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On the continuation of analytic sets

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§ 1. Introduction.

1. It is well known as Hartogs-Osgood's theorem that for a relatively compact domain D in C^n $(n \geqq 2)$ with the connected boundary ∂D every holomorphic function in a connected neighborhood of ∂D is continuable to D. In [**21**], Rothstein gave an analogous continuation theorem of analytic sets in domains in C^n with suitable convexity conditions. In this paper, we attempt to generalize his results to the case of analytic sets in complex spaces[1)].

As in the proof of Hartogs-Osgood's theorem [**7**], we consider a real-valued function v such that for any p an analytic set M in $\{v > v(p)\}$ is continuable to a neighborhood of p (local continuability) and assert $\inf \Lambda = -\infty$ for the set Λ of all λ satisfying that M is continuable to $\{v > \lambda\}$ (global continuability). For the study of local continuability, Rothstein restricted himself to the case of grad $v \neq 0$. His results are insufficient for the study of the global continuability of analytic sets in complex spaces. With some improvements in his arguments, we shall first prove the following local continuation theorem.

If an open set D in a complex space X is $*$-strongly s-concave at p in X (see Definition 2.8), *every purely* $(s+1)$*-dimensional analytic set in D is continuable to a neighborhood of p.*

The first four sections are devoted to the proof of this theorem. In § 2, we define several kinds of convex functions and convex open sets in a complex space and give some elementary properties and the relations of these convexities. In § 3, we state the definition of the continuation of analytic sets in order to avoid misuse and ambiguity of terminology. As in case of holomorphic functions, we need the theorem of identity for irreducible analytic sets. Using this, we give some general properties on the continuation of analytic sets (§ 3, 2° and 3°). The proof of the above local continuation theorem can be reduced to the study of a special complex space X which is

1) In this paper, a complex space means "β-Raum" in the sense of Grauert-Remmert [**10**] and is always assumed to be σ-compact.

mapped into a polydisc

$$G' = \{|z_1| < \rho_1, \cdots, |z_n| < \rho_n\}$$

in C^n by a proper nowhere degenerate holomorphic mapping ψ and its open subset $D = \psi^{-1}(G)$ with

$$\begin{aligned} G = \{\delta < |z_1| < \rho_1, |z_2| < \rho_2, \cdots, |z_n| < \rho_n\} \\ \cup \{|z_1| < \rho_1, |z_2 - a_2| < \varepsilon_2, \cdots, |z_{s+1} - a_{s+1}| \\ < \varepsilon_{s+1}, |z_{s+2}| < \rho_{s+2}, \cdots, |z_n| < \rho_n\}. \end{aligned}$$

We investigate the continuability of analytic sets in D to X in § 4 and accomplish the proof of the above theorem in § 5.

2. Using these arguments, we can generalize the local continuation theorem of holomorphic functions given in the previous paper [7] to the case of complex spaces as follows:

Let D be an open set in a complex space and $*$-strongly s-concave at p in X. If X is normal and $\dim_p X \geqq s+1$, *or* $\mathrm{dih}_p X \geqq s+1$, *every holomorphic function in D is continuable to a neighborhood of p.*

Moreover, we can prove similar continuation theorems of vector-valued holomorphic functions by the author [6], meromorphic functions using E. E. Levi's continuity theorem and its improvement by H. Kneser and holomorphic mappings into a Stein space by Kerner's Lemma ([13]) or into a relatively compact weakly 1-convex open set in a K-complete space by the method of Andreotti-Stoll [3] in § 6, 1° and 2°.

Rothstein treated the continuation of Cousin-II distributions in his paper [20]. With a Cousin-II distribution $\mathfrak{U}$ on a complex manifold we can associate two holomorphic distributions $\mathfrak{U}^n$ and $\mathfrak{U}^d$, which are identified with the coherent analytic sheaves of principal ideals of holomorphic functions. The study of the continuability of $\mathfrak{U}$ is reduced to that of principal analytic sets, disregarding the vanishing order of holomorphic functions. We prove the following proposition:

If X is a complex manifold and an open set D in X is $*$-strongly $(\dim_p X - 2)$*-concave at p, every Cousin-II distribution on D is continuable to a neighborhood of p* (§ 6, 3°).

It is desirable that we shall have the local boundary conditions for the continuability of coherent analytic sheaves over complex spaces.

3. For the global continuability, we get the following theorem:

Let X be a complex space, v a $*$-strongly s-convex function on X (see Definition 2.5) *and B an open subset of X satisfying that* 1° *for any λ $\bar{B} \cap \{v > \lambda\} \Subset X$ i.e. $\bar{B} \cap \{v > \lambda\}$ is relatively compact in X and* 2° *for $p \in \partial B$ and any locally analytic set M with* $\dim_p M \geqq s+1$ *$M - B$ intersects* $\{v > v(p)\}$. *Then* *every purely* $(s+1)$*-dimensional analytic set in a neighborhood* U *of* ∂B *is continuable to* $U\cup B$.

For example, if a finite-dimensional complex space X is K-complete, there exists a $*$-strongly 1-convex function on X and every relatively compact weakly 1-convex open set B (see Definition 2.8) has the desired properties. Therefore, for these X and B, every analytic set M in $X-B$ is continuable to X if M has no irreducible component of dimension at most one.

We have also the analogous continuation theorems of holomorphic mappings, meromorphic functions and Cousin-II distributions under the suitable conditions corresponding to the above local boundary condition (§ 7).

According to Kasahara [12], we call a sheaf $\mathcal{A}$ to be hard if the theorem of identity holds for each sections of $\mathcal{A}$ (Definition 8.1). Holomorphic functions, holomorphic mappings and meromorphic functions etc. give examples of hard sheaves. In § 8, putting together our results and Kasahara's methods ([7] and [12]), we get the following generalization of Hartogs-Osgood's theorem for these sheaves.

Let X *be a purely n-dimensional normal complex space and* v *a* $*$*-strongly* $(n-1)$*-convex function on* X *which satisfies that each connected component of the set* $\{\lambda\leqq v\leqq\mu\}$ *is compact for any* λ, μ $(\lambda<\mu)$ *and is represented as* $v=v'\tau$ *by a nowhere degenerate holomorphic mapping* τ *of* X *into a purely n-dimensional complex manifold* Y. *If for an open set* D *in* X *and a compact subset* K *of* D, $D-K$ *is connected, every holomorphic function and meromorphic function etc. in* $D-K$ *are continuable to* D.

In case that X is not normal, we have some similar results by considering the normalization (Theorem 8.5).

§ 2. Convex functions and convex open sets in a complex space.

1. Let X be a complex manifold of dimension n and v a real-valued differentiable function of class C^2 on X. At a point p in X the number of positive or negative proper values of the Hermitian matrix $\left(\left(\frac{\partial^2 v}{\partial z_i\partial\bar{z}_j}\right)_p\right)_{1\leqq i,j\leqq n}$ is invariant under an arbitrary holomorphic change of local coordinates at p, because we have

$$\frac{\partial^2 v}{\partial w_i\partial\bar{w}_j}=\sum_{k,l}\frac{\partial^2 v}{\partial z_k\partial\bar{z}_l}\left(\frac{\partial z_k}{\partial w_i}\right)\overline{\left(\frac{\partial z_l}{\partial w_j}\right)} \tag{1}$$

by Cauchy-Riemann's equations $\frac{\partial\bar{z}_k}{\partial w_i}=\overline{\left(\frac{\partial z_k}{\partial\bar{w}_i}\right)}=0$ for another system of local coordinates $w_1,\cdots,w_n$ at p.

Definition 2.1. v is said to be *strongly s-convex* at p $(1\leqq s\leqq n)$ if

$\left(\left(\frac{\partial^2 v}{\partial z_i \partial \bar{z}_j}\right)_p\right)_{1 \leqq i,j \leqq n}$ has at least $n-s+1$ positive proper values and to be strongly s-convex on an open set D if v is strongly s-convex at each point of D.

A strongly 1-convex function is said to be strongly plurisubharmonic. Obviously, a strongly plurisubharmonic function is plurisubharmonic.

A differentiable function of class C^2 is strongly s-convex at p if and only if $\left(\left(\frac{\partial^2 v}{\partial z_i \partial \bar{z}_j}\right)_p\right)_{s \leqq i,j \leqq n}$ is positive definite or v is strongly plurisubharmonic on $\{z_1 = \cdots = z_{s-1} = 0\}$ for a suitable system of local coordinates $z_1, \cdots, z_n$ at p $(p=(0))$. We give the following lemmas on strongly s-convex functions for later uses.

LEMMA 2.2. *Let v be a differentiable function of class C^2 in a neighborhood of the origin in C^n for which $\left(\left(\frac{\partial^2 v}{\partial z_i \partial \bar{z}_j}\right)_o\right)_{s \leqq i,j \leqq n}$ is positive definite. Then we can take a neighborhood U of the origin and a positive real number δ such that for any holomorphic functions $w_1, \cdots, w_n$ on U with $|w_i - z_i| \leqq \delta$ $(1 \leqq i \leqq n)$ $\{w_1, \cdots, w_n\}$ is a system of local coordinates at q and $\left(\left(\frac{\partial^2 v}{\partial w_i \partial \bar{w}_j}\right)_q\right)_{s \leqq i,j \leqq n}$ is positive definite for any q in a suitable neighborhood V of the origin $(V \subset U)$.*

PROOF. By Cauchy's inequality, for any domains U' and U'' with $U' \Subset U'' \Subset U$ there exists a positive real number η such that $\left|\frac{\partial f}{\partial z_i}\right| \leqq \eta \sup_{z \in U''} |f(z)|$ on U' $(1 \leqq i \leqq n)$ for any holomorphic function f on U. Thus, holomorphic functions $w_1, \cdots, w_n$ sufficiently near to $z_1, \cdots, z_n$ satisfy $\left(\frac{\partial(w_1, \cdots, w_n)}{\partial(z_1, \cdots, z_n)}\right)_o \neq 0$ and hence they constitute a system of local coordinates in a neighborhood of the origin. On the other hand, the proper values of a Hermitian matrix vary continuously with its components. The last assertion follows from the relation (1).

LEMMA 2.3. *Under the same assumption as above, we take a function*

$$v' = \operatorname{Re} f + \kappa(|z_s|^2 + \cdots + |z_n|^2) + u(z_1, \cdots, z_n) \qquad (\kappa > 0) \tag{2}$$

satisfying the conditions $v(p) = v'(p)$ and $v' \leqq v$ in a neighborhood U of the origin, where Ref *denotes the real part of the holomorphic function*

$$f = v(p) + 2 \times \left(\sum_i \left(\frac{\partial v}{\partial z_i}\right)_o z_i\right) + \sum_{1 \leqq i,j \leqq n} \left(\frac{\partial^2 v}{\partial z_i \partial z_j}\right)_o z_i z_j \tag{3}$$

and u is a differentiable function of class C^2 on U with $u(0, \cdots, 0, z_s, \cdots, z_n) = 0$.

PROOF. Using the above holomorphic function f, we have the Taylor expansion

$$v(z)=\mathrm{Re}\, f+\sum_{1\leqq i,j\leqq n}\Big(\frac{\partial^2 v}{\partial z_i\partial \bar z_j}\Big)_o z_i\bar z_j+o(|z|^2)\,,$$

where $\eta=o(|z|^2)$ means $\eta/|z_1|^2+\cdots+|z_n|^2\to 0$ as $z\to 0$. As is well known,

$$\sum_{s\leqq i,j\leqq n}\Big(\frac{\partial^2 v}{\partial z_i\partial \bar z_j}\Big)_o z_i z_j\geqq \delta(|z_s|^2+\cdots+|z_n|^2)$$

with the minimum δ of proper values of $\Big(\Big(\frac{\partial^2 v}{\partial z_i\partial \bar z_j}\Big)_o\Big)_{s\leqq i,j\leqq n}$. Taking a sufficiently small neighborhood U of the origin, we have

$$\begin{aligned} v(z) &\geqq \mathrm{Re}\, f+\sum_{1\leqq i,j\leqq n}\Big(\Big(\frac{\partial^2 v}{\partial z_i\partial \bar z_j}\Big)_o\Big) z_i z_j-\varepsilon(|z_1|^2+\cdots+|z_n|^2)\\ &\geqq \mathrm{Re}\, f+\delta(|z_s|^2+\cdots+|z_n|^2)+\sum_{\min(i,j)\leqq s-1}\Big(\frac{\partial^2 v}{\partial z_i\partial \bar z_j}\Big)_o z_i z_j\\ &\quad -\varepsilon(|z_1|^2+\cdots+|z_n|^2)\qquad (0<\varepsilon<\delta)\end{aligned}$$

on U. If we take $\kappa=\delta-\varepsilon$ and

$$u(z):=\sum_{\min(i,j)\leqq s-1}\Big(\frac{\partial^2 v}{\partial z_i\partial \bar z_j}\Big)_o z_i\bar z_j-\varepsilon(|z_1|^2+\cdots+|z_{s-1}|^2)\,,$$

the function v' defined by (2) has the desired properties. q. e. d.

REMARK. As is easily seen, the above f defined by (3) is invariant under an arbitrary linear transformation of coordinates in C^n.

2. DEFINITION 2.4. A real-valued function v on a complex space X is said to be *strongly s-convex* at p in X ($s\geqq 1$) if there exist a biholomorphic mapping φ of a neighborhood U of p onto an analytic subset of a domain D in C^n and a strongly s-convex function $\tilde v$ on D such that $\tilde v\varphi=v$ on U.

REMARK. For a strongly s-convex function on a complex manifold in the sense of Definition 2.1, the restriction of v to a regular submanifold Y of X is also strongly s-convex on Y. Therefore, Definition 2.4 is consistent with Definition 2.1 in case of complex manifolds.

DEFINITION 2.5. We shall say v to be *$*$-strongly s-convex* at p if in Definition 2.4 we can take a nowhere degenerate holomorphic mapping of U into D instead of the above biholomorphic mapping φ.

EXAMPLES 2.6. (i) $v=|z_s|^2+\cdots+|z_n|^2+u(z_1,\cdots,z_{s-1})$ is a strongly s-convex function on C^n, where $u(z)$ is an arbitrary differentiable function on C^n in variables $z_1,\cdots,z_{s-1}$ only.

(ii) If v is a strongly (or $*$-strongly) s-convex function on a complex space X, the restriction $v|Y$ of v to an arbitrary complex subspace Y of X is also strongly (or $*$-strongly) s-convex on Y.

(iii) Let τ be a holomorphic mapping of a complex space X into another complex space Y and v a $*$-strongly s-convex function on Y. If $r=\dim_p \tau^{-1}\tau(p)$,

$v\tau$ is $*$-strongly $(r+s)$-convex at p.

Indeed, we can take a nowhere degenerate holomorphic mapping φ' of a neighborhood U of p into a domain D' in $C^{n'}$ and a strongly $(r+s)$-convex function v' on D' with $v=v'\varphi'$ on U' as follows. By Definition 2.5, v is represented as $v=\tilde{v}\varphi$ for a suitable nowhere degenerate holomorphic mapping φ of a neighborhood U of p into a domain D in C^n and a strongly s-convex function $\tilde{v}$ on D. Since $\dim_p(\varphi\tau)^{-1}(\varphi\tau)(p)=\dim_p\tau^{-1}\tau(p)=r$, we can find r holomorphic functions $\varphi'_{n+1},\cdots,\varphi'_{n+r}$ in a neighborhood U' of p $(U'\subset U)$ such that the mapping $\varphi'=\varphi\tau\times(\varphi'_{n+1},\cdots,\varphi'_{n+r})$ of U' into C^{n+r} is nowhere degenerate on U'. Now we define the canonical extension v' of $\tilde{v}$ putting $v'(p_1,p_2)=\tilde{v}(p_1)$ for any $(p_1,p_2)\in D\times C^n$, which is strongly $(r+s)$-convex on $D'=D\times C^n$. These φ', v' and D' have the desired properties.

