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On the variety of orbits with respect to an algebraic group of birational transformations

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For an algebraic variety V and an algebraic group G operating on V, we can construct the variety V_G of G-orbits on V and the natural rational mapping f of V to V_G (cf. [8]). The variety V_G is obtained as a model of the subfield of all the G-invariant elements in the field of rational functions on V.

The purpose of this paper is to prove several results concerning on the relations between the Albanese varieties (and the spaces of linear differential forms of the first kind) of V and of V_G. Denoting by G_0 the connected component of G containing the identity element, we see that the finite group G/G_0 operates on the variety V_{G_0} of G_0-orbits on V and V_G is naturally birationally equivalent to the variety $(V_{G_0})_{G/G_0}$ of (G/G_0)-orbits on V_{G_0}. Hence we may restrict ourselves to the two cases: (i) G is connected and (ii) G is a finite group; and the second case (ii) has already been treated in our previous paper [3].

In § 1, we shall give the definition of the variety V_G and prove several preliminary results.

In § 2, we shall first construct the Albanese variety $\mathrm{Alb}(V_G)$[1] of V_G as a quotient abelian variety of the Albanese variety $A=\mathrm{Alb}(V)$ of V (Theorem 1). In particular, for the connected algebraic group G_0, we define a rational homomorphism φ of G_0 into A and it will be proved that $A_1=A/\varphi(G_0)$ is the Albanese variety of V_{G_0} (Theorem 2). Then we shall also prove that $\mathrm{Alb}(V)$ is isogenous to the direct product of $\mathrm{Alb}(V_{G_0})$ and the Albanese variety of the generic G_0-orbit $\overline{G_0P}$[2] on V (Theorem 3) and we have the inequality $0\leqq \dim \mathrm{Alb}(V)-\dim \mathrm{Alb}(V_{G_0})\leqq \dim V-\dim V_{G_0}$. Moreover, by means of the l-adic representations $M_l^{(A)}$ and $M_l^{(A^*)}$ of the rings of endomorphisms of A and $A^*=\mathrm{Alb}(G_0)$, we define the two matrix representations of the finite group G/G_0. Then, if G operates regularly and effectively on V, we shall show that the dimension of $\mathrm{Alb}(V_G)$ is equal to the half of the difference of the multi-

1) For a variety W, $\mathrm{Alb}(W)$ denotes an Albanese variety of W.
2) Cf. § 1.

plicities of the identity representation in $(M_l^{(A)} \mid G/G_0)$ and in $(M_l^{(A^*)} \mid G/G_0)$ (Theorem 4).

In § 3, we suppose that V and V_G are complete and nonsingular. Then, under some assumptions on the index $(G:G_0)$ and on the homomorphism φ, we shall prove the inequality $0 \leqq \dim \mathfrak{D}_0(V_G) - \dim \mathfrak{D}_0(\mathrm{Alb}\,(V_G)) \leqq \dim \mathfrak{D}_0(V) - \dim \mathfrak{D}_0(\mathrm{Alb}\,(V))$[3] (Theorem 5). Next, for the inclusion mapping ι of $\overline{G_0P}$ into V, we shall decide the image and the kernel of the adjoint mapping $\delta\iota$ of $\mathfrak{D}_0(V)$ into $\mathfrak{D}_0(\overline{G_0P})$ (Theorem 6) and then shall give a necessary and sufficient condition for an element ω of $\mathfrak{D}_0(V)$ to belong to $\delta f(\mathfrak{D}_0(V_G))$ (Theorem 7).

Finally, as an appendix, we shall consider a complete homogeneous space V for a connected algebraic group $\tilde{G}$. It will be proved that V is birationally equivalent to the direct product of $\mathrm{Alb}\,(V)$ and of a rational variety and so we have $\dim \mathfrak{D}_0(V) = \dim \mathfrak{D}_0(\mathrm{Alb}\,(V))$.

§ 1. The variety of orbits

Let V be an algebraic variety and let G be an algebraic group operating on V; let k be a field of definition for V, G and the operation of G on V. This implies that, for each component G_i of G, there exists a rational mapping $(g_i, P) \to g_iP$ of $G_i \times V$ to V defined over k such that if (g_i, g_j, P) is a generic point of $G_i \times G_j \times V$ over k, then we have $g_i(g_jP) = (g_ig_j)P$ and $k(g_i, g_iP) = k(g_i, P)$ (cf. [8], [**10**]).

Then we can construct *the variety V_G of G-orbits on V* and *the natural rational mapping f of V to V_G*, both defined over k, which are characterized to within a birational correspondence by the following properties: f is a generically surjective and separable mapping and, for generic points P_1, P_2 of V over k, we have $f(P_1) = f(P_2)$ if and only if we have $g_1P_1 = g_2P_2$ with generic points g_1, g_2 of some components of G over $k(P_1, P_2)$ (cf. [8]). If we identify $k(V_G)$ with a subfield of $k(V)$ by f, then $k(V_G)$ consists of all G-invariant functions. Let P be a generic point of V over k and g_i a generic point of a component G_i of G over $k(P)$. Then g_iP is also a generic point of V over k and, as $(gg_i^{-1})g_iP = gP$ with a generic point g of a component of G over $k(g_i, P)$, we have $f(g_iP) = f(P)$.

Let N be a normal algebraic subgroup of G defined over k. Let π_N be the canonical rational homomorphism of G to G/N and f' the natural rational mapping of V to the variety V_N of N-orbits on V. Then, by the rule $\pi_N(g_i)f'(P) = f'(g_iP)$ for a generic point (g_i, P) of each $G_i \times V$ over k, G/N operates on the variety V_N and the variety $(V_N)_{G/N}$ of (G/N)-orbits on V_N is

3) For a complete, nonsingular variety W, $\mathfrak{D}_0(W)$ denotes the space of linear differential forms of the first kind on W.

naturally birationally equivalent to V_G (cf. [8]). On the other hand, let λ be a surjective rational homomorphism of G to an algebraic group G' operating on V. We suppose that G', λ and the operation of G' on V are defined over k. If the operation of G on V is the composite of λ and that of G', i. e. we have $g_iP=\lambda(g_i)P$ for a generic point (g_i, P) of each $G_i\times V$ over k, then the variety $V_{G'}$ of G'-orbits on V is considered as the variety of G-orbits on V and conversely.

