J. Math. Soc. Japan
Vol. 19, No. 2, 1967

On the equivalence of Gaussian measures

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(Received April 15, 1966)
(Revised Oct. 21, 1966)

§ 1. Introduction.

Let P be a Gaussian measure on the function space $(\boldsymbol{R}^T, \mathscr{B})$, where T is an interval and $\mathscr{B}$ is the σ-algebra generated by all cylinder sets. Then the family of w-functions:

$$X(t, w) = \text{the } t\text{-coordinate of } w,\ w \in \boldsymbol{R}^T,\ t \in T,$$

defines a Gaussian process on the probability measure space $(\boldsymbol{R}^T, \mathscr{B}, P)$. Conversely, every Gaussian process on an arbitrary probability measure space has a representation of such type (coordinate representation). In this paper we shall use only the coordinate representation, unless stated otherwise. Thus we have a one-to-one correspondence between Gaussian processes with the time parameter t in T and Gaussian measures on the function space $\boldsymbol{R}^T$. Two Gaussian processes are said to be *equivalent*, if their corresponding Gaussian measures are equivalent, i. e. mutually absolutely continuous.

J. Hajek [1] and J. Feldman [2] found independently that two Gaussian measures are either equivalent or singular, and Yu. Rozanov [3] established a criterion for the equivalence in terms of the linear operator on $L^2(X)$, Hilbert space spanned by $\{X(t, w)\}$ (the precise definition is given in section 2).

D. Varberg [7] has established a necessary and sufficient condition for a class of Gaussian processes to be equivalent to the Brownian motion. He treats the '*factorable*' Gaussian processes, the covariance function of which can be written in the form

$$r(t, s) = \int_T R(t, u)R(s, u)du,$$

where T is a finite interval $[0, b]$. Further he gives conditions on the kernel function of the linear transformation acting on the Brownian path.

Lately L. Shepp [10] has solved many problems concerning the *B-equivalence* (the equivalence to the Brownian motion $\{B(t, w)\}$) of a Gaussian process. He has given a simple necessary and sufficient condition on the mean and covariance function for the *B-equivalence*[1], and has obtained explicit expressions of Radon-Nicodym derivative. Further he has shown that any B-equivalent Gaussian process can be realized by a linear transformation of $\{B(t, w)\}$ such that

$$B(t, w)+\int_T\int_0^t g(v, u)dvdB(u, w)+\int_0^t m'(u)du . \tag{1.1}$$

In the present paper, it is shown that any Gaussian process equivalent to a Gaussian process $\{X(t, w)\}$ can be realized by a linear transformation of $\{X(t, w)\}$ such that

$$\mathfrak{F}X(t, w)=FX(t, w)+\mathfrak{f}[X(t, w)] , \tag{1.2}$$

where F is an invertible linear operator on $L^2(X)$, $F-I$ is of Hilbert-Schmidt type and $\mathfrak{f}$ is a bounded linear functional on $L^2(X)$ (Theorem 2). In case of the Brownian motion, we obtain the same expression of the linear transformation (1.2) with (1.1) of L. Shepp using a different method from his (Theorem 3). Our method is based on the works of Yu. Rozanov [3]. We extend this result in case of a certain class of Gaussian processes including purely non-deterministic stationary Gaussian processes (Theorem 4). Section 5 is devoted to some remarks, one of which enables us to extend the Skorokhod's results on the equivalence of two Gaussian additive processes.

The author wishes to express his hearty thanks to those who helped him in the course of this paper. Professor T. Hida always encouraged him and gave him many valuable suggestions. Professor S. Ito gave him some remarks on the theory of Hilbert space. Mr. I. Kubo and other members of the Probability Seminar helped him with valuable discussions.

§ 2. General theory.

Let $\{X(t, w)\}$ be a Gaussian process defined on a probability space $(\boldsymbol{R}^T, \mathcal{B}, P)$, where T is a finite or infinite interval. We may assume that

$$EX(t, w)=\int_{\boldsymbol{R}^T} X(t, w)dP(w)=0, \qquad t\in T , \tag{2.1}$$

without loss of generality.