PROPOSITION 2.7 (Maximum Principle). *A $*$-strongly s-convex function v on a complex space X cannot take its maximum at any interior point p of X with* $\dim_p X\geqq s$.

PROOF. Assume that X is of dimension at least s at p. According to Definition 2.5, we take a nowhere degenerate holomorphic mapping φ of a neighborhood U of p into D in C^n and a strongly s-convex function $\tilde{v}$ on D with $v=\tilde{v}\varphi$. Making U and D sufficiently small, we may assume that φ maps U properly onto an analytic set in D (cf. Remmert [**17**]). By the assumption, we have $\dim_p X=\dim_{\varphi(p)}\varphi(U)\geqq s$. On the other hand, $\tilde{v}$ is strongly plurisubharmonic on $L=\{z_1=\cdots=z_{s-1}=0\}$ for suitable local coordinates $z_1,\cdots,z_n$ $(\varphi(p)=(0))$ and hence on the analytic subset $\varphi(U)\cap L$. Since $\dim_{\varphi(p)}\varphi(U)\cap L\geqq s-(s-1)=1$ and $\tilde{v}$ is not constant on any irreducible component of $\varphi(U)\cap L$, $\tilde{v}$ cannot take its maximum on $\varphi(U)\cap L$ at the origin by the maximum principle of a plurisubharmonic functions (cf. [**9**] and [**7**] Lemma 3 p. 185). Consequently, v cannot take its maximum on U at p. q. e. d.

Now we define several types of open sets in a complex space X.

DEFINITION 2.8. For a positive integer s, an open set D in X is said to be

1°, *strongly s-convex* (or *s-concave*) at a point p in X if we can take a strongly s-convex function on a neighborhood U of p such that $D\cap U=\{v<v(p)\}$ (or $D\cap U=\{v>v(p)\}$, respectively),

2°, *$*$-strongly s-convex* (or *s-concave*) at p if in 1° we can take a $*$-strongly s-convex function on U instead of the above v,

3°, *analytically s-convex* (or *s-concave*) at p if there exist s holomorphic functions $f_1,\cdots,f_s$ in a neighborhood U of p such that for the level set

$$L_p(f_1,\cdots,f_s)=\{q\in U: f_1(q)=f_1(p),\cdots,f_s(q)=f_s(p)\}$$

$L_p(f_1,\cdots,f_s)\cap\bar{D}=\{p\}$ (or $L_p(f_1,\cdots,f_s)-\{p\}\subseteqq D$),

4°, *Rothstein s-convex* (or *s-concave*) at p if any locally analytic set M with

$\dim_p M \geqq s$ intersects $\bar{D}^c$ (or D), and

5°, *weakly s-convex* (or *s-concave*) at p if the above M intersects always $D^c-\{p\}$ (or $\bar{D}-\{p\}$).

If an open set D in X is strongly ($*$-strongly or Rothstein etc.) s-convex at each boundary point of D, we call it a strongly ($*$-strongly or Rothstein etc.) s-convex set.

PROPOSITION 2.9. *Among the above open sets we have the following relations;*

strongly s-convex $\rightarrow$ *$*$-strongly s-convex* $\rightarrow$ *Rothstein s-convex*
$\rightarrow$ *weakly s-convex*

and

$*$-strongly s-convex $\rightarrow$ *analytically s-convex* $\rightarrow$ *Rothstein* $(s+1)$*-convex*.

The analogous implications are also valid for concave open sets.

PROOF. For brevity, we denote "strongly s-convex", "$*$-strongly s'-convex" and "analytically s''-convex" etc. by the symbols 1_s°, $2_{s'}^\circ$ and $3_{s''}^\circ$, etc..

$1_s^\circ \rightarrow 2_s^\circ$ and $4_s^\circ \rightarrow 5_s^\circ$ are trivial.

$2_s^\circ \rightarrow 4_s^\circ$. Assume D is $*$-strongly s-convex at p. Then there exists a $*$-strongly s-convex function v on a neighborhood U of p such that $U \cap D = \{v < v(p)\}$. For an arbitrary locally analytic set M with $\dim_p M \geqq s$ the restriction $v|M$ of v to M cannot take its maximum at p by Proposition 2.7. This shows $M \cap \bar{D} \neq \phi$.

$2_s^\circ \rightarrow 3_s^\circ$. Under the same assumption as above, the $*$-strongly s-convex function v with $D \cap U = \{v < v(p)\}$ may be represented as $v = \tilde{v}\varphi$ for a nowhere degenerate holomorphic mapping φ of U into a domain $\tilde{D}$ in C^n and a strongly s-convex function $\tilde{v}$ satisfies the assumption of Lemma 2.3 for a suitable system of coordinates $z_1, \cdots, z_n$ in $\tilde{D}$ ($\varphi(p) = (0)$). Applying Lemma 2.3 to $\tilde{v}$, we take the function f defined by (3) and $\tilde{v}'$ defined by (2) in some neighborhood W of $\varphi(p)$ ($W \subseteqq \tilde{D}$) and we have

$$\tilde{v} \geqq \tilde{v}' = \kappa(|z_s|^2 + \cdots + |z_n|^2) > v(p)$$

on $W \cap \{z_1 = \cdots = z_{s-1} = f - v(p) = 0\} - \{(0)\}$. Then, the holomorphic functions $f_1 = z_1\varphi, \cdots, f_{s-1} = z_{s-1}\varphi$, $f_s = f\varphi$ satisfy the condition $L_p(f_1, \cdots, f_s) \cap \bar{D} \cap V = \{p\}$ for a neighborhood V of p with $V \subseteqq U$ and $\varphi(V) \subseteqq W$. Therefore, D is analytically s-convex at p.

$3_s^\circ \rightarrow 4_{s+1}^\circ$. Assume D is analytically s-convex at p. Then there exist holomorphic functions $f_1, \cdots, f_s$ in a neighborhood V of p such that $L_p(f_1, \cdots, f_s) \cap \bar{D} \cap V = \{p\}$. For any locally analytic set M with $\dim_p M \geqq s+1$, we have $\dim_p M \cap L_p(f_1, \cdots, f_s) \geqq (s+1) - s = 1$. Therefore $M \cap L_p(f_1, \cdots, f_s)$ contains at least one point different from p and hence $M \cap \bar{D}^c \neq \phi$. q. e. d.

§ 3. General consideration on the continuation of analytic sets.

1. For the description of the continuability conditions of analytic sets, it is convenient to introduce the notion of analytic set germs on an arbitrary set. For a set M in a complex space X we shall say M to be analytic at a point p of X if there exists a neighborhood U of p such that $U \cap M$ is an analytic subset of U and to be analytic on a set E ($\subseteqq X$) if M is analytic at each point of E. An analytic set on E is also analytic in some neighborhood of E. A locally analytic set M is nothing but an analytic set on M itself. For two analytic sets M and N on E, if there exists a neighborhood D of E such that both M and N are analytic in D and $M \cap D = N \cap D$, we shall say M and N define the same germ on E, or M is equal to N on E, and denote this by $M|E = N|E$ or simply $M|p = N|p$ in case of $E = \{p\}$. If E is open, we may consider $M|E = M \cap E$. Without ambiguity, we can define the intersection $M \cap N$ and the inclusion $M \subseteqq N$ of two analytic sets M and N on E. Moreover, if a family $\{M_\alpha\}$ of analytic sets on E is locally finite, namely, for each point p of E there exists a neighborhood U of p such that $M_\alpha \cap U \neq \phi$ for only finitely many α, the union $\bigcup_\alpha M_\alpha$ is also an analytic set on E which is uniquely determined independently on the choice of representatives.

If an analytic set on E cannot be decomposed into the union $M = M_1 \cup M_2$ with $M_i|E \neq M|E$ $(i = 1, 2)$, we call M to be irreducible on E. By an irreducible component of $M|E$ we mean a maximal element of those analytic subsets of $M|E$ which are irreducible on E. As is well known, we can decompose an analytic set in an open set into the locally finite union of its irreducible components and an analytic set at a point into the finite union of them.

For an analytic set M in an open set D and its normalization $(\tilde{M}, \mu)$ with the projection μ, it is well known that each irreducible component M_i of M determines exactly one connected component $\mu^{-1}(M_i)$ of $\tilde{M}$. In particular, M is irreducible if and only if $\tilde{M}$ is connected. We can generalize this to the case of an analytic set on a locally closed set as follows.

Take an analytic set M on a locally closed set E and an open set D such that E is closed in D and M is considered as an analytic set in D. Then $M|E$ is irreducible if and only if $\mu^{-1}(M \cap E)$ is a connected subset of $\tilde{M}$, where $(\tilde{M}, \mu)$ is the normalization of the analytic set M in D.

Indeed, if M is reducible on E, we can decompose it as $M \cap D' = M_1 \cup M_2$ for non-empty two analytic sets M_1 and M_2 in an open set D' ($E \subseteqq D' \subseteqq D$) satisfying that M_1 and M_2 do not include any irreducible component in common. Easily we see $\mu^{-1}(M \cap E) = \mu^{-1}(M_1 \cap E) \cup \mu^{-1}(M_2 \cap E)$ and $\mu^{-1}(M_1 \cap E) \cap \mu^{-1}(M_2 \cap E) = \phi$. Since each $\mu^{-1}(M_i \cap E)$ is closed in $\tilde{M}$, this shows $\mu^{-1}(M \cap E)$ is disconnected. Conversely, if $\mu^{-1}(M \cap E)$ is disconnected in $\tilde{M}$ we can find two non-empty disjoint closed sets F_1 and F_2 with $\mu^{-1}(M\cap E)=F_1\cup F_2$ and two disjoint open sets O_1 and O_2 in $\tilde{M}$ with $O_1\cup O_2=\tilde{M}$ and $F_i\subseteqq O_i$ $(i=1,2)$. For each point p in E, we take a neighborhood $V(p)$ of p $(V(p)\subseteqq D)$ such that $\mu^{-1}(V(p))\subseteqq O_1\cup O_2$ if $p\in M$ and $V(p)\cap M=\phi$ if $p\notin M$. Then for a neighborhood $W=\bigcup_{p\in E}V(p)$ of E we have $\widetilde{M\cap W}=\mu^{-1}(W)=(\mu^{-1}(W)\cap O_1)\cup(\mu^{-1}(W)\cap O_2)$ and hence $M=M_1\cup M_2$, where $M_i:=\mu(\mu^{-1}(W)\cap O_i)$ $(i=1,2)$ are analytic sets in W. Since $\mu^{-1}(W)\cap O_i\supseteqq F_i$ $(i=1,2)$, we can conclude $M_i|E\neq M|E$ $(i=1,2)$. Consequently, M is reducible on E.

The following lemma plays an important role in this section.

LEMMA 3.1 (Theorem of identity for analytic sets). *Let M and M' be two purely k-dimensional analytic sets on a set E. If $M|p$ includes an irreducible component of $M'|p$ for a point p in $E\cap M'$ and M' is irreducible in E, $M|E$ includes $M'|E$.*

PROOF. By definition, $M\cap D$ and $M'\cap D$ are analytic sets in D for a suitable neighborhood D of E. As is well known in case that D is open, an irreducible component N of $M'\cap D$ including an irreducible component of $M'|p$ $(p\in M'\cap D)$ is an irreducible component of M in D (Abhyankar [1] (34, 14), p. 296). By N' we denote the union of all irreducible components of $M'\cap D$ with the exception of N. From the assumption of the irreducibility of $M'|E$, we conclude $M'|E=N|E\cup N'|E=N|E\subseteqq M|E$. q. e. d.

2. DEFINITION 3.2. Let X be a complex space and E a subset of X. We shall call *an analytic set M on E to be continuable to a set E'* if there exists an analytic set M' on $E\cup E'$ with $M'|E=M|E$. Then $M'|E'$ is called a continuation of M to E'.

If $\overline{M\cap E}\cap(E'-E)=\phi$, M is obviously continuable to E'. Indeed, we can take a neighborhood U of $E'-E$ with $U\cap M\cap E=\phi$ and a neighborhood D of E such that $M\cap D$ is analytic in D. Then $M\cap D$ is analytic in the neighborhood $D'=D\cup U$ of $E\cup E'$ and $(M\cap D)|E'$ is a continuation of M to E'.

An analytic set M on E can be decomposed as the locally finite union $M=\bigcup_k M^k$ of purely k-dimensional analytic sets M^k on E and hence M is continuable to E' if and only if each M^k is continuable to E'. For the study of the continuability of analytic sets, there is no loss of generality if we restricted ourselves to pure-dimensional analytic sets and any continuation of a purely k-dimensional analytic set is assumed to be of pure-dimension k.

In the rest of this section, we study the continuability of analytic sets in an open set D to another open set D' $(D\subseteqq D')$ or to a boundary point p of D. We assume all analytic sets to be of pure-dimension k $(\geqq 1)$.

PROPOSITION 3.3. *Suppose an analytic set M in D is continuable to D'. Then there exists one and only one continuation M^* of M to D' satisfying that* *each irreducible component of* M^* *intersects* D.

We shall call it the irredundant continuation of M *to* D'.

PROOF. Existence. By the assumption, there exists a continuation M' of M to D'. We take the union of all irreducible components M_i of M' with $M_i \cap D \neq \phi$. Obviously, M^* is analytic in D' and $M^* \cap D \subseteqq M' \cap D = M$. On the other hand, any point q in M is contained in some irreducible component of M'. Thus $M^* \cap D = M$. Namely, M^* is a continuation of M to D' and satisfies the desired condition.

Uniqueness. Let M_1^* and M_2^* be two continuations of M with the above condition. For each irreducible component N of M_1^* we have $M_2^* \cap D = M_1^* \cap D \supseteqq N \cap D \neq \phi$ and therefore M_2^* includes an irreducible component of $N|q$ for some q in $N \cap D$ by the assumption of dimension. According to Lemma 3.1, N is included in M_2^*. Thus $M_1^* \subseteqq M_2^*$ and similarly $M_2^* \subseteqq M_1^*$. q. e. d.

COROLLARY 3.4. *Suppose that any analytic set in* D' *intersects* D. *Then a continuation of any analytic set in* D *to* D' *is unique if it exists.*

PROOF. In this case, any continuation is the irredundant continuation. Therefore it is uniquely determined. q. e. d.

Conversely, if there exists an analytic set N in D' with $N \cap D = \phi$, for a continuation M' of M in D to D' $M' \cup N$ is also a continuation of M to D' which is not equal to M'. We cannot assert the uniqueness of the continuation of any analytic set in D to D'.

COROLLARY 3.5. *If* D *is Rothstein* k*-concave at* p, *any analytic set in* D *has at most one continuation of it to* p.

PROOF. It suffices to show that for two analytic sets M and N at p, $M \cap D = N \cap D$ implies $M|p = N|p$. To this end, we take a neighborhood V of p such that both M and N are analytic in V and each irreducible component M_i of M and N in V contains p. Then by the assumption each M_i intersects D. Both $M \cap V$ and $N \cap V$ are the irredundant continuations of $M \cap D = N \cap D$ to V. According to Proposition 3.3, we conclude $M \cap V = N \cap V$.

q. e. d.

PROPOSITION 3.6. *If an analytic set* M *in* D *satisfies the conditions* 1° *there exists an analytic set* M' *in* D' *with* $M' \cap D \supseteqq M$ *and* 2° *any irreducible component of* M' *including some irreducible component of* $M|q$ $(q \in M \cap D)$ *is irreducible in* D, *then* M *is continuable to* D'.