Let G_0 be the component of G containing the identity element e and let f_0 be the natural rational mapping of V to the variety V_{G_0} of G_0-orbits on V. Let P be a generic point of V over k and put $Q=f_0(P)$, which is a generic point of V_{G_0} over k. Then, as $k(Q)$ is algebraically closed in $k(P)$ and f_0 is separable (cf. [8]), P has a locus X over $k(Q)$. For a generic point g_0 of G_0 over $k(P)$, g_0P is a generic point of V over k and we have $f_0(g_0P)=f_0(P)=Q$ and the locus of g_0P over $k(Q)$ coincides with X. When G_0 operates regularly on V, X is equal to the Zariski closure $\overline{G_0P}$ of the G_0-orbit of P (i. e. the locus of g_0P over $k(P)$). In fact, let P' be a generic point of X over $k(P)$. Then P' is a generic specialization of P over $k(Q)$ and so we have $f_0(P')=f_0(P)=Q$, which implies that we have $g_0P=g_0'P'$ with generic points g_0, g_0' of G_0. As G_0 operates regularly on V, we have $P'=g_0'^{-1}g_0P$ and so X is contained in $\overline{G_0P}$. Conversely, we have $\dim\overline{G_0P}=\dim_{k(P)} g_0P\leqq\dim_{k(Q)} g_0P=\dim X$. Hence we have $X=\overline{G_0P}$. In the following, whether or not G_0 operates regularly on V, we denote by $\overline{G_0P}$ the locus X of P over $k(Q)$. Hence we have

$$\dim\overline{G_0P}=\dim V-\dim V_{G_0}\,. \tag{1}$$

Let V' be an algebraic variety birationally equivalent to V such that G_0 operates regularly on V' and the operations of G_0 on V and on V' commute with the birational transformation T (cf. [**10**]). Then $X=\overline{G_0P}$ is birationally equivalent to the locus of $T(P)$ over the G_0-invariant subfield $k(Q)$ of $k(P)=k(T(P))$, which coincides with the closure of the orbit $G_0T(P)$ in V'. Hence we see that $\overline{G_0P}$ is birationally equivalent to a prehomogeneous space (and so to a homogeneous space) for G_0.

§ 2. Albanese varieties

Let A be an Albanese variety of V and α a canonical mapping of V into A, both defined over k.

Let $G_0, G_1, \cdots, G_n$ be the components of G, all defined over k, and let (g_i, P) be a generic point of $G_i\times V$ over k $(i=0, 1, \cdots, n)$. Let W_i be the locus of $\alpha(g_iP)-\alpha(P)$ over k and let C be the intersection of all the closed subgroups of A containing $W_0, W_1, \cdots, W_n$. Then C is an algebraic subgroup of A, defined over k (cf. [**9**]).

LEMMA 1. *Let λ' be a rational homomorphism of A into an abelian variety A', defined over $K\supset k$, such that $\lambda'(C)=0$. Then there exists a rational mapping α' of V_G into A', defined over K, such that $\lambda'\circ\alpha=\alpha'\circ f$. Moreover, if λ' is surjective, then $\alpha'(V_G)$ generates A'.*

PROOF. Let P be a generic point of V over K. For any generic point g_i' of G_i over $K(P)$, we have $\lambda'(\alpha(g_i'P))=\lambda'(\alpha(P))$, which implies that $\lambda'(\alpha(P))$ is rational over $K(f(P))$. Hence there exists a rational mapping α' such that $\lambda'\circ\alpha=\alpha'\circ f$. If λ' is surjective, then any point y' of A' can be written in the form $y'=\sum^{t}\lambda'\circ\alpha(P_i')$ with some P_i' in V. Then y' is a specialization of $y=\sum^{t}\lambda'\circ\alpha(P_i)=\sum^{t}\alpha'\circ f(P_i)$ over K, where $P_1,\cdots,P_t$ are independent generic points of V over K.

LEMMA 2. *Let α' be a rational mapping of V_G into an abelian variety A'. Then there exists a rational homomorphism λ' of A into A' such that $\lambda'\circ\alpha=\alpha'\circ f+\text{constant}$ and $\lambda'(C)=0$.*

PROOF. From the universal mapping property of α, it follows the existence of the rational homomorphism λ' such that $\lambda'\circ\alpha=\alpha'\circ f+\text{constant}$. Let λ' and α' be defined over $K\supset k$ and (g_i, P) a generic point of $G_i\times V$ over K. Then we have $f(g_iP)=f(P)$ (cf. § 1) and so $\lambda'(\alpha(g_iP)-\alpha(P))=\alpha'(f(g_iP))-\alpha'(f(P))=0$. Since $\alpha(g_iP)-\alpha(P)$ is a generic point of W_i over K and λ' is defined over K, we have $\lambda'(C)=0$.

THEOREM 1. *The abelian variety $A_0=A/C$ is an Albanese variety of V_G, defined over k. Moreover, there exists a canonical mapping α_0 of V_G into A_0 such that*

$$\alpha_0\circ f=\mu\circ\alpha\,, \tag{2}$$

where μ is the canonical homomorphism of A onto A/C.

PROOF. Since we have $\mu(C)=0$, there exists a rational mapping α_0 of V_G into A_0, defined over k, such that $\mu\circ\alpha=\alpha_0\circ f$ and $\alpha_0(V_G)$ generates A_0 (see Lemma 1). On the other hand, for a rational mapping α' of V_G into an abelian variety A', there exists a rational homomorphism λ' of A into A' such that $\lambda'\circ\alpha=\alpha'\circ f+\text{constant}$ and $\lambda'(C)=0$ (see Lemma 2). Then we have a rational homomorphism ρ of $A_0=A/C$ into A' such that $\lambda'=\rho\circ\mu$, from which we have $\alpha'=\rho\circ\alpha_0+\text{constant}$.