Let $X(t)$ denote the P-equivalent class containing the random variable $X(t, w)$ and let $L^2(X)$ be a Hilbert space spanned by $\{X(t) ; t\in T\}$ with the inner product

$$\langle X(t), X(s)\rangle=EX(t, w)X(s, w) , \qquad t, s\in T , \tag{2.2}$$

1) Dr. H. Oodaira informed to the author that he had obtained the analogous result on the mean and covariance function.

and the norm

(2.3) $$\| X(t) \|^2 = EX(t, w)^2, \qquad t \in T.$$

Every element X in $L^2(X)$ is therefore a P-equivalent class of w-functions and we denote a representative w-function belonging to X by $X(w)$.

We assume, in this paper, that $L^2(X)$ is separable.

If $\{X(t, w)\}$ is continuous in the mean, then this assumption is satisfied.

Let $\{X_1(t, w)\}$ be another Gaussian process defined on $(\boldsymbol{R}^T, \mathcal{B}, P_1)$ with the mean function $m(t)$ and the covariance function $r_1(t, s)$.

DEFINITION. Two Gaussian processes are said to be *equivalent* if their corresponding measures P and P_1 are equivalent.

We shall first restate Rozanov's theorem using Feldman's terminology.

DEFINITION (according to J. Feldman [2]). An invertible bounded linear transformation F from a Hilbert space onto itself is called an *equivalence operator*, if F^*F-I (I = identity operator) is of Hilbert-Schmidt type (or equivalently if $\sqrt{F^*F}-I$ is of Hilbert-Schmidt type).

THEOREM 1 (Yu. Rozanov [3]). *$\{X_1(t, w\}$ is equivalent to $\{X(t, w)\}$ if and only if there exists an equivalence operator F and a bounded linear functional $\mathfrak{f}$ on $L^2(X)$ such that*

(A) $$\langle FX(t), FX(s) \rangle = r_1(t, s), \qquad t, s \in T,$$

(B) $$\mathfrak{f}[X(t)] = m(t), \qquad t \in T.$$

REMARK. The equivalence operator F can be replaced by $\sqrt{F^*F}$, so that F can be assumed to be a positive definite self-adjoint operator.

Given a C. O. N. S. $\{f_k\}$, we shall define the Hilbert-Schmidt norm of a bounded linear operator F by

(2.4) $$\| F \|_{H.S.} = \sqrt{\sum_k \| Ff_k \|^2};$$

it is well-known that the right side is independent of the choice of $\{f_k\}$, and so $\| F \|_{H.S.}$ is well defined. It is evident that F is of Hilbert-Schmidt type if and only if $\| F \|_{H.S.} < +\infty$. The following lemma will be useful later.

LEMMA 1.

(i) *If F is of Hilbert-Schmidt type, then*

(2.5) $$\sum_k \| Ff_k \|^2 \leqq \| F \|_{H.S.}^2$$

for any O. N. S. $\{f_k\}$.

(ii) *Suppose that $\mathcal{H}_n$, $n = 1, 2, 3, \cdots$, be an increasing sequence of finite dimensional subspaces of a Hilbert space $\mathcal{H}$ such that $\mathcal{H}$ is the least closed linear manifold containing all $\mathcal{H}_n$'s. Let $\{f_i^n; i = 1, 2, \cdots, N_n\}$ be a C. O. N. S. in $\mathcal{H}_n$ for each $n = 1, 2, 3, \cdots$. Then*

$$\|F\|_{H.S.}^2 = \sup_n \sum_{i=1}^{N_n} \|Ff_i^n\|^2 . \tag{2.6}$$

PROOF. (i) is clear by the definition of $\|F\|_{H.S.}$. To prove (ii), let $\{f_i\}$ be a C. O. N. S. in $\mathscr{H}$ such that $\{f_i, i=1, 2, \cdots, N_n\}$ spans $\mathscr{H}_n$ for each n. Writing f_i as $f_i = \sum a_{ij}^n f_j^n$, then $(a_{ij}^n)_{i,j=1}^{N_n}$ will be an orthogonal $N_n \times N_n$ matrix.

$$\begin{aligned} \|F\|_{H.S.}^2 &= \sup_n \sum_{i=1}^{N_n} \|Ff_i\|^2 \\ &= \sup_n \sum_{i=1}^{N_n} \sum_{j=1}^{N_n} \sum_{k=1}^{N_n} a_{ij}^n a_{ik}^n \langle Ff_j^n, Ff_k^n \rangle \\ &= \sup_n \sum_{j=1}^{N_n} \sum_{k=1}^{N_n} \sum_{i=1}^{N_n} a_{ij}^n a_{ik}^n \langle Ff_j^n, Ff_k^n \rangle \\ &= \sup_n \sum_{j=1}^{N_n} \|Ff_j^n\|^2 . \end{aligned} \tag{2.7}$$

Noting the fact that the Gaussian measure on $(\boldsymbol{R}^T, \mathscr{B})$ is completely determined by its mean function and its covariance function, we can derive the following theorem immediately from Theorem 1.

