic form $\tilde{\mathfrak{g}}\ni A\to\varphi(A, \theta A)$ are negative definite, this equality means that the quadratic form $\mathfrak{g}\ni X\to\varphi(X, \theta X)\in\boldsymbol{R}$ is also negative definite.

Let θ be an involutive isomorphism of L onto L^*. We shall denote by $\mathfrak{g}_\theta$ the eigen space of the involutive automorphism θ of $\mathfrak{g}$ corresponding to the eigen value 1. $\mathfrak{g}_\theta$ is, as usual, a subalgebra of $\mathfrak{g}$ and, putting $\tilde{\mathfrak{g}}_\theta=\tilde{\mathfrak{g}}\cap\mathfrak{g}_\theta$ and

$\mathfrak{m}_\theta = (\mathfrak{m}+\mathfrak{m}^*) \cap \mathfrak{g}_\theta$, we find that the system $(\mathfrak{g}_\theta, \mathfrak{m}_\theta, \tilde{\mathfrak{g}}_\theta)$ forms an s-system and that $\dim \mathfrak{m}_\theta = \dim \mathfrak{m}$. Noting that θ is an element of $A(\mathfrak{g})$, we have $ad\theta \tilde{G} = \tilde{G}$ and $ad\theta G = G$, where $ad\theta a = \theta a \theta^{-1}$ for all $a \in A(\mathfrak{g})$. We shall denote by G_θ the subgroup of G consisting of all fixed points ($\in G$) of $ad\theta$. The Lie algebra of G_θ is given by $\mathfrak{g}_\theta$. We put $\tilde{G}_\theta = G_\theta \cap \tilde{G}$. Then we have easily $\tilde{G}_\theta = G_\theta \cap G'$ and

PROPOSITION 3.2. (1) *The homogeneous space $G_\theta/\tilde{G}_\theta$ is an affine symmetric homogeneous space, and is naturally an open submanifold of G/G'.* (2) *If θ is a $*$-isomorphism, $G_\theta/\tilde{G}_\theta$ is a compact Riemannian symmetric homogeneous space, and $G_\theta/\tilde{G}_\theta = G/G'$.*

PROOF. (1) Let α be the involutive automorphism of $\mathfrak{g}$ associated with the s-system $(\mathfrak{g}, \mathfrak{m}+\mathfrak{m}^*, \tilde{\mathfrak{g}})$, α being an element of $A(\mathfrak{g})$. We have $ad\alpha \tilde{G} = \tilde{G}$ and $ad\alpha G = G$. Since $\theta\alpha = \alpha\theta$, it follows that $ad\alpha G_\theta = G_\theta$. Let H be the subgroup of G_θ consisting of all fixed points ($\in G_\theta$) of $ad\alpha$. Then we have $\tilde{G}_\theta \subset H$ and we see that the Lie algebra of H coincides with that of $\tilde{G}_\theta$, i.e., $\tilde{\mathfrak{g}}_\theta$. Hence $G_\theta/\tilde{G}_\theta$ is an affine symmetric homogeneous space with respect to the involution: $G_\theta \ni a \to ad\alpha a \in G_\theta$. We have $\dim G_\theta/\tilde{G}_\theta = \dim \mathfrak{m}_\theta = \dim \mathfrak{m}$ and $\tilde{G}_\theta = G_\theta \cap G'$, which indicates that $G_\theta/\tilde{G}_\theta$ is naturally an open submanifold of G/G'. (2) Since the quadratic form $\mathfrak{g} \ni X \to \varphi(X, \theta X) \in \boldsymbol{R}$ is negative definite, we know that G_θ is compact. Therefore $G_\theta/\tilde{G}_\theta$ is a compact Riemannian symmetric homogeneous space. Since $G_\theta/\tilde{G}_\theta$ is an open and closed submanifold of G/G' and since G/G' is connected, we get $G_\theta/\tilde{G}_\theta = G/G'$.

REMARK 1. If L is an irreducible l-system of type (C) and if θ is a $*$-isomorphism, then $G_\theta/\tilde{G}_\theta$ can be proved to be a compact hermitian symmetric homogeneous space and G to be the group of all complex automorphisms of $G_\theta/\tilde{G}_\theta$, cf. § 4.

REMARK 2. Let θ be an involutive isomorphism of L onto L^*. We define the *dual* θ^* of θ, being again an involutive isomorphism of L onto L^*, as follows: $\theta^* X = -\theta X$ if $X \in \mathfrak{m}+\mathfrak{m}^*$ and $\theta^* X = \theta X$ if $X \in \tilde{\mathfrak{g}}$. We have $\mathfrak{m}_\theta + \mathfrak{m}_{\theta*}$ (direct sum) $= \mathfrak{m}+\mathfrak{m}^*$ and $\tilde{G}_\theta = \tilde{G}_{\theta*}$. Under the hypothesis that θ is a $*$-isomorphism, $G_{\theta*}/\tilde{G}_\theta$ may be regarded as the non-compact form of $G_\theta/\tilde{G}_\theta$.

Finally we shall prove the uniqueness of $*$-isomorphisms. Hereafter, the symbol G° will denote the connected component of the identity of a Lie group G. If θ is a $*$-isomorphism of L onto L^*, we know that G_θ° is a maximal compact subgroup of the adjoint group $A(\mathfrak{g})^\circ = G^\circ$ of $\mathfrak{g}$.

LEMMA 3.2. *Let θ be a $*$-isomorphism of L onto L^*. Then $\tilde{G}_\theta^\circ$ is a maximal compact subgroup of $\tilde{G}^\circ$.*

PROOF. Let $(,)$ be the positive definite inner product on $\mathfrak{m}$ defined by $(\xi, \xi') = -\langle \xi, \theta\xi' \rangle$ for all $\xi, \xi' \in \mathfrak{m}$ and denote by ${}^t a$ the transpose of an endomorphism a of $\mathfrak{m}$ with respect to this inner product. Then we have ${}^t a^{-1}$

$=ad\theta a\in\tilde{G}$ for all $a\in\tilde{G}$, and $\tilde{G}_\theta$ consists of all elements $a\in\tilde{G}$ such that $a={}^t a^{-1}$. Lemma 3.2 follows from these and the fact that $\tilde{G}=N(\tilde{\mathfrak{g}}, GL(\mathfrak{m}))$ is an algebraic subgroup of $GL(\mathfrak{m})$, cf. [**6**].

PROPOSITION 3.3. *Let* θ_i $(i=1, 2)$ *be a* $*$*-isomorphism of* L *onto* L^*. *Then there exists an* $a\in\tilde{G}^\circ$ *such that* $a\theta_1 a^{-1}=\theta_2$.

PROOF. By Prop. 2.1, we may assume without loss of generality that L is irreducible. By Lemma 3.2, $\tilde{G}^\circ_{\theta_i}$ is a maximal compact subgroup of $\tilde{G}^\circ$. Hence we can find an $a\in\tilde{G}^\circ$ such that $ada\tilde{G}^\circ_{\theta_1}=\tilde{G}^\circ_{\theta_2}$ or equivalently $ada\tilde{\mathfrak{g}}_{\theta_1}=\tilde{\mathfrak{g}}_{\theta_2}$. Since $a\theta_1 a^{-1}$ is again a $*$-isomorphism of L onto L^* and since $\tilde{\mathfrak{g}}_{a\theta_1 a^{-1}}=ada\tilde{\mathfrak{g}}_{\theta_1}$, we may assume that $\tilde{\mathfrak{g}}_{\theta_1}=\tilde{\mathfrak{g}}_{\theta_2}$. It follows from Prop. 1.1 that $\theta_1 A=\theta_2 A$ for all $A\in\tilde{\mathfrak{g}}$. We now put $u=\theta_1^{-1}\theta_2$, which is an element of $\tilde{G}:\theta_1(u\xi)=\theta_2\xi$ for all $\xi\in\mathfrak{m}$. We see easily that u is in the center C of $\tilde{G}$. Therefore by Prop. 2.3, (3), u is of the form λE_L $(\lambda\neq 0)$ or $\lambda E_L+\mu I_L$ $((\lambda, \mu)\neq 0)$ according as L is of type (R) or of type (C). But we have $\langle\xi, \theta_i\xi\rangle<0$ for all $\xi\in\mathfrak{m}$ $(\neq 0)$, and if L is of type (C), θ_i is an anti-automorphism of $\mathfrak{g}$ (see Prop. 4.1). It follows that u is of the form λE_L with a $\lambda>0$ in either case. Finally we set $b=\frac{1}{\sqrt{\lambda}}E_L$, which is an element of $\tilde{C}^\circ$. Then we have $b\theta_1 b^{-1}(\xi)=\theta_2\xi$ for all $\xi\in\mathfrak{m}$ and consequently $b\theta_1 b^{-1}=\theta_2$.

§4. Classification of irreducible *l*-systems of type (*C*)

PROPOSITION 4.1. *Let* $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ *be an irreducible l-system of type* (C) *and let* θ *be a* $*$*-isomorphism of* L *onto* L^*. *The notation being as in* §3, *we have*: (1) $\mathfrak{g}_\theta$ *is a simple compact real form of* $\mathfrak{g}$, *where* $\mathfrak{g}$ *should be regarded as a complex Lie algebra as in Prop.* 2.4; (2)[3] $\tilde{\mathfrak{g}}_\theta$ *is a real form of* $\tilde{\mathfrak{g}}$; (3) I_L *is in the center of* $\tilde{\mathfrak{g}}_\theta$; (4)[4] $\mathfrak{m}_\theta$ *is a real form of* $\mathfrak{m}+\mathfrak{m}^*$, *more precisely*, $\mathfrak{m}$ *(resp.* $\mathfrak{m}^*$*) consists of all elements* $X-\sqrt{-1}[I_L, X]$ *(resp.* $X+\sqrt{-1}[I_L, X]$*), where* $X\in\mathfrak{m}_\theta$.

PROOF. (1) Since the quadratic form $\mathfrak{g}\ni X\rightarrow\varphi(X, \theta X)\in\boldsymbol{R}$ is negative definite, θ is an anti-automorphism of $\mathfrak{g}$ and hence $\mathfrak{g}_\theta$ is a compact real form of $\mathfrak{g}$. By Prop. 2.2, $\mathfrak{g}_\theta$ is a simple Lie algebra. (2) is clear, because $\theta\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}$. (3) We have $\theta E_L=-E_L$ and $I_L=\sqrt{-1}\,E_L$. Since θ is an anti-automorphism, we get $\theta I_L=I_L$, i.e., $I_L\in\tilde{\mathfrak{g}}_\theta$. I_L is clearly in the center of $\tilde{\mathfrak{g}}_\theta$. (4) Take any $\xi\in\mathfrak{m}$ and set $X=\xi+\theta\xi$ $(\in\mathfrak{m}_\theta)$. We have $[I_L, X]=\sqrt{-1}\,\xi-\sqrt{-1}\,\theta\xi$, whence $\xi=\frac{1}{2}(X-\sqrt{-1}\,[I_L, X])$ and $\theta\xi=\frac{1}{2}(X+\sqrt{-1}\,[I_L, X])$. Since $\dim_R\mathfrak{m}_\theta=\dim_R\mathfrak{m}$, we get (4).

We shall say that an s-system $S=(\mathfrak{g}_0, \mathfrak{m}_0, \tilde{\mathfrak{g}}_0)$ is *simple* (resp. *compact*) if $\mathfrak{g}_0$

---

3) and 4) By Prop. 2.4, $\mathfrak{m}$, $\mathfrak{m}^*$ and $\tilde{\mathfrak{g}}$ are complex subalgebras of $\mathfrak{g}$.

is simple (resp. compact). A simple compact s-system $S=(\mathfrak{g}_0, \mathfrak{m}_0, \tilde{\mathfrak{g}}_0)$ will be called *hermitian* if the center of $\tilde{\mathfrak{g}}_0$ is not trivial. The notation being as in Prop. 4.1, we find from Prop. 4.1 that the system $S=(\mathfrak{g}_\theta, \mathfrak{m}_\zeta, \tilde{\mathfrak{g}}_\theta)$ is a simple compact hermitian s-system.

PROPOSITION 4.2. *Let $S=(\mathfrak{g}_0, \mathfrak{m}_0, \tilde{\mathfrak{g}}_0)$ be a simple compact hermitian s-system and set $\mathfrak{g}=\mathfrak{g}_0^C$ and $\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}_0^C$. There are complex subalgebras $\mathfrak{m}$ and $\mathfrak{m}^*$ of $\mathfrak{g}$ satisfying the following conditions:* (1) *If we consider $\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*$ and $\tilde{\mathfrak{g}}$ as real algebras, then the system $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ forms an irreducible l-system of type (C)*; (2) *The conjugation θ of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ gives a $*$-isomorphism of L onto L^* such that $S=(\mathfrak{g}_\theta, \mathfrak{m}_\theta, \tilde{\mathfrak{g}}_\theta)$.*

For a proof of Prop. 4.2, see Helgason [**2**].

To each irreducible l-system of type (C), $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$, we now associate a fixed $*$-isomorphism θ of L onto L^*, and we set $S=(\mathfrak{g}_\theta, m_\theta, \tilde{\mathfrak{g}}_\theta)$ which is a simple compact hermitian s-system. In virtue of Props. 3.3, 4.1 and 4.2, we arrive at the following conclusion: The assignment $L \rightarrow S$ gives a one-to-one correspondence between the set of all isomorphism classes of irreducible l-system of type (C) and the set of all isomorphism classes of simple compact hermitian s-systems.

Here is the list of classification of simple compact hermitian s-systems and hence of irreducible l-systems of type (C).

Type of S (or L)	$\mathfrak{g}_\theta$	$\tilde{\mathfrak{g}}_\theta$
$I_{m, m'}$ $(m \geqq m' \geqq 1)$	$\mathfrak{su}(m+m')$	$\mathfrak{su}(m)\times\mathfrak{su}(m')\times \boldsymbol{R}$
II_m $(m \geqq 3)$	$\mathfrak{so}(2m, \boldsymbol{R})$	$\mathfrak{u}(m)$
III_m $(m \geqq 2)$	$\mathfrak{sp}(m)$	$\mathfrak{u}(m)$
IV_m $(m \geqq 3)$	$\mathfrak{so}(m+2, \boldsymbol{R})$	$\mathfrak{so}(m, \boldsymbol{R})\times\mathfrak{so}(2, \boldsymbol{R})$
V	$\mathfrak{e}_6$	$\mathfrak{so}(10, \boldsymbol{R})\times\mathfrak{so}(2, \boldsymbol{R})$
VI	$\mathfrak{e}_7$	$\mathfrak{e}_6\times\mathfrak{so}(2, \boldsymbol{R})$

One notes that an irreducible l-system of type $I_{m,1}$ means an l-system of type $P^m(\boldsymbol{C})$.