Let L be the maximal connected linear algebraic subgroup of G_0, $A^*=G_0/L$ the Albanese variety of G_0 and π the canonical homomorphism of G_0 onto A^*, all assumed to be defined over k. By a well-known theorem on abelian varieties, there exist a rational homomorphism φ of G_0 into A, defined over k, an endomorphism η of A and a constant point c of A such that we have

$\alpha(g_0P)-\alpha(P)=\varphi(g_0)+\eta(\alpha(P))+c$. For $g_0=e$, we have $\eta(\alpha(P))+c=0$ and, as $\alpha(V)$ generates A, we see that $\eta=0$ and $c=0$, i.e.

$$\alpha(g_0P)-\alpha(P)=\varphi(g_0)\,. \tag{3}$$

Since $\varphi(L)=0$, there exists a rational homomorphism φ^* of A^* into A, defined over k, such that $\varphi=\varphi^*\circ\pi$. If G_0 operates faithfully on V, then it is proved that φ^* is an isogeny of A^* to an abelian subvariety of A (cf. [7]).

THEOREM 2. *The abelian variety $A_1=A/\varphi^*(A^*)$ is an Albanese variety of V_{G_0} defined over k.*

PROOF. From the definition, the locus W_0 of $\alpha(g_0P)-\alpha(P)$ over k coincides with the abelian subvariety $\varphi(G_0)=\varphi^*(A^*)$ of A (see (3)).

COROLLARY 1. *If $G_0=L$ is linear, then* Alb(V_L) *is isomorphic to* Alb(V)[4].

COROLLARY 2. *If G_0 operates faithfully on V, then we have*

$$\dim \mathrm{Alb}\,(V_{G_0})=\dim \mathrm{Alb}\,(V)-\dim \mathrm{Alb}\,(G_0)\,. \tag{4}$$

PROOF. A^* is an Albanese variety of G_0 and φ^* is an isogeny by our assumption (cf. [7]). Hence we have $\dim\varphi^*(A^*)=\dim \mathrm{Alb}\,(G_0)$.

Let P be a generic point of V over k and let (B,β) be an Albanese variety of the variety $\overline{G_0P}$. For the inclusion mapping ι of $\overline{G_0P}$ into V, there exists a rational homomorphism ψ of B into A such that $\alpha\circ\iota=\psi\circ\beta+$constant. Hence we have, for a generic point g_0 of G_0 over $k(P)$,

$$\varphi(g_0)=\alpha\circ\iota(g_0P)-\alpha\circ\iota(P)=\psi\circ\beta(g_0P)+\text{constant}\,. \tag{5}$$

LEMMA 3. *ψ is an isogeny of B onto $\varphi^*(A^*)$.*

PROOF. Let G' be a connected algebraic group which is the image of G_0 by a rational homomorphism λ and operates faithfully on V such that the operation of G_0 on V is the composite of λ and that of G' (cf. [**10**]). Let $A^{*\prime}=G'/L'$ be the Albanese variety of G' and $\varphi^{*\prime}$ the rational homomorphism of $A^{*\prime}$ into A defined in a similar way as φ^*. Then we have $V_{G_0}=V_{G'}$, $\overline{G_0P}=\overline{G'P}$ and $\varphi^*(A^*)=\varphi^{*\prime}(A^{*\prime})$. We have $\psi(B)\supset\varphi^*(A^*)=\varphi^{*\prime}(A^{*\prime})$ (see (5)) and so $\dim B\geqq\dim\psi(B)\geqq\dim\varphi^{*\prime}(A^{*\prime})$. On the other hand, as $\overline{G'P}=\overline{G_0P}$ is birationally equivalent to a homogeneous space for G' (cf. §1), we have $\dim B\leqq\dim \mathrm{Alb}\,(G')=\dim A^{*\prime}$ (cf. [**6**]). Since $\varphi^{*\prime}$ is an isogeny (cf. [7]), we have $\dim A^{*\prime}=\dim B=\dim\psi(B)=\dim\varphi^{*\prime}(A^{*\prime})$ and so ψ is an isogeny of B onto $\varphi^{*\prime}(A^{*\prime})=\varphi^*(A^*)$.

Since A is isogenous to the direct product of A_1 and $\varphi^*(A^*)$ (see Theorem 2), we have the following

THEOREM 3. *$A=$* Alb(V) *is isogenous to the direct product of $A_1=$*Alb(V_{G_0}) *and of $B=$* Alb$(\overline{G_0P})$: *$A\sim A_1\times B$. In particular, we have*

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4) See footnote 1).

(6) $$\dim \mathrm{Alb}(V)=\dim \mathrm{Alb}(V_{G_0})+\dim \mathrm{Alb}(\overline{G_0P}).$$

COROLLARY 1. *We have*

(7) $$0\leqq \dim \mathrm{Alb}(V)-\dim \mathrm{Alb}(V_{G_0})\leqq \dim V-\dim V_{G_0}.$$

PROOF. Since $\overline{G_0P}$ is birationally equivalent to a homogeneous space for G_0, we have $\dim \mathrm{Alb}(\overline{G_0P})\leqq \dim \overline{G_0P}$ (cf. [**6**]) and so the inequality (7) (see (1), (6)).

COROLLARY 2. *In Corollary* 1,

(i) *If* $G_0=L$ *is linear, then the equality* $\dim \mathrm{Alb}(V)-\dim \mathrm{Alb}(V_{G_0})=0$ *holds. When* G_0 *operates faithfully on* V, *the converse is also true. In this case, we have* $\mathrm{Alb}(V)\cong \mathrm{Alb}(V_{G_0})$.