THEOREM 2. *$\{X_1(t, w)\}$ is equivalent to $\{X(t, w)\}$ if and only if $\{X_1(t, w)\}$ has a representation*

$$X_1(t, w) \underset{(L)}{=} FX(t, w) + \mathfrak{f}[X(t, w)] \tag{2.8}$$

with an equivalence operator F and a bounded linear functional $\mathfrak{f}$ on $L^2(X)$.

REMARK 1. "$\underset{(L)}{=}$" means the two stochastic processes yield the same probability measure on $(\boldsymbol{R}^T, \mathscr{B})$.

REMARK 2. F can be assumed to be positive definite selfadjoint (see the remark after Theorem 1).

§ 3. Gaussian processes equivalent to the Brownian motion.

We call a Gaussian process *B-equivalent*, if it is equivalent to the Brownian motion $\{B(t, w); t \in T\}$, $0 \in T$. Let $L^2(B)$ be the Hilbert space spanned by $\{B(t)\}$ with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$ as in Section 2. Then every element Z of $L^2(B)$ is expressed in the form

$$Z = \int_T F(u) dB(u) , \tag{3.1}$$

where $F(u)$ is a real function defined on T satisfying

$$\int_T |F(u)|^2 du < +\infty . \tag{3.2}$$

From Theorem 2, we can prove that every *B-equivalent* process has a representation

$$X_1(t, w) \underset{(L)}{=} FB(t, w) + \mathfrak{f}[B(t, w)], \qquad t \in T, \tag{3.3}$$

where $FB(t, w)$ should be of the form

$$\int_T F(t, u) dB(u, w),$$

and we have $\mathfrak{f}[B(t, w)] = m(t)$, $t \in T$.

In this section, we shall determine a condition for the *B-equivalence* of $\{X_1(t, w)\}$ in terms of kernel function $F(t, u)$ and $m(t)$.

First we prove two lemmas.

DEFINITION. Let $\mathcal{H}$ be a Hilbert space and $Z(t)$ be a $\mathcal{H}$-valued function defined on an interval T. Then $Z(t)$ is called $\mathcal{S}$*-absolutely continuous,* if there exists a $\mathcal{H}$-valued function $Z'(s)$ defined for almost all $s \in T$ such that

$$Z(t) - Z(u) = \int_u^t Z'(s) ds, \qquad \text{for every} \quad t, u \in T, \tag{3.4}$$

in sense of Bochner integral and

$$\int_T \| Z'(s) \|^2 ds < +\infty. \tag{3.5}$$

LEMMA 2. *Let K be a linear operator on $L^2(B)$ and put*

$$Z(t) = KB(t), \qquad t \in T. \tag{3.6}$$

Then K is of Hilbert-Schmidt type if and only if $Z(t)$ is $\mathcal{S}$-absolutely continuous.

PROOF. For simplicity, we prove the lemma in case of $T = [0, +\infty)$, since the other cases can be treated in the same way.

Suppose that $Z(t)$ is $\mathcal{S}$*-absolutely continuous* and let

$$\begin{aligned} B_k^n &= \sqrt{2^n}\,[B(t_k^n) - B(t_{k-1}^n)], \\ Z_k^n &= \sqrt{2^n}\,[Z(t_k^n) - Z(t_{k-1}^n)], \end{aligned} \tag{3.7}$$

where $t_k^n = 2^{-n}k$, $k = 0, 1, 2, \cdots, 2^n n$, $n = 1, 2, 3, \cdots$, and let $\mathcal{H}_n$ be the closed linear subspace spanned by $\{B_k^n;\ k = 1, 2, \cdots, 2^n n\}$. Then $\mathcal{H}_n$'s and $L^2(B)$ satisfies the hypothesis of (ii) of Lemma 1 and $\{B_k^n;\ k = 1, 2, \cdots, 2^n n\}$ is a C. O. N. S. in $\mathcal{H}_n$ for each n. From (3.4) and (3.5) and noting that $KB(0) = 0$,