PROPOSITION 4.3. *Let L be an irreducible l-system of type (R). Then L is of type $P^m(\boldsymbol{R})$ if and only if L^C is of type $I_{m,1}$.*

PROOF. This follows easily from Lemma 3.1.

§ 5. $\tilde{G}$-structures

Throughout this and subsequent four sections, we shall study a fixed l-system $L=(\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ and use the notations and identifications given in § 1. We set $n=\dim \mathfrak{m}$.

Let M be a manifold of dimension n. The bundle of frames of M, F, is a principal fiber bundle over the base space M with the general linear group $GL(n, \boldsymbol{R})$ of degree n as structure group. If we identify the two groups $GL(n, \boldsymbol{R})$ and $GL(\mathfrak{m})$ with respect to a fixed base of $\mathfrak{m}$, the principal fiber bundle F may be defined as follows: The point-set of F is the totality of all isomorphisms of $\mathfrak{m}$ onto $T_p(M)$, where p runs over M; the action of $GL(\mathfrak{m})$ on F is given by $F\times GL(\mathfrak{m})\ni(x, a)\rightarrow xa=x\circ a\in F$; the projection π_F of F onto M is defined by $\pi_F(x)=p$ if x maps $\mathfrak{m}$ onto $T_p(M)$.

Considering the (identity) representation of $\tilde{G}$ on $\mathfrak{m}$, we give the following definition.

DEFINITION 5.1. Let M be an n-dimmensional manifold. A $\tilde{G}$-structure on M is a principal fiber bundle $\tilde{P}$ over the base space M with structure group $\tilde{G}$ which is a subbundle of the bundle of frames of M, F.

Let $\tilde{P}$ be a $\tilde{G}$-structure on a manifold M and let $\tilde{\pi}$ be the projection of $\tilde{P}$ onto M, $\tilde{\pi}$ being the restriction of π_F to $\tilde{P}$. The *basic form* $\tilde{\theta}$ of $\tilde{P}$ is, by definition, the $\mathfrak{m}$-valued 1-form on $\tilde{P}$ defined by $\tilde{\theta}(X)=x^{-1}\tilde{\pi}X$ for all $x\in\tilde{P}$ and $X\in T_x(\tilde{P})$.

NOTATION. Let P be a principal fiber[5] bundle over a manifold M with a Lie group G as structure group. R_a (resp. A^*) will denote the right translation (resp. the vertical vector field) on P corresponding to an $a\in G$ (resp. an $A\in\mathfrak{g}$), $\mathfrak{g}$ being the Lie algebra of G.

PROPOSITION 5.1. (1) *Let X be a tangent vector to $\tilde{P}$ at $x\in\tilde{P}$. $\tilde{\theta}(X)=0$ if and only if X is vertical, i.e., of the form A_x^* with a (unique) $A\in\tilde{\mathfrak{g}}$.* (2) *$R_a^*\tilde{\theta}=a^{-1}\tilde{\theta}$ for all $a\in\tilde{G}$.*

DEFINITION 5.2. Let $\tilde{P}_i$ $(i=1, 2)$ be a $\tilde{G}$-structure on a manifold M_i and let $\tilde{\theta}_i$ be the basic form of $\tilde{P}_i$. An isomorphism of $\tilde{P}_1$ onto $\tilde{P}_2$ is a bundle isomorphism φ of $\tilde{P}_1$ onto $\tilde{P}_2$ such that $\varphi^*\tilde{\theta}_2=\tilde{\theta}_1$. A homeomorphism f of M_1 onto M_2 is called a $\tilde{G}$-homeomorphism of M_1 onto M_2 if it is covered by an isomorphism φ of $\tilde{P}_1$ onto $\tilde{P}_2$.

PROPOSITION 5.2. *The notation being as in Def.* 5.2, *every $\tilde{G}$-homeomorphism f of M_1 onto M_2 is covered by a unique isomorphism φ of $\tilde{P}_1$ onto $\tilde{P}_2$.*

Assuming that a $\tilde{G}$-structure is given on a manifold M, let us define the notion of a tensor field on M in terms of the $\tilde{G}$-structure.

---

5) As for a principal fiber bundle, we use the notations and terminologies given in [5].

NOTATION. $\mathcal{T}_s^r(\mathfrak{m})$ will denote the vector space of all tensors of type $\binom{r}{s}$ on $\mathfrak{m}$, i.e., $\mathcal{T}_s^r(\mathfrak{m})=(\otimes^r\mathfrak{m})\times(\otimes^s\mathfrak{m}^*)$. Given vector spaces $V_1, \cdots, V_l$ and V, $\mathcal{L}(V_1, \cdots, V_l; V)$ will denote the vector space of all multi-linear mappings of $V_1\times\cdots\times V_l$ into V. We may naturally identify $\mathcal{L}(\mathfrak{m}^*, \cdots, \mathfrak{m}^*, \mathfrak{m}, \cdots, \mathfrak{m}; \mathcal{T}_s^r(\mathfrak{m}))$ (with $\mathfrak{m}^*$ p times and $\mathfrak{m}$ q times) with $\mathcal{T}_{s+q}^{r+p}(\mathfrak{m})$; for example, $\mathcal{L}(\mathfrak{m}; \mathfrak{m})=\mathfrak{gl}(\mathfrak{m})=\mathfrak{m}\otimes\mathfrak{m}^*$ and $\mathcal{L}(\mathfrak{m}, \mathfrak{m}; \boldsymbol{R})=\mathcal{L}(\mathfrak{m}; \mathfrak{m}^*)=\mathfrak{m}^*\otimes\mathfrak{m}^*$. The group $GL(\mathfrak{m})$ linearly acts on $\mathcal{T}_s^r(\mathfrak{m})$ through the mapping $GL(\mathfrak{m})\times\mathcal{T}_s^r(\mathfrak{m})\ni(a, X)\rightarrow X^a\in\mathcal{T}_s^r(\mathfrak{m})$, where $(\xi_1\otimes\cdots\otimes\xi_r\otimes\omega_1\otimes\cdots\otimes\omega_s)^a=(a\xi_1)\otimes\cdots\otimes(a\xi_r)\otimes({}^ta^{-1}\omega_1)\otimes\cdots\otimes({}^ta^{-1}\omega_s)$ for all $\xi_1, \cdots, \xi_r\in\mathfrak{m}$ and $\omega_1, \cdots, \omega_s\in\mathfrak{m}^*$.

DEFINITION 5.3. Let $\tilde{P}$ be a $\tilde{G}$-structure on a manifold M. A tensor field of type $\binom{r}{s}$ on M is a mapping Φ of $\tilde{P}$ into $\mathcal{T}_s^r(\mathfrak{m})$ satisfying the equality

$$\Phi_{xa}=(\Phi_x)^{a^{-1}}$$

for all $x\in\tilde{P}$ and $a\in\tilde{G}$.

As is clear, this definition of a tensor field on M is equivalent to the usual one.

EXAMPLES. (1) A tensor field of type $\binom{1}{0}$ (a vector field) on M is a mapping X of $\tilde{P}$ into $\mathfrak{m}$ such that $X_{xa}=a^{-1}X_x$. (2) A tensor field of type $\binom{0}{1}$ (a 1-form) on M is a mapping E of $\tilde{P}$ into $\mathfrak{m}^*$ such that $E_{xa}={}^taE_x$. (3) A tensor field of type $\binom{1}{1}$ on M is a mapping U of $\tilde{P}$ into $\mathfrak{gl}(\mathfrak{m})$ such that $U_{xa}=a^{-1}U_xa$. (3) We define a mapping Ψ_L of $\mathfrak{m}\otimes\mathfrak{m}^*$ into $\tilde{\mathfrak{g}}\subset\mathfrak{gl}(\mathfrak{m})$ by $\Psi_L(\xi, \omega)=[\xi, \omega]$ for all $\xi\in\mathfrak{m}$ and $\omega\in\mathfrak{m}^*$, which is a tensor of type $\binom{2}{2}$ on $\mathfrak{m}$. We have $\Psi_L^a=\Psi_L$ for all $a\in\tilde{G}$, i.e.,

$$a\cdot[a^{-1}\xi, {}^ta\omega]\cdot a^{-1}=[\xi, \omega]$$

for all $\xi\in\mathfrak{m}$ and $\omega\in\mathfrak{m}^*$. This means that the constant mapping $\tilde{P}\ni x\rightarrow\Psi_L\in\mathcal{T}_2^2(\mathfrak{m})$ is a tensor field of type $\binom{2}{2}$ on M. It can be shown that[6] $\tilde{G}$ consists of all elements $a\in GL(\mathfrak{m})$ leaving Ψ_L invariant.

In what follows, we shall consider a fixed $\tilde{G}$-structure $\tilde{P}$ on a manifold M.

DEFINITION 5.4. An affine connection in $\tilde{P}$ is a linear mapping B of $\mathfrak{m}$ into the vector space $\mathfrak{X}(\tilde{P})$ of all vector fields on $\tilde{P}$ satisfying the conditions:

(B. 1) $\tilde{\theta}(B(\xi))=\xi$ for all $\xi\in\mathfrak{m}$, $\tilde{\theta}$ being the basic form of $\tilde{P}$;

(B. 2) $R_aB(\xi)=B(a^{-1}\xi)$ for all $a\in\tilde{G}$ and $\xi\in\mathfrak{m}$.

By condition (B. 1) and Prop. 5.1, 1, we get

LEMMA 5.1. *Let B be an affine connection in $\tilde{P}$. Every tangent vector X*

---

6) Therefore, we see that a $\tilde{G}$-structure $\tilde{P}$ on a manifold M is a tensor structure on M defined by a suitable tensor field of type $\binom{2}{2}$ on M.

to $\tilde{P}$ *at* $x \in \tilde{P}$ *is uniquely written in the form* $B(\xi)_x + A_x^*$, *where* $\xi \in \mathfrak{m}$ *and* $A \in \tilde{\mathfrak{g}}$.

Let B be an affine connection in $\tilde{P}$. By Lemma 5.1, we can find, for each $x \in \tilde{P}$ and $\xi, \xi' \in \mathfrak{m}$, a unique pair $(T_x(\xi, \xi'), R_x(\xi, \xi'))$ of elements of $\mathfrak{m}$ and $\tilde{\mathfrak{g}}$ respectively as follows:

$$[B(\xi), B(\xi')]_x = B(T_x(\xi, \xi'))_x + R_x(\xi, \xi')_x^* .$$

The elements $T_x(\xi, \xi')$ and $R_x(\xi, \xi')$ are bilinear and anti-symmetric with respect to the two variables ξ and ξ'. By condition (B. 2), the mappings $T: \tilde{P} \ni x \to T_x \in \mathcal{L}(\mathfrak{m}, \mathfrak{m}; \mathfrak{m})$ and $R: \tilde{P} \ni x \to R_x \in \mathcal{L}(\mathfrak{m}, \mathfrak{m}; \tilde{\mathfrak{g}})$ are tensor fields of type $\binom{1}{2}$ and of type $\binom{1}{3}$ on M respectively, i.e., $T_{xa}(\xi, \xi') = a^{-1}T_x(a\xi, a\xi')$ and $R_{xa}(\xi, \xi') = a^{-1}R_x(a\xi, a\xi')a$. The tensor field T and R are called the torsion and curvature tensor fields of B respectively. The Ricci tensor field of B is, by definition, the tensor field $R^*: \tilde{P} \ni x \to R_x^* \in \mathcal{L}(\mathfrak{m}; \mathfrak{m}^*)$ of type $\binom{0}{2}$ on M defined by $\langle \xi', R_x^*(\xi) \rangle =$ the trace of the endomorphism $\eta \to R_x(\eta, \xi)\xi'$ of $\mathfrak{m}$. We have $R_x^*(\xi) = \sum_i [R_x(\xi, e_i), \omega^i]$ for all $\xi \in \mathfrak{m}$, where (e_i) is a base of $\mathfrak{m}$ and (ω^i) is the dual base of (e_i). We now define the covariant derivative of a tensor field Φ of type $\binom{r}{s}$ on M. At each $x \in \tilde{P}$, let $\nabla\Phi_x$ be the linear mapping of $\mathfrak{m}$ into $\mathcal{T}_s^r(\mathfrak{m})$ defined by $\nabla_\xi \Phi_x = B(\xi)_x \Phi$. By condition (B. 2), we have

$$\nabla_\xi \Phi_{xa} = (\nabla_{a\xi} \Phi_x)^{a^{-1}} .$$

This indicates that the mapping $\nabla\Phi: \tilde{P} \ni x \to \nabla\Phi_x \in \mathcal{L}(\mathfrak{m}; \mathcal{T}_s^r(\mathfrak{m})) = \mathcal{T}_{s+1}^r(\mathfrak{m})$ is a tensor field of type $\binom{r}{s+1}$ on M, which is called the covariant derivative of Φ.

The following proposition will be usefull in our later arguments.

PROPOSITION 5.3. (1) *Let* B_i $(i=1, 2)$ *be an affine connection in* $\tilde{P}$ *and let* T_i *be the torsion tensor field of* B_i. *Then there is a unique tensor field* $U: \tilde{P} \ni x \to U_x \in \mathcal{L}(\mathfrak{m}; \tilde{\mathfrak{g}})$ *of type* $\binom{1}{2}$ *on* M *such that*

$$B_2(\xi)_x = B_1(\xi)_x + U_x(\xi)_x^* \tag{5.1}$$

for all $x \in \tilde{P}$ *and* $\xi \in \mathfrak{m}$. *In this case, we have*

$$T_2(\xi, \xi') = T_1(\xi, \xi') + U(\xi)\xi' - U(\xi')\xi$$

for all $\xi, \xi' \in \mathfrak{m}$. (2) *If* B_1 *is an affine connection in* $\tilde{P}$ *and if* U *is a tensor field of type* $\binom{1}{2}$ *on* M, *then the linear mapping* B_2 *of* $\mathfrak{m}$ *into* $\mathfrak{X}(\tilde{P})$ *defined by* (5.1) *is an affine connection in* $\tilde{P}$.

§ 6. G'-bundles

DEFINITION 6.1. Let $\tilde{P}$ be a $\tilde{G}$-structure on a manifold M. We say that a system $(P, \bar{l})$ is a G'-bundle associated to the $\tilde{G}$-structure $\tilde{P}$, if P is a principal fiber bundle over the base space M with structure group G' and if $\bar{l}$ is a base preserving bundle homomorphism of P onto $\tilde{P}$ corresponding to the homomorphism l of G' onto $\tilde{G}$.

Let $\tilde{P}$ be a $\tilde{G}$-structure on a manifold M and let $(P, \bar{l})$ be a G'-bundle associated to $\tilde{P}$. The $\mathfrak{m}$-valued 1-form $\theta = \bar{l}^*\theta$ will be called the *basic form* of $(P, \bar{l})$, $\hat{\theta}$ being the basic form of $\tilde{P}$.