PROOF. Take an analytic set M' with the conditions 1° and 2°. We consider the union M^* of all irreducible components M_i' of M' satisfying that each M_i' includes an irreducible component of $M|q$ for some $q \in D \cap M$. We want to show the analytic set M^* is a continuation of M. For a point q in M each irreducible component of $M|q$ is also an irreducible component of $M'|q$ by the assumption that M and M' are purely k-dimensional. Hence we can take the irreducible component of M' including an irreducible component of $M|q$, which is also an irreducible component of M^* by the definition. Thus we get $M^* \cap D \supseteqq M$. On the other hand, each irreducible component M_i' of M^* is irreducible in D by the condition 2° and includes an irreducible component of $M|q$ ($q \in M \cap D$) which is also an irreducible component $M_i'|q$. By Lemma 3.1, we see $M_i' \cap D \subseteqq M$. This asserts $M^* \cap D \subseteqq M$ and consequently $M^* \cap D = M$. q. e. d.

REMARK. As a partial converse of this proposition we can assert that if every analytic set on a set E is continuable to E' ($E \subseteqq E'$), then every irreducible analytic set on E' is irreducible on E.

Indeed, if some irreducible analytic set M' in E' is decomposed as $M = M_1 \cup M_2$ on E with $M_i|E \neq M|E$ ($i = 1, 2$), the analytic set M_1 on E is not continuable to E'. For, any analytic set M_1' on E' with $M_1'|E \supseteqq M|E$ must include the total set $M|E'$ by Lemma 3.1. We cannot take any M_1' with $M_1'|E = M_1|E$.

3. Even if an analytic set M in D is continuable to each point in a subset E of ∂D, we cannot necessarily assert that M is continuable to E (Stoll [**23**] p. 213). But we can prove

PROPOSITION 3.7. *Let D be an open set in X and E be a closed subset of ∂D. If D is Rothstein k-concave at any point of E and an analytic set M in D is continuable to each point of E, then M is continuable to E* (cf. Rothstein [**21**] Hilfssatz 5.1 p. 126).

PROOF. By definition, for any p in E there exists an analytic set $M(p)$ in a neighborhood $U(p)$ of p with $M(p) \cap D = M \cap U(p)$. Then we can take two locally finite open coverings $\mathfrak{U} = \{U_i\}$ and $\mathfrak{U}' = \{U_i'\}$ of E such that $U_i' \Subset U_i$ for any i and each U_i is included in some $U(p)$ ($p \in E$), say $p = p_i$. We denote the analytic set $M(p_i) \cap U_i'$ by M_i'. We want to show the set $M' = \bigcup_i M_i'$ is a continuation of M to E. Obviously, $M' \cap D = M$. It suffices to show M' is analytic on E. Take an arbitrary point q of E. Rewriting the indices of U_i, we may assume $q \in U_1' \cap \bigcap_{2 \leqq i \leqq l} \bar{U}_i'$ and $q \notin \bigcup_{i > l} \bar{U}_i'$. Then we can find a neighborhood $V(q)$ of q satisfying that $V(q) \subseteqq U_1' \cap \bigcap_{2 \leqq i \leqq l} U_i$, $V(q) \cap U_i' = \phi$ for any i ($> l$) and furthermore each irreducible component of $M(p_i) \cap V(q)$ ($1 \leqq i \leqq l$) contains q. Since D is Rothstein k-concave at q, each irreducible component of $M(p_i) \cap V(q)$ ($1 \leqq i \leqq l$) intersects D. According to Lemma 3.1, the condition $M(p_i) \cap D = M = M(p_j) \cap D$ on $U(p_i) \cap U(p_j)$ implies $M(p_i) \cap V(q) = M(p_j) \cap V(q)$ for any i, j ($1 \leqq i, j \leqq l$). Therefore, we obtain

$$M' \cap V(q) = (\bigcup_j M_j') \cap V(q) \subseteqq \bigcup_{1 \leqq i \leqq l} M(p_j) \cap V(q)$$

$$= M(p_1) \cap V(q) \subseteqq M' \cap V(q) .$$

The set $M' \cap V(q) = M(p_1) \cap V(q)$ is analytic in $V(q)$. This completes the proof. q. e. d.

§ 4. Some properties of special complex spaces.

1. Before we state general continuation theorems, we give some properties of holomorphic functions and analytic sets in special complex spaces.

We take domains;

$$G' = \{|z_1| < \rho_1, \cdots, |z_k| < \rho_k\}$$

$$G \text{ (or precisely } G_a) = \{\delta < |z_1| < \rho_1, |z_2| < \rho_2, \cdots, |z_k| < \rho_k\} \cup \{|z_1| < \rho_1, |z_i - a_i| < \varepsilon_i ; 2 \leqq i \leqq k\}$$

in C^k, where $\rho_i > 0$ $(1 \leqq i \leqq k)$, $\rho_1 > \delta > 0$, $\varepsilon_i > 0$ $(2 \leqq i \leqq k)$, $a = (a_2, \cdots, a_k)$ and they are chosen as $G \subseteqq G'$.

We give the following fundamental lemma, which is an improvement of [7] Lemma 5 p. 192.

LEMMA 4.1. *Suppose M is a thin analytic subset of G'. Then any curve $\gamma(t)$ $(0 \leqq t \leqq 1)$ in $G'-M$ with the end points $\gamma(0)$, $\gamma(1)$ in $G-M$ is homotopic in $G'-M$ to a curve contained in $G-M$.*

PROOF. Firstly, for an arbitrarily fixed point $b = (b_1, \cdots, b_k) \in G-M$ the given curve $\gamma(t)$ may be assumed to satisfy $\gamma(0) = \gamma(1) = b$. To see this, we take a curve α joining $\gamma(0)$ with b and β joining $\gamma(1)$ with b in $G-M$. If $\alpha^{-1}\gamma\beta$ is homotopic to γ' in $G-M$, γ is homotopic to $\alpha\gamma'\beta^{-1}$, which is contained in $G-M$. Thus it suffices to prove Lemma 4.1 for the curve $\gamma(t)$ with $\gamma(0) = \gamma(1) = b$.

Moreover, M may be assumed to be a principal analytic set in G' defined as zeros of a holomorphic function f in G'. Indeed, since G' is a domain of holomorphy, we can take a holomorphic function $f(z_1, \cdots, z_k)$ in G' with $f \not\equiv 0$ and $M \subseteqq \{f=0\}$. By the above argument, the given curve $\gamma(t)$ satisfies $\gamma(0) = \gamma(1) \notin \{f=0\}$. We cover $\gamma(t)$ by finitely many simply connected subdomains $U_1, \cdots, U_t$ of G' with $U_i \cap M = \phi$ and $U_i \cap U_{i+1} \neq \phi$ $(1 \leqq i \leqq t-1)$. Since each $U_i - \{f=0\}$ is a connected open subset of G', it is easy to take a curve contained in $G' - \{f=0\}$ homotopic to $\gamma(t)$. Without loss of generality, we may assume $M = \{f=0\}$.

The holomorphic function f is expanded in the Hartogs series

$$f(z_1, \cdots, z_k) = \sum_{\nu=0} \alpha_\nu(z_2, \cdots, z_k) z_1^\nu$$

in G', where $\alpha_\nu(z_2, \cdots, z_k)$ are holomorphic on $\tilde{G} = \{|z_2| < \rho_2, \cdots, |z_k| < \rho_k\}$. The assumption of $f \neq 0$ implies some $\alpha_{\nu_0}(z_2, \cdots, z_k)$ does not vanish identically. Then we see easily the canonical projection π of M into the $(z_2, \cdots, z_k)$-space is nowhere degenerate on $M \cap \{\alpha_{\nu_0}(z_2, \cdots, z_k) \neq 0\}$. By S, we denote the union of the singular locus of M and the set of all regular points of M at which the Jacobian of π vanishes. The set S is a thin analytic subset of $M \cap \{\alpha_{\nu_0}$

$\neq 0\}$ and π is locally biholomorphic on $M \cap \{\alpha_{\nu_0} \neq 0\} - S$.

Next, we shall show the given curve $\gamma(t) = (\gamma_1(t), \cdots, \gamma_k(t))$ may be assumed to have the properties (i) $\gamma(0) = \gamma(1)$ and $|\gamma_i(0) - a_i| < \varepsilon_i$ $(2 \leqq i \leqq k)$ (ii) $M \cap \{z_2 - \gamma_2(t) = \cdots = z_k - \gamma_k(t) = 0\}$ is a discrete set and (iii) for some ρ' with $\rho_1 > \rho' > \delta$ $\{|z_1| \leqq \rho'\} \cap M \cap \{z_2 - \gamma_2(t) = \cdots = z_k - \gamma_k(t) = 0\} \cap \pi^{-1}\pi(S) = \phi$. According to Grauert [8] (Satz 10 p. 251) $\pi(S)$ cannot cover any non-empty open subset of $\tilde{G}$. We can find a point $b = (b_1, \cdots, b_k)$ in G with $|b_i - a_i| < \varepsilon_i$ $(2 \leqq i \leqq k)$ and $b \notin \pi^{-1}\pi(S)$. We may assume the given curve $\gamma(t)$ satisfies $\gamma(0) = \gamma(1) = b$ by the above argument and hence the prorerty (i). Moreover, as in the proof of the first part, $\gamma(t)$ is homotopic in $G' - M$ to a curve contained in $\{\alpha_{\nu_0} \neq 0\} \cap (G' - M)$. We may consider the curve $\gamma(t)$ to satisfy this condition and hence the property (ii). Now for the given curve $\gamma(t) = (\gamma_1(t), \cdots, \gamma_k(t))$ we take a real number ρ' with $\rho_1 > \rho' > \sup_{0 \leqq t \leqq 1} |\gamma_1(t)|$ $(\rho' > \delta)$ and a relatively compact subdomain D of $\tilde{G} \cap \{\alpha_{\nu_0} \neq 0\}$ including $\pi(\{\gamma(t); 0 \leqq t \leqq 1\})$. Since every point p of $M \cap \{\alpha_{\nu_0} \neq 0\}$ has a neighborhood U of p such that $U \cap \pi^{-1}\pi(S)$ is a thin analytic set, the set $\pi^{-1}\pi(S) \cap (\{|z_1| \leqq \rho'\} \times D)$ is included in a finite union of locally thin analytic subsets of G'. We can find easily a curve γ' in $(\{|z_1| < \rho'\} \times D) - M$ homotopic to γ with $\gamma'(t) \notin \pi^{-1}\pi(S)$. This shows the given curve may be assumed to have the property (iii).

For the curve with the above properties the proof of Lemma 4.1 is similar to [7] Lemma 5. Take the set $\mathcal{T}$ of parameters $(0 \leqq \tau \leqq 1)$ such that there exists a curve $\gamma^\tau(t) = (\gamma_1^\tau(t), \cdots, \gamma_k^\tau(t))$ in $G' - M$ homotopic to $\gamma(t)$ with the above properties (i)$\sim$(iii) and moreover (iv) for some sequence $\tau_0 = 0, \tau_1, \cdots, \tau_{2r} = \tau (\tau_j \leqq \tau_{j+1})$ $\gamma^\tau(t) \in G - M$ if $\tau_{2j} \leqq t \leqq \tau_{2j+1}$ $(0 \leqq j \leqq r-1)$ and $\pi\gamma^\tau(t) = \pi\gamma(\tau)$ if $\tau_{2j-1} \leqq t \leqq \tau_{2j}$ $(1 \leqq j \leqq r)$ and $\gamma^\tau(t) = \gamma(t)$ if $t \geqq \tau$. Suppose $\tau_0 = \sup \mathcal{T} < 1$. By the assumption (ii) $\{|z_1| \leqq \rho'\} \cap M \cap \pi^{-1}\pi(\gamma(\tau_0))$ has only finitely many points. We denote them by $p_l = (c_l, \gamma_2(\tau_0), \cdots, \gamma_k(\tau_0))$ $(1 \leqq l \leqq s)$. According to the assumption (iii), in a suitable neighborhood of each p_l, M is represented as

$$M: z_1 = \psi^{(l)}(z_2, \cdots, z_k)$$

where $\psi^{(l)}$ is a holomorphic function in a neighborhood of $\pi(\gamma(\tau_0))$. Then we can take a neighborhood $T = \{t; |t - \tau_0| < \varepsilon\}$ of τ_0 and sufficiently small open discs B_l with centers c_l $(1 \leqq l \leqq s)$ in the z_1-space such that (a) $B_l \cap B_m = \phi$ $(l \neq m)$, (b) $\{(z_1, \gamma_2(t), \cdots, \gamma_k(t)); |z_1| < \rho', z_1 \notin B_l, t \in T\} \cap M = \phi$ for any l, (c) $\gamma_1(t) \notin B_l$ for any l and $t \in T$, and (d) $\bar{B}_l \subseteqq \{|z_1| < \rho_1\}$ for any l, $\bar{B}_l \subseteqq \{\delta < |z_1| < \rho_1\}$ if $|c_l| > \delta$ and $\bar{B}_l \subseteqq \{|z_1| < \rho'\}$ if $|c_l| < \rho'$. By the definition of τ_0 we can find a curve $\gamma^\tau(t)$ with the above properties (i)$\sim$(iv) for some $\tau \in T$. After a suitable deformation $\gamma^\tau(t)$ may be assumed to satisfy $\gamma_1^\tau(t) \notin \bigcup_{1 \leqq l \leqq s} B_l$ for any $t \in [\tau_{2j-1}, \tau_{2j}]$ $(1 \leqq j \leqq r)$ and $|\gamma_1(\tau_j)| > \delta$ $(0 \leqq j \leqq 2r)$. Moreover, we can reparametrize so that $|\gamma_1(\tau_j)| < \rho'$ for any $t \in [\tau_{2j-1}, \tau_{2j}]$ by the addition of some τ_j.

Now we define a new curve $\gamma^{\tau'}(t)$ homotopic to $\gamma(t)$ for some $\tau' \in (\tau_0, \tau_0+\varepsilon)$ as follows. We join the curve segment $\gamma^{\tau}(t)$ $(0 \leqq t \leqq \tau_1)$ with the segments

$$
\begin{array}{lll}
(\gamma_1(\tau_1), \gamma_2(t), \cdots, \gamma_k(t)) & \tau \leqq t \leqq \tau', & \\
(\gamma_1^{\tau}(t), \gamma_2(\tau'), \cdots, \gamma_k(\tau')) & \tau_1 \leqq t \leqq \tau_2, & \\
(\gamma_1^{\tau}(\tau_2), \gamma_2(t), \cdots, \gamma_k(t)) & \tau' \geqq t \geqq \tau & \text{decreasingly}, \\
(\gamma_1^{\tau}(t), \gamma_2^{\tau}(t), \cdots, \gamma_k^{\tau}(t)) & \tau_2 \leqq t \leqq \tau_3, & \\
\cdots\cdots\cdots\cdots & & \\
(\gamma_1^{\tau}(t), \gamma_2^{\tau}(t), \cdots, \gamma_k^{\tau}(\mathrm{t})) & \tau_{2r-2} \leqq t \leqq \tau_{2r-1}, & \\
(\gamma_1^{\tau}(\tau_{2r-1}), \gamma_2(t), \cdots, \gamma_k(t)) & \tau \leqq t \leqq \tau', & \\
(\gamma_1^{\tau}(t), \gamma_2(\tau'), \cdots, \gamma_k(\tau')) & \tau_{2r-1} \leqq t \leqq \tau' &
\end{array}
$$

and $\gamma(t)$ $t \geqq \tau'$. Then we can parametrize the obtained curve so as to satisfy $\tau' \in \mathfrak{T}$. This contradicts the definition of τ_0. Thus $\tau_0 = 1$.

For $\tau' \in \mathfrak{T}$ sufficiently near to 1, the curve $\gamma^{\tau'}$ is contained in $G-M$. This shows Lemma 4.1. q. e. d.