$$\begin{aligned} \sum_{k=1}^{2^n n} \| KB_i^n \|^2 &= \sum_{k=1}^{2^n n} \| Z_k^n \|^2 \\ &= 2^n \sum_k \left\| \int_{t_{k-1}^n}^{t_k^n} Z'(s) ds \right\|^2 \end{aligned}$$

$$\tag{3.8} \begin{aligned} &\leqq 2^n \sum_k \left| \int_{t_{k-1}^n}^{t_k^n} \| Z'(s) \| ds \right|^2 \\ &\leqq \int_0^n \| Z'(s) \|^2 ds \\ &\leqq \int_T \| Z'(s) \|^2 ds < +\infty . \end{aligned}$$

Hence, by Lemma 1, we see that

$$\| K \|_{H.S.}^2 = \sup_n \sum_{k=1}^{2^n n} \| KB_k^n \|^2 \leqq \int_T \| Z'(s) \|^2 ds < +\infty ,$$

and therefore K is of Hilbert-Schmidt type.

Conversely, suppose that K is of Hilbert-Schmidt type. For every sequence of disjoint intervals (a_k, b_k) in T, define

$$\tag{3.9} B_k = (b_k - a_k)^{-\frac{1}{2}} [B(b_k) - B(a_k)] , \quad k = 1, 2, \cdots .$$

Then $\{B_k\}$ is an O. N. S. in $L^2(B)$. By (i) of Lemma 1,

$$\tag{3.10} \sum_k \| KB_k \|^2 = \sum_k (b_k - a_k)^{-1} \| Z(b_k) - Z(a_k) \|^2 \leqq M ,$$

where $M = \| K \|_{H.S.}^2$.

Hence, for every choice of disjoint intervals, we have

$$\tag{3.11} \begin{aligned} \sum_k \| Z(b_k) - Z(a_k) \| &= \sum_k (b_k - a_k)^{\frac{1}{2}} (b_k - a_k)^{-\frac{1}{2}} \| Z(b_k) - Z(a_k) \| \\ &\leqq \left[\left\{ \sum_k (b_k - a_k) \right\} \left\{ \sum_k (b_k - a_k)^{-1} \| Z(b_k) - Z(a_k) \|^2 \right\} \right]^{\frac{1}{2}} \\ &\leqq \sqrt{M} [\sum_k (b_k - a_k)]^{\frac{1}{2}} . \end{aligned}$$

Let $\{\varphi_j\}$ be a C. O. N. S., and let

$$\tag{3.12} z_j(t) = \langle Z(t), \varphi_j \rangle , \quad j = 1, 2, 3, \cdots .$$

Then by (3.11), for every choice of disjoint intervals, we have

$$\tag{3.13} \begin{aligned} \sum_k | z_j(b_k) - z_j(a_k) | &= \sum_k | \langle Z(b_k) - Z(a_k), \varphi_j \rangle | \\ &\leqq \sum_k \| Z(b_k) - Z(a_k) \| \leqq \sqrt{M} [\sum_k (b_k - a_k)]^{\frac{1}{2}} , \end{aligned}$$

so that $z_j(t)$ is absolutely continuous in t. Noting that $Z(0) = KB(0) = 0$, we have

$$\tag{3.14} z_j(t) = \int_0^t z_j'(s) ds , \quad j = 1, 2, \cdots ,$$

where $z_j'(s)$ is the density, which is defined for almost all $s \in T$.

Let n be any positive integer and put

$$(3.15)\qquad z_j^n(t)=\begin{cases} 2^n\displaystyle\int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} z_j'(s)ds, & \left(\begin{array}{l}\frac{k-1}{2^n}\leqq t<\frac{k}{2^n}\\ k=1,2,\cdots,2^n n\end{array}\right)\\ 0, & t\geqq n,\qquad j=1,2,\cdots.\end{cases}$$