By Prop. 5.1, we have

PROPOSITION 6.1. (1) *Let X be a tangent vector to P at $z \in P$. $\theta(X) = 0$ if and only if X is vertical, i.e., of the form A_z^* with a (unique) $A \in \mathfrak{g}'$.* (2) *$R_a^*\theta = l(a)^{-1}\theta$ for all $a \in G'$.*

We show that to each $\tilde{G}$-structure $\tilde{P}$ on a manifold M there is associated at least one G'-bundle, say $(P, \bar{l})$. Indeed, the $\tilde{G}$-structure $\tilde{P}$ and the injection of $\tilde{G}$ into G' give rise to a principal fiber bundle P over the base space M with structure group G' in such a way that $\tilde{P}$ is a subbundle of P. Furthermore, there is a unique homomorphism $\bar{l}$ of P onto $\tilde{P}$ subject to the condition $\bar{l}(x) = x$ at each $x = \tilde{P}$.

DEFINITION 6.2. Let $\tilde{P}_i$ $(i = 1, 2)$ be a $\tilde{G}$-structure on a manifold M_i, let $(P_i, \bar{l}_i)$ be a G'-bundle associated to $\tilde{P}_i$ and let θ_i be the basic form of $(P_i, \bar{l}_i)$. An isomorphism of $(P_1, \bar{l}_1)$ onto $(P_2, \bar{l}_2)$ is a bundle isomorphism φ of P_1 onto P_2 such that $\varphi^*\theta_2 = \theta_1$.

PROPOSITION 6.2. *The notation being as in Def. 6.2, let φ be a bundle isomorphism of P_1 onto P_2. Then φ is an isomorphism of $(P_1, \bar{l}_1)$ onto $(P_2, \bar{l}_2)$ if and only if there is a (unique) isomorphism $\tilde{\varphi}$ of $\tilde{P}_1$ onto $\tilde{P}_2$ such that $\tilde{\varphi} \circ \bar{l}_1 = \bar{l}_2 \circ \varphi$.*

PROOF. First suppose that φ is an isomorphism of $(P_1, \bar{l}_1)$ onto $(P_2, \bar{l}_2)$. There is a unique bundle isomorphism $\tilde{\varphi}$ of $\tilde{P}_1$ onto $\tilde{P}_2$ such that $\tilde{\varphi} \circ \bar{l}_1 = \bar{l}_2 \circ \varphi$. Since $\varphi^*\theta_2 = \theta_1$, we have $\bar{l}_1^*\tilde{\theta}_1 = \bar{l}_1^*(\tilde{\varphi}^*\tilde{\theta}_2)$. Hence $\tilde{\theta}_1 = \tilde{\varphi}^*\tilde{\theta}_2$. The converse is easy.

PROPOSITION 6.3. *The notation being as in Def. 6.2, let $\tilde{\varphi}$ be an isomorphism of $\tilde{P}_1$ onto $\tilde{P}_2$. Then there is at least one isomorphism φ of $(P_1, \bar{l}_1)$ onto $(P_2, \bar{l}_2)$ such that $\bar{l}_2 \circ \varphi = \tilde{\varphi} \circ \bar{l}_1$.*

PROOF. By Prop. 1.7 the homogeneous space $G'/\tilde{G}$ is homeomorphic with the vector space $\mathfrak{m}^*$. Therefore we can find at least one bundle homomorphism h_i of $\tilde{P}_i$ into P_i (corresponding to the injection of $\tilde{G}$ into G') such that $\bar{l}_i \circ h_i = 1$ (the identity transformation of $\tilde{P}_i$). Hence there is a unique bundle isomorphism φ of P_1 onto P_2 such that $\varphi \circ h_1 = h_2 \circ \tilde{\varphi}$. We have clearly $\bar{l}_2 \circ \varphi$

$\tilde{\varphi} \circ \bar{l}_1$.

Let $\tilde{P}$ be a $\tilde{G}$-structure on a manifold M and let $(P, \bar{l})$ be a G'-bundle associated to $\tilde{P}$. We shall say that a bundle homomorphism h of $\tilde{P}$ into P (corresponding to the injection of $\tilde{G}$ into G') is *admissible* if it satisfies the condition $\bar{l} \circ h = 1$. By the proof of Prop. 6.3, there is at least one admissible homomorphism of $\tilde{P}$ into P.

PROPOSITION 6.4. (1) *The notation being as above, let h_i $(i=1, 2)$ be an admissible homomorphism of $\tilde{P}$ into P. Then there exists a unique 1-form $F: \tilde{P} \to \mathfrak{m}^*$ on M such that*

$$h_2(x) = h_1(x) \cdot \exp F_x \tag{6.1}$$

at each $x \in \tilde{P}$. (2) *If h_1 is an admissible homomorphism of $\tilde{P}$ into P and if $F: \tilde{P} \to \mathfrak{m}^*$ is a 1-form on M, then the mapping h_2 of $\tilde{P}$ into P defined by* (6.1) *is an admissible homomorphism of $\tilde{P}$ into P.*

PROOF. (1) At each $x \in \tilde{P}$, $h_1(x)$ and $h_2(x)$ lie in the same fiber of P. Hence there is a unique $\tau \in G'$ such that $h_2(x) = h_1(x)\tau$. Since $\bar{l} \circ h_1 = \bar{l} \circ h_2 = 1$, we have $x = x \cdot l(\tau)$, i.e., $l(\tau) = e$. By Prop. 1.6, τ is of the form $\exp F_x$ with a unique $F_x \in \mathfrak{m}^*$. We must prove that the mapping $F: \tilde{P} \ni x \to F_x \in \mathfrak{m}^*$ is a 1-form on M. We have $\exp F_{xa} = a^{-1}(\exp F_x)a = \exp(ada^{-1}F_x)$ for all $x \in \tilde{P}$ and $a \in \tilde{G}$, whence $F_{xa} = {}^t a F_x$. (2) can be analogously proved.

We here state a lemma concerning a tensor field on a manifold with a $\tilde{G}$-structure.

LEMMA 6.1. *Let $\tilde{P}$ be a $\tilde{G}$-structure on a manifold M and let $(P, \bar{l})$ be a G'-bundle associated to $\tilde{P}$. If $\tilde{\Phi}: \tilde{P} \to \mathfrak{T}^r_s(\mathfrak{m})$ is a tensor field of type $\binom{r}{s}$ on M, then the mapping $\Phi = \tilde{\Phi} \circ \bar{l}: P \to \mathfrak{T}^r_s(\mathfrak{m})$ satisfies the equality:*

$$\Phi_{za} = (\Phi_z)^{l(a)^{-1}} \tag{6.2}$$

for all $z \in P$ and $a \in G'$. Conversely, every mapping $\Phi: P \to \mathfrak{T}^r_s(\mathfrak{m})$ satisfying (6.2) *induces a unique tensor field $\tilde{\Phi}: \tilde{P} \to \mathfrak{T}^r_s(\mathfrak{m})$ of type $\binom{r}{s}$ on M such that $\Phi = \tilde{\Phi} \circ \bar{l}$.*

Lemma 6.1 enables us to define a tensor field of type $\binom{r}{s}$ on M to be a mapping $\Phi: P \to \mathfrak{T}^r_s(\mathfrak{m})$ satisfying (6.2).

Example (the prototype of G'-bundles). Let us consider the homogeneous space $M_L = G/G'$. As usual, G may be considered as a principal fiber bundle over the base space M_L with structure group G': The action of G' on G is given by the mapping $G \times G' \ni (z, a) \to za \in G$, where za stands for the product of z and a in the group G, and the projection π of G onto M_L is defined by $\pi(z) = zo$, o being the origin of M_L, i.e., the coset G' of G/G'. The group G acts on the bundle of frames F of M_L as follows: $(zx)\xi = z(x\xi)$ for all $z \in G$, $x \in F$ and $\xi \in \mathfrak{m}$, where z in the right side should be confounded with the transformation on M_L induced by z. Now we have $\mathfrak{g}=\mathfrak{m}+\mathfrak{g}'$ (direct sum). This being said, we define an isomorphism x_0 of $\mathfrak{m}$ onto $T_0(M_L)$ by $x_0\xi=\pi\xi_e$ for all $\xi\in\mathfrak{m}$, where ξ_e means the value at e taken by the left invariant vector field ξ on G. Note that we are identifying $\mathfrak{g}$ with the Lie algebra of all left invariant vector fields on G. x_0 being a point of F, let $\bar{l}$ be the mapping of G into F defined by $\bar{l}(z)=zx_0$ for all $z\in G$.

LEMMA 6.2. *The mapping $\bar{l}$ is a base preserving bundle homomorphism of G into F corresponding to the homomorphism l of G' into $GL(\mathfrak{m})$.*

PROOF. We first show that $ax_0=x_0l(a)$ for all $a\in G'$. In fact, we have $(ax_0)\xi=a(\pi\xi_e)=\pi((ada\xi)_e)=\pi((l(a)\xi)_e)=(x_0l(a))\xi$ for all $\xi\in\mathfrak{m}$, whence $ax_0=x_0l(a)$. It follows that $\bar{l}(za)=\bar{l}(z)l(a)$ for all $z\in G$ and $a\in G'$. Moreover we have $\pi_F(\bar{l}(z))=\pi_F(zx_0)=\pi(z)$ for all $z\in G$, π_F being the projection of F onto M_L.

Lemma 6.2 indicates that the image $\tilde{P}_L$ of G by $\bar{l}$ is a $\tilde{G}$-structure on M_L and that the system $(G, \bar{l})$ is a G'-bundle associated to $\tilde{P}_L$.

PROPOSITION 6.5. *Let θ be the basic form of $(G, \bar{l})$. Then we have $\theta(\xi)=\xi$ for all $\xi\in\mathfrak{m}$.*

PROOF. Let $\tilde{\pi}$ be the projection of $\tilde{P}_L$ onto M_L and let $\tilde{\theta}$ be the basic form of $\tilde{P}_L$. Since $\theta=\bar{l}^*\tilde{\theta}$, we have $\bar{l}(z)\theta(\xi_z)=\bar{l}(z)\tilde{\theta}(\bar{l}(\xi_e))=\tilde{\pi}(\bar{l}\xi_z)=\pi\xi_z=z(\pi\xi_e)$ $=(zx_0)\xi=\bar{l}(z)\xi$ for all $z\in G$ and $\xi\in\mathfrak{m}$, whence $\theta(\xi_z)=\xi$.

§ 7. Connections of type (L)

For all $a\in G'$ and $\xi\in\mathfrak{m}$, we shall denote by $D(a, \xi)$ the $\mathfrak{g}'$-component of $ada\xi$ in the decomposition: $\mathfrak{g}=\mathfrak{m}+\mathfrak{g}'$. We have $ada\xi=l(a)\xi+D(a, \xi)$.

DEFINITION 7.1. Let $\tilde{P}$ be a $\tilde{G}$-structure on a manifold M and let $(P, \bar{l})$ be a G'-bundle associated to $\tilde{P}$. A connection of type (L) in $(P, \bar{l})$ is a linear mapping C of $\mathfrak{m}$ into the vector space $\mathfrak{X}(P)$ of all vector fields on P satisfying the following conditions:

(C. 1) $\theta(C(\xi))=\xi$ for all $\xi\in\mathfrak{m}$, θ being the basic form of $(P, \bar{l})$;

(C. 2) $R_aC(\xi)=C(l(a)^{-1}\xi)+D(a^{-1}, \xi)^*$ for all $a\in G'$ and $\xi\in\mathfrak{m}$.

EXAMPLE (the prototype of connections of type (L)). Let us consider the $\tilde{G}$-structure $\tilde{P}_L$ on the hemogeneous space $M_L=G/G'$ and the G'-bundle $(G, \bar{l})$ associated to $\tilde{P}_L$. We define a linear mapping C_L of $\mathfrak{m}$ into $\mathfrak{X}(G)$ by $C_L(\xi)=\xi$ for all $\xi\in\mathfrak{m}$. By Prop. 6.5, C_L satisfies condition (C. 1). We have $R_a\xi=ada^{-1}\xi$, showing that C_L satisfies condition (C. 2). Hence C_L is a connection of type (L) in $(G, \bar{l})$.

The notation being as in Def. 7.1, we study the fundamental properties of a connection of type (L).

I. From condition (C. 1) and Prop. 6.1, (1), we get

LEMMA 7.1. *Every tangent vector X to P at $z\in P$ is uniquely written in* *the form:* $C(\xi)_z+A_z^*$, *where* $\xi\in\mathfrak{m}$ *and* $A\in\mathfrak{g}'$.

For each $X\in\mathfrak{g}$, we define a vector field X^* on P by $X^*=C(\xi)+A^*$ if $X=\xi+A$, $\xi\in\mathfrak{m}$ and $A\in\mathfrak{g}'$. The mapping $\mathfrak{g}\ni X\to X^*\in\mathfrak{X}(P)$ is linear and we have

PROPOSITION 7.1.

(1) *The mapping* $X\to X_z^*$ *gives an isomorphism of* $\mathfrak{g}$ *onto* $T_z(P)$ *at each* $z\in P$;

(2) $\theta(X^*)\equiv X \bmod \mathfrak{g}'$ *for all* $X\in\mathfrak{g}$;

(3) $R_aX^*=(ada^{-1}X)^*$ *for all* $X\in\mathfrak{g}$ *and* $a\in G'$;

(4) $[A^*, X^*]=[A, X]^*$ *for all* $X\in\mathfrak{g}$ *and* $A\in\mathfrak{g}'$.

This is easy from conditions (C. 1), (C. 2) and Lemma 7.1.

By Lemma 7.1, we can find, for each $z\in P$ and $\xi, \xi'\in\mathfrak{m}$, a unique pair $(S_z(\xi, \xi'), K_z(\xi, \xi'))$ of elements of $\mathfrak{m}$ and $\mathfrak{g}'$ respectively as follows:

$$[C(\xi), C(\xi')]_z=C(S_z(\xi, \xi'))_z+K_z(\xi, \xi')_z^*$$

or equivalently

$$[\xi^*, \xi'^*]_z=(S_z(\xi, \xi')+K_z(\xi, \xi'))_z^*.$$

The elements $S_z(\xi, \xi')$ and $K_z(\xi, \xi')$ are bilinear and anti-symmetric with respect to the two variables ξ and ξ'.