2. LEMMA 4.2. *Let X be a complex space and ϕ a proper nowhere degenerate holomorphic mapping of X onto G'. If X is irreducible, then $\phi^{-1}(G)$ is irreducible and therefore connected.*

PROOF. By the assumption, ϕ is locally biholomorphic on $X-E$, where E denotes the thin analytic subset $\phi^{-1}\phi(S)$ of X for the union S of the singular locus of X and the set of those regular points of X at which the Jacobian of ϕ vanishes. From the irreducibility of X, the set $X-E$ is connected and hence any point p and q in $\phi^{-1}(G)-E$ can be joined by a curve $\alpha(t)$ contained in $X-E$. Applying Lemma 4.1 to the analytic set $\phi(S)$, we can find a curve γ' contained in $G-\phi(S)$ which is homotopic to $\gamma = \phi\alpha$ in $G'-\phi(S)$. Obviously, the lift of γ' to $X-E$ is a curve in $\phi^{-1}(G)-E$ joining p and q. This shows the connectivity of $\phi^{-1}(G)-E$ and hence the irreducibility of $\phi^{-1}(G)$. q. e. d.

PROPOSITION 4.3. *In the same situation as above, if X is normal, every holomorphic function on $\phi^{-1}(G)$ is continuable to the whole space X.*

PROOF. We may assume X is irreducible. As in the proof of Lemma 4.2, we note ϕ is locally biholomorphic on $X-E$ for a thin analytic subset E of X. Furthermore, for any point p in $G'-\phi(E)$, $\phi^{-1}(p)$ consists of finitely many points $\sigma_1(p), \cdots, \sigma_t(p)$, where t is independent of p. Take a holomorphic function f on $\phi^{-1}(G)$. As usual, constructing elementary symmetric functions of some of $\{f\sigma_1, \cdots, f\sigma_t\}$, we can find a pseudopolynomial

$$P(w; z_1, \cdots, z_k) = w^s + a_1(z)w^{s-1} + \cdots + a_s(z)$$

with coefficients a_l holomorphic in G such that the discriminant $d(z)$ of P does not vanish identically and $P(f(p); \phi(p)) \equiv 0$ in $\phi^{-1}(G)$. It is easily shown by

Laurent expansion that every holomorphic function in G is continuable to G'. Therefore, each a_l has its continuation a_l' to G' and we obtain a pseudopolynomial

$$P'(w\,;\,z_1,\cdots,z_k)=w^s+a_1'(z)w^{s-1}+\cdots+a_s'(z).$$

Then the discriminant $d'(z)$ of P' does not vanish identically, because $d(z)=d'(z)\neq 0$ in G. We put $M=\{d'=0\}\cup\phi(E)$. The function f is holomorphically continuable along any path in $X-\phi^{-1}(M)$ remaining a solution of the equation $P'=0$. Consequently, we get a possibly many-valued holomorphic continuation f' of f to $X-\phi^{-1}(M)$.

Assume f' is not single-valued. Then we can take a closed curve $\alpha(t)$ $(0\leqq t\leqq 1)$ in $X-\phi^{-1}(M)$ such that the continuation of a branch of f' along α from $\alpha(0)$ to $\alpha(1)$ has the germ at $\alpha(1)$ distinct from the original germ at $\alpha(0)$. Without loss of generality, we may assume $\alpha(0)=\alpha(1)\in\phi^{-1}(G)$. Then, by Lemma 4.1, the curve $\gamma=\phi\alpha$ in $G'-M$ is homotopic in $G'-M$ to a curve γ' contained in $G-M$ and hence α is homotopic in $X-\phi^{-1}(M)$ to the lift β of γ' which is contained in $\phi^{-1}(G-M)$. The continuation of the above branch of f' at $\alpha(0)=\beta(0)$ along β to $\beta(1)$ has the same germ at $\alpha(1)=\beta(1)$ as the continuation along α. This contradicts the single-valuedness of the original f in $\phi^{-1}(G)$. Therefore f' is a single-valued holomorphic function in $X-\phi^{-1}(M)$. Since f' is locally bounded in X, we get a holomorphic function f' in X by Riemann's theorem on removable singularities. f' is equal to f in $\phi^{-1}(G)$ by Lemma 4.2 and the theorem of identity. q. e. d.

COROLLARY 4.4. *If a complex space X is mapped onto G' by a proper nowhere degenerate holomorphic mapping ϕ, we have $f(X)=f(\phi^{-1}(G))$ for each holomorphic function f on X.*

PROOF. Take the normalization $\tilde{X}$ of X with the projection μ. The normal complex space $\tilde{X}$ is mapped onto G' by a proper nowhere degenerate holomorphic mapping $\phi\mu$. If $a\notin f\phi^{-1}(G)=(f\mu)(\phi\mu)^{-1}(G)$, $\dfrac{1}{f\mu-a}$ is a holomorphic function of $(\phi\mu)^{-1}(G)$ which is continuable to $\tilde{X}$ by Proposition 4.3. This shows $a\notin f\mu(\tilde{X})=f(X)$. Consequently we have $f(X)\subseteqq f\phi^{-1}(G)$ and hence $f(X)=f\phi^{-1}(G)$. q. e d.

PROPOSITION 4.5. *Let X be a complex space and ϕ a proper nowhere degenerate holomorphic mapping of X into $G'\times Z$, where $Z=\{|z_{k+1}|<\rho_{k+1},\cdots,|z_n|<\rho_n\}$ $(\rho_i>0)$. If a purely k-dimensional analytic set M in $\phi^{-1}(G\times Z)$ satisfies $\overline{\phi(M)}\cap(G\times\partial Z)=\phi$, M is continuable to the whole space X.*

Then the irredundant continuation M^* of M to X satisfies the condition $\overline{\phi(M^*)}\cap(G'\times\partial Z)=\phi$.

PROOF. According to Remmert [17], $\phi(M)$ is a purely k-dimensional analytic set in $G\times Z$. By the assumption $\overline{\phi(M)}\cap(G\times\partial Z)=\phi$, for any $a=(a_1,$

$\cdots, a_k)$ in G $\{z_1 = a_1, \cdots, z_k = a_k\} \cap \psi(M)$ is considered as a relatively compact analytic set in Z and hence contains only a finite number of points. Therefore the canonical projection π of $\psi(M)$ into G is proper nowhere degenerate. As in the proof of Proposition 4.3, each coordinate function z_l $(k+1 \leqq l \leqq n)$ satisfies $P_l(z_l; z_1, \cdots, z_k) = 0$ on $\psi(M)$ for a suitable pseudopolynomial P_l with coefficients holomorphic in G. And each P_l is continuable to a pseudopolynomial P'_l with coefficients holomorphic in G'. By these P'_l we define the set $N = \{P'_l(z_l; z_1, \cdots, z_k) = 0;\ k+1 \leqq l \leqq n\}$ in $G' \times C^{n-k}$, which is a purely k-dimensional analytic set. The set $M' = \psi^{-1}(N)$ is also a k-dimensional analytic set in X and satisfies $M' \cap \psi^{-1}(G \times Z) \supseteqq M$. Thus M satisfies the condition 1° of Proposition 3.6.

Now we take an irreducible component M'_i of M' which includes an irreducible component of $M|p$ for some p in M. Since $\psi(M'_i)$ is an irreducible k-dimensional analytic set in $G \times Z$, we can find an irreducible component N_i of N including $\psi(M'_i)$, which includes an irreducible component of $\psi(M)|\psi(p)$. Obviously the projection π of N_i into G' is proper and nowhere degenerate. By Lemma 4.2, $\pi^{-1}(G) \cap N_i$ is irreducible in $G \times C^{n-k}$. On the other hand, by the assumption $\overline{\psi(M)} \cap (G \times \partial Z) = \phi$, the set $\psi(M)$ may be considered as an analytic subset of $G \times C^{n-k}$. Applying Lemma 3.1 to analytic sets N_i and $\psi(M)$ in $G \times C^{n-k}$, we conclude $N_i \cap (G \times C^{n-k}) \subseteqq \psi(M) \subseteqq G \times Z$. Furthermore, we assert $N_i \subseteqq G' \times Z$. Indeed, for the coordinate function z_l $(k+1 \leqq l \leqq n)$ it follows from Corollary 4.4 that $|z_1| < \rho_1$ on N_i because $|z_1| < \rho_1$ on $N_i \cap (G \times Z)$. Consequently $\psi(M'_i) = N_i$ and the holomorphic mapping $\pi\psi$ is proper nowhere degenerate on M'_i. By Lemma 4.1, $\psi^{-1}(G \times Z) \cap M'_i = (\pi\psi)^{-1}(G) \cap M'_i$ is irreducible. The condition 2° of Proposition 3.6 is thus satisfied. Therefore M is continuable to the whole space X.

The irredundant continuation M^* of M is the finite union of the above M'_i, each of which satisfies $\overline{\psi(M'_i)} \cap (G' \times \partial Z) = \phi$. The set M^* satisfies also $\overline{\psi(M^*)} \cap (G' \times \partial Z) = \phi$. q. e. d.

§ 5. A local continuation theorem.

1. Let D be an open subset of a complex space X and $*$-strongly s-concave at a point p in X. Our object in this section is to study the continuability of analytic sets in D to p.

To this end, we need the following

LEMMA 5.1 (Grauert). *Let $\mathfrak{M}$ be a countable family of locally analytic sets in a complex space X and $\varphi = (\varphi_1, \cdots, \varphi_n)$ a nowhere degenerate holomorphic mapping of X into C^n. Then for any positive real number ε there exists a non-singular matrix (a_{ij}) of type (n, n) such that*

$$|a_{ij}|<\varepsilon\ (i\neq j),\qquad |a_{ii}-1|<\varepsilon\ (i,j=1,2,\cdots,n)$$

and for holomorphic functions $\varphi'_i=\sum_{1\leqq j\leqq n} a_{ij}\varphi_j\ (1\leqq i\leqq n)$ *the holomorphic mapping defined by* $\varphi'_1,\cdots,\varphi'_k\ (1\leqq k\leqq n)$ *is nowhere degenerate on each k-dimensional set in* $\mathfrak{M}$.

For the proof see Grauert [8], Satz 11, p. 252.

LEMMA 5.2. *Let M be a purely k-dimensional analytic set in D satisfying* $p\in\bar{M}$ *and* $k=s+1$. *Then M can be decomposed into the union of two analytic sets* M_1 *and* M_2 *satisfying the following conditions:*

For each M_i $(i=1,2)$, *if* $M_i\neq\phi$, *there exist a neighborhood* U_i *of p in X and a proper nowhere degenerate holomorphic mapping* ψ_i *of* U_i *into* $G'\times Z$ *in the* $(z_1,\cdots,z_n)$*-space such that* $\psi_i^{-1}(G\times Z)\subseteqq D$ *and* $\overline{\psi_i(M_i)}\cap(G\times\partial Z)=\phi$, *and the* $*$*-strongly s-convex function v with* $U_i\cap D=\{v>v(p)\}$ *is plurisubharmonic on the level set* $L_q(z_2\psi_i,\cdots,z_k\psi_i)\cap U_i$ *for any* $q\in U_i$, *where G, G′ and Z are domains as defined in* §4, 1°.

PROOF. By Definition 2.5 and 2.8, there exist a nowhere degenerate holomorphic mapping φ of a neighborhood U of p into a domain W in C^N and a strongly s-convex function $\tilde{v}$ on W with $D\cap U=\{\tilde{v}\varphi>\tilde{v}\varphi(p)=0\}$. For an arbitrarily fixed local coordinates $u_1,\cdots,u_N$ in a neighborhood of $\varphi(p)$ (say, $\varphi(p)=(0)$), we take the holomorphic function f defined by (3) in Lemma 2.3 and decompose M into the union of two analytic sets M_1 and M_2 such that $f\varphi$ does not vanish identically on each irreducible component of M_1 and vanish identically on M_2. We shall show that these M_i $(i=1,2)$ satisfy the imposed conditions.

First, we examine the condition for M_1. By Lemma 5.1 and Lemma 2.2, we can choose a new system of local coordinates $w_i=\sum_{1\leqq j\leqq N} a_{ij}u_j\ (1\leqq i\leqq N)$ for a suitable non-singular matrix $(a_{ij})_{1\leqq i,j\leqq N}$ such that for $\varphi_i=w_i\varphi$ the holomorphic mapping $(\varphi_1,\cdots,\varphi_s)$ of a neighborhood U of p into C^s is nowhere degenerate on $U\cap M_1\cap\{f\varphi=0\}$ and $\left(\left(\frac{\partial^2\tilde{v}}{\partial w_i\partial\bar{w}_j}\right)_q\right)_{1\leqq i,j\leqq N}$ is positive definite for any q in a sufficiently small W' of the origin $(W'\subseteqq W)$, because $M_1\cap\{f\varphi=0\}$ is purely s-dimensional. Apply Lemma 2.3 to the function $\tilde{v}$. We get a function

$$\tilde{v}'=\mathrm{Re}\,f+\kappa(|w_s|^2+\cdots+|w_N|^2)+u(w_1,\cdots,w_N)$$

with the properties in Lemma 2.3, where we can use the same f as above by Remark to Lemma 2.3.

Now, we map W' into C^n $(n=N+1)$ by the holomorphic mapping

$$\begin{aligned}\tau:\ & z_1=w_{k-1}\\ & z_2=f(w_1,\cdots,w_N)\\ & z_i=w_{i-2} \qquad\qquad (3\leqq i\leqq k)\end{aligned}$$

$$z_j = w_{j-1} \qquad (k+1 \leqq j \leqq n)$$

and define the canonical extension $\tilde{v}$ to $W' \times C$, which we denote again by $\tilde{v}$. Then $\tilde{v}$ is strongly plurisubharmonic on $\{z_2 = a_2, \cdots, z_k = a_k\}$ for each $(a_1, \cdots, a_n) \in W' \times C$. Therefore $v = \tilde{v}\varphi_1$ is plurisubharmonic on the level set $L_q(z_2\varphi_1, \cdots, z_k\varphi_1) \cap U_1$ for any q in a sufficiently small neighborhood U_1 of p, where $\psi_1 = \tau\varphi$. By the definition of φ and τ, $\psi_1(M_1 \cap U_1) \cap \{z_1 = \cdots = z_k = 0\}$ is a countable discrete set. Accordingly, we can take a domain $Z = \{|z_i| < \rho_i;\ k+1 \leqq i \leqq n\}$ with $\overline{\psi_1(M_1 \cap U_1)} \cap (\{z_1 = \cdots = z_k = 0\} \times \partial Z) = \phi$ and hence $G' = \{|z_i| < \rho_i;\ 1 \leqq i \leqq k\}$ with $\overline{\psi_1(M_1 \cap U_1)} \cap (G' \times \partial Z) = \phi$ by the compactness of ∂Z, where $G' \times Z \subseteqq W' \times C$. Taking the trivial extension $\tilde{u}(z)$ of $u(w)$ to $W' \times C$, we see the function

$$v' = \mathrm{Re}\,(z_2) + \kappa(|z_1|^2 + |z_{k+1}|^2 + \cdots + |z_n|^2) + \tilde{u}(z)$$

satisfies $v'\psi_1 = \tilde{v}'\varphi \leqq v$, $v'\psi_1(p) = v(p)$ in U_1 and $\tilde{u}(z_1, z_2, 0, \cdots, 0, z_{k+1}, \cdots, z_n) \equiv 0$ in $W' \times C$. If we choose sufficiently small ρ_i $(2 \leqq i \leqq k)$, it holds that $|u(z)| < \frac{\kappa\delta}{2}$ in $G' \times Z$ for an arbitrarily fixed δ with $0 < \delta < \rho_1$. Taking a suitable point $a = (a_1, \cdots, a_n)$ (e.g. $\mathrm{Re}\, a_2 > 0$, $a_3 = \cdots = a_k = 0$), sufficiently small ε_i and the above G', δ, we have $v' > v'(0) = 0$ on $G \times Z$ with the domain $G = \{\delta < |z_1| < \rho_1, |z_2| < \rho_2, \cdots, |z_k| < \rho_k\} \cup \{|z_1| < \rho_1, |z_i - a_i| < \varepsilon_i;\ 2 \leqq i \leqq k\}$ and $G \subseteqq G'$. Then, taking U_1 and $G' \times Z$ sufficiently small, we may assume ψ_1 is a proper nowhere degenerate holomorphic mapping of U_1 into $G' \times Z$. Obviously, $\psi_1^{-1}(G \times Z) \subseteqq D$ and $\overline{\psi_1(M_1)} \cap (G \times \partial Z) = \phi$.