Then, by Lebesgue's theorem we have

$$(3.16)\qquad \lim_n z_j^n(t)=z_j'(t),\qquad \text{for every } t\in T-N_j,$$

where N_j is a null set; N_j can be taken independently of j, since $\bigcup_j N_j$ is also a null set. Hence, by Fatou's lemma and (3.10), we have

$$(3.17)\qquad \begin{aligned}\int_T\sum_{j=1}^{+\infty} z_j'(s)^2ds &\leqq \liminf_n \int_T\sum_j z_j^n(s)^2ds\\ &=\liminf\sum_{k=1}^{2^n n}\sum_{j=1}^{+\infty}2^n\Big[z_j\Big(\frac{k}{2^n}\Big)-z_j\Big(\frac{k-1}{2^n}\Big)\Big]^2\\ &=\liminf\sum_k 2^n\Big\|Z\Big(\frac{k}{2^n}\Big)-Z\Big(\frac{k-1}{2^n}\Big)\Big\|^2\leqq M<+\infty.\end{aligned}$$

Put

$$(3.18)\qquad Z'(s)=\sum_{j=1}^{+\infty}z_j'(s)\varphi_j.$$

Then, by (3.17), $Z'(s)$ is a $L^2(B)$-valued function defined for almost all $s\in T$ and we have

$$(3.19)\qquad \int_T\|Z'(s)\|^2ds=\int_T\sum_j z_j'(s)^2ds<+\infty.$$

Therefore the Bochner integral $\int_0^t Z'(s)ds$ exists, and from (3.12) and (3.14), it follows that

$$(3.20)\qquad \langle Z(t)-\int_0^t Z'(s)ds,\ \varphi_j\rangle=0,$$

for each $j=1,2,3,\cdots$. (3.19) and (3.20) imply (3.4) and (3.5) and therefore $Z(t)$ is *$\mathcal{S}$-absolutely continuous.*

Thus we have proved the lemma.

LEMMA 3. *In order that there exists a bounded linear functional $\mathfrak{f}$ in $L^2(B)$ with $\mathfrak{f}[B(t)]=m(t)$, it is necessary and sufficient that $m(t)$ is absolutely continuous in t and that*

$$(3.21)\qquad \int_T m'(s)^2ds<+\infty,$$

where $m'(s)$ is its density.

PROOF. If such $\mathfrak{f}$ exists, then $\mathfrak{f}$ can be written as $\mathfrak{f}(\cdot)=\langle\cdot,Y\rangle$ by Riesz-

Fisher theorem. Let $(a_k, b_k) = 1, 2, \cdots$, be any system of disjoint intervals in T. Then

$$\sum_k |m(b_k)-m(a_k)| = \sum_k |\langle B(b_k)-B(a_k), Y\rangle|$$
$$= \sum_k \sqrt{(b_k-a_k)}|\langle B_k, Y\rangle| \leqq \sqrt{\sum_k (b_k-a_k)}\sqrt{\sum_k \langle B_k, Y\rangle^2}$$

where B_k's are defined in (3.9). Noting that $\{B_k\}$ is an O. N. S. in $L(B)$, we can see that

$$\sum_k \langle B_k, Y\rangle^2 \leqq \| Y \|^2 .$$

Therefore $m(t)$ is absolutely continuous in t. The rest of the proof is the same as that of Lemma 2.

THEOREM 3. *$\{X_1(t, w)\}$ is B-equivalent if and only if it has a representation*

$$X_1(t, w) \underset{(L)}{=} B(t, w) + \int_T \int_0^t g(v, u) dv dB(u, w) + \int_0^t m'(u) du , \tag{3.22}$$

where $g(v, u)$ and $m'(u)$ are real functions which satisfy the following conditions (C.1)-(C.3) *and* (3.21).

$$\int_T \int_T g(v, u)^2 dv du < +\infty . \tag{C.1}$$

(C.2) *The linear operator F determined by*

$$FB(t) = B(t) + \int_T \int_0^t g(v, u) dv dB(u) , \qquad t \in T . \tag{3.23}$$

is invertible.

$$g(v, u) = g(u, v) , \qquad \textit{for almost all } (v, u) \in T \times T . \tag{C.3}$$

PROOF. If $\{X_1(t, w)\}$ is *B-equivalent*, then it has a representation (2.8) of Theorem 2. By Remark 2 after Theorem 2, we may assume that F is a self-adjoint equivalence operator. Since $F-I$ is of Hilbert-Schmidt type, by Lemma 2, $Z(t) = (F-I)B(t)$ is *$\mathcal{S}$-absolutely continuous*. Let

$$Z'(s) = \int_T g(s, u) dB(u) \tag{3.24}$$

be its density. Then from (3.5), we have

$$\int_T \| Z'(s) \|^2 ds = \int_T \int_T g(v, u)^2 dv du < +\infty . \tag{3.25}$$

Hence, we have

$$F[B(t)] = B(t) + \int_T \int_0^t g(v, u) dv dB(u) , \qquad t \in T , \tag{3.26}$$

and the invertibility of an equivalence operator implies (C.2). (C.3) immediately derives from the self-adjointness of F.