(ii) *If* $G_0=A^*$ *is an abelian variety, then the equality* $\dim \mathrm{Alb}(V)-\dim \mathrm{Alb}(V_{G_0})=\dim V-\dim V_{G_0}$ *holds. When* G_0 *operates faithfully on* V, *the converse is also true.*

PROOF. (i) The equality holds if and only if $\varphi^*(A^*)=0$ (see Theorem 2); hence we have the assertion. (ii) The equality holds if and only if we have $\dim \mathrm{Alb}(\overline{G_0P})=\dim \overline{G_0P}$, i.e. $\overline{G_0P}$ is birationally equivalent to a homogeneous space for an abelian variety (cf. [**6**]); hence the first part is clear. Conversely, if the equality holds, we have $(\dim \mathrm{Alb}(V)-\dim \mathrm{Alb}(V_L))+(\dim \mathrm{Alb}(V_L)-\dim \mathrm{Alb}(V_{G_0}))=(\dim V-\dim V_L)+(\dim V_L-\dim V_{G_0})$. Then we have $\dim \mathrm{Alb}(V)-\dim \mathrm{Alb}(V_L)=0$ (see (i)) and, as $V_{G_0}=(V_L)_{A^*}$, $\dim \mathrm{Alb}(V_L)-\dim \mathrm{Alb}(V_{G_0})=\dim V_L-\dim V_{G_0}$ (see the first part of (ii)). Hence we have $\dim V-\dim V_L=\dim \overline{LP}=0$ (see (1)); so, if G_0 operates faithfully on V, then, as L is defined over k, we have $L=\{e\}$.

We suppose that G_0 operates regularly on V. Then, for a point P_0 on V, we denote by $\overline{G_0P_0}$ the Zariski closure of the G_0-orbit of P_0, i.e. the locus of g_0P_0 over $k(P_0)$ with a generic point g_0 of G_0 over $k(P_0)$, and by (B_0, β_0) the Albanese variety of $\overline{G_0P_0}$. Clearly $\overline{G_0P_0}$ is also a prehomogeneous space for G_0 and we have $\dim B_0\leqq \dim \overline{G_0P_0}$ (cf. [**6**]). If α is defined at P_0 (for example, if P_0 is simple on V), we have $\varphi(g_0)=\alpha(g_0P_0)-\alpha(P_0)$ for a generic point g_0 of G_0 over $k(P_0)$ and we can also prove, in a similar way as the proof of Lemma 3 and (5), that there exists an isogeny ψ_0 of B_0 onto $\varphi^*(A^*)$ such that $\varphi(g_0)=\psi_0\circ\beta_0(g_0P_0)+\text{constant}$[5]. Hence we have

THEOREM 3′. *If* G_0 *operates regularly on* V *and* $\alpha(P_0)$ *is defined, then* $A=\mathrm{Alb}(V)$ *is isogenous to the direct product of* $A_1=\mathrm{Alb}(V_{G_0})$ *and of* $B_0=\mathrm{Alb}(\overline{G_0P_0})$: $A\sim A_1\times B_0$.

COROLLARY. *We have*

5) Hence all the closures $\overline{G_0P_0}$ of G_0-orbits (such that $\alpha(P_0)$ is defined) have the Albanese varieties isogenous to each other.

$$0 \leqq \dim \mathrm{Alb}\,(V) - \dim \mathrm{Alb}\,(V_{G_0}) \leqq \dim \overline{G_0 P_0}\,. \tag{8}$$

In the rest of this section, we assume that G operates regularly and effectively on V. Then, for the homomorphism φ^* of $A^* = \mathrm{Alb}\,(G_0)$ into A, we have, using the notations in the proof of Lemma 3, $\varphi^{*\prime} \circ \lambda = \varphi^*$. As G_0 operates effectively on V, the kernel of λ must be trivial and so we see that φ^* is also an isogeny.

For a point g in G, we have $g G_0 g^{-1} \subset G_0$ and, as π is a canonical mapping of G_0 into the Albanese variety A^*, there exists an element ξ_g of $\mathcal{A}(A^*)$[6] such that

$$\pi(g g_0 g^{-1}) = \xi_g \circ \pi(g_0) \tag{9}$$

for all g_0 in G_0. Since π is a homomorphism, we have $\xi_{g_0} = \delta_{A^*}$[6] for g_0 in G_0 and so the mapping $g \to \xi_g \to M_l^{(A^*)}(\xi_g)$[6] defines a matrix representation of the finite group G/G_0.

On the other hand, for a point g in G, there exist an element η_g of $\mathcal{A}(A)$ and a constant point a_g in A such that

$$\alpha(gP) = \eta_g \circ \alpha(P) + a_g \tag{10}$$

for a generic point P of V. We have $\eta_{g_0} = \delta_A$ for g_0 in G_0 and so the mapping $g \to \eta_g \to M_l^{(A)}(\eta_g)$ defines also a matrix representation of the finite group G/G_0.

THEOREM 4. *Let $G = G_0 g_0 \cup G_0 g_1 \cup \cdots \cup G_0 g_n$ $(g_0 = e)$ be the decomposition of G into the cosets of G_0. Then we have*

$$\begin{aligned} \dim \mathrm{Alb}\,(V_G) &= \frac{1}{2}\,\mathrm{rank}\, M_l^{(A)}\Big(\sum_{i=0}^{n} \eta_{g_i}\Big) - \frac{1}{2}\,\mathrm{rank}\, M_l^{(A^*)}\Big(\sum_{i=0}^{n} \xi_{g_i}\Big) \\ &= \frac{1}{2}(\textit{the multiplicity of } \mathrm{id} \textit{ in } M_l^{(A)} \mid G/G_0) \\ &\quad - \frac{1}{2}(\textit{the multiplicity of } \mathrm{id} \textit{ in } M_l^{(A^*)} \mid G/G_0)\,, \end{aligned} \tag{11}$$

where id *is the identity representation.*

PROOF. For a point g in G and a point g_0 in G_0, we have $\eta_g \circ \varphi(g_0) = \eta_g(\alpha(g_0 P) - \alpha(P)) = \alpha((g g_0 g^{-1}) g P) - \alpha(gP) = \varphi(g g_0 g^{-1})$ (see (3)). Hence we have $\eta_g(\varphi(G_0)) \subset \varphi(G_0)$ and so η_g induces an element η_g^* of $\mathcal{A}(\varphi^*(A^*))$. Since $\varphi = \varphi^* \circ \pi$, we see that $\varphi^* \circ \pi(g g_0 g^{-1}) = \varphi^* \circ \xi_g \circ \pi(g_0)$ is equal to $\varphi(g g_0 g^{-1})$

6) For an abelian variety A, we use the following notations: $\mathcal{A}(A)$ = the ring of endomorphisms of A, δ_A = the identity element of $\mathcal{A}(A)$, $M_l^{(A)}$ = the l-adic representation of $\mathcal{A}(A)$.