For the analytic set M_2, we can choose a system of local coordinates $z_i = \sum_{1 \leqq j \leqq N} b_{ij} u_j$ $(1 \leqq i \leqq N)$ such that $\left(\left(\frac{\partial^2 v}{\partial z_i \partial \bar{z}_j}\right)_q\right)$ is positive definite on $\{z_3 = z_3(q), \cdots, z_k = z_k(q)\}$ for any q in a neighborhood W of $\varphi(p)$ and the holomorphic mapping $(\varphi_1, \cdots, \varphi_k)$, $\varphi_i = z_i\varphi$, is nowhere degenerate on $U_2 \cap M_2$ for some neighborhood U_2 of p. Without recourse to the mapping τ, we can define the mapping $\psi_2 = \varphi$ and domains $G \times Z$ and $G' \times Z$ in C^n $(n = N)$ with the desired properties by the same method as above. q. e. d.

REMARK. In Lemma 5.2, we can find $G = G_a(\nu)$ $(\nu = 1, 2, \cdots)$ as the above domain G such that $\lim a_i^{(\nu)} = 0$ $(2 \leqq i \leqq k)$ for $a^{(\nu)} = (a_2^{(\nu)}, \cdots, a_k^{(\nu)})$. In fact, it suffices to take $a_2^{(\nu)} = \frac{1}{\nu + \nu_0}$, $a_3^{(\nu)} = \cdots = a_k^{(\nu)} = 0$ for sufficiently large ν_0.

2. Now, we have the first main theorem.

THEOREM 5.3. *Let D be an open set which is $*$-strongly s-concave at p. If $k \geqq s+1$, every purely k-dimensional analytic set M in D is uniquely continuable to p.*

PROOF. To prove the continuability of M to p, we may assume $k = s+1$ and $p \in \bar{M}$. Then M can be decomposed into the union of two analytic subsets

M_1 and M_2 in Lemma 5.2. If each M_i is continuable to p, $M=M_1\cup M_2$ is obviously continuable to p. Therefore, we may regard M as M_1 or M_2, namely, there exists a proper nowhere degenerate holomorphic mapping ϕ of a neighborhood U of p into $G'\times Z$ such that $\Delta:=U\cap\phi^{-1}(G\times Z)\subseteqq D$, $\overline{\phi(M)}\cap(G\times\partial Z)=\phi$ and the $*$-strongly s-convex function v with $D\cap U=\{v>v(p)=0\}$ is plurisubharmonic on the level set $N(q)=L_q(z_1\phi,\cdots,z_k\phi)\cap U$ for any q in U, where G, G' and Z are domains as defined in §4. Now we can apply Proposition 4.5. There exists a purely k-dimensional analytic set M^* in U with $M^*\cap\Delta=M\cap\Delta$ and $\overline{\phi(M^*)}\cap(G'\times\partial Z)=\phi$.

We want to show $M^*\cap D=M$. To this end, we prove $N(q)\cap N$ intersects Δ for any purely k-dimensional analytic set N in $D\cap U$ with $\overline{\phi(N)}\cap(G\times\partial Z)=\phi$ and any q in N. Assume that some purely k-dimensional analytic set N in $D\cap U$ satisfies $\overline{\phi(N)}\cap(G\times\partial Z)=\phi$ and for a point q in N, $N':=N(q)\cap N$ does not intersect Δ. Then N' is of dimension at least one and relatively compact in U, because $\phi(N')\subseteqq((G'-G)\times Z)\cap\{z_2-z_2\phi(q)=\cdots=z_k-z_k\phi(q)=0\}\subseteqq\{|z_1|\leqq\delta, z_2-z_2\phi(q)=\cdots=z_k-z_k\phi(q)=0\}\times Z$ and ϕ is proper. Since $v=0$ on $\partial D\cap U$, $v|N'$ attains its maximum in an interior point of N'. By the maximum principle, v must be identically equal to zero on N'. This is a contradiction.

Especially, each irreducible component of both $M^*\cap D$ and M intersects Δ. By Lemma 3.1 and the condition $M^*\cap\Delta=M\cap\Delta$, we conclude $M^*\cap D=M$. Consequently, M is continuable to $D\cup U$.

The uniqueness of the continuation of M to p is an immediate consequence of Proposition 2.9 and Corollary 3.5. q. e. d.

COROLLARY 5.4. *In the above situation, if at a point p, X is irreducible and of dimension $k=s+1$, there exists an arbitrarily small irreducible neighborhood U of p such that U is mapped onto G' by a proper nowhere degenerate holomorphic mapping ϕ and $\phi^{-1}(G)\subseteqq D$ is valid, where G and G' are domains as defined in* §4, 1°.

Moreover, for such neighborhood U of p, $D\cap U$ is irreducible and any s-dimensional analytic subset of U intersects D.

PROOF. In Lemma 5.2, we take D for the analytic set M. Then Lemma 5.2 implies that there exists a (not necessarily open) neighborhood U of p which can be decomposed as $U=U_1\cup U_2$ such that for each i $(=1, 2)$ U_i is mapped onto G' by a proper nowhere degenerate holomorphic mapping ϕ_i with $\phi_i^{-1}(G)\subseteqq D$. Since X is irreducible at p, we can take an irreducible neighborhood U of p with the above properties. Then, by Lemma 4.2, $\Delta=\phi^{-1}(G)$ is irreducible. On the other hand, each irreducible component of $D\cap U$ intersects Δ as in the proof of Theorem 5.3. Thus $D\cap U$ is irreducible by Lemma 3.1.

Now we shall prove the last assertion. Take an s-dimensional analytic set N in U. The image $\phi(N)$ of N by the nowhere degenerate proper holomorphic mapping ϕ is also an s-dimensional analytic set in G'. If $\phi(N) \cap \{\delta < |z_1| < \rho_1, |z_2| < \rho_2, \cdots, |z_k| < \rho_k\} = \phi$, the canonical projection π of $\phi(N)$ into the domain $\tilde{G} = \{|z_2| < \rho_2, \cdots, |z_k| < \rho_k\}$ in C^{k-1} is proper nowhere degenerate. Then, since π is open and closed, we have $\pi\phi(N) = \{|z_2| < \rho_2, \cdots, |z_k| < \rho_k\}$. Therefore, $\phi(N)$ intersects $\{|z_1| < \rho_1, |z_2 - a_2| < \varepsilon_2, \cdots, |z_k - a_k| < \varepsilon_k\}$. In any way, $\phi(N) \cap G \neq \phi$ and hence $N \cap D \cap U \neq \phi$. q. e. d.

3. In Theorem 5.3, provided $k = s$, the conclusion is false. We give some counter examples.

EXAMPLES 5.5. (i) We consider an open set $D = \{|z|^2 + |w|^2 > 2\}$ in C^2. The set $M = \{(z, w); zw = 1, |z| > 1\} \cap D$ is an analytic set in $\{|z| > 1\} \cap D$ and $\bar{M}$ does not intersect $\{|z| \leqq 1\} \cap D$. Therefore M is an analytic set in D. Obviously, D is strongly 1-concave at each boundary point of D. Nevertheless, a purely 1-dimensional analytic set M in D is not continuable to a boundary point $p = (1, 1)$ of D. Because, for any analytic set M' in a neighborhood U of p, $M' \supseteqq M \cap U$ implies necessarily $M' \cap \{zw = 1, |z| \leqq 1\} \neq \phi$. There is no analytic set M' in a neighborhood V of p with $M' \cap D = M \cap V$.

(ii) Take a sequence $\{a_k\}$ in the z-plane satisfying that $|a_k| > 1$, $\sum_k (|a_k| - 1) < \infty$ and the set of all accumulation points of $\{a_k\}$ is the set $\{|z| = 1\}$. The Blaschke product

$$B(z) = \prod_k \frac{|a_k|(z - a_k)}{a_k(z\bar{a}_k - 1)}$$

is a holomorphic function in $\{|z| > 1\}$ with $|B(z)| < 1$ and the set $\{|z| = 1\}$ is the natural boundary. Now we consider the set $N := \{(z, w); w = B(z) \text{ in } D\}$ which is a purely 1-dimensional analytic set in D. We can find a real number λ $(0 < \lambda < 2)$ and a point p in $\{|z|^2 + |w|^2 = \lambda\}$ such that N has a continuation N' in the domain $D = \{|z|^2 + |w|^2 > \lambda\}$ and N' is not continuable to a strongly 1-concave boundary point p of D. Otherwise, N is uniquely continuable to the whole space C^2 (c.f. the proof of Theorem 7.1 in §7). By N^*, we denote the continuation of N to C^2. Then the canonical projection π of N^* to the z-plane is locally biholomorphic on N^* except an at most countable set $\{b_i; i = 1, 2, \cdots\}$. In a neighborhood of each $q = (z_0, w_0) \in N^* - \{b_i\}$ N^* is represented as $w = \chi_q(z)$, where $\chi_q(z)$ is a holomorphic function on some neighborhood of z_0 with $\chi_q(z_0) = w_0$. For a point q in M, we have to take $\chi_q(z) = B(z)$. Therefore we see $\chi_r(z) = B(z)$ at any point r which can be joined with q by a curve in N^* disjoint $\{b_i\}$. Consequently, the function $B(z)$ is holomorphically continuable along a curve intersecting $\{|z| = 1\}$. This is a contradiction.

From the above proof, we can also assert that there is no purely 1-dimensional analytic set which includes N. The analytic set N in D is not continuable to C^2 even if we consider the possibly many-valued continuation.

§6. The continuation of holomorphic mappings, meromorphic functions and Cousin-II distributions.

1. Holomorphic functions. Let D be an open set in a complex space X which is $*$-strongly s-concave at a point p in X.

PROPOSITION 6.1. *If X is normal and of dimension at least $s+1$ at p, there exists a neighborhood U of p such that every holomorphic function in D is continuable to $D\cup U$.*

PROOF. Since X is irreducible at p and $\dim_p X \geqq s+1$, we can find an irreducible neighborhood U of p and a proper nowhere degenerate holomorphic mapping ϕ of U onto G' with $\Delta=\phi^{-1}(G)\subseteqq D$ by Corollary 5.4, where G and G' are domains as defined in §4, 1° ($k=\dim_p X$). Take a holomorphic function f in D. By Proposition 4.3, we have the continuation f' of the restriction $f|\Delta$ of f to U. On the other hand, $U\cap D$ is connected by Corollary 5.4. By the theorem of identity, we conclude $f'=f$ in $U\cap D$. q. e. d.

A weakly holomorphic function f on a complex space Z is by definition a holomorphic function on the complex manifold $\mathring{Z}$ consisting of all regular points of Z such that for any p in Z, f is bounded in some neighborhood of p. With a weakly holomorphic function f on Z we can associate one and only one holomorphic function $\tilde{f}$ in the normalization $\tilde{Z}$ of Z with the projection μ by the relation $\tilde{f}=f\mu$. Then, denoting by $S_N(f)$ the set of all points p in Z such that f is not the trace of any holomorphic function in a neighborhood of p, we know $S_N(f)$ is an analytic subset of Z (Kasahara [**12**] Lemma 5).

PROPOSITION 6.2. *If* $\operatorname{dih}_p X \geqq s+1$, *namely, there exists a prime sequence consisting of at least* $s+1$ *elements in the maximal ideal of the local ring* $\mathcal{O}_p$, *then for some neighborhood U of p every holomorphic function in D is continuable to $D\cup U$* (cf. Andreotti-Grauert [**2**] Théorèm 10, p. 232).

PROOF. Let $(\tilde{X}, \mu)$ be the normalization of X with the projection μ. Then $\tilde{D}=\mu^{-1}(D)$ is $*$-strongly s-concave at each point $\tilde{p}_i$ in $\mu^{-1}(p)$. By the assumption of $\operatorname{dih}_p X\geqq s+1$, $\dim_q X \geqq \operatorname{dih}_q X \geqq s+1$ for any q in some neighborhood of p, whence $\dim_{\tilde{p}_i}\tilde{X}\geqq s+1$ for each $\tilde{p}_i$. By Proposition 6.1, each $\tilde{p}_i$ has a neighborhood $\tilde{V}_i$ such that $\tilde{V}_i\cap\tilde{V}_j=\phi$ $(i\neq j)$ and every holomorphic function in D is continuable to $\tilde{V}_i$. There exists a neighborhood V of p with $\mu^{-1}(V)\subseteqq\bigcup_i \tilde{V}_i$. According to Corollary 5.4, for each $\tilde{p}_i$ there exists a neighborhood $\tilde{V}'_i$ of $\tilde{p}_i$ $(\tilde{V}'_i\subseteqq\mu^{-1}(V))$ satisfying that any s-dimensional analytic subset of $\tilde{V}'_i$ intersects $\tilde{D}$. A neighborhood U of p with $\mu^{-1}(U)\subseteqq\bigcup_i\tilde{V}'_i$ has the property that each s-dimensional analytic set N in V with $N\cap D=\phi$ does not intersect

U, where we may assume $\dim_q X \geqq s+1$ for any $q \in U$. Take a holomorphic function f in D. We get a continuation $\tilde{f}'$ of a holomorphic function $\tilde{f}=f\mu$ on $\tilde{D}$ to $\tilde{D} \cup \mu^{-1}(V)$. Then there exists one and only one weakly holomorphic function f' in $D \cup V$ with $\tilde{f}'=f'\mu$. Obviously, the analytic set $S_N(f')$ in V does not intersect D.

If $\dim (S_N(f') \cap U) \geqq s$, then $S_N(f') \cap D \neq \phi$ by the above assumption. This is a contradiction. Thus $\dim_q S_N(f') \leqq s-1 \leqq \mathrm{dih}_q X-2$ for any q in U. Then the holomorphic function f' in $U-S_N(f')$ is continuable to U by Scheja's generalization of Riemann's theorem on removable singularities (Scheja [22], p. 359). Consequently, $S_N(f') \cap U=\phi$, namely, f' is holomorphic on U. q. e. d.

COROLLARY 6.3. *If X is normal and $\dim_p X \geqq s+1$, or $\mathrm{dih}_p X \geqq s+1$, for an arbitrary Fréchet space F every F-valued holomorphic function in D is continuable to a neighborhood of p.*

This is an immediate consequence from the above propositions and [6] Corollary 1 to Theorem 2 (cf. Bungart-Rossi [5], Appendix). q. e. d.

COROLLARY 6.4. *Under the same assumption as above, for an arbitrary Stein space Y every holomorphic mapping of D into Y is continuable to a neighborhood of p.*

For the proof, it is sufficient to prove the following

LEMMA 6.5. *Let φ be a holomorphic mapping of a complex space X_1 into another complex space X_2 such that each holomorphic function f_1 in X_1 corresponds to exactly one holomorphic function f_2 in X_2 with $f_1=f_2\varphi$. Then each holomorphic mapping τ_1 of X_1 into a Stein space Y corresponds to exactly one holomorphic mapping τ_2 of X_2 into Y with $\tau_1=\tau_2\varphi$.*

PROOF. This was given by H. Kerner ([13] Satz 2) in the case of normal complex space. In his proof, the normality of Y is used only to prove the fact that a normal Stein space Y is homeomorphic to the space of all closed maximal ideals of $H(Y)$ endowed with the weak topology by the canonical correspondence, where $H(Y)$ denotes the topological C-algebra of all holomorphic functions on Y endowed with the compact convergence topology. This is also valid for an arbitrary Stein space (Iwahashi [11]). q. e. d.