From Lemma 3 and the fact that $B(0)=0$, it follows that $m(t)=\mathfrak{f}B(t)$ has the form

$$m(t)=\int_0^t m'(u)du\,, \qquad t\in T\,, \tag{3.27}$$

with $m'(u)$ satisfying (3.12).

Thus we have proved the necessity of the theorem. The sufficiency can easily be proved in the same manner.

NOTE 1. As we mentioned in Remark 2 after Theorem 2, Theorem 3 is valid even if (C.3) is omitted.

NOTE 2. (C.2) is not an elegant condition, but we have two different sufficient conditions (3.28) and (3.29), each of which implies (C.2):

$$\int_T\int_T g(v,u)^2 dvdu<1\,. \tag{3.28}$$

(3.29) The representation appeared in the right side of (3.23) is proper canonical (T. Hida [**4**]).

In the considerations above, we viewed the Wiener measure on $(\boldsymbol{R}^T, \mathscr{B})$. However, the Wiener measure is also a measure on the space of continuous functions $(\boldsymbol{C}, \mathscr{B}_C)$, where $\mathscr{B}_C$ is the σ-algebra generated by the cylinder sets. Using Kolmogorov-Prokhorov's theorem [**5**], the process $\{X_1(t, w)\}$ in (3.22) has a continuous version, because we have

$$E_1|X_1(t)-X_1(s)|^4\leqq cE|B(t)-B(s)|^2=3c|t-s|^2 \tag{3.30}$$

with some constant c by virtue of the boundedness of F and $\mathfrak{f}$. Therefore P can be considered as a measure on $(\boldsymbol{C}, \mathscr{B}_C)$ and $\mathfrak{F}=F+\mathfrak{f}$ will give a linear transformation from $(\boldsymbol{C}, \mathscr{B}_C)$ into itself which transforms the Wiener measure P on $(\boldsymbol{C}, \mathscr{B}_C)$ to the measure P_1 on $(\boldsymbol{C}, \mathscr{B}_C)$.

EXAMPLE 1. Let $\{U(t, w)\}$ be the Ornstein-Uhlenbeck's Brownian motion on $(\boldsymbol{C}, \mathscr{B}_C)$ where T is the interval $[0, 1]$. Then a process $\{U(t, w)-\exp(-t)U(0, w)\}$ is *B-equivalent*.

In fact, this process has the proper canonical representation

$$\begin{aligned} &U(t,w)-\exp(-t)U(0,w)\\ &=\int_0^t \exp(-t+u)dB(u,w) \\ &=B(t,w)-\int_0^t\int_u^t \exp(-v+u)dvdB(u,w)\,, \qquad t\in T\,. \end{aligned} \tag{3.31}$$

This is the case where $g(v, u)$ and $m'(u)$ in (3.22) have the form:

$$g(v, u)=\begin{cases} \exp(-v+u), & \text{if } 1\geqq v\geqq u\geqq 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$m'(u)=0, \qquad u\in T.$$

This example shows that the path of the Ornstein-Uhlenbeck's Brownian motion and that of the Brownian motion (Wiener process) have the same local continuity.

§ 4. Processes equivalent to C-processes.

A process with zero mean is called a *C-process*, if it has a proper canonical representation with respect to the Brownian motion $\{B(t, w)\}$, that is, $X(t)$ can be expressed in the form

$$X(t)=\int^{t} c(t, u)dB(u), \qquad t\in T, \tag{4.1}$$

where $c(t, u)$ is the proper canonical kernel (T. Hida [4]) satisfying

$$\int_{T}|c(t, u)|^{2}du<+\infty, \qquad t\in T, \tag{4.2}$$

and $\{B(t, w)\}$ is the Brownian motion such that

$$L^{2}(X)=L^{2}(B). \tag{4.3}$$

It is well-known that a purely non-deterministic stationary Gaussian process is a C-process.