$=\eta_g\circ\varphi(g_0)=\eta_g\circ\varphi^*\circ\pi(g_0)$. Hence we have $\varphi^*\circ\xi_g=\eta_g\circ\varphi^*$ for all g in G and, as φ^* is an isogeny of A^* to $\varphi^*(A^*)$, we have

$$M_l^{(A^*)}(\xi_g)=M\cdot M_l^{(\varphi^*(A^*))}(\eta_g^*)\cdot M^{-1} \tag{12}$$

with a nonsingular matrix M independent of g. Let D be an abelian subvariety of A such that $A=\varphi^*(A^*)+D$ and $\varphi^*(A^*)\cap D$ is a finite group. We take a rational prime l which does not divide the order of $\varphi^*(A^*)\cap D$ and fix it. Then, taking suitable l-adic coordinates of A, we may assume that we have

$$M_l^{(A)}(\eta_g)=\begin{pmatrix} M_l^{(\varphi^*(A^*))}(\eta_g^*) & * \\ 0 & N_g\end{pmatrix}. \tag{13}$$

Moreover we may assume that V_G and V_{G_0} are normal and so we have a Galois covering $\bar{f}\colon V_{G_0}\to V_G$ with the Galois group $\bar{G}=G/G_0$. (cf. § 1). Let f_0 be the natural rational mapping of V to V_{G_0} and α_1 the canonical mapping of V_{G_0} into $A_1=\mathrm{Alb}\,(V_{G_0})=A/\varphi^*(A^*)$ and let μ_1 be the canonical homomorphism of A onto $A_1=A/\varphi^*(A^*)$. Then we may assume that we have

$$\mu_1\circ\alpha=\alpha_1\circ f_0 \tag{14}$$

(see (2)). Let $\bar{g}$ be an element of $\bar{G}$, which is the coset of G_0 containing an element g of G. Then we have $\bar{g}(f_0(P))=f_0(gP)$ (cf. § 1). Moreover there exist an element $\bar{\eta}_{\bar{g}}$ of $\mathcal{A}(A_1)$ and a constant point $\bar{a}_{\bar{g}}$ of A_1 such that we have $\alpha_1(\bar{g}f_0(P))=\bar{\eta}_{\bar{g}}\circ\alpha_1(f_0(P))+\bar{a}_{\bar{g}}=\bar{\eta}_{\bar{g}}\circ\mu_1\circ\alpha(P)+\bar{a}_{\bar{g}}$, which is also equal to $\alpha_1(f_0(gP))=\mu_1\circ\alpha(gP)=\mu_1\circ\eta_g\circ\alpha(P)+\mu_1(a_g)$. Since $\alpha(V)$ generates A, we have

$$\mu_1\circ\eta_g=\bar{\eta}_{\bar{g}}\circ\mu_1 \tag{15}$$

for all g in G. As μ_1 is the canonical homomorphism of A onto A_1 with the kernel $\varphi^*(A^*)$, we have

$$M_l^{(A_1)}(\sum_{i=0}^{n}\bar{\eta}_{\bar{g}_i})=N\cdot(\sum_{i=0}^{n}N_{g_i})\cdot N^{-1}$$

with a nonsingular matrix N (see (13), (15)); and as $(\sum_{i=0}^{n}\bar{\eta}_{\bar{g}_i})(A_1)$ is isogenous to $A_0=\mathrm{Alb}\,(V_G)$ and $\dim\mathrm{Alb}\,(V_G)=\frac{1}{2}\operatorname{rank} M_l^{(A_1)}(\sum_{i=0}^{n}\bar{\eta}_{\bar{g}_i})$ (cf. [3]), we have

$$\begin{aligned}\dim\mathrm{Alb}\,(V_G)&=\frac{1}{2}\operatorname{rank}(\sum_{i=0}^{n}N_{g_i})\\ &=\frac{1}{2}\operatorname{rank} M_l^{(A)}(\sum_{i=0}^{n}\eta_{g_i})-\frac{1}{2}\operatorname{rank} M_l^{(\varphi^*(A^*))}(\sum_{i=0}^{n}\eta_{g_i}^*)\end{aligned}$$

(see (13)). Hence we have the first formula of Theorem (see (12)). The second formula follows from the first by a group-theoretical lemma in [3].

COROLLARY. (i) *If G is a finite group, then we have*

(16) $$\dim \mathrm{Alb}\,(V_G)=\frac{1}{2}(\textit{the multiplicity of } \mathrm{id} \textit{ in } M_\iota^{(A)}\mid G)^{7)}.$$

(ii) *If* $G=G_0$ *is connected, then we have*

(17) $$\dim \mathrm{Alb}\,(V_{G_0})=\frac{1}{2}\deg M_\iota^{(A)}-\frac{1}{2}\deg M_\iota^{(A^*)}$$
$$=\dim \mathrm{Alb}\,(V)-\dim \mathrm{Alb}\,(G_0)^{8)}.$$

§ 3. Linear differential forms of the first kind

Let ι be the inclusion mapping of $\overline{G_0P}$ into V, where P is a generic point of V over k, and let ι^* be also the inclusion mapping of $\varphi^*(A^*)$ into A.