LEMMA 7.2. *Let* $a\in G'$, $\xi, \xi'\in\mathfrak{m}$ *and* $z\in P$. *Then we have*

$$S_{za}(\xi, \xi')+K_{za}(\xi, \xi')=ada^{-1}S_z(l(a)\xi, l(a)\xi')+ada^{-1}K_z(l(a)\xi, l(a)\xi').$$

PROOF. We have

$$\begin{aligned}R_{a^{-1}}\cdot[\xi^*, \xi'^*]_{za}&=[R_{a^{-1}}\xi^*, R_{a^{-1}}\xi'^*]_z=[(ada\xi)^*, (ada\xi')^*]_z\\&=[(l(a)\xi)^*+D(a, \xi)^*, (l(a)\xi')^*+D(a, \xi')^*]_z\\&=(S_z(l(a)\xi, l(a)\xi')+K_z(l(a)\xi, l(a)\xi')+[D(a, \xi), l(a)\xi']\\&\quad+[l(a)\xi, D(a, \xi')]+[D(a, \xi), D(a, \xi')])_z^*.\end{aligned}$$

We have

$$\begin{aligned}ada[\xi, \xi']&=[ada\xi, ada\xi']=[l(a)\xi+D(a, \xi), l(a)\xi'+D(a, \xi')]\\&=[D(a, \xi), l(a)\xi']+[l(a)\xi, D(a, \xi')]+[D(a, \xi), D(a, \xi')]\\&=0.\end{aligned}$$

Hence we get

$$R_{a^{-1}}\cdot[\xi^*, \xi'^*]_{za}=(S_z(l(a)\xi, l(a)\xi')+K_z(l(a)\xi, l(a)\xi'))_z^*. \tag{7.1}$$

On the other hand, we get

$$R_{a^{-1}}\cdot[\xi^*, \xi'^*]_{za}=(adaS_{za}(\xi, \xi')+adaK_{za}(\xi, \xi'))_z^*. \tag{7.2}$$

Lemma 7.2 follows immediately from (7.1) and (7.2).

We define, at each $z\in P$, a linear mapping S_z^* of $\mathfrak{m}$ into $\tilde{\mathfrak{g}}$ by $S_z^*(\xi)$

$=\sum_i [S_z(\xi, e_i), \omega^i]$ for all $\xi \in \mathfrak{m}$, where (e_i) is a base of $\mathfrak{m}$ and (ω^i) is the dual base of (e_i). Moreover, we denote by $W_z(\xi, \xi')$ the $\tilde{\mathfrak{g}}$-component of $K_z(\xi, \xi')$ in the decomposition: $\mathfrak{g}'=\mathfrak{m}^*+\tilde{\mathfrak{g}}$, and define, at each $z \in P$, a linear mapping of $\mathfrak{m}$ into $\mathfrak{m}^*$ by $W_z^*(\xi)=\sum_i [W_z(\xi, e_i), \omega^i]$ for all $\xi \in \mathfrak{m}$, (e_i) and (ω^i) being just as above. It is clear that S_z^* and W_z^* are well defined.

PROPOSITION 7.2. *Let $a \in G'$, $\xi, \xi' \in \mathfrak{m}$ and $z \in P$.*

(1) $$S_{za}(\xi, \xi')=l(a)^{-1}S_z(l(a)\xi, l(a)\xi');$$

(2) *Assume that $S_w=0$ at each $w \in P$. Then,*
$$W_{za}(\xi, \xi')=l(a)^{-1}W_z(l(a)\xi, l(a)\xi')l(a);$$

(3) *Assume that $S_w^*=0$ at each $w \in P$. Then,*
$$W_{za}^*(\xi)={}^t l(a)W_z^*(l(a)\xi).$$

PROOF. There is a unique element ω of $\mathfrak{m}^*$ such that $a=l(a)\cdot \exp \omega$ (Prop. 1.6). Then we have, from Lemma 7.2,
$$S_{za}(\xi, \xi')+W_{za}(\xi, \xi') \equiv l(a)^{-1}S_z(l(a)\xi, l(a)\xi')$$
$$-[\omega, l(a)^{-1}S_z(l(a)\xi, l(a)\xi')]+l(a)^{-1}W_z(l(a)\xi, l(a)\xi')l(a) \mod \mathfrak{m}^*.$$
(1) and (2) are immediate from this equality. We have
$$W_{za}(\xi, \xi')=-[\omega, l(a)^{-1}S_z(l(a)\xi, l(a)\xi')]+l(a)^{-1}W_z(l(a)\xi, l(a)\xi')l(a),$$
and hence it follows that
$$\begin{aligned}W_{za}^*(\xi)&=-\sum_i [[\omega, l(a)^{-1}S_z(l(a)\xi, l(a)e_i)], \omega^i]\\&\quad+\sum_i [l(a)^{-1}W_z(l(a)\xi, l(a)e_i)l(a), \omega^i]\\&=-[\omega, ad\,l(a)^{-1}\sum_i [S_z(l(a)\xi, l(a)e_i), {}^t l(a)^{-1}\omega^i]]\\&\quad+ad\,l(a)^{-1}\sum_i [W_z(l(a)\xi, l(a)e_i), {}^t l(a)^{-1}\omega^i].\end{aligned}$$
Since $(l(a)e_i)$ forms a base of $\mathfrak{m}$ and $({}^t l(a)^{-1}\omega^i)$ is the dual base of $(l(a)e_i)$, we get
$$W_{za}^*(\xi)=-[\omega, ad\,l(a)^{-1}S_z^*(l(a)\xi)]+{}^t l(a)W_z^*(l(a)\xi)={}^t l(a)W_z^*(l(a)\xi).$$

Prop. 7.2, (1) shows that the mapping $S: P \ni z \to S_z \in \mathcal{L}(\mathfrak{m}, \mathfrak{m}; \mathfrak{m})$ is a tensor field of type $\binom{1}{2}$ on M (see Lemma 6.1), which will be called the torsion tensor field of C. It follows that the mapping $S^*: P \ni z \to S_z^* \in \mathcal{L}(\mathfrak{m}; \tilde{\mathfrak{g}})$ is a tensor field of type $\binom{1}{2}$ on M. Similarly, Prop. 7.2, (2) (resp. (3)) means that the mapping $W: P \ni z \to W_z \in \mathcal{L}(\mathfrak{m}, \mathfrak{m}; \tilde{\mathfrak{g}})$ (resp. $W^*: P \ni z \to W_z^* \in \mathcal{L}(\mathfrak{m}; \mathfrak{m}^*)$) is a tensor field of type $\binom{1}{3}$ (resp. of type $\binom{0}{2}$) on M under the condition that $S=0$

(resp. $S^*=0$).

II. Let us fix an admissible homomorphism h of $\tilde{P}$ into P.

PROPOSITION 7.3. *There exists a unique pair* (B, J) *of an affine connection* B *in* $\tilde{P}$ *and a tensor field* $J:\tilde{P}\ni x\rightarrow J_x\in\mathcal{L}(\mathfrak{m};\mathfrak{m}^*)$ *of type* $\binom{0}{2}$ *on* M *such that*

$$h\cdot B(\xi)_x=C(\xi)_z+J_x(\xi)^*_z \tag{7.3}$$

for all $x\in\tilde{P}$ *and* $\xi\in\mathfrak{m}$, *where* $z=h(x)$.

PROOF. First we make a general remark: Let X be a tangent vector to P at $z\in P$. Then, $\bar{l}X=0$ if and only if X is of the form ω^*_z with a unique $\omega\in\mathfrak{m}^*$. Uniqueness: By (7.3), we have $\bar{l}\circ h\cdot B(\xi)_x=\bar{l}\cdot C(\xi)_z+\bar{l}\cdot J_x(\xi)^*_z=\bar{l}\cdot C(\xi)_z$. Since $\bar{l}\circ h=1$, this means that B and hence J are uniquely determined by C. Existence: First we define a linear mapping B of $\mathfrak{m}$ into $\mathfrak{X}(\tilde{P})$ by $B(\xi)_x=\bar{l}\cdot C(\xi)_z$ for all $x\in\tilde{P}$ and $\xi\in\mathfrak{m}$, where $z=h(x)$. By condition (C. 1), we have $\tilde{\theta}(B(\xi)_x)=\theta(C(\xi)_z)=\xi$, showing that B satisfies condition (B. 1). By condition (C. 2), we get $R_a\cdot B(\xi)_x=R_a\circ\bar{l}\cdot C(\xi)_z=\bar{l}\circ R_a\cdot C(\xi)_z=\bar{l}\cdot C(a^{-1}\xi)_{za}=B(a^{-1}\xi)_{xa}$ for all $a\in\tilde{G}$. Hence B satisfies condition (B. 2). Thus B is an affine connection in $\tilde{P}$. Next we have $\bar{l}\cdot(h\cdot B(\xi)_x-C(\xi)_z)=B(\xi)_x-B(\xi)_x=0$. Hence, $h\cdot B(\xi)_x-C(\xi)_z$ is of the form $J_x(\xi)^*_z$ with a unique $J_x(\xi)\in\mathfrak{m}^*$. It is clear that $J_x(\xi)$ is linear with respect to the variable ξ. We must prove that the mapping $J:\tilde{P}\ni x\rightarrow J_x\in\mathcal{L}(\mathfrak{m};\mathfrak{m}^*)$ is a tensor field of type $\binom{0}{2}$ on M. For all $a\in\tilde{G}$, we have $R_a\circ h\cdot B(\xi)_x=h\circ R_a\cdot B(\xi)_x=h\cdot B(a^{-1}\xi)_{xa}=C(a^{-1}\xi)_{za}+J_{xa}(a^{-1}\xi)^*_{za}$ and $R_a\circ h\cdot B(\xi)_x=R_a\cdot C(\xi)_z+R_a\cdot J_x(\xi)^*_z=C(a^{-1}\xi)_{za}+(ada^{-1}J_x(\xi))^*_{za}$. It follows that $J_{xa}(a^{-1}\xi)=ada^{-1}J_x(\xi)={}^taJ_x(\xi)$, which proves that J is a tensor field of type $\binom{0}{2}$ on M.

The affine connection B and the tensor field J in Prop. 7.3 will be called *induced* from C by h.

Let z be a point of P and set $x=\bar{l}(z)$. Since z and $h(x)$ lie in the same fiber of P, there is a unique $a\in G'$ such that $z=h(x)a$. Since $x=\bar{l}(z)=x\cdot l(a)$, we have $l(a)=e$, i.e., a is of the form $\exp\omega$ with a unique $\omega\in\mathfrak{m}^*$.

In what follows, B (resp. J) will denote the affine connection (resp. the tensor field) induced from C by h.

PROPOSITION 7.4. *The notation being as above, we have*

$$C(\xi)_z=R_a\circ h\cdot B(\xi)_x-(J_x(\xi)+D(a^{-1},\xi))^*_z \tag{7.4}$$

for all $z\in P$ *and* $\xi\in\mathfrak{m}$.

PROOF. We put $y=h(x)$. By (7.3), we have $R_a\circ h\cdot B(\xi)_x=R_a\cdot C(\xi)_y+R_a\cdot J_x(\xi)^*_y=C(\xi)_z+D(a^{-1},\xi)^*_z+(ada^{-1}J_x(\xi))^*_z=C(\xi)_z+(J_x(\xi)+D(a^{-1},\xi))^*_z$.

As for the affine connection B, we shall use the notations in §5.

LEMMA 7.3. *Let* $x\in\tilde{P}$, $\xi,\xi'\in\mathfrak{m}$ *and set* $z=h(x)$. *We have*

$$T_x(\xi, \xi')+J_x(T_x(\xi, \xi'))+R_x(\xi, \xi')$$
$$=S_z(\xi, \xi')+K_z(\xi, \xi')+[J_x(\xi), \xi']+[\xi, J_x(\xi')]+\nabla_\xi J_x(\xi')-\nabla_{\xi'}J_x(\xi) .$$

PROOF. For all $\xi\in\mathfrak{m}$, we define a vector field $H(\xi)$ on P by $H(\xi)_w =(\xi+J_y(\xi))_w^*$ at each $w\in P$, where $y=\bar{l}(w)$. Since $\bar{l}\circ h=1$, (7.3) means that $H(\xi)$ is h-related to $B(\xi)$, i. e., $h\cdot B(\xi)_x=H(\xi)_x$ at each $x\in\tilde{P}$, where $z=h(x)$. It follows that $[H(\xi), H(\xi')]$ is also h-related to $[B(\xi), B(\xi')]$, i. e.,

$$h\cdot[B(\xi), B(\xi')]_x=[H(\xi), H(\xi')]_z . \tag{7.5}$$

First we have

$$\begin{aligned} h\cdot[B(\xi), B(\xi')]_x &= h\cdot B(T_x(\xi, \xi'))_x+h\cdot R_x(\xi, \xi')_x^* \\ &= H(T_x(\xi, \xi'))_z+R_x(\xi, \xi')_z^* \\ &= (T_x(\xi, \xi')+J_x(T_x(\xi, \xi'))+R_x(\xi, \xi'))_z^* . \end{aligned} \tag{7.6}$$

Let f be any function defined on a neighborhood U of z. We have

$$(H(\xi')f)(w)=(\xi'+J_y(\xi'))_w^*f$$

at each $w\in U$, where $y=\bar{l}(w)$. Since $\bar{l}\cdot H(\xi)_z=B(\xi)_x$, it follows easily that

$$H(\xi)_zH(\xi')f=(\nabla_\xi J_x(\xi'))_z^*f+(\xi+J_x(\xi))_z^*((\xi'+J_x(\xi'))^*f) .$$

From this and an analogous equality, we get

$$\begin{aligned} [H(\xi), H(\xi')]_z &= (\nabla_\xi J_x(\xi')-\nabla_{\xi'}J_x(\xi))_z^* \\ &\quad +[(\xi+J_x(\xi))^*, (\xi'+J_x(\xi'))^*]_z \\ &= (S_z(\xi, \xi')+K_z(\xi, \xi')+[\xi, J_x(\xi')] \\ &\quad +[J_x(\xi), \xi']+\nabla_\xi J_x(\xi')-\nabla_{\xi'}J_x(\xi))_z^* . \end{aligned} \tag{7.7}$$

Lemma 7.3 follows from (7.5), (7.6) and (7.7).

PROPOSITION 7.5. *The notation being as in Lemma* 6.1, *we have*

(1) $$\hat{S}=T;$$

(2) *Assume that* $S=0$. *Then,*

$$\tilde{W}(\xi, \xi')=R(\xi, \xi')-[J(\xi), \xi']+[J(\xi'), \xi];$$

(3) *Assume that* $S^*=0$. *Then,*

$$(\tilde{W}^*)(\xi)=R^*(\xi)-\frac{1}{2}J(\xi)+\sum_i [[J(e_i), \xi], \omega^i] ,$$

where (e_i) *is a base of* $\mathfrak{m}$ *and* (ω^i) *is the dual base of* (e_i).