2. Holomorphic mapping into a relatively compact weakly 1-convex domain in a *K*-complete space. The following proposition is essentially due to Andreotti-Stoll [3] § 2.

PROPOSITION 6.6. *Let Y be a relatively compact weakly 1-convex domain in a K-complete space Z and D an open subset of a complex space X satisfying the following condition at a boundary point p of D:*

There exists a fundamental system of connected neighborhoods $\mathfrak{U}$ of p such that for each $U \in \mathfrak{U}$ every bounded holomorphic function in $U \cap D$ is continuable to U.

If D is analytically $(k-1)$-concave at p $(k=\dim_p X)$, every holomorphic mapping of D into Y is continuable to a neighborhood of p.

PROOF. Take a holomorphic mapping τ of D into Y. We consider the set $\Gamma=\bigcap_{U\in\mathfrak{U}}\overline{\tau(U\cap D)}$. Since Y is relatively compact in Z, $\overline{\tau(U\cap D)}$ is a compact subset of $\bar{Y}$ for each $U\in\mathfrak{U}$. Moreover, it is connected. Otherwise, $U\cap D=O_1\cup O_2$ for two non-empty disjoint open sets O_1 and O_2. The bounded holomorphic fuction $f=1$ on O_1 and $=0$ on O_2 is not continuable to U. Thus Γ is the intersection of a directed family of non-empty connected compact sets, whence it is also a non-empty connected compact subset of $\bar{Y}$.

On the other hand, for an arbitrarily fixed q in Γ, there exist finitely many holomoprhic functions $f_1, \cdots, f_l$ in Z such that q is an isolated point of $L_q(f_1, \cdots, f_l)$ by the assumption of the K-completeness of Z. Since each $f_i\tau$ is a bounded holomorphic function, $f_i\tau$ is continuable to an arbitrary U in $\mathfrak{U}$. Particularly, each $f_i\tau$ is continuous at p. Therefore, $\bigcap_{D\to U\in\mathfrak{U}}\overline{f_i\tau(U\cap D)}=f_i(\Gamma)$ consists of one and only one point, which is nothing but $f_i(q)$. This asserts $\Gamma\subseteqq L_q(f_1, \cdots, f_l)$.

As Γ is connected and contains q as an isolated point, we deduce $\Gamma=\{q\}$. This implies that for any neighborhood W of q there exists some U in $\mathfrak{U}$ with $\tau(U\cap D)\subseteqq W$. Taking W sufficiently small, we may regard W as an analytic set in the unit polydisc P^n in the $(z_1, \cdots, z_n)$-space. Then $z_i\tau$ $(1\leqq i\leqq n)$ are bounded holomorphic functions in $U\cap D$ and hence continuable to U by the assumption. We denote the continuation of $z_i\tau$ by τ_i and define the holomorphic mapping $\tau'=(\tau_1, \cdots, \tau_n)$ of U into C^n, which is equal to τ in $U\cap D$. As in the proof of Corollary 4.4, we see $|\tau_i|\leqq 1$ in U and furthermore $|\tau_i|<1$ on U by the maximum principle of holomorphic functions. Thus the range $\tau'(U)$ of τ' is included in P^n. On the other hand, we can write $W=\{g_i(z_1, \cdots, z_n)=0\,;\ i=1, 2, \cdots\}$, where g_i is holomorphic in P^n. $g_i(\tau_1, \cdots, \tau_n)=0$ in $U\cap D$ implies $g_i(\tau_1, \cdots, \tau_n)=0$ in the whole set U. This shows $\tau(U)\subseteqq W$.

To complete the proof, it suffices to show that the above q is contained in Y. In fact, in this case, for a sufficiently small W we have $\tau'(U)\subseteqq W\subseteqq Y$, that is, τ' is a holomorphic mapping of U into Y. By the assumption, there exists an at least one-dimensional analytic set M in a neighborhood of p such that $M-\{p\}\subseteqq D$. Then we may assume M is irreducible and of dimension 1. If $\tau'(M)=\{q\}$, we can find easily a point p' contained in $M\cap D$. Hence $q=\tau'(p')=\tau(p')\in Y$. If $\tau'(M)$ contains a point different from q, it includes a 1-dimensional locally analytic set N passing through q. Obviously, $N-\{q\}\subseteqq Y$. It cannot happen to be $q\in\partial Y$ by the weak 1-convexity of Y. q.e.d.

COROLLARY 6.7. *Let D be an open set in a complex space X which is $*$-strongly s-concave at p. If X is normal and $\dim_p X\geqq s+1$, or $\operatorname{dih}_p X\geqq s+1$,* *every holomorphic mapping of D into the above Y is continuable to a neighborhood of p.*

COROLLARY 6.8. *Let X be a normal complex space and M be a thin analytic subset of X. Then every holomorphic mapping of $X-M$ into Y is continuable to X.*

PROOF. For a k-dimensional analytic set M at p we can find easily k holomorphic functions $f_1, \cdots, f_k$ in a neighborhood of p such that $L_p(f_1, \cdots, f_k)$ contains p as an isolated point. Thus $X-M$ is analytically k-concave at p. If $k \leqq \dim_p X-1$ and X is normal, $X-M$ satisfies the conditions of Proposition 6.6. q. e. d.

3. Meromorphic functions. For meromorphic functions, we have also

PROPOSITION 6.9. *Let D be an open set and $*$-strongly s-concave at p. If $\dim_q X \geqq s+1$ for any q in some neighborhood of p, every meromorphic function is continuable to a neighborhood of p.*

PROOF. Accordig to Kasahara [**12**], Lemma 6, the set of all meromorphic functions in a complex space is canonically isomorphic to the set of all meromorphic functions in its normalization. Without loss of generality, we may assume X is normal.

As in the proof of Proposition 6.1, we take an irreducible neighborhood U of p and a proper nowhere degenerate holomorphic mapping ϕ of U onto G' with $\Delta_\nu = \phi^{-1}(G_{a^{(\nu)}}) \subseteqq D$, where $G_{a^{(\nu)}}$ and G' are domains as defined in §4, 1° and $G_{a^{(\nu)}}$ can be chosen as $a^{(\nu)}$ converges to zero by Remark to Lemma 5.2. For a meromorphic function f in D, similarly to the case of holomorphic functions, there exists a pseudopolynomial

$$P(w\,;\, z_1, \cdots, z_k) = w^t + a_1(z) w^{t-1} + \cdots + a_t(z)$$

with coefficients meromorphic in $\bigcup_\nu G_{a^{(\nu)}}$ such that $P(f(q)\,;\,\phi(q)) = 0$ for any q in $\bigcup_\nu \Delta_\nu$, where it is defined, and the discriminant $d(z)$ of P does not vanish identically (cf. the proof of Proposition 4.3 and Grauert-Remmert [**10**] p. 269). Then each a_l is meromorphically continuable to a neighborhood W of the origin by Levi-Kneser's continuity theorem of meromorphic functions (Levi [**15**], Kneser [**14**] and Okuda-Sakai [**16**]). Thus we get the pseudopolynomial

$$P'(w\,;\, z) = w^t + a_1'(z) w^{t-1} + \cdots + a_t'(z)$$

with coefficients meromorphic in W and a possibly many-valued holomorphic function f' in $\phi^{-1}(W-S)$ with $f' = f$ in some non-empty subset of $\bigcup_\nu \Delta_\nu$ as a root of $P' = 0$, where S is a thin analytic subset of W. By Lemma 4.1 and Corollary 5.4, f' is single-valued in $U' = \phi^{-1}(W-S)$ and $f' = f$ in $D \cap U'$. Then it is easily to show f' is meromorphic in W (cf. [**10**] p. 269). This completes the proof. q. e. d.

4. Cousin-II distributions on a complex manifold. By definition a Cousin-II distribution on a complex manifold X is a family $\mathfrak{U}=\{(U_i, f_i)\}$ of open sets U_i and not identically vanishing meromorphic functions f_i on U_i such that $X=\bigcup_i U_i$ and $\frac{f_i}{f_j}$ is holomorphic in $U_i \cap U_j$ for any i, j ($U_i \cap U_j \neq \phi$). For an arbitrary set E, we shall say a Cousin-II distribution on some neighborhood D of E a Cousin-II distribution on E and two Cousin-II distributions $\mathfrak{U}$ and $\mathfrak{V}$ on E to be equivalent on E if there exists a neighborhood D of E such that both $\mathfrak{U}$ and $\mathfrak{V}$ are defined on D and $\frac{f_i}{g_j}$ and $\frac{g_j}{f_i}$ are holomorphic in $U_i \cap V_j \cap D$ for each $(U_i, f_i) \in \mathfrak{U}$, $(V_j, g_j) \in \mathfrak{V}$ ($U_i \cap V_j \cap D \neq \phi$), which we denote by $\mathfrak{U}|E=\mathfrak{V}|E$. A Cousin-II distribution $\mathfrak{U}$ on E defines canonically the restriction $\mathfrak{U}|E'$ of $\mathfrak{U}$ to a subset E' of E. For $\mathfrak{U}=\{(U_i, f_i)\}$ and $\mathfrak{V}=\{(V_j, g_j)\}$ on an open set D, we can define the product $\mathfrak{U}\cdot\mathfrak{V}=\{(U_i \cap V_j, f_i g_j);\ U_i \cap V_j \neq \phi\}$ of $\mathfrak{U}$ and $\mathfrak{V}$ and the inverse $\frac{1}{\mathfrak{U}}=\left\{\left(U_i, \frac{1}{f_i}\right)\right\}$ of $\mathfrak{U}$. If for $\mathfrak{U}=\{(U_i, f_i)\}$ each f_i is holomorphic in U_i, we shall say $\mathfrak{U}$ a *holomorphic* Cousin-II distribution. Then, without ambiguity, we can define the set of those points at which $f_i=0$ for some i. We denote it by $\mathrm{Supp}(\mathfrak{U})$.

A meromorphic function f in an open set D can be represented as $f=\frac{f^1}{f^2}$ in a neighborhood V of each point p in D with f^1, f^2 holomorphic in V, where we may assume $(f^1, f^2)_q=1$ or the germs of f^1 and f^2 at q are coprime in the local ring $\mathcal{O}_q$ for any $q \in U$. For a Cousin-II distribution $\mathfrak{U}$ on an open set D, we get $\mathfrak{V}=\left\{\left(V_k, \frac{f_k^1}{f_k^2}\right)\right\}$ equivalent to $\mathfrak{U}$ by a suitable refinement such that f_k^1 and f_k^2 have the above property on V_k for any k. Then $\mathfrak{U}^n=\{(V_k, f_k^1)\}$ and $\mathfrak{U}^d=\{(V_k, f_k^2)\}$ are both holomorphic Cousin-II distributions and we have $\mathfrak{U}=\frac{\mathfrak{U}^n}{\mathfrak{U}^d}$ and $\mathfrak{U}^n=\left(\frac{1}{\mathfrak{U}}\right)^d$. For two $\mathfrak{U}$ and $\mathfrak{V}$, they are equivalent if and only if $\mathrm{Supp}\left(\frac{\mathfrak{U}}{\mathfrak{V}}\right)^n=\mathrm{Supp}\left(\frac{\mathfrak{V}}{\mathfrak{U}}\right)^n=\phi$.

DEFINITION 6.10. A Cousin-II distribution $\mathfrak{U}$ on a set E is said to be continuable to another set E' if there exists a Cousin-II distribution $\mathfrak{U}'$ on $E \cup E'$ such that $\mathfrak{U}'|E=\mathfrak{U}|E$.

PROPOSITION 6.11. *Let D be an open set in a complex manifold X and $*$-strongly s-concave at a point p. If $n=\dim_p X \geqq s+2$, every Cousin-II distribution on D is uniquely continuable to p.*

PROOF. Take a Cousin-II distribution $\mathfrak{U}$ on D. If both $\mathfrak{U}^n$ and $\mathfrak{U}^d$ are continuable to p, $\mathfrak{U}$ is also continuable. For the proof of the continuability, we may assume that $\mathfrak{U}$ is holomorphic. The set $M=\mathrm{Supp}(\mathfrak{U})$ is a purely $(n-1)$-dimensional analytic set in D if it is not empty. By the assumption, we can find the unique continuation M' of M to a neighborhood U' of p (Theorem 5.3). We may assume $p \in M'$. Otherwise, the proof is obvious. By $\mathcal{I}_q(M')$, we denote the set of those germs of holomorphic functions at $q \in M'$ which vanish identically on M'. As is well known, $\mathcal{I}_q(M')$ is a principal ideal of $\mathcal{O}_q$ $(q \in M')$. By the coherence of the analytic sheaf $\mathcal{I}(M') = \bigcup_{q \in U'} \mathcal{I}_q(M')$, there exists a holomorphic function f' in a neighborhood U'' of p $(U'' \subseteqq U')$ such that for any $q \in U''$ the germ f'_q defined by f' at q is a generator of $\mathcal{I}_q(M')$. Now we decompose M' as

$$M' = M'_1 \cup \cdots \cup M'_t$$

in a neighborhood U of p $(U \subseteqq U'')$, where each M'_i is irreducible in U and defines an irreducible component of $M'|p$. Then for each M'_i there exists a neighborhood V_i of p $(V_i \subseteqq U)$ such that $V_i \cap M'_i \cap D$ is irreducible by Corollary 5.4. We may consider that to each M'_i corresponds exactly one prime factor f'_i of f' such that f'_{iq} is a generator of $\mathcal{I}_q(M_i)$ for any $q \in U$. Since D is Rothstein $(n-1)$-concave at p by Proposition 2.9, we can take a point q_i in $(M'_i - \bigcap_{i \neq j} M'_j) \cap \bigcap_{1 \leqq j \leqq t} V_j \cap D$ and some element, say (U_i, f_i), in $\mathfrak{U}$ with $q_i \in U_i$. As $f_i = 0$ on $\mathrm{Supp}\,(\mathfrak{U}) = M = M' \cap D$, there exists a poistive integer h_i such that $f_i = (f'_i)^{h_i} g_i$ in a neighborhood of q_i for a holomorphic function g_i with $g_i \not\equiv 0$ on M'_i. Using these h_i $(1 \leqq i \leqq t)$, we define a holomorphic function $f = (f'_1)^{h_1} \cdots (f'_t)^{h_t}$. We shall show that for the Cousin-II distribution $\mathfrak{V} = \{(U, f)\}$ on U, $\mathfrak{V}$ is equivalent to $\mathfrak{U}$ of $V \cap D$, where $V = \bigcap_{1 \leqq i \leqq t} V_i$. We have $N_1 = \mathrm{Supp}\left(\dfrac{\mathfrak{U}}{\mathfrak{V}}\right)^n \subseteqq \mathrm{Supp}\,(\mathfrak{U}) = M' \cap D$. Since both N_1 and $M' \cap D$ are purely $(n-1)$-dimensional, if not empty, N_1 is the union of some irreducible components of $M' \cap D$. On the other hand, since $g_i \not\equiv 0$ on M'_i, there exists a point $q'_i \in M'_i$ arbitrarily near to q_i with $g_i(q'_i) \neq 0$. Then $q'_i \notin N_1$ and hence N_1 does not include any irreducible component of $M'_i \cap D$ which includes an irreducible component of $M'_i | q'_i$, especially any $M'_i \cap V_i \cap D$. Thus we conclude $N_1 \cap V = \phi$. Next we consider the analytic set $N_2 = \mathrm{Supp}\left(\dfrac{\mathfrak{V}}{\mathfrak{U}}\right)^n$. Then we have $q'_i \notin N_2$ and $N_2 \subseteqq \mathrm{Supp}\,(\mathfrak{V}) = M' \cap D$. By the same argument as above we conclude $N_2 \cap V = \phi$. Consequently, $\mathfrak{V}$ is equivalent to $\mathfrak{U}$ on $D \cap V$ or $\mathfrak{U}' = \mathfrak{U} \cup \{(V, f)\}$ is a continuation of $\mathfrak{U}$ to $D \cup V$.