LEMMA 4. *Let f_0 be the natural rational mapping of V to V_{G_0}. For a differential form ω_1 on V_{G_0}, we have $\delta\iota\circ\delta f_0(\omega_1)=0$.*

PROOF. Let b be a rational function on V_{G_0} defined over k. Then, for a generic point g_0 of G_0 over $k(P)$, $(b\circ f_0)(g_0P)=(b\circ f_0)(P)$ is rational over $k(f_0(P))$, which implies that the rational function $\delta f_0(b)$ induces a constant function on $\overline{G_0P}$. Hence we have $\delta\iota\circ\delta f_0(db)=d(\delta\iota\circ\delta f_0(b))=0$.

LEMMA 5. *Let μ_1 be the canonical rational homomorphism of A onto $A_1=A/\varphi^*(A^*)$. Then $\delta\iota^*$ induces an isomorphism of $\mathfrak{D}_0(A)/\delta\mu_1(\mathfrak{D}_0(A_1))$ onto $\mathfrak{D}_0(\varphi^*(A^*))$*[9].

PROOF. Clearly $\delta\iota^*$ maps $\mathfrak{D}_0(A)$ onto $\mathfrak{D}_0(\varphi^*(A^*))$ surjectively and, as $\mu_1\circ\iota^*=0$, the kernel of $\delta\iota^*$ in $\mathfrak{D}_0(A)$ contains $\delta\mu_1(\mathfrak{D}_0(A_1))$. Since μ_1 is separable, we have $\dim\delta\mu_1(\mathfrak{D}_0(A_1))=\dim\mathfrak{D}_0(A_1)=\dim\mathfrak{D}_0(A)-\dim\mathfrak{D}_0(\varphi^*(A^*))$, which proves Lemma.

In the following, we assume that

1) *the characteristic p of the universal domain does not divide the index* $(G:G_0)$.

2) *the rational homomorphism φ is separable.*

We note that, as we have $\varphi=\varphi^*\circ\pi$ and π is generically surjective and separable, the assumption 2) is equivalent to

2′) *the rational homomorphism φ^* is separable.*

Let α' be the restriction of the rational mapping α to $\overline{G_0P}$, i.e. $\alpha'=\alpha\circ\iota$. Then, as we have $\varphi(g_0)=\alpha'(g_0P)-\alpha(P)$, $\alpha'-\alpha(P)$ defines a generically surjective rational mapping of $\overline{G_0P}$ to $\varphi^*(A^*)$ defined over $k(P)$ and we have

(18) $$(\alpha-\alpha(P))\circ\iota=\iota^*\circ(\alpha'-\alpha(P)).$$

Moreover, as φ is the composite of the generically surjective rational mapping $g_0\to g_0P$ of G_0 to $\overline{G_0P}$ and of $\alpha'-\alpha(P)$, the rational mapping $\alpha'-\alpha(P)$ is also

7) Cf. [**3**].
8) See Cor. 2 of Theorem 2.
9) See footnote 3).

separable by our assumption 2)[10].

THEOREM 5. *Let ω_0 be a linear differential form on V_G. If $\delta f(\omega_0)$ belongs to $\delta\alpha(\mathfrak{D}_0(A))$, then ω_0 belongs to $\delta\alpha_0(\mathfrak{D}_0(A_0))$.*

PROOF. We have $\delta f(\omega_0)=\delta\alpha(\theta)$ with an element θ in $\mathfrak{D}_0(A)$ and so we have $\delta\iota\circ\delta f(\omega_0)=\delta\iota\circ\delta\alpha(\theta)=\delta\alpha'\circ\delta\iota^*(\theta)$ (see (18)), which is equal to $\delta\iota\circ\delta f_0\circ\delta\bar{f}(\omega_0)=0$ (see Lemma 4). Then, as $\alpha'-\alpha(P)$ is separable, we see that $\delta\iota^*(\theta)=0$ and so there exists an element θ_1 of $\mathfrak{D}_0(A_1)$ such that $\theta=\delta\mu_1(\theta_1)$ (see Lemma 5). Hence we have $\delta f_0\circ\delta\bar{f}(\omega_0)=\delta\alpha\circ\delta\mu_1(\theta_1)=\delta f_0\circ\delta\alpha_1(\theta_1)$ (see (14)) and, as f_0 is separable, $\delta\bar{f}(\omega_0)=\delta\alpha_1(\theta_1)$. As the characteristic p does not divide the degree $(G:G_0)$ of the Galois covering $\bar{f}: V_{G_0}\to V_G$ by our assumption 1), we have $\omega_0=\delta\alpha_0(\theta_0)$ with an element θ_0 of $\mathfrak{D}_0(A_0)$ (cf. [4]).

Since $\delta\alpha$ (resp. $\delta\alpha_0$) maps $\mathfrak{D}_0(A)$ (resp. $\mathfrak{D}_0(A_0)$) injectively into $\mathfrak{D}_0(V)$ (resp. $\mathfrak{D}_0(V_G)$), we have the following

THEOREM 5′. *Let V and V_G be complete and nonsingular. Then we have*

$$0\leqq \dim\mathfrak{D}_0(V_G)-\dim\mathfrak{D}_0(A_0)\leqq \dim\mathfrak{D}_0(V)-\dim\mathfrak{D}_0(A)\,. \tag{19}$$

COROLLARY. *If we have $\delta\alpha(\mathfrak{D}_0(A))=\mathfrak{D}_0(V)$, then we have also $\delta\alpha_0(\mathfrak{D}_0(A_0))=\mathfrak{D}_0(V_G)$.*

For the Albanese variety (B,β) of $\overline{G_0P}$, there exists an isogeny ψ of B onto $\varphi^*(A^*)$ such that $\alpha'-\alpha(P)=\psi\circ\beta+\text{constant}$ (see Lemma 3, (5)). Since $\overline{G_0P}$ is birationally equivalent to a homogeneous space for G_0, β is generically surjective (cf. [6]) and so ψ is separable by our assumption.