PROOF. First note that W (resp. W^*) is a tensor field on M under the condition that $S=0$ (resp. $S^*=0$). (1) and (2) are immediate from Lemma 7.3. Let us prove (3). From 7.3, we get $(\tilde{W}^*)(\xi)=R^*(\xi)-\sum_i [[J(\xi), e_i], \omega^i] + \sum_i [[J(e_i), \xi], \omega^i]$. But, for all $\xi'\in\mathfrak{m}$, we have $\langle\xi', \sum_i [[J(\xi), e_i], \omega^i]\rangle$

$=\sum_i \varphi([\xi', J(\xi)], [e_i, \omega^i]) = \sum_i \langle [\xi', J(\xi)]e_i, \omega^i \rangle = Tr[\xi', J(\xi)]$. Since $Tr[\xi', J(\xi)] = \frac{1}{2}\langle \xi', J(\xi)\rangle$ (Prop. 1.4), we get $\sum_i [[J(\xi), e_i], \omega^i] = \frac{1}{2} J(\xi)$.

III. PROPOSITION 7.6. *Let h_i $(i=1, 2)$ be an admissible homomorphism of $\tilde{P}$ into P and let B_i be the affine connection in $\tilde{P}$ induced from C by h_i. Let F be the* 1*-form on M defined by* (6.1). *Then we have*

$$B_2(\xi)_x = B_1(\xi)_x + [F_x, \xi]^*_x$$

for all $x \in \tilde{P}$ and $\xi \in \mathfrak{m}$.

PROOF. Let x be a point of $\tilde{P}$. If we set $z = h_2(x)$ and $a = \exp F_x$, then we have $x = \bar{l}(z)$ and $z = h_1(x)\cdot a$. Hence from Prop. 7.4, we get

$$C(\xi)_z = R_a \circ h_1 \cdot B_1(\xi) - (J_x(\xi) + D(a^{-1}, \xi))^*_z, \tag{7.8}$$

J being the tensor field on M induced from C by h_1. We have

$$\bar{l} \circ R_a \circ h_1 = 1,\ l(D(a^{-1}, \xi)) = -[F_x, \xi] \text{ and } B_2(\xi)_x = \bar{l}\cdot C(\xi)_z.$$

Therefore by applying $\bar{l}$ to the both sides of (7.8), we obtain the desired equality.

We shall denote by $\mathfrak{A}(C)$ the family of affine connections in $\tilde{P}$ which are induced from C by all admissible homomorphisms of $\tilde{P}$ into P.

PROPOSITION 7.7. *Each affine connection in $\mathfrak{A}(C)$ is induced from C by a unique admissible homomorphism of $\tilde{P}$ into P.*

PROOF. Suppose that an affine connection B in $\tilde{P}$ is induced from C by two admissible homomorphisms, say h_1 and h_2, of $\tilde{P}$ into P. Let F be the 1-form on M defined by (6.1). Then by Prop. 7.6, we have $[F_x, \xi]=0$ for all $x \in \tilde{P}$ and $\xi \in \mathfrak{m}$. Therefore we get $F=0$ (Prop. 1.2, (2)).

§8. The torsion tensor field of a $\tilde{G}$-structure

Let $\tilde{P}$ be a $\tilde{G}$-structure on a manifold M. Given a tensor field $T: \tilde{P} \to \mathcal{L}(\mathfrak{m}, \mathfrak{m}; \mathfrak{m})$ of type $\binom{1}{2}$ on M, we define a new tensor field $T^*: \tilde{P} \to \mathcal{L}(\mathfrak{m}; \mathfrak{g})$ of type $\binom{1}{2}$ on M by $T^*_x(\xi) = \sum_i [T_x(\xi, e_i), \omega^i]$ for all $\xi \in \mathfrak{m}$ and $x \in \tilde{P}$, where (e_i) is a base of $\mathfrak{m}$ and (ω^i) is the dual base of (e_i).

THEOREM 8.1. *Let $\tilde{P}$ be a $\tilde{G}$-structure on a manifold M. Then there exists a unique tensor field $T: \tilde{P} \to \mathcal{L}(\mathfrak{m}, \mathfrak{m}: \mathfrak{m})$ of type $\binom{1}{2}$ on M satisfying the following conditions:*

(1) $$T^* = 0;$$

(2) *There exists at least one affine connection B in $\tilde{P}$ whose torsion tensor*

field is equal to T.

DEFINITION 8.1. Let $\tilde{P}$ be a $\tilde{G}$-structure on a manifold M. The tensor field $T: \tilde{P} \rightarrow \mathcal{L}(\mathfrak{m}, \mathfrak{m}; \mathfrak{m})$ of type $\binom{1}{2}$ on M whose unique existence is assured by Th. 8.1, is called the torsion tensor field of the $\tilde{G}$-structure $\tilde{P}$.

The proof of Th. 8.1 is preceeded by several lemmas. Let $\mathcal{T}$ be the subspace of $\mathcal{L}(\mathfrak{m}, \mathfrak{m}; \mathfrak{m})$ consisting of all elements T such that $T(\xi, \xi') = -T(\xi', \xi)$ for all $\xi, \xi' \in \mathfrak{m}$. We set $\mathcal{E} = \mathcal{L}(\mathfrak{m}; \tilde{\mathfrak{g}})$ and define a linear mapping Δ of $\mathcal{E}$ into $\mathcal{T}$ by $\Delta(U)(\xi, \xi') = U(\xi)\xi' - U(\xi')\xi$ for all $U \in \mathcal{E}$ and $\xi, \xi' \in \mathfrak{m}$. Furthermore we define a linear mapping Δ^* of $\mathcal{T}$ into $\mathcal{E}$ by $\Delta^*(T)(\xi) = \sum_i [T(\xi, e_i), \omega^i]$ for all $T \in \mathcal{T}$ and $\xi \in \mathfrak{m}$, where (e_i) is a base of $\mathfrak{m}$ and (ω^i) is the dual base of (e_i). $\mathcal{T}$ (resp. $\mathcal{E}$) is clearly a $\tilde{G}$-stable subspace of $\mathcal{L}(\mathfrak{m}, \mathfrak{m}; \mathfrak{m})$ (resp. $\mathcal{L}(\mathfrak{m}; \mathfrak{gl}(\mathfrak{m}))$).

We have easily

LEMMA 8.1. *Let* $T \in \mathcal{T}$, $U \in \mathcal{E}$ *and* $a \in \tilde{G}$.

(1) $$\Delta(U^a) = \Delta(U)^a;$$

(2) $$\Delta^*(T^a) = \Delta^*(T)^a.$$

Let θ be a fixed $*$-isomorphism of L onto L^*. We define positive definite inner products $(\,,)$ on $\mathfrak{m}$, $\mathcal{T}$ and $\mathcal{E}$ respectively as follows: $(\xi, \xi') = -\langle \xi, \theta\xi' \rangle$ for all $\xi, \xi' \in \mathfrak{m}$; $(T, T') = \frac{1}{2} \sum_{i,j} (T(e_i, e_j), T'(e_i, e_j))$ for all $T, T' \in \mathcal{T}$, where (e_i) is an orthonormal base with respect to the inner product $(\,,)$ on $\mathfrak{m}$; $(U, U') = -\sum_i \varphi(U(e_i), \theta U'(e_i))$ for all $U, U' \in \mathcal{E}$, (e_i) being just as above.

LEMMA 8.2.

$$(T, \Delta(U)) = (\Delta^*(T), U)$$

for all $T \in \mathcal{T}$ *and* $U \in \mathcal{E}$.

PROOF.

$$\begin{aligned}(T, \Delta(U)) &= -\frac{1}{2} \sum_{i,j} \langle T(e_i, e_j), \theta\Delta(U)(e_i, e_j) \rangle \\ &= -\frac{1}{2} \sum_{i,j} \varphi(T(e_i, e_j), [\theta U(e_i), \theta e_j] - [\theta U(e_j), \theta e_i]) \\ &= \sum_{i,j} \varphi([T(e_i, e_j), \theta e_j], \theta U(e_i)) \\ &= -\sum_i \varphi(\Delta^*(T)(e_i), \theta U(e_i)) \\ &= (\Delta^*(T), U).\end{aligned}$$

We put $\Theta = \Delta^* \circ \Delta$. Then we have easily, from Lemma 8.2,

LEMMA 8.3.

(1) $$\Theta(\mathcal{E}) = \Delta^*(\mathcal{T});$$

(2) $$\Theta^{-1}(0)=\Delta^{-1}(0)\,;$$

(3) $$\mathcal{E}=\Theta^{-1}(0)+\Theta(\mathcal{E})\quad (\textit{direct sum}).$$

PROOF OF THEOREM 8.1. We first prove uniqueness. Let B (resp. B') be an affine connection in $\tilde{P}$ and let T (resp. T') be the torsion tensor field of B (resp. B'). Assuming that $T^*=T'^*=0$, we must prove $T=T'$. By Prop. 5.3, we can find a tensor field $U:\tilde{P}\to\mathcal{E}$ of type $\binom{1}{2}$ on M such that $T'=T+\Delta(U)$. Since $\Delta^*(T)=T^*=\Delta^*(T')=T'^*=0$, we have $\Theta(U)=0$. Therefore by Lemma 8.3, (2), we get $\Delta(U)=0$, whence $T=T'$. Let us now prove existence. We see from the proof of Lemma 3.2 that the group $\tilde{G}_\theta$ is a (maximal) compact subgroup of $\tilde{G}$ and that the homogeneous space $\tilde{G}/\tilde{G}_\theta$ is homeomorphic with a euclidean space. Hence there is at least one affine connection, say B', in $\tilde{P}$, cf. [5]. Denoting by T' the torsion tensor field of B', we can find, at each $x\in\tilde{P}$, a unique element U_x of $\Theta(\mathcal{E})$ such that $\Delta^*(T'_x)=\Theta(U_x)$ (Lemma 8.3). For all $x\in\tilde{P}$ and $a\in\tilde{G}$, we have $\Delta^*(T'_{xa})=\Theta(U_{xa})=\Theta(U_x^{a^{-1}})$ and $U_x^{a^{-1}}\in\Theta(\mathcal{E})$ (Lemma 8.1), whence $U_{xa}=U_x^{a^{-1}}$. This means that the mapping $U:\tilde{P}\ni x\to U_x\in\Theta(\mathcal{E})$ is a tensor field of type $\binom{1}{2}$ on M. Therefore by Prop. 5.3, there is an affine connection B in $\tilde{P}$ whose torsion tensor field is given by $T=T'-\Delta(U)$. We have $T^*=\Delta^*(T)=\Delta^*(T')-\Theta(U)=0$, completing the proof of Th. 8.1.

REMARK. The notion of the torsion tensor field of a $\tilde{G}$-structure generalizes the notion of the Nijenhuis tensor field of an almost complex structure. In fact, let L be an irreducible l-system of type $I_{m,1}$, i.e., of type $P^m(\boldsymbol{C})$. In this case, the associated representation $(\tilde{G}, \mathfrak{m})$ is equivalent to the representation $(GL(m,\boldsymbol{C}), \boldsymbol{C}^m)$, and it can be proved that the torsion tensor field of a $\tilde{G}$-structure coincides with the Nijenhuis tensor field of the corresponding almost complex structure.

§ 9. Normal connections of type (L)

Let $\tilde{P}$ be a $\tilde{G}$-structure on a manifold M. We introduce an equivalence relation in the set of all affine connections in $\tilde{P}$ as follows: Let B_i $(i=1,2)$ be an affine connection in $\tilde{P}$. $B_1\sim B_2$ if and only if there is a 1-form $F:\tilde{P}\to\mathfrak{m}^*$ on M such that

(9.1) $$B_2(\xi)_x=B_1(\xi)_x+[F_x,\xi]_x^*$$

for all $x\in\tilde{P}$ and $\xi\in\mathfrak{m}$. One notes that the 1-form F in (9.1) is uniquely determined by B_1 and B_2 (cf. Prop. 7.7) and that, given an affine connection B_1 in $\tilde{P}$ and a 1-form F on M, the linear mapping B_2 of $\mathfrak{m}$ into $\mathfrak{X}(\tilde{P})$ defined by (9.1) is an affine connection in $\tilde{P}$ (Prop. 5.3). Moreover since $[[\omega,\xi],\xi']$

$=[[\omega, \xi'], \xi]$ for all $\xi, \xi' \in \mathfrak{m}$ and $\omega \in \mathfrak{m}^*$, it follows from Prop. 5.3 that if $B_1 \sim B_2$, then the torsion tensor fields of the two connections coincide.

DEFINITION 9.1. We say that two affine connections B_1 and B_2 in $\tilde{P}$ are mutually L-equivalent if $B_1 \sim B_2$. Let $\mathfrak{A}$ be a class of mutually L-equivalent affine connections in $\tilde{P}$. The torsion tensor field T of $\mathfrak{A}$ is defined to be the torsion tensor field of some affine connection in $\mathfrak{A}$. The class $\mathfrak{A}$ is called admissible if $T^*=0$ or equivalently if T coincides with the torsion tensor field of $\tilde{P}$.

From Props. 6.4, 7.5 and 7.6, we get

PROPOSITION 9.1. *Let $(P, \bar{l})$ be a G'-bundle associated to $\tilde{P}$ and let C be a connection of type (L) in $(P, \bar{l})$. The family $\mathfrak{A}(C)$ of affine connections in $\tilde{P}$ induced from C forms a class of mutually L-equivalent affine connections in $\tilde{P}$. The class $\mathfrak{A}(C)$ is admissible if and only if $S^*=0$.*

DEFINITION 9.2. Let $\tilde{P}_i$ $(i=1, 2)$ be a $\tilde{G}$-structure on a manifold M_i and let $\mathfrak{A}_i$ be a class of mutually L-equivalent affine connections in $\tilde{P}_i$. An isomorphism $\tilde{\varphi}$ of $\tilde{P}_1$ onto $\tilde{P}_2$ is called an isomorphism of $(\tilde{P}_1, \mathfrak{A}_1)$ onto $(\tilde{P}_2, \mathfrak{A}_2)$ if $\tilde{\varphi}\mathfrak{A}_1=\mathfrak{A}_2$. A homeomorphism f of M_1 onto M_2 is called a homeomorphism of $(M_1, \mathfrak{A}_1)$ onto $(M_2, \mathfrak{A}_2)$ if there is a (unique) isomorphism $\tilde{\varphi}$ of $(\tilde{P}_1, \mathfrak{A}_1)$ onto $(\tilde{P}_2, \mathfrak{A}_2)$ which covers f.

Now let us define an endomorphism Q_L of $\mathcal{L}(\mathfrak{m}; \mathfrak{m}^*)$ by

$$Q_L(J)(\xi)=\sum_i [[J(e_i), \xi], \omega^i]$$

for all $J \in \mathcal{L}(\mathfrak{m}; \mathfrak{m}^*)$ and $\xi \in \mathfrak{m}$, where (e_i) is a base of $\mathfrak{m}$ and (ω^i) is the dual base of (e_i).