To see the uniqueness, we take two continuations $\mathfrak{U}'$ and $\mathfrak{U}''$ of $\mathfrak{U}$ to p. The set $N_3 = \mathrm{Supp}\left(\dfrac{\mathfrak{U}'}{\mathfrak{U}''}\right)^n$ is a purely $(n-1)$-dimensional analytic set in a neighborhood of p if it is not empty. If $p \in N_3$, N_3 intersects D by Proposition 2.9. This contradicts the fact $\mathfrak{U}'|D = \mathfrak{U}''|D$. Thus $p \notin N_3$ and hence we have a neighborhood W_1 of p with $N_3 \cap W_1 = \phi$. Similarly, there exists another neighborhood W_2 of p with $\mathrm{Supp}\left(\frac{\mathfrak{U}''}{\mathfrak{U}'}\right)^n \cap W_2 = \phi$. Consequently $\mathrm{Supp}\left(\frac{\mathfrak{U}'}{\mathfrak{U}''}\right)^n$ $= \mathrm{Supp}\left(\frac{\mathfrak{U}''}{\mathfrak{U}'}\right)^n = \phi$ on $W = W_1 \cap W_2$ or $\mathfrak{U}'$ is equivalent to $\mathfrak{U}''$ on W. q. e. d.

COROLLARY 6.12. *Let M be an $(n-2)$-dimensional analytic subset of a complex manifold X of pure-dimension n. Then every Cousin-II distribution on $X-M$ is uniquely continuable to the whole space X.*

PROOF. According to Remmert-Stein [18], every purely $(n-1)$-dimensional analytic set in $X-M$ is continuable to X. On the other hand, $X-M$ is Rothstein $(n-1)$-concave at any point in M and, for any purely $(n-1)$-dimensional irreducible locally analytic set N, $N-M$ is also irreducible. Using these facts, we can prove Corollary 6.12 by the same argument as in the proof of Proposition 6.11. q. e. d.

§ 7. Global continuation theorems.

1. Let X be a complex space, where there exists a $*$-strongly s-convex function v. In this section, we give some global continuation theorems for such a complex space. For example, if X is a K-complete complex space of dimension n, there exists a nowhere degenerate holomorphic mapping $f=(f_1, \cdots, f_n)$ of X into C^n. The function $v=|f_1|^2+\cdots+|f_n|^2$ is $*$-strongly 1-convex on X. And for another complex space Y, if there exists a holomorphic mapping τ of Y into X with $\dim_p \tau^{-1}\tau(p) \leqq r$ for any $p \in Y$, the function $u=v\tau$ is a $*$-strongly $(r+1)$-convex function in Y for the above v (see, Example 2.6 (iii)).

THEOREM 7.1. *Let B be an open subset of X satisfying the following conditions $(B)_k$;*

1° $\bar{B} \cap \{v > \lambda\} \Subset X$ *for any real number λ,*

2° *for any $p \in \partial B$ and any locally analytic set M with* $\dim_p M \geqq k$ *$M-B$ intersects $\{v > v(p)\}$.*

If $k \geqq s+1$, every k-dimensional analytic set on ∂B is uniquely continuable to B.

PROOF. Take a purely k-dimensional analytic set M on ∂B, which is also analytic in a neighborhood U of ∂B. For brevity, we put $D=B \cup U$ and $D_\lambda = D \cap \{v > \lambda\}$ for a real number λ.

We prove first the uniqueness of the continuation. Let M_1 and M_2 be two purely k-dimensional analytic sets in D_λ for some λ such that $M_1 \cap (D_\lambda - B) = M_2 \cap (D_\lambda - B)$. If some irreducible component N of M_1 or M_2 does not intersect $D_\lambda - B$, the restriction of v to N takes its maximum at an interior point of N. This contradicts the maximum principle for $v|N$. Consequently, any irreducible component of M_1 and M_2 intersects $D_\lambda - B$ and hence $M_1 = M_2$

by Lemma 3.1. As we may choose λ arbitrarily, this shows also the continuation of M to B is unique.

Next, we shall show the existence of the continuation M^* of M. To this end, we consider the set Λ of all λ such that $-\infty \leqq \lambda \leqq \lambda_0 = \sup v(\bar{B})$ and there exists a purely k-dimensional analytic set M_λ in D with $M_\lambda|(D_\lambda - B) = M|(D_\lambda - B)$. Obviously $\lambda_0 \in \Lambda$ and hence $\Lambda \neq \phi$. Assume $\lambda_1 = \inf \Lambda > -\infty$. Then there exists a monotone decreasing sequence $\{\lambda_n\}$ in Λ converging to λ_1. If $n \geqq m$, $M_{\lambda_n}|(D_{\lambda_m} - B) = M|(D_{\lambda_m} - B) = M_{\lambda_m}|(D_{\lambda_m} - B)$. As is shown in the proof of the uniqueness, we have $M_{\lambda_n} \cap D_{\lambda_m} = M_{\lambda_m}$. Therefore, the set $M_{\lambda_1} = \bigcup_n M_{\lambda_n}$ is a purely k-dimensional analytic set in D_{λ_1} and satisfies $M_{\lambda_1}|(D_{\lambda_1} - B) = M|(D_{\lambda_1} - B)$. By the definition, $\lambda_1 \in \Lambda$. Now we define the analytic set $M' = M \cup M_{\lambda_1}$ is $(D - \bar{B}) \cup D_{\lambda_1}$. It is necessary to show that M' is continuable to each point in the set $E = \bar{B} \cap \{v = \lambda_1\}$. For a point $p \notin \bar{M}'$, the continuability is evident. For a point p in $\bar{M}' \cap B \cap E$, there exists a neighborhood $U(p)$ of p and an analytic set $M(p)$ in $U(p)$ such that $U(p) \subseteqq B$ and $M(p) \cap D = M_{\lambda_1} \cap U(p) = M' \cap U(p)$, by Theorem 5.3. And for a point p in $\partial B \cap \bar{M}' \cap E$, there exist two analytic sets $M_1(p)$ and $M_2(p)$ in a neighborhood $U(p)$ of p such that $M_1(p) = M \cap U(p)$ and $M_2(p) \cap D_{\lambda_1} = M_{\lambda_1} \cap U(p)$. Then we have $M_1(p)|(D_{\lambda_1} - B) = M_2|(D_{\lambda_1} - B)$. On the other hand, by the assumption, every k-dimensional locally analytic set N with $\dim_p N \geqq k$ intersects $D_{\lambda_1} - B$. Especially, each irreducible component of $M_1|p$ and $M_2|p$ intersects $D_{\lambda_1} - B$. In view of Lemma 3.1, we can conclude $M_1(p) \cap U'(p) = M_2(p) \cap U'(p)$ for a sufficiently small neighborhood $U'(p)$ of p, which is equal to M' in $((D - \bar{B}) \cup D_{\lambda_1}) \cap U'(p)$. This shows M' is continuable to p. Since $(D - \bar{B}) \cup D_{\lambda_1}$ is Rothstein k-concave at each point of E, we can apply Proposition 3.7. We get the continuation M'' of M' to a neighborhood V of E with $M''|(V - B) = M'|(V - B) = M|(V - B)$. We see easily $\lambda_2 = \sup v(\bar{B} - V) < \lambda_1$ and $\lambda_2 \in \Lambda$. This is a contradiction. Thus we have $\lambda_1 = -\infty$. The analytic set $M^* = M_{-\infty}$ is a continuation of M to D.

q. e. d.

COROLLARY 7.2. *If B is a relatively compact weakly l-convex subset of X, then every purely $(s+l)$-dimensional analytic set on $X - B$ is uniquely continuable to X.*

PROOF. According to Proposition 2.9, we can take s holomorphic functions $f_1, \cdots, f_s$ in a neighborhood of p such that $L_p(f_1, \cdots, f_s) - \{p\} \subseteqq \{v > v(p)\}$. For any locally analytic set M with $\dim_p M \geqq s+1$, $M \cap L_p(f_1, \cdots, f_s)$ is of dimension at least $(s+1) - s = 1$ at p and hence contains a point in B^c different from p by the assumption. Thus $(M - B) \cap \{v > v(p)\} \neq \phi$. The set B satisfies the condition $(B)_{s+1}$ of Theorem 7.1. q. e. d.

COROLLARY 7.3. *If a $*$-strongly s-convex function v on X satisfies that $\{\lambda < v < \mu\}$ is relatively compact in X for any λ, μ $(\lambda < \mu)$, every purely $(s+1)$-*

dimensional analytic set in $X_\lambda=\{v>\lambda\}$ *is uniquely continuable to* X.

PROOF. A purely $(s+1)$-dimensional analytic set M in X is considered as an analytic set on $X-B$ for $B=\{v<\mu\}$ $(\mu>\lambda)$. Obviously, B satisfies the conditions $(B)_{s+1}$. The unique continuation of M in $X-B$ to X is also the unique continuation of M in X_λ to X. q. e. d.

We can show another application of Theorem 7.1.

COROLLARY 7.4. *If an open set* B *satisfies the conditions* $(B)_k$ *of Theorem* 7.1 *and* $k\geqq s+1$, *each purely* k*-dimensional irreducible analytic set on* $\bar{B}$ *is irreducible on* ∂B.

PROOF. This is an immediate consequence of Theorem 7.1 and Remark to Proposition 3.6. q. e. d.

REMARK. By definition, an analytic polyhedron P in X is a relatively compact open set consisting of some connected components of the set $\{|f_1|<1,\cdots,|f_N|<1\}$ defined by holomorphic functions $f_1,\cdots,f_N$ in X. An arbitrary boundary point p of P satisfies $|f_i(p)|=1$ for some f_i. Then, for any locally analytic set N with $\dim_p N\geqq 1$, there exists a point q on N different from p with $|f_i(q)|\geqq 1$ by the maximum principle of f_i on N. This shows every analytic polyhedron is weakly 1-convex. Theorem 6.3 in Rossi [**19**] p. 464 is a special case of Corollary 7.4 (cf. § 3, 1°).

2. For the continuation of holomorphic mappings, meromorphic functions and Cousin-II distributions, we have the analogous results to Theorem 7.1.

LEMMA 7.5. *Let* D *be an open set in* X *and Rothstein* $k(p)$*-concave at each point in a closed subset* E *of* ∂D, *where* $k(p)=\varliminf_{q\to p}\dim_q X$. *If a holomorphic (or meromorphic) function* f *in* D *is continuable to each point of* E, f *is continuable to some neighborhood of* $D\cup E$ (cf. Proposition 3.7).

PROOF. By the assumption, there exist locally finite open coverings $\{U_i\}$ and $\{U_i'\}$ of E and holomorphic (or meromorphic) functions f_i on U_i such that $U_i'\Subset U_i$ and $f_i=f=f_j$ on $U_i\cap U_j\cap D$ if $U_i\cap U_j\cap D\neq\phi$. Then, as in the proof of Proposition 3.7, for each point p in E we take a neighborhood $V(p)$ of p such that $V(p)\subseteqq U_i'$, $V(p)\subseteqq U_i$ if $p\in\bar{U}_i'$ and $V(p)\cap U_i'=\phi$ if $p\notin\bar{U}_i'$. Since D is Rothstein $k(p)$-concave at p, we may assume that each irreducible component of $V(p)$ intersects D. We put $D'=D\cup\bigcup_{p\in E}V(p)$. We can define a holomorphic (or meromorphic) function f' in D' by putting $f'=f_i$ in $V(p)\subseteqq U_i'$ without ambiguity. Then f' is obviously a continuation of f to D'. q. e. d.

THEOREM 7.6. *Assume an open set* B *satisfies the conditions* $(B)_k$ *of Theorem* 7.1 *and* $\varliminf_{q\to p}\dim_q X\geqq k$ *for any* p *in* ∂B.

If for any point p *in* X, X *is normal at* p *and* $\dim_p X\geqq s+1$ *or* $\mathrm{dih}_p X\geqq s+1$, *every holomorphic function on a connected neighborhood* U *of* ∂B *is uniquely continuable to* $U\cup B$.

The analogous conclusions are valid for holomorphic functions with values in a Fréchet space and holomorphic mappings of U into a Stein space or a relatively compact weakly 1-convex open set in a K-complete space under the above assumption, and for meromorphic functions under the only assumption $\dim_p X \geqq s+1$ *for any* p.

PROOF. The proof is similar to Theorem 7.1. We have to apply the results in § 6, $1°\sim 3°$ and Lemma 7.5 instead of Theorem 5.3 and Proposition 3.7. We omit the details. q. e. d.

As special cases of Theorem 7.6, we have

COROLLARY 7.7. *Let X be a complex space admitting $*$-strongly s-convex function, B a relatively compact weakly l-convex open set and* $\dim_p X \geqq l+s$ *for any $p \in \partial B$. If for any p in X, X is normal at p and* $\dim_p X \geqq s+1$ *or* $\mathrm{dih}_p X \geqq s+1$, *every holomorphic function in a neighborhood U of ∂B is uniquely continuable to $U \cup B$* (cf. Fujimoto-Kasahara [7] Theorem 3).

COROLLARY 7.8. *Suppose a $*$-strongly s-convex function v on X satisfies* $\{\lambda < v < \mu\} \Subset X$ *for any λ, μ $(\lambda < \mu)$. If for any point p in X, X is normal at p and* $\dim_p X \geqq s+1$, *or* $\mathrm{dih}_p X \geqq s+1$, *then every holomorphic function in* $X = X \cap \{v > \lambda\}$ *is continuable to X* (cf. Andreotti-Grauert [2] Théorèm 1.5).

3. LEMMA 7.9. *Let D be an open subset of a complex manifold X of pure-dimension n and Rothstein $(n-1)$-concave at each point of a closed subset E of ∂D. If a Cousin-II distribution $\mathfrak{U}$ in D is continuable to each point of E, $\mathfrak{U}$ is continuable to E.*

PROOF. As in the proof of Proposition 3.7 and Lemma 7.5, we take locally finite open coverings $\{U_i\}$, $\{U_i'\}$ of E and Cousin-II distributions $\mathfrak{U}_i$ on U_i such that $U_i' \Subset U_i$ and $\mathfrak{U}_i | U_j \cap D = \mathfrak{U} \ | U_i \cap U_j = \mathfrak{U}_j | U_i \cap D$ if $U_i \cap U_j \cap D \neq \phi$. Let p be a point of $U_1' \cap \bigcap_{2 \leqq i \leqq r} \bar{U}_i'$ with $p \notin \bigcap_{j > r} \bar{U}_j'$. Then there exists a neighborhood $V(p)$ of p with $V(p) \subseteqq U_1' \cap \bigcap_{2 \leqq i \leqq r} U_i$ and $V(p) \cap U_j' = \phi$ $(j > r)$. Making $V(p)$ sufficiently small, we have $\mathfrak{U}_i | V(p) = \mathfrak{U}_j | V(p)$ for any i, j $(1 \leqq i, j \leqq r)$. Indeed, each set $M_{ij} = \mathrm{Supp}\left(\frac{\mathfrak{U}_i}{\mathfrak{U}_j}\right)$ is purely $(n-1)$-dimensional analytic set in $\bigcap_{1 \leqq i \leqq r} U_i$ or empty, and cannot intersect D. Since D is Rothstein $(n-1)$-concave at p, we conclude $p \notin M_{ij}$ $(1 \leqq i, j \leqq r)$. If we take $V(p)$ with $V(p) \cap M_{ij} = \phi$ for any i, j $(1 \leqq i, j \leqq r)$, it has the desired properties. Thus we get a Cousin-II distribution $\mathfrak{U}'$ on $D' = D \cup \bigcup_{p \in E} V(p)$ with $\mathfrak{U}' | V(p) = \mathfrak{U}_i | V(p)$ if $V(p) \subseteqq U_i'$. This shows $\mathfrak{U}$ has a continuation $\mathfrak{U}'$ to D'. q. e. d.