LEMMA 6. *$\delta\iota$ induces a surjective homomorphism of $\delta\alpha(\mathfrak{D}_0(A))$ onto $\delta\beta(\mathfrak{D}_0(B))$ with the kernel $\delta\alpha\circ\delta\mu_1(\mathfrak{D}_0(A_1))=\delta f_0\circ\delta\alpha_1(\mathfrak{D}_0(A_1))$.*

PROOF. As ψ is a separable isogeny, we have $\delta\psi(\mathfrak{D}_0(\varphi^*(A^*))=\mathfrak{D}_0(B)$ and so $\delta\iota\circ\delta\alpha(\mathfrak{D}_0(A))=\delta\beta\circ\delta\psi\circ\delta\iota^*(\mathfrak{D}_0(A))=\delta\beta(\mathfrak{D}_0(B))$. On the other hand, for an element θ of $\mathfrak{D}_0(A)$, $\delta\iota\circ\delta\alpha(\theta)=0$ if and only if $\delta\alpha'\circ\delta\iota^*(\theta)=0$ (see (18)), i.e. $\delta\iota^*(\theta)=0$. Hence we have the assertion (see Lemma 5 and (14)).

Therefore we have the following

THEOREM 6. *Let V and V_{G_0} be complete and nonsingular. If we have $\delta\alpha(\mathfrak{D}_0(A))=\mathfrak{D}_0(V)$, then the adjoint mapping $\delta\iota$ induces an isomorphism of $\mathfrak{D}_0(V)/\delta f_0(\mathfrak{D}_0(V_{G_0}))$ onto $\delta\beta(\mathfrak{D}_0(B))$.*

We suppose that G_0 operates regularly on V. Then, for a point P_0 on V and the inclusion mapping ι_0 of $\overline{G_0P_0}$ into V (cf. § 2), we have also $(\alpha-\alpha(P_0))\circ\iota_0=\iota^*\circ(\alpha^{(0)}-\alpha(P_0))$, where $\alpha^{(0)}$ is the restriction of α to $\overline{G_0P_0}$[11]. Moreover, for the Albanese variety (B_0,β_0) of $\overline{G_0P_0}$, the isogeny ψ_0 of B_0 onto $\varphi^*(A^*)$ defined in § 2 is also separable by the assumption 2) and so we can

10) As seen in the following arguments, we can replace the assumption 2) by the weak one: $\alpha'-\alpha(P)$ is separable.

11) Since V is assumed to be nonsingular, α is everywhere defined.

prove, in a similar way as the proof of Lemma 6, that $\delta\iota_0$ induces a surjective homomorphism of $\delta\alpha(\mathfrak{D}_0(A))$ onto $\delta\beta_0(\mathfrak{D}_0(B_0))$ with the kernel $\delta\alpha\circ\delta\mu_1(\mathfrak{D}_0(A_1))$. In particular, if the orbit G_0P_0 is closed, then the variety G_0P_0 is a complete homogeneous space for G_0 and so we have $\mathfrak{D}_0(G_0P_0)=\delta\beta_0(\mathfrak{D}_0(B_0))$ (cf. Appendix). Hence we have

THEOREM 6'. *If G_0 operates regularly on V, then, under the same assumption as in Theorem 6 the adjoint mapping $\delta\iota_0$ induces an isomorphism of $\mathfrak{D}_0(V)/\delta f_0(\mathfrak{D}_0(V_{G_0}))$ onto $\delta\beta_0(\mathfrak{D}_0(B_0))$. In particular, if G_0P_0 is closed, $\delta\iota_0$ induces an isomorphism of $\mathfrak{D}_0(V)/\delta f_0(\mathfrak{D}_0(V_{G_0}))$ onto $\mathfrak{D}_0(G_0P_0)$.*

We note that there exists always a closed G_0-orbit on V, i.e. the G_0-orbit having the smallest dimension (cf. [1]). Moreover, if the quotient space V/G_0 exists, then all the G_0-orbits on V are closed. If $G_0=A^*$ is an abelian variety, then all the G_0-orbits are also closed (cf. [7]).

When we have $\delta\alpha(\mathfrak{D}_0(A))=\mathfrak{D}_0(V)$, Lemma 6 implies that an element ω of $\mathfrak{D}_0(V)$ belongs to $\delta f_0\circ\delta\alpha_1(\mathfrak{D}_0(A_1))$ if and only if $\delta\iota(\omega)=0$. On the other hand, we know, under the assumption 1), an element ω_1 of $\delta\alpha_1(\mathfrak{D}_0(A_1))$ belongs to $\delta\bar{f}\circ\delta\alpha_0(\mathfrak{D}_0(A_0))$ if and only if $\delta\bar{g}(\omega_1)=\omega_1$ for all the elements $\bar{g}$ of the Galois group $\bar{G}=G/G_0$ of the Galois covering $\bar{f}: V_{G_0}\to V_G$ (cf. [5])[12]. Moreover we have

$$\delta g\circ\delta f_0=\delta f_0\circ\delta\bar{g} \tag{20}$$

for all g in G, where $\bar{g}$ is the coset containing g[12].

THEOREM 7. *Let V and V_G be complete and nonsingular. When we have $\delta\alpha(\mathfrak{D}_0(A))=\mathfrak{D}_0(V)$, an element ω of $\mathfrak{D}_0(V)$ belongs to the subspace $\delta f(\mathfrak{D}_0(V_G))$ if and only if*

$$\delta\iota(\omega)=0 \quad \text{and} \quad \delta g(\omega)=\omega \tag{21}$$

for all g in G[13].