LEMMA 9.1.

$$Q_L(J^a)=Q_L(J)^a$$

for all $J \in \mathcal{L}(\mathfrak{m}; \mathfrak{m}^*)$ and $a \in \tilde{G}$.

Furthermore we define an endomorphism Φ_L of $\mathcal{L}(\mathfrak{m}; \mathfrak{m}^*)$ to be

$$\Phi_L(J)=\frac{1}{2}J-Q_L(J)$$

for all $J \in \mathcal{L}(\mathfrak{m}; \mathfrak{m}^*)$.

From now on (Th. 9.1–Prop. 9.4), we assume that the endomorphism Φ_L is an automorphism.

THEOREM 9.1. *Let $\tilde{P}$ be a $\tilde{G}$-structure on a manifold M, let $\mathfrak{A}$ be an admissible class of mutually L-equivalent affine connections in $\tilde{P}$ and let $(P, \bar{l})$ be a G'-bundle associated to $\tilde{P}$. The notation being as in § 7, there exists a connection C of type (L) in $(P, \bar{l})$ satisfying the condition:*

$$\text{(9.2)} \qquad \mathfrak{A}(C)=\mathfrak{A}, \quad S^*=0 \quad \textit{and} \quad W^*=0.$$

PROOF. We fix an affine connection B in $\mathfrak{A}$ and an admissible homomorphism h of $\tilde{P}$ into P. Since Φ_L is assumed to be an automorphism, we can find, at each $x \in \tilde{P}$, a unique linear mapping J_x of $\mathfrak{m}$ into $\mathfrak{m}^*$ such that

$$R_x^* = \Phi_L(J_x), \tag{9.3}$$

where R^* is the Ricci tensor field of B. It follows from Lemma 9.1 that the mapping $J: \tilde{P} \ni x \to J_x \in \mathcal{L}(\mathfrak{m}; \mathfrak{m}^*)$ is a tensor field of type $\binom{0}{2}$ on M. Using h, B and J thus obtained, we define a linear mapping C of $\mathfrak{m}$ into $\mathfrak{X}(P)$ by (7.4). From Prop. 6.1 and conditions (B. 1) and (B. 2), we infer that C satisfies conditions (C. 1) and (C. 2), i. e., C is a connection of type (L) in $(P, \bar{l})$. It is clear that B (resp. J) coincides with the affine connection (resp. the tensor field), induced from C by h. Hence $\mathfrak{A}(C) = \mathfrak{A}$. Since $\mathfrak{A}$ is admissible, we see from Prop. 7.5, (1) that $S^* = 0$. Finally from (9.3) and Prop. 7.5, (3), we get $W^* = 0$.

DEFINITION 9.3. Let $\tilde{P}$ be a $\tilde{G}$-structure on a manifold M and let $\mathfrak{A}$ be an admissible class of mutually L-equivalent affine connections in $\tilde{P}$. We say that a system $(P, \bar{l}, C)$ is a normal connection of type (L) associated to $\mathfrak{A}$, if $(P, \bar{l})$ is a G'-bundle associated to $\tilde{P}$ and if C is a connection of type (L) in (P, l) satisfying condition (9.2).

DEFINITION 9.4. Let $\tilde{P}_i$ $(i = 1, 2)$ be a $\tilde{G}$-structure on a manifold M_i and let $(P_i, \bar{l}_i, C_i)$ be a normal connection of type (L) associated to a certain admissible class of mutually L-equivalent affine connections in $\tilde{P}_i$. An isomorphism φ of $(P_1, \bar{l}_1)$ onto $(P_2, \bar{l}_2)$ is called an isomorphism of $(P_1, \bar{l}_1, C_1)$ onto $(P_2, \bar{l}_2, C_2)$ if $\varphi C_1 = C_2$.

THEOREM 9.2[7]. *Let $\tilde{P}_i$ $(i = 1, 2)$ be a $\tilde{G}$-structure on a manifold M_i, let $\mathfrak{A}_i$ be an admissible class of mutually L-equivalent affine connections in $\tilde{P}_i$ and let $(P_i, \bar{l}_i, C)$ be a normal connection of type (L) associated to $\mathfrak{A}_i$. If $\tilde{\varphi}$ is an isomorphism of $(\tilde{P}_1, \mathfrak{A}_1)$ onto $(\tilde{P}_2, \mathfrak{A}_2)$, there corresponds to $\tilde{\varphi}$ a unique isomorphism φ of $(P_1, \bar{l}_1, C_1)$ onto (P_2, l_2, C_2) which induces $\tilde{\varphi}$ i. e., $l_2 \circ \varphi = \tilde{\varphi} \circ l_1$. Conversely every isomorphism φ of (P_1, l_1, C_1) onto (P_2, l_2, C_2) induces a unique isomorphism $\tilde{\varphi}$ of $(\tilde{P}_1, \mathfrak{A}_1)$ onto $(\tilde{P}_2, \mathfrak{A}_2)$.*

PROOF. First suppose that there is given an isomorphism $\tilde{\varphi}$ of $(\tilde{P}_1, \mathfrak{A}_1)$ onto $(\tilde{P}_2, \mathfrak{A}_2)$. We take a fixed affine connection B_1 in $\mathfrak{A}_1$ and set $B_2 = \tilde{\varphi} B_1$. Since $\tilde{\varphi}\mathfrak{A}_1 = \mathfrak{A}_2$, B_2 belongs to $\mathfrak{A}_2$. Since $\mathfrak{A}_i = \mathfrak{A}(C_i)$, it follows from Prop. 7.7 that the affine connection B_i is induced from C_i by a unique admissible homomorphism h_i of $\tilde{P}_i$ into P_i. By the proof of Prop. 6.3, there is a unique bundle isomorphism φ of P_1 onto P_2 such that $\varphi \circ h_1 = h_2 \circ \tilde{\varphi}$. We have $\bar{l}_2 \circ \varphi = \tilde{\varphi} \circ \bar{l}_1$ and hence φ is an isomorphism of $(P_1, \bar{l}_1)$ onto $(P_2, \bar{l}_2)$. Let J_i be the tensor

7) This allows us to speak of "the" normal connection of type (L) associated to a $\tilde{G}$-structure $\tilde{P}$.

field on M_i induced from C_i by h_i. Since $B_2=\tilde{\varphi}B_1$ and since Φ_L is an automorphism, we have $(J_2)_{\tilde{\varphi}(x)}=(J_1)_x$ at each $x\in\tilde{P}_1$ (Prop. 7.5, (3)). Therefore, it follows from Prop. 7.4 that $\varphi C_1=C_2$, which shows that φ is an isomorphism of $(P_1, \bar{l}_1, C_1)$ onto $(P_2, \bar{l}_2, C_2)$. Let us now prove the uniqueness of φ. Let φ' be a second isomorphism of $(P_1, \bar{l}_1, C_1)$ onto $(P_2, \bar{l}_2, C_2)$ such that $\bar{l}_2\circ\varphi'=\tilde{\varphi}\circ\bar{l}_1$. We put $h_2'=\varphi'\circ h_1\circ\tilde{\varphi}^{-1}$, being an admissible homomorphism of $\tilde{P}_2$ into P_2. Then we have $B_2(\xi)_y=\tilde{\varphi}\cdot B_1(\xi)_x=\tilde{\varphi}\circ\bar{l}_1\cdot C_1(\xi)_z=\bar{l}_2\circ\varphi'\cdot C_1(\xi)_z=\bar{l}_2\cdot C_2(\xi)_w$ at each $y\in\tilde{P}_2$, where $x=\tilde{\varphi}^{-1}(y)$, $z=h_1(x)$ and $w=h_2'(y)$. This means that the affine connection B_2 is induced from C_2 by h_2'. Hence we have $h_2'=h_2$ by Prop. 7.7. Since $\varphi'\circ h_1=h_2\circ\tilde{\varphi}=\varphi\circ h_1$, we get $\varphi'=\varphi$. Now suppose that there is given an isomorphism φ of $(P_1, \bar{l}_1, C_1)$ onto $(P_2, \bar{l}_2, C_2)$. By Prop. 6.2, φ induces a unique isomorphism $\tilde{\varphi}$ of $\tilde{P}_1$ onto $\tilde{P}_2$. We take any affine connection B_1 in $\mathfrak{A}_1$ and denote by h_1 the corresponding admissible homomorphism of $\tilde{P}_1$ into P_1. If we put $h_2=\varphi\circ h_1\circ\tilde{\varphi}^{-1}$, being an admissible homomorphism of $\tilde{P}_2$ into P_2, then we have $(\tilde{\varphi}B_1(\xi))_y=\tilde{\varphi}\cdot B_1(\xi)_x=\tilde{\varphi}\circ\bar{l}_1\cdot C(\xi)_z=\bar{l}_2\circ\varphi\cdot C_1(\xi)_z=\bar{l}_2\cdot C_2(\xi)_w$ at each $y\in\tilde{P}_2$, where $x=\tilde{\varphi}^{-1}(y)$, $z=h_1(x)$ and $w=h_2(y)$. This means that the affine connection $\tilde{\varphi}B_1$ in $\tilde{P}_2$ is induced from C_2 by h_2. Thus $\tilde{\varphi}B_1\in\mathfrak{A}_2$. Therefore $\tilde{\varphi}\mathfrak{A}_1\subset\mathfrak{A}_2$ and hence $\tilde{\varphi}\mathfrak{A}_1=\mathfrak{A}_2$.

Let $\tilde{P}$ be a $\tilde{G}$-structure on a manifold M and let $\mathfrak{A}$ be a class of mutually L-equivalent affine connections in $\tilde{P}$. We denote by $G(M, \mathfrak{A})$ the group of all transformations of $(M, \mathfrak{A})$.

PROPOSITION 9.2. *The notation being as above, we assume that M is connected and that the class $\mathfrak{A}$ is admissible. Then the group $G(M, \mathfrak{A})$ is a Lie group of dimension $\leqq\dim\mathfrak{g}$ with respect to the natural topology.*

PROOF. By Th. 9.1, there is associated to $\mathfrak{A}$ a normal connection $(P, \bar{l}, C)$ of type (L). By Th. 9.2 and Prop. 5.2, $G(M, \mathfrak{A})$ may be identified with the group of all automorphisms of $(P, \bar{l}, C)$; the notation being as in §7, $G(M, \mathfrak{A})$ consists of all transformations φ of P satisfying the equalities: $\varphi X^*=X^*$, $R_a\circ\varphi=\varphi\circ R_a$ for all $X\in\mathfrak{g}$ and $a\in G'$. Therefore by Prop. 7.1, (1) and a theorem of S. Kobayashi [4], $G(M, \mathfrak{A})$ becomes a Lie group of dimension $\leqq\dim\mathfrak{g}$ in such a way that it is a Lie transformation group on P and hence on M.

Let us now consider the $\tilde{G}$-structure $\tilde{P}_L$ on the homogeneous space $M_L=G/G'$, the G'-bundle $(G, \bar{l})$ associated to $\tilde{P}_L$ and the connection C_L of type (L) in $(G, \bar{l})$ which have been observed in §6 and §7. We have $[C_L(\xi), C_L(\xi')]=[\xi, \xi']=0$ for all $\xi, \xi'\in\mathfrak{m}$. Hence the system $(G, \bar{l}, G_L)$ forms a normal connection of type (L) associated to the admissible class $\mathfrak{A}_L=\mathfrak{A}(C_L)$.

PROPOSITION 9.3. *We have naturally $G=G(M_L, \mathfrak{A}_L)$.*

PROOF. For all $a\in G$, we denote by L_a (resp. T_a) the left translation on G (resp. the transformation of M_L) induced by a. Now let a be any element of G. We have $L_a X = X$ for all $X \in \mathfrak{g}$. Hence we see that L_a is an automorphism of $(G, \bar{l}, C_L)$. Since L_a induces the transformation T_a of M_L, it follows from Th. 9.2 that T_a is a transformation of $(M_L, \mathfrak{A}_L)$, i.e., $T_a \in G(M_L, \mathfrak{A}_L)$. Since T_a is covered by a unique automorphism of $(G, \bar{l}, C_L)$, we deduce that the homomorphism $G \ni a \to T_a \in G(M_L, \mathfrak{A}_L)$ is injective. Thus we have proved $G \subset G(M_L, \mathfrak{A}_L)$. Conversely we shall prove $G(M_L, \mathfrak{A}_L) \subset G$. We take any element f of $G(M_L, \mathfrak{A}_L)$. By Th. 9.2, f is induced by a unique isomorphism φ of $(G, \bar{l}, C_L)$. We have $\varphi X^* = X^*$ for all $X \in \mathfrak{g}$. Since $X^* = X$, there is a unique element a of G such that $\varphi = L_a$. Consequently we get $f = T_a$, which proves our assertion.

PROPOSITION 9.4. *Let U_i $(i = 1, 2)$ be a connected open set of M_L. Every homeomorphism f of $(U_1, \mathfrak{A}_L)$ onto $(U_2, \mathfrak{A}_L)$ is extended to a unique transformation of M_L of the form $T_a (a \in G)$.*

The proof of Prop. 9.4 is entirely similar to that of Prop. 9.3 and therefore it is omitted.

REMARK. Prop. 9.3 implies that the action of G on the homogeneous space $M_L = G/G'$ is effective under the hypothesis that Φ_L is an automorphism. However this hypothesis is unnecessary, as is seen from the proof of Th. 9.2.

The following discussions will be concerned with the equivalence problems associated with $\tilde{G}$-structures.

The notation being as in § 8, we put

$$\delta(L) = \dim \Delta^{-1}(0).$$

For each $\omega \in \mathfrak{m}^*$, let $\tilde{\omega}$ be the element of $\mathcal{E}$ defined by $\tilde{\omega}(\xi) = [\omega, \xi]$ for all $\xi \in \mathfrak{m}$. Since $[[\omega, \xi], \xi'] = [[\omega, \xi'], \xi]$ for all $\xi, \xi' \in \mathfrak{m}$, it follows from Prop. 1.2, (2) that the assignment $\omega \to \tilde{\omega}$ gives an injective linear mapping of $\mathfrak{m}^*$ into $\Delta^{-1}(0)$. Hence we have

LEMMA 9.2. $\delta(L) \geqq n = \dim \mathfrak{m}$, *and the equality holds good if and only if the mapping $\omega \to \tilde{\omega}$ gives an isomorphism of $\mathfrak{m}^*$ onto $\Delta^{-1}(0)$.*

From Prop. 5.3 and Lemma 9.2, we get

PROPOSITION 9.5. *Assume that $\delta(L) = n$. Every $\tilde{G}$-structure $\tilde{P}$ on a manifold M admits a unique admissible class, say $\langle P \rangle$, of mutually L-equivalent affine connections.*

Hereafter we assume that Φ_L is an automorphism and that $\delta(L) = n$.