THEOREM 7.10. *If an open set B in a complex manifold X of pure-dimension n satisfies the conditions $(B)_{n-1}$ of Theorem 7.1 for a $*$-strongly s-convex function v and $n \geqq s+2$, then every Cousin-II distribution on ∂B is continuable to $\bar{B}$.*

PROOF. We proceed as in the proof of Theorem 7.1. Take a Cousin-II distribution $\mathfrak{U}$ on a neighborhood U of ∂B. For an arbitrary λ, if two Cousin-II distributions $\mathfrak{U}_1$ and $\mathfrak{U}_2$ on $D_\lambda=\{v>\lambda\}\cap(B\cup U)$ are equivalent to $\mathfrak{U}$ on $U-\bar{B}$, the sets $M_{12}=\mathrm{Supp}\left(\frac{\mathfrak{U}_1}{\mathfrak{U}_2}\right)^n$ and $M_{21}=\mathrm{Supp}\left(\frac{\mathfrak{U}_2}{\mathfrak{U}_1}\right)^n$ are purely $(n-1)$-dimensional analytic sets in D_λ, if not empty. By the assumption each of them does not intersect $U-\bar{B}$. Thus we conclude $M_{12}=M_{21}=\phi$, using the maximum principle for v. Thus $\mathfrak{U}_1|D_\lambda=\mathfrak{U}_2|D_\lambda=\mathfrak{U}|D_\lambda$. Consequently, the continuation of $\mathfrak{U}$ to $\bar{B}$ is unique.

To see the continuability of $\mathfrak{U}$, we take the set Λ of all λ such that there exists a Cousin-II distribution $\mathfrak{U}_\lambda$ on D_λ with $\mathfrak{U}_\lambda|(U-\bar{B})=\mathfrak{U}$. Obviously, $\lambda_0=\sup v(\bar{B})\in\Lambda$ and $\lambda_1=\inf\Lambda\in\Lambda$ by the above argument. Thus we get a Cousin-II distribution $\mathfrak{U}'$ on $(U-\bar{B})\cup D_{\lambda_2}$ with $\mathfrak{U}'|U-\bar{B}=\mathfrak{U}$. It is continuable to each point of $E=\bar{B}\cap\{v=\lambda_1\}$. This is easily shown by using Proposition 6.11 and the fact that for a point $p\in\partial B$ two distributions $\mathfrak{U}_1$ and $\mathfrak{U}_2$ at p with $\mathfrak{U}_1|(U-\bar{B})\cap D_{\lambda_1}=\mathfrak{U}_2|(U-\bar{B})\cap D_{\lambda_1}$ satisfies also $\mathfrak{U}_1|p=\mathfrak{U}_2|p$ because $p\notin\mathrm{Supp}\left(\frac{\mathfrak{U}_1}{\mathfrak{U}_2}\right)^n\cup\mathrm{Supp}\left(\frac{\mathfrak{U}_2}{\mathfrak{U}_1}\right)^n$. By Lemma 7.9, $\mathfrak{U}_{\lambda_1}$ is continuable to E, whence we can find easily $\lambda_2\in\Lambda$ with $\lambda_2<\lambda_1$. This is a contradiction. The distribution $\mathfrak{U}_{-\infty}$ is a continuation of $\mathfrak{U}$ to $U\cup B$. q. e. d.

§ 8. The continuation of sections of hard sheaves.

1. For the simultaneous treatment of the continuation of holomorphic mappings and meromorphic functions etc., Kasahara introduced the notion of a hard sheaf in his paper [**12**] as follows;

DEFINITION 8.1. A sheaf $\mathcal{A}$ of sets over a topological space X is said to be *hard* if for any pair of a connected open set U and its open subset U', the restriction map $\rho^U_{U'}:\Gamma(U,\mathcal{A})\to\Gamma(U',\mathcal{A})$ is injective, where $\Gamma(U,\mathcal{A})$ denotes the set of all sections of $\mathcal{A}$ on U.

REMARK. For a locally connected topological space X

(i) if there exists a basis $\mathfrak{U}=\{U\}$ of connected open sets such that for any $U, U'\in\mathfrak{U}$ with $U'\subseteqq U$, $\rho^U_{U'}$ is injective, then $\mathcal{A}$ is a hard sheaf,

(ii) if there exists an open covering $\mathfrak{V}=\{V\}$ of X such that the restriction $\mathcal{A}|V$ of $\mathcal{A}$ to V is hard over V for any $V\in\mathfrak{V}$, then $\mathcal{A}$ is hard over X.

In order to state Kasahara's theorem ([**12**], Theorem 1), we recall his definition of an admissible function v and a domain good for v. Let $\mathcal{A}$ be a hard sheaf of sets over a locally arcwise connected Hausdorff space X. According to [**12**], a continuous real-valued function v on X is said to be *pre-admissible* if (1) for any λ, μ $(\lambda<\mu)$ each connected component of $\{p\in X;\ \lambda\leqq v(p)\leqq\mu\}$ is compact and (2) for any $p\in X$ there exists a fundamental system $\mathfrak{U}$ of connected neighborhoods of p such that $U \cap \{v > v(p)\}$ is a non-empty connected set for each $U \in \mathfrak{U}$ and to be *admissible* for $\mathcal{A}$ if, furthermore, for any $U \in \mathfrak{U}$ the restriction mapping $\Gamma(U, \mathcal{A}) \rightarrow \Gamma(U \cap \{v > v(p)\}, \mathcal{A})$ is surjective. A relatively compact open set B, B is said to be good for v if it satisfies the followings:

(1) Taking an arbitrary point p in ∂B, we denote one of the sets $\partial B \cap \{v > v(p)\}$, $B \cap \{v > v(p)\}$ and $(X-\bar{B}) \cap \{v < v(p)\}$ by Δ. For any neighborhood U of p we can find another neighborhood V of p $(V \subseteqq U)$ such that each point of $V \cap \Delta$ can be joined to p by a curve in U.

(2) For any real number ρ except finitely many $\rho_0 > \cdots > \rho_s$, each point $p \in \partial B \cap \{v = \rho\}$ has arbitrarily small neighborhoods W' and W'' such that $W' \cap (X-\bar{B}) \cap \{v > \rho\}$, $W' \cap B \cap \{v > \rho\}$, $W'' \cap (X-\bar{B})$ and $W'' \cap B$ are all non-empty connected.

THEOREM (Kasahara). *Let $\mathcal{A}$ be a hard sheaf over a locally arcwise connected, locally compact Hausdorff space X with a countable basis of open sets and v an admissible function for $\mathcal{A}$ on X. Take an open set D and its compact subset K satisfying that $D-K$ is connected. If there exists an open set B such that $K \subset B \subseteqq D$ and the boundary ∂B is good for v, the restriction mapping $\Gamma(D, \mathcal{A}) \rightarrow \Gamma(D-K, \mathcal{A})$ is bijective.*

As consequences of the previous sections we can give several examples of hard sheaves and admissible functions on normal complex spaces.

EXAMPLES 8.2. Let X be a normal complex space of pure-dimension n. Then a $*$-strongly $(n-1)$-convex function v on X satisfying the condition (1) for pre-admissible function is admissible for the following sheaves:

(i) The structure sheaf $\mathcal{O}$ of all germs of holomorphic functions on X and more generally the sheaf $\mathcal{O}^F$ of all germs of F-valued holomorphic functions on X for a Fréchet space F (Proposition 6.1 and Corollary 6.3).

(ii) $\mathcal{O}^Y$; the sheaf of all germs of holomorphic mappings of X into Y if Y is a Stein space or a relatively compact weakly 1-convex subdomain of a K-complete space (Corollary 6.4 and Corollary 6.7).

(iii) $\mathfrak{M}$; the sheaf of all germs of meromorphic functions on X (Proposition 6.9).

(iv) An analytic sheaf $\mathcal{A}$ which is locally isomorphic to $\mathcal{O}^F$ for a Fréchet space F, namely, there exists an open covering $\mathfrak{U} = \{U\}$ of X such that $\mathcal{A}|U$ is isomorphic to $\mathcal{O}^F|U$ for each $U \in \mathfrak{U}$ (Remark to Definition 8.1). For example, the sheaf of the germs of all cross-sections of a holomorphic vector bundle of dimension n over X is locally isomorphic to $\mathcal{O}^n = \mathcal{O}^{C^n}$. Moreover, we take a locally trivial fiber space (B, π, X, Y), where B, X and Y are all complex spaces, π is a holomorphic mapping of B onto X and there exists an open covering $\mathfrak{U}$ of X such that we can find a biholomorphic mapping $\varphi_U : \pi^{-1}(U)$

$\to U\times Y$ with $Pr_U\varphi_U=\pi$ for each $U\in\mathfrak{U}$ and the canonical projection Pr_U of $U\times Y$ onto U. Then the direct image $\pi_*(\mathcal{O})$ of the structure sheaf $\mathcal{O}$ of B is locally isomorphic to $\mathcal{O}^{H(Y)}$ for the Fréchet space $H(Y)$ of all holomorphic functions on Y with the compact convergence topology. Indeed, we have the isomorphisms

$$\Gamma(V, \pi_*(\mathcal{O}))=\Gamma(\pi^{-1}(V), \mathcal{O})\cong\Gamma(V\times Y, \mathcal{O})\cong\Gamma(V, \mathcal{O}^{H(Y)})$$

for any open subset V of U ($U\in\mathfrak{U}$) and these isomorphisms commute the restriction map of sections over V to another V' (Fujimoto [5], Theorem 9).

2. In the above Kasahara's theorem, for some special complex spaces and real analytic admissible function v for the structure sheaves, we can take off the assumption of the existence of an open set B with the boundary good for v by the following

LEMMA 8.3. *Let X be a purely n-dimensional normal complex space and τ nowhere degenerate holomorphic mapping of X into a purely n-dimensional complex manifold Y. Then for a real-valued real analytic function $\tilde{v}$ on Y, an open set D in X and a compact subset K of D, there exists an open set B such that $K\subset B\subseteqq D$ and B is good for $v=\tilde{v}\tau$.*

PROOF. This was shown by Fujimoto-Kasahara ([7] § 7) in case that X is a complex manifold and by Kasahara ([12]) in case that X is a normal complex space and τ is a nowhere degenerate holomorphic mapping of X into C^n ($n=\dim X$). Joining their methods together, we can prove easily Lemma 8.3. q. e. d.

Thus we have

THEOREM 8.4. *Let X be a purely n-dimensional complex space and v a pre-admissible $*$-strongly s-convex function v on X which is represented as $v=v'\tau$ by a suitable nowhere degenerate holomorphic mapping τ of X into a purely n-dimensional complex manifold Y and a real analytic function v'. Take an open set D and its compact subset K such that for each connected component D_i, D_i-K is connected. If X is normal and $n\geqq s+1$, the restriction mapping $\rho^D_{D-K}:\Gamma(D, \mathcal{A})\to\Gamma(D-K, \mathcal{A})$ is bijective, where $\mathcal{A}$ denotes one of the sheaves of Examples 8.2.*

For an arbitrary complex space X, using the normalization of X, we have also

THEOREM 8.5. *In the same situation as in Theorem 8.4, take an open set D and its compact subset K such that for each irreducible component D_i of D, D_i-K is irreducible. Then the restriction mapping $\Gamma(D, \mathcal{A})\to\Gamma(D-K, \mathcal{A})$ is bijective if $n\geqq s+1$ for any $p\in X$ in case of $\mathcal{A}=\mathcal{M}$ (Example 8.2 (iii)) or if $\mathrm{dih}_p X\geqq s+1$ for any $p\in X$ in case that $\mathcal{A}$ is either the sheaf of Examples 8.2 (i) or (ii).*

PROOF. Take the normalization $(\tilde{X}, \mu)$ of X, where $\tilde{X}$ is a normal complex space and the projection μ is proper nowhere degenerate holomorphic and has the holomorphic inverse on the set $\mathring{X}$ of all regular points of X. By the assumption, there is a pre-admissible $*$-strongly s-convex function $v=v'\tau$ on X, where the function v' and the mapping τ satisfy the conditions of Theorem 8.4. Obviously, $\tilde{v}=v\mu=v'\tau\mu$ satisfies also the conditions of Theorem 8.4 and for each connected component $\tilde{D}_i$ of $\tilde{D}$, $\tilde{D}_i-\tilde{K}$ is connected, where $\tilde{D}=\mu^{-1}(D)$ and $\tilde{K}=\mu^{-1}(K)$. Since X is purely n-dimensional and $n\geqq s+1$ in any case, we can apply Theorem 8.4 to the complex space $\tilde{X}$.

By $\mathcal{A}_X$ and $\mathcal{A}_{\tilde{X}}$ we denote one of the sheaves of Examples 8.2 defined over X and $\tilde{X}$, respectively. In case of $\mathcal{A}_X=\mathcal{M}_X$, by Kasahara [**12**] Lemma 6 and Theorem 8.4 above we get the canonical isomorphisms

$$\Gamma(D, \mathcal{M}_X)\cong\Gamma(\tilde{D}, \mathcal{M}_{\tilde{X}})\cong\Gamma(\tilde{D}-\tilde{K}, \mathcal{M}_{\tilde{X}})\cong\Gamma(D-K, \mathcal{M}_X)\,.$$

In the other cases, we have also the isomorphisms $\Gamma(\tilde{D}, \mathcal{A}_{\tilde{X}})\cong\Gamma(\tilde{D}-\tilde{K}, \mathcal{A}_{\tilde{X}})$. Take a section $f\in\Gamma(D-K, \mathcal{A}_X)$. In any case, f defines a section $\tilde{f}=f\mu\in\Gamma(\tilde{D}-\tilde{K}, \mathcal{A}_{\tilde{X}})$, which has a continuation $\tilde{f}'\in\Gamma(\tilde{D}, \mathcal{A}_{\tilde{X}})$ with $\tilde{f}=\tilde{f}'$ on $\tilde{D}-\tilde{K}$ by the above isomorphism. Then, since μ has the holomorphic inverse on $\mathring{X}$, we can find a section $f'\in\Gamma(D\cap\mathring{X}, \mathcal{A}_X)$ with $f'\mu=\tilde{f}'$. It suffices to show f' is continuable to D. To this end, as in the proof of Theorem 7.1, we consider the set Λ of all λ satisfying that f' is continuable to $D_\lambda=D\cap\{v>\lambda\}$. In any case, if a section $f'\in\Gamma(D\cap\mathring{X}, \mathcal{A}_X)$ is continuable to $D_{v(p)}$ for a point p in K, f' has a continuation $f''\in\Gamma(U, \mathcal{A}_X)$ with $f'=f''$ on $D_{v(p)}$ for some neighborhood U of p. Since we may assume any connected component of $U\cap\mathring{X}$ intersects $D_{v(p)}$ and $\mathcal{A}_X$ is a hard sheaf, we see $f'=f''$ in $U\cap\mathring{X}$. Using this fact, we can conclude easily $\inf\Lambda=-\infty$. This shows the restriction mapping $\Gamma(D, \mathcal{A}_X)\to\Gamma(D-K, \mathcal{A}_X)$ is bijective. q. e. d.

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