PROOF. If ω belongs to $\delta f(\mathfrak{D}_0(V_G))\subset\delta f_0(\mathfrak{D}_0(V_{G_0}))$, we have $\delta\iota(\omega)=0$ (see Lemma 4), which implies that there exists an element θ_1 of $\mathfrak{D}_0(A_1)$ such that $\omega=\delta f_0\circ\delta\alpha_1(\theta_1)$ (see Lemma 6). Then we have $\delta\alpha_1(\theta_1)=\delta\bar{f}(\omega_0)$ with ω_0 in $\mathfrak{D}_0(V_G)=\delta\alpha_0(\mathfrak{D}_0(A_0))$ and so $\delta\bar{g}(\delta\bar{f}(\omega_0))=\delta\bar{f}(\omega_0)$ for all $\bar{g}$ in $\bar{G}$ (cf. [5]), i.e. $\delta g(\omega)=\omega$ for all g in G (see (20)). Conversely if we have $\delta\iota(\omega)=0$, there exists an element θ_1 of $\mathfrak{D}_0(A_1)$ such that $\omega=\delta f_0\circ\delta\alpha_1(\theta_1)$ (see Lemma 6). Moreover, if $\delta g(\omega)=\omega$, we have $\delta\bar{g}\circ\delta\alpha_1(\theta_1)=\delta\alpha_1(\theta_1)$ (see (20)), which implies that we have $\delta\alpha_1(\theta_1)=\delta\bar{f}(\omega_0)$ with some ω_0 in $\mathfrak{D}_0(V_G)$ (cf. [5]). Hence we have $\omega=\delta f_0\circ\delta\bar{f}(\omega_0)=\delta f(\omega_0)$.

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12) We denote by δg (resp. $\delta\bar{g}$) the adjoint mapping of the birational mapping of V to $V: P\to gP$ (resp. of V_{G_0} to $V_{G_0}: Q\to\bar{g}Q$) for an element g of G (resp. $\bar{g}$ of $\bar{G}$).

13) For any ω in $\mathfrak{D}_0(V)$ and any g_0 in G_0, we have $\delta g_0(\omega)=\omega$ (cf. [7]).

We suppose that G is a linear algebraic group. Then we have $G_0=L$ and $\varphi(G_0)=\{0\}$ and so clearly the assumption 2) is satisfied. Moreover the Albanese variety B of $\overline{LP}$ is trivial, i.e. $\dim B=0$. Therefore we have the following results (see Theorems 5′, 6, 7).

Corollary. *If G is linear and we have $\delta\alpha(\mathfrak{D}_0(A))=\mathfrak{D}_0(V)$, then, under the assumption* 1), *we have $\delta\alpha_0(\mathfrak{D}_0(A_0))=\mathfrak{D}_0(V_G)$ and $\delta f(\mathfrak{D}_0(V_G))=\{\omega\in\mathfrak{D}_0(V)\mid \delta g(\omega)=\omega$ for all g in $G\}$. In particular, we have $\delta f_0(\mathfrak{D}_0(V_{G_0}))=\mathfrak{D}_0(V)$.*

Appendix. Complete homogeneous spaces

In this appendix, we shall consider a complete homogeneous space V with respect to a connected algebraic group $\tilde{G}$ and the space $\mathfrak{D}_0(V)$.

Proposition 1. *V is birationally equivalent to the direct product of the Albanese variety $A=\mathrm{Alb}(V)$ and of a rational variety.*

Proof. We may assume that $\tilde{G}$ is generated by an abelian variety $\tilde{A}$ and a connected linear algebraic group $\tilde{L}$ (cf. [**6**]). Let $\tilde{B}$ be a Borel subgroup of $\tilde{L}$. Then, since V is complete, there exists a point P_0 on V which is fixed by all the elements of $\tilde{B}$ (cf. [**1**]). Let K be a field of definition for V, $\tilde{G}$, $\tilde{A}$, $\tilde{L}$, the operation of $\tilde{G}$ on V and the solvability for $\tilde{B}$, over which P_0 is rational. Let $(\tilde{a}_1, \tilde{a}_2, \tilde{l}_1, \tilde{l}_2)$ be a generic point of $\tilde{A}\times\tilde{A}\times\tilde{L}\times\tilde{L}$ over K. Then, as $\pi_{\tilde{B}}(\tilde{l}_1)$ and $\pi_{\tilde{B}}(\tilde{l}_2)$[14] are independent generic points of $\tilde{L}/\tilde{B}$ over K and $\tilde{L}/\tilde{B}$ is a prehomogeneous space for $\tilde{B}$ (cf. [**2**]), i.e. there exists a $\tilde{B}$-orbit on $\tilde{L}/\tilde{B}$ which contains an open set of $\tilde{L}/\tilde{B}$, we have $\pi_{\tilde{B}}(\tilde{l}_2)=\tilde{b}\pi_{\tilde{B}}(\tilde{l}_1)$ with some $\tilde{b}$ in $\tilde{B}$. As $\tilde{A}$ is contained in the center of $\tilde{G}$ (cf. [8]), we have $\pi_{\tilde{B}}(\tilde{a}_2\tilde{l}_2)=(\tilde{a}_2\tilde{a}_1^{-1}\tilde{b})\pi_{\tilde{B}}(\tilde{a}_1\tilde{l}_1)$, which implies that $\tilde{G}/\tilde{B}$ is a prehomogeneous space for a connected algebraic group $\tilde{A}\tilde{B}$. Then, considering a surjective rational mapping $\tilde{g}\to\tilde{g}P_0$ of $\tilde{G}$ to V, we see that there exists a surjective rational mapping of $\tilde{G}/\tilde{B}$ to V, which commutes with the operations of $\tilde{G}$ on $\tilde{G}/\tilde{B}$ and on V. Hence V is also a prehomogeneous space for $\tilde{A}\tilde{B}$ defined over K. Then there exists a homogeneous space V^* for $\tilde{A}\tilde{B}$, which is birationally equivalent to V. Since $\tilde{A}\tilde{B}/\tilde{B}$ is an abelian variety, the solvable group $\tilde{B}$ is the maximal connected linear subgroup of $\tilde{A}\tilde{B}$. Hence we see that V^* is birationally equivalent to the direct product of the Albanese variety and of a rational variety (cf. [**6**]).

Then we have easily the following

Proposition 2. *Let α be a canonical mapping of V into $A=\mathrm{Alb}(V)$. Then we have $\mathfrak{D}_0(V)=\delta\alpha(\mathfrak{D}_0(A))$ and $\mathfrak{D}_0(V)$ is the set of all the $\tilde{G}$-invariant linear differential forms on V.*

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14) $\pi_{\tilde{B}}$ denotes the canonical rational mapping of $\tilde{G}$ to $\tilde{G}/\tilde{B}$.

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