DEFINITION 9.5. Let $\tilde{P}$ be a $\tilde{G}$-structure on a manifold M. We say that a system $(P, \bar{l}, C)$ is a normal connection of type (L) associated to $\tilde{P}$ if it is a normal connection of type (L) associated to the unique class $\langle P \rangle$.

THEOREM 9.3. *To every $\tilde{G}$-structure $\tilde{P}$ on a manifold M there is associated at least one normal connection $(P, \bar{l}, C)$ of type (L).*

THEOREM 9.4. *Let $\tilde{P}_i$ $(i = 1, 2)$ be a $\tilde{G}$-structure on a manifold M_i and let*

$(P_i, \bar{l}_i, C_i)$ be a normal connection of type (L) associated to $\tilde{P}_i$. If $\tilde{\varphi}$ is an isomorphism of $\tilde{P}_1$ onto $\tilde{P}_2$, there corresponds to $\tilde{\varphi}$ a unique isomorphism φ of $(P_1, \bar{l}_1, C_1)$ onto $(P_2, \bar{l}_2, C_2)$ which induces $\tilde{\varphi}$. Conversely, every isomorphism φ of $(P_1, \bar{l}_1, C_1)$ onto $(P_2, \bar{l}_2, C_2)$ induces a unique isomorphism $\tilde{\varphi}$ of $\tilde{P}_1$ onto $\tilde{P}_2$.

Ths. 9.3 and 9.4 follow from Ths. 9.1 and 9.2 respectively.

Given a $\tilde{G}$-structure $\tilde{P}$ on a manifold M, we shall denote by $G(M)$ the group of all $\tilde{G}$-transformations of M.

Prop. 9.2 yields

PROPOSITION 9.6. *The notation being as above, we assume that M is connected. Then the group $G(M)$ is a Lie group of dimension $\leqq \dim \mathfrak{g}$ with respect to the natural topology.*

Props. 9.3 and 9.4 yield

PROPOSITION 9.7. $G = G(M_L)$.

PROPOSITION 9.8. *Let U_i $(i=1, 2)$ be a connected open set of M_L. Every $\tilde{G}$-homeomorphism f of U_1 onto U_2 is extended to a unique transformation of M_L of the form $T_a (a \in G)$.*

EXAMPLES. (1) Let L be an l-system of type $P^m(R)$. If $m \geqq 2$, the endomorphism Φ_L is an automorphism (Th. 10.1). The normal connection of type (L) is nothing but the normal projective connection of degree m. (2) The m-dimensional Möbius space gives rise to an l-system L such that the complexification L^C of L is irreducible of type IV_m [**10**]. If $m \geqq 3$, the endomorphism Φ_L is an automorphism and $\delta(L) = n$ (Ths. 10.1 and 10.2). The normal connection of type (L) is nothing but the normal conformal connection of degree m.

§ 10. The endomorphism Φ_L and the integer $\delta(L)$

THEOREM 10.1. *Let L be an l-system and let $L \cong L_1 \times \cdots \times L_s$ be a decomposition of L into irreducible l-systems. Then Φ_L is an automorphism if and only if none of L_i is of type $P^1(\boldsymbol{R})$ or $P^1(\boldsymbol{C})$.*

In general, let $L = (\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ be an l-system. We identify $\mathcal{L}(\mathfrak{m}; \mathfrak{m}^*)$ and $\mathfrak{m}^* \otimes \mathfrak{m}^*$ as follows: $(\omega \otimes \omega')\xi = \langle \xi, \omega' \rangle \omega$ for all $\omega, \omega' \in \mathfrak{m}^*$ and $\xi \in \mathfrak{m}$. The duality $\langle , \rangle$ between $\mathfrak{m}$ and $\mathfrak{m}^*$ yields a duality $\langle , \rangle$ between $\mathfrak{m} \otimes \mathfrak{m}$ and $\mathfrak{m}^* \otimes \mathfrak{m}^*$, and the endomorphism Q_L is then defined by

$$\langle \xi \otimes \xi', Q_L(\omega \otimes \omega') \rangle = \varphi([\xi, \omega], [\xi', \omega'])$$

for all $\xi, \xi' \in \mathfrak{m}$ and $\omega, \omega' \in \mathfrak{m}^*$.

LEMMA 10.1. *Let L_i $(1 \leqq i \leqq s)$ be an l-system and let L be the product of $L_1, \cdots, L_s$. Then Φ_L is injective if and only if each Φ_{L_i} is injective.*

PROOF. We set $L_i = (\mathfrak{g}_i, \mathfrak{m}_i, \mathfrak{m}_i^*, \tilde{\mathfrak{g}}_i)$ and $L = (\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ and identify $\mathfrak{g}_i$ with an ideal of $\mathfrak{g}$. We have $\mathfrak{m}^*\otimes\mathfrak{m}^*=\sum_{i,j}\mathfrak{m}_i^*\otimes\mathfrak{m}_j^*$ (direct sum). Lemma 10.1 follows from the following facts: (1) $Q_L(\mathfrak{m}_i^*\otimes\mathfrak{m}_i^*)\subset\mathfrak{m}_i^*\otimes\mathfrak{m}_i^*$ and $Q_L(\mathfrak{m}_i^*\otimes\mathfrak{m}_j^*)=\{0\}$ $(i\neq j)$; (2) Q_{L_i} is identical with the restriction of Q_L to $\mathfrak{m}_i^*\otimes\mathfrak{m}_i^*$.

LEMMA 10.2. *Let L be an l-system and let L^C be the complexification of L. Then Φ_L is injective if and only if Φ_{L^C} is injective.*

PROOF. Setting $L=(\mathfrak{g},\mathfrak{m},\mathfrak{m}^*,\tilde{\mathfrak{g}})$, we have $L^C=(\mathfrak{g}^C,\mathfrak{m}^C,\mathfrak{m}^{*C},\tilde{\mathfrak{g}}^C)$. We identify $\mathfrak{m}^{*C}\otimes_C\mathfrak{m}^{*C}$ with a subspace of $\mathfrak{m}^{*C}\otimes_R\mathfrak{m}^{*C}$ as follows:

$$\omega\otimes\omega'=\frac{1}{2}(\omega\otimes\omega'-(\sqrt{-1}\,\omega)\otimes(\sqrt{-1}\,\omega'))$$

for all $\omega,\omega'\in\mathfrak{m}^{*C}$. Furthermore we define an injective linear mapping ρ of $\mathfrak{m}^*\otimes\mathfrak{m}^*$ into $\mathfrak{m}^{*C}\otimes_C\mathfrak{m}^{*C}$ by $\rho(\omega\otimes\omega')=\omega\otimes_C\omega'$. We have $\mathfrak{m}^{*C}\otimes_C\mathfrak{m}^{*C}=\rho(\mathfrak{m}^*\otimes\mathfrak{m}^*)+\sqrt{-1}\,\rho(\mathfrak{m}^*\otimes\mathfrak{m}^*)$ (direct sum). Now Lemma 10.2 is an immediate consequence from the followings: (1) $Q_{L^C}(\mathfrak{m}^{*C}\otimes\mathfrak{m}^{*C})\subset\mathfrak{m}^{*C}\otimes_C\mathfrak{m}^{*C}$; (2) Q_{L^C} restricted to $\mathfrak{m}^{*C}\otimes_C\mathfrak{m}^{*C}$ is complex linear; (3) $Q_{L^C}\circ\rho=\rho\circ Q_L$.

Let $L=(\mathfrak{g},\mathfrak{m},\mathfrak{m}^*,\tilde{\mathfrak{g}})$ be an irreducible l-system of type (C). By Prop. 2.4, $\mathfrak{g}$ becomes a complex Lie algebra in such a way that $\mathfrak{m}$, $\mathfrak{m}^*$ and $\tilde{\mathfrak{g}}$ are complex subalgebras of $\mathfrak{g}$. Let us identify $\mathfrak{m}^*\otimes_C\mathfrak{m}^*$ with a subspace of $\mathfrak{m}^*\otimes\mathfrak{m}^*$as above. Then we have easily $Q_L(\mathfrak{m}^*\otimes\mathfrak{m}^*)\subset\mathfrak{m}^*\otimes_C\mathfrak{m}^*$. Now take a $*$-isomorphism θ of L onto L^* and let $S=(\mathfrak{g}_\theta,\mathfrak{m}_\theta,\tilde{\mathfrak{g}}_\theta)$ be the corresponding simple compact hermitian s-system. We have $\mathfrak{g}=\mathfrak{g}_\theta^C$, $\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}_\theta^C$ and $\mathfrak{m}+\mathfrak{m}^*=\mathfrak{m}_\theta^C$ (Prop. 4.1). Let φ_θ be the Killing form of $\mathfrak{g}_\theta$ and let $\langle,\rangle_\theta$ be the positive definite inner product on $\mathfrak{m}_\theta$ defined by $\langle X,Y\rangle_\theta=-\varphi_\theta(X,Y)$ for all $X,Y\in\mathfrak{m}_\theta$; the inner product $\langle,\rangle_\theta$ on $\mathfrak{m}_\theta$ gives rise to a positive definite inner product $\langle,\rangle_\theta$ on $\mathfrak{m}_\theta\otimes\mathfrak{m}_\theta$. This being said, we define an endomorphism P[8] of $\mathfrak{m}_\theta\otimes\mathfrak{m}_\theta$ by

$$\langle X\otimes X',P(Y\otimes Y')\rangle_\theta=\varphi_\theta([X,Y],[X',Y'])$$

for all $X,X',Y,Y'\in\mathfrak{m}_\theta$, cf. [**11**]. It is shown that P is self-adjoint with respect to the inner product $\langle,\rangle_\theta$. Now the endomorphism P is naturally extended to a complex endomorphism P^C of $(\mathfrak{m}_\theta\otimes\mathfrak{m}_\theta)^C=\mathfrak{m}_\theta^C\otimes_C\mathfrak{m}_\theta^C$. Then we can

8) Let $X_1,\cdots,X_n$ be an orthonormal base of $\mathfrak{m}_\theta$ with respect to the inner product $<,>_\theta$. Set

$$[[X_h,X_k],X_j]=\sum_i R_{ijhk}X_i.$$

Then R_{ijhk} may be considered as the components of the curvature tensor field of the (compact irreducible hermition) symmetric space $M_L=G_\theta/\tilde{G}_\theta$. If $f=\sum_{i,j}f_{ij}X_i\otimes X_j$, we get

$$P(f)=\sum_{i,j}(\sum_{h,k}R_{ihjk}f_{hk})X_i\otimes X_j.$$

prove that P^C leaves $\mathfrak{m}^* \underset{C}{\otimes} \mathfrak{m}^*$ stable and that Q_L and P^C coincide on $\mathfrak{m}^* \underset{C}{\otimes} \mathfrak{m}^*$.

LEMMA 10.3. *Let L be an irreducible l-system of type* (C). *Then Φ_L is injective except the case when L is of type $I_{1,1}$.*

PROOF. We first remark that L is of type $I_{1,1}$ if and only if $\dim \mathfrak{g}_\theta = 3$. Let P^* be the restriction of P^C to $\mathfrak{m}^* \underset{C}{\otimes} \mathfrak{m}^*$. Then we see from the above argument that Φ_L is injective if and only if $\frac{1}{2}$ is not an eigen value of P^*. If $\dim \mathfrak{g}_\theta = 3$, we have easily $\Phi_L = 0$. If $\dim \mathfrak{g}_\theta \geqq 3$, we infer from [11] that the maximal eigen value of P and hence of P^* is smaller than $\frac{1}{2}$, from which follows that Φ_L is injective.

Theorem 10.1 follows from Lemmas 10.1, 10.2, 10.3 and Props. 2.1, 2.2, 2.5, 4.3.

Now we give our attention to the integer $\delta(L)$.

Given an l-system $L = (\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$, $n(L)$ will denote the dimension of $\mathfrak{m}$.

THEOREM 10.2. *Let L be an l-system and let $L \cong L_1 \times \cdots \times L_s$ be a decomposition of L into irreducible l-systems. We assume that each L_i is of classical type. Then $\delta(L) = n(L)$ if and only if none of L_i is of type $P^m(\boldsymbol{R})$ or $P^m(\boldsymbol{C})$* $(m \geqq 2)$.

LEMMA 10.4. *Let L_i $(1 \leqq i \leqq s)$ be an l-system and let L be the product of $L_1, \cdots, L_s$. Then $\delta(L) = n(L)$ if and only if $\delta(L_i) = n(L_i)$ for each i.*

PROOF. $\delta(L) = \sum_i \delta(L_i)$, $n(L) = \sum_i n(L_i)$ and $\delta(L_i) \geqq n(L_i)$.

LEMMA 10.5. *Let L be an l-system and let L^C be the complexification of L. Then $\delta(L) = n(L)$ if and only if $\delta(L^C) = n(L^C)$.*

PROOF. $\delta(L^C) = 2\delta(L)$ and $n(L^C) = 2n(L)$.

A direct calculation gives

LEMMA 10.6. *Let L be an irreducible l-system of type* (C). *Assuming that L is of classical type, we have $\delta(L) = n(L)$ except the case when L is of type $I_{m,1}$ $(m \geqq 2)$ or III_3.*

Since $III_3 \cong I_{3,1}$, Th. 10.2 follows from Lemmas 10.4, 10.5, 10.6 and Props. 2.1, 2.2, 2.5, 4.3.

REMARK. Let $L = (\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ be an l-system. Set $\mathcal{E}' = \mathcal{L}(\mathfrak{m}; \mathfrak{m}^*)$ and denote by $\mathcal{T}'$ the subspace of $\mathcal{L}(\mathfrak{m}, \mathfrak{m}; \tilde{\mathfrak{g}})$ consisting of all elements W such that $W(\xi, \xi') = -W(\xi', \xi)$ for all $\xi, \xi' \in \mathfrak{m}$. Define linear mappings Δ' of $\mathcal{E}'$ into $\mathcal{T}'$ and Δ'^* of $\mathcal{T}'$ into $\mathcal{E}'$ respectively as follows: $\Delta'(J)(\xi, \xi') = [J(\xi), \xi'] - [J(\xi'), \xi]$ for all $J \in \mathcal{E}'$ and $\xi, \xi' \in \mathfrak{m}$; $\Delta'^*(W)(\xi) = \sum_i [W(\xi, e_i), \omega^i]$ for all $W \in \mathcal{T}'$ and $\xi \in \mathfrak{m}$, where (e_i) is a base of $\mathfrak{m}$ and (ω^i) is the dual base of (e_i). Then we have

$$\Phi_L = \Delta'^* \circ \Delta' .$$

Now take a $*$-isomorphism θ of L onto L^* and define positive definite inner products $(,)$ in $\mathcal{E}'$ and $\mathcal{T}'$ respectively as follows: $(J, J') = -\sum_i \langle \theta J(e_i), J'(e_i)\rangle$ for all $J, J' \in \mathcal{E}'$ where (e_i) is an orthonormal base of $\mathfrak{m}$ with respect to the inner product $(,)$ introduced in § 8, i.e., $\langle e_i, \theta e_j\rangle = -\delta_{ij}$;

$$(W, W') = -\frac{1}{2} \sum_{i,j} \varphi(W(e_i, e_j), \theta W'(e_i, e_j))$$

for all $W, W' \in \mathcal{T}'$, (e_i) being just as above. Then, analogously to Lemma 8.2, we get

$$(W, \Delta'(J)) = (\Delta'^*(W), J)$$

for all $W \in \mathcal{T}'$ and $J \in \mathcal{E}'$, from which follows that

(1) $$\Phi_L^{-1}(0) = \Delta'^{-1}(0),$$

(2) $$(\Phi_L(J), J) = (\Delta'(J), \Delta'(J)) \geqq 0.$$

We mention that we can give a direct proof of Th. 10.1 by using (1).

Appendix

The cohomology group associated with an l-system

As we have observed in the text, the operators Δ, Δ^*, Θ, Δ', Δ'^*, Φ_L and the integer $\delta(L)$ play very important roles in the construction of the normal connections of type (L). In this appendix, we shall give cohomological interpretations of these operators and integer[9].

Let $L = (\mathfrak{g}, \mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}})$ be an l-system. We use the notations and identifications given in the text. Put $\mathcal{T}(\mathfrak{m}) = \sum_{r,s} \mathcal{T}_s^r(\mathfrak{m})$ (for the definition of $\mathcal{T}_s^r(\mathfrak{m})$, see § 5). Then the group $\tilde{G}(\subset GL(\mathfrak{m}))$ linearly acts on $\mathcal{T}(\mathfrak{m})$ through the mapping $\tilde{G} \times \mathcal{T}(\mathfrak{m}) \ni (a, X) \to X^a \in \mathcal{T}(\mathfrak{m})$ (§ 5). Note that $\mathfrak{m}, \mathfrak{m}^*, \tilde{\mathfrak{g}}(\subset \mathfrak{gl}(\mathfrak{m}))$ and hence $\mathfrak{g}$ are $\tilde{G}$-stable subspaces of $\mathcal{T}(\mathfrak{m})$.

For each integer p, define a subspace $\mathfrak{g}_p$ of $\mathfrak{g}$ as follows: $\mathfrak{g}_p = \mathfrak{m}(p = -1)$, $= \tilde{\mathfrak{g}}(p = 0)$, $= \mathfrak{m}^*$ $(p = 1)$ and $= 0$ $(p \neq -1, 0, 1)$. Then the family $(\mathfrak{g}_p)$ satisfies the followings:

(1) $$\mathfrak{g} = \sum_p \mathfrak{g}_p \text{ (direct sum)},$$

(2) $$[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}.$$

(1) and (2) mean that $\mathfrak{g}$ is a graded Lie algebra. By utilizing the family $(\mathfrak{g}_p)$ thus obtained, we shall define the cohomology group $H(L) = \sum_{p,q} H^{p,q}(L)$ associated with the l-system L.

Let $C^{p,q}(L)$ be the vector space of all $\mathfrak{g}_{p-1}$-valued q-forms on $\mathfrak{m} = \mathfrak{g}_{-1}$ and

9) We owe to the referee these cohomological interpretations.

put

$$C(L)=\sum_{p,q} C^{p,q}(L).$$

Then, $C^{p,q}(L)$ and hence $C(L)$ are $\tilde{G}$-stable subspaces of $\mathcal{T}(\mathfrak{m})$: $(c^a)(x_1, \cdots, x_q) = (c(a^{-1}x_1, \cdots, a^{-1}x_q))^a$ for all $c \in C^{p,q}(L)$, $a \in \tilde{G}$ and $x_1, \cdots, x_q \in \mathfrak{m}$. More precisely, $C^{0,q}(L)$ (resp. $C^{1,q}(L)$, resp. $C^{2,q}(L)$) is a $\tilde{G}$-stable subspace of $\mathcal{T}_q^1(\mathfrak{m})$ (resp. $\mathcal{T}_{q+1}^1(\mathfrak{m})$, resp. $\mathcal{T}_{q+1}^0(\mathfrak{m})$), and $C^{p,q}(L)=0$ $(p \neq 0, 1, 2)$. Now define coboundary operator $\partial: C(L) \to C(L)$ to be $\partial C^{p,q}(L) \subset C^{p-1,q+1}(L)$ and

$$(\partial c)(x_1, \cdots, x_{q+1}) = \sum_{i=1}^{q+1} (-1)^i [c(x_1, \cdots, \hat{x}_i, \cdots, x_{q+1}), x_i]$$

for all $c \in C^{p,q}(L)$ and $x_1, \cdots, x_{q+1} \in \mathfrak{m}$.

We have easily

LEMMA 1. $\partial(c^a) = (\partial c^a)$, $c \in C(L)$, $a \in \tilde{G}$.

LEMMA 2. $\partial^2 = 0$.

PROOF. Take any $c \in C^{p,q}(L)$ and $x_1, \cdots, x_{q+2} \in \mathfrak{m}$. Then we have

$$\begin{aligned}
(\partial^2 c)(x_1, \cdots, x_{q+2}) &= \sum_i (-1)^i [(\partial c)(x_1, \cdots, \hat{x}_i, \cdots, x_{q+2}), x_i] \\
&= \sum_{j<i} (-1)^{i+j} [[c(x_1, \cdots, \hat{x}_j, \cdots, \hat{x}_i, \cdots, x_{q+2}), x_j], x_i] \\
&\quad + \sum_{j>i} (-1)^{i+j-1} [[c(x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{q+2}), x_j], x_i] \\
&= \sum_{j<i} (-1)^{i+j} [[c(x_1, \cdots, \hat{x}_j, \cdots, \hat{x}_i, \cdots, x_{q+2}), x_j], x_i] \\
&\quad + \sum_{j<i} (-1)^{i+j-1} [[c(x_1, \cdots, \hat{x}_j, \cdots, \hat{x}_i, \cdots, x_{q+2}), x_i], x_j].
\end{aligned}$$

Since $[\mathfrak{m}, \mathfrak{m}]=0$, we get $(\partial^2 c)(x_1, \cdots, x_{q+2})=0$.

As usual, to the complex $(C(L)=\sum_{p,q} C^{p,q}(L), \partial)$ there is associated the cohomology group $H(L)=\sum_{p,q} H^{p,q}(L)$:

$$H^{p,q}(L) = \partial^{-1}(0) \cap C^{p,q}(L)/C^{p+1,q-1}(L).$$

We can easily verify that $H^{p,q}(L)=0$ $(p \neq 0, 1, 2)$, $H^{0,0}(L)=\mathfrak{m}$, $H^{1,0}(L)=H^{2,0}(L)=0$ and $H^{0,1}(L)=\mathfrak{gl}(\mathfrak{m})/\tilde{\mathfrak{g}}$ (Lemma 1.2).

We now define the (formal) adjoint operator $\partial^*: C(L) \to C(L)$ of ∂ to be $\partial^* C^{p,q}(L) \subset C^{p+1,q-1}(L)$ and

$$(\partial^* c)(x_1, \cdots, x_{q-1}) = (-1)^q \sum_{i=1}^{n} [c(x_1, \cdots, x_{q-1}, e_i), \omega^i]$$

for all $c \in C^{p,q}(L)$ and $x_1, \cdots, x_{q-1} \in \mathfrak{m}$, where (e_i) is a base of $\mathfrak{m}$ and (ω^i) is the dual base of (e_i). It is evident that $\partial^* c$ is well defined.

We get easily

LEMMA 3. $\partial^*(c^a) = (\partial^* c)^a$, $c \in C(L)$, $a \in \tilde{G}$.

Let us now show that ∂^* is really the adjoint operator of ∂ with respect to a certian positive definite inner product (,) on $C(L)$. Take any $*$-isomorphism θ of L onto L^* and define a positive definite inner product (,) on $\mathfrak{g}$ by $(x, y)=-\varphi(x, \theta y)$ for all $x, y\in\mathfrak{g}$. This being said, we define a positive definite inner product (,) on $C^{p,q}(L)$ by

$$(c, c')=\frac{1}{q!}\sum_{i_1,\cdots,i_q}(c(e_{i_1}, \cdots, e_{i_q}), c'(e_{i_1}, \cdots, e_{i_q}))$$

for all $c, c'\in C^{p,q}(L)$, where (e_i) is an orthonormal base of $\mathfrak{m}$ with respect to the inner product (,). The inner products (,) on $C^{p,q}(L)$ naturally give rise to a positive definite inner product (,) on $C(L)$.

LEMMA 4. *∂^* is the adjoint operator of ∂ with respect to the inner product (,) on $C(L)$: $(\partial c, c')=(c, \partial^*c')$, $c, c'\in C(L)$.*

PROOF. Let $c\in C^{p,q}(L)$ and $c'\in C^{p-1,q+1}(L)$. Then we get

$$\begin{aligned}(\partial c, c')&=-\frac{1}{(q+1)!}\sum_{i_1,\cdots,i_{q+1}}\varphi((\partial c)(e_{i_1}, \cdots, e_{i_{q+1}}), \theta\cdot c'(e_{i_1}, \cdots, e_{i_{q+1}}))\\ &=-\frac{1}{(q+1)!}\sum_{i_1,\cdots,i_{q+1}}\sum_j(-1)^j\varphi([c(e_{i_1}, \cdots, \hat{e}_{i_j}, \cdots, e_{i_{q+1}}), e_{i_j}], \theta\cdot c'(e_{i_1}, \cdots, e_{i_{q+1}}))\\ &=\frac{(-1)^{q+1}}{(q+1)!}\sum_j\sum_{i_1,\cdots,\hat{i}_j,\cdots,i_{q+1}}\sum_{i_j}\varphi(c(e_{i_1}, \cdots, \hat{e}_{i_j}, \cdots, e_{i_{q+1}}),\\ &\qquad\qquad \theta[c'(e_{i_1}, \cdots, \hat{e}_{i_j}, \cdots, e_{i_{q+1}}, e_{i_j}), \theta e_{i_j}])\\ &=\frac{1}{q+1}\sum_{j}\sum_{i_1,\cdots,\hat{i}_j,\cdots,i_{q+1}}(c(e_{i_1}, \cdots, \hat{e}_{i_j}, \cdots, e_{i_{q+1}}), (\partial^*c')(e_{i_1}, \cdots, \hat{e}_{i_j}, \cdots, e_{i_{q+1}}))\\ &=(c, \partial^*c'),\end{aligned}$$

where we have used the fact that $(-\theta e_i)$ is the dual base of (e_i).

An important consequence of Lemma 4 is that there is defined the notion of a harmonic form in $C(L)$. We put $\square=\partial^*\partial+\partial\partial^*$ (Laplace Bertrami operator). Then a form in $\square^{-1}(0)=\mathcal{H}=\sum_{p,q}\mathcal{H}^{p,q}$ is called harmonic. As usual we get $\mathcal{H}^{p,q}=\partial^{-1}(0)\cap\partial^{*-1}(0)\cap C^{p,q}(L)$ and the orthogonal decompositions:

$$\begin{aligned}C^{p,q}(L)&=\mathcal{H}^{p,q}+\square\, C^{p,q}(L),\\ &=\mathcal{H}^{p,q}+\partial C^{p+1,q-1}(L)+\partial^*C^{p-1,q+1}(L),\end{aligned}$$

$$\partial^{-1}(0)\cap C^{p,q}(L)=\mathcal{H}^{p,q}+\partial C^{p+1,q-1}(L).$$

Therefore we get $H^{p,q}(L)=\mathcal{H}^{p,q}$.

Relations between our previous notations and cohomology:

(A) $\mathcal{E}=C^{1,1}(L)$, $\mathcal{T}=C^{0,2}(L)$, $\varDelta=\partial\mid C^{1,1}(L)$, $\varDelta^*=\partial^*\mid C^{0,2}(L)$, $\Theta=\partial^*\partial\mid C^{1,1}(L)$ (§ 8);

(B) $\mathcal{E}'=C^{2,1}(L)$, $\mathcal{T}'=C^{1,2}(L)$, $\varDelta'=\partial\mid C^{2,1}(L)$, $\varDelta'^*=\partial^*\mid C^{1,2}(L)$, $\Phi_L=\partial^*\partial\mid C^{2,1}(L)=\square\mid C^{2,1}(L)$ (§ 9 and § 10);

(C) $\tilde{\omega}=-\partial\omega$, $\omega\in\mathfrak{m}^*=C^{2,0}(L)$, $\delta(L)=\dim(\partial^{-1}(0)\cap C^{1,1}(L))$, $n(L)=\dim\partial C^{2,0}(L)$ (§ 9 and § 10);

(D) $R^*=\partial^*R$, $S^*=\partial^*S$, $W^*=\partial^*W$, $T^*=\partial^*T$ (§ 5, § 7 and § 8).

Finally we add

PROPOSITION. (1) *Φ_L is an automorphism if and only if $H^{2,1}(L)=0$.* (2) *$\delta(L)=n(L)$ if and only if $H^{1,1}(L)=0$.*

PROOF. (1) follows from (B) and the fact that $H^{2,1}(L)=\mathcal{H}^{2,1}$. (2) is immediate from (C).

Mathematical Institute
Nagoya University

Bibliography

[1] M. Berger, Les espaces symétriques non compacts, Ann. Ec. Norm. Sup., **74** (1957), 85–177.

[2] S. Helgason, Differential geometry and symmetric spaces, Academic Press, 1962.

[3] N. Iwahori, On real irreducible representations of Lie algebras, Nagoya Math. J., **14** (1959), 59–83.

[4] S. Kobayashi, Theory of connections, Annali di Math., **43** (1957), 119–194.

[5] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Interscience Publishers, 1963.

[6] S. Murakami, On the automorphisms of a real semi-simple Lie algebra, J. Math. Soc. Japan, **4** (1952), 103–133.

[7] T. Nagano, Transformation groups on compact symmetric spaces, to appear.

[8] Séminaire "Sophus Lie", Théorie des algèbres de Lie, Topologie des groupes de Lie, Paris, 1955.

[9] N. Tanaka, Projective connections and projective transformations, Nagoya Math J., **12** (1957), 1–24.

[10] N. Tanaka, Conformal connections and conformal transformations, Trans. Amer. Math. Soc., **92** (1959), 168–190.

[11] A. Weil, On discrete subgroups of Lie groups, II, Ann. of Math., **75** (1962), 578–602.