In this section, we investigate a necessary and sufficient condition imposed on the linear transformation F and functional $\mathfrak{f}$ on $L^{2}(X)$ for which a Gaussian process is equivalent to a given C-process, when $T=[0, T_{1}]$ or $(-\infty, +\infty)$.

THEOREM 4. *A Gaussian process* $\{X_{1}(t, w)\}$ *is equivalent to the C-process which has a proper canonical representation* (4.1) *if and only if there exists a B-equivalent process* $\{Y(t, w)\}$ *which has the representation* (3.22) *and* $\{X_{1}(t, w)\}$ *has the representation*

$$\begin{aligned} X_{1}(t, w) &\underset{(L)}{=}\int^{t} c(t, u)dY(u, w) \\ &=\int^{t} c(t, u)dB(u, w)+\int_{T}\int^{t} c(t, z)g(z, u)dz dB(u, w) \\ &\quad +\int^{t} c(t, u)m'(u)du, \qquad t\in T. \end{aligned} \tag{4.4}$$

PROOF. If $\{X_{1}(t, w)\}$ is equivalent to the C-process represented as (4.1), then by Theorem 2, $\{X_{1}(t, w)\}$ has a representation (2.8) with the equivalence operator F and the bounded linear functional $\mathfrak{f}$. By (4.3), Lemma 2 and Lemma 3, there exist real functions $g(v, u)$ and $m'(u)$ satisfying the conditions of Theorem 3 such that

$$FB(t, w)=B(t, w)+\int_T\int_0^t g(v, u)dvdB(u, w),$$

$$\mathfrak{f}[B(t, w)]=\int_0^t m'(u)du, \qquad t\in T.$$

Put

$$Y(t, w)=FB(t, w)+\mathfrak{f}[B(t, w)], \qquad t\in T.$$

Then by Theorem 3, $\{Y(t, w)\}$ is *B-equivalent*. By the boundedness of F and $\mathfrak{f}$, we get

$$(4.5) \qquad FX(t, w)=F\Big[\int^t c(t, u)dB(u, w)\Big]$$

$$=\int^t c(t, u)\Big\{dB(u, w)+\int_T g(u, z)dB(z, w)du\Big\}, \qquad t\in T,$$

$$(4.6) \qquad \mathfrak{f}[X(t, w)]=\mathfrak{f}\Big[\int^t c(t, u)dB(u, w)\Big]$$

$$=\int^t c(t, u)m'(u)du, \qquad t\in T.$$

Therefore, $\{X_1(t, w)\}$ has the representation (4.4).

Similarly we can prove the converse.

EXAMPLE 2. (See Example 1 in Section 3.) The Brownian motion $\{B(t, w)\}$ is equivalent to a C-process the proper canonical representation of which is given by (3.31) for $T=[0, 1]$.

In fact, $\{B(t, w)\}$ has a representation

$$(4.7) \qquad B(t, w)=\int_0^t \exp(-t+u)dB(u, w)+\int_T\int_u^t \exp(-t+z)dzdB(u, w).$$

This is the case where

$$g(v, u)=\begin{cases} 1, & \text{if } 1\geqq v\geqq u\geqq 0, \\ 0, & \text{otherwise,}\end{cases}$$

and $m'(u)\equiv 0$.

§ 5. Concluding remarks.

(1) Equivalence of two additive processes.

A Gaussian additive process with mean zero and $T=[0, T_1]$, (T_1 may be infinite), has a representation

$$(5.1)\qquad X=(t, w)=X(0, w)+\int_0^t c(u)dB(u, w)+\sum_{t_j\leq t} a_j Y_{t_j}(w).$$

(See Corollary of Theorem 1.6 of T. Hida [4].) Here $L^2(X)$ can be decomposed as

$$(5.2)\qquad L^2(X)=L^2(B)\oplus[\sum_{t_j\in T}\oplus M(Y_{t_j})]\oplus M(X(0)),$$

where Y_{t_j}'s are O. N. S. of $L^2(X)$, a_j's are real constants, $c(u)$ is a real function such that

$$\sum_{t_j\leq t} a_j^2+\int_0^t c(u)^2du<+\infty, \qquad \text{for every } t\in T,$$

and $M(Y)$, $Y\in L^2(X)$, denotes the closed linear subspace of $L^2(X)$ spanned by Y.

Let $L_t^2(X)$ be the closed linear subspace of $L^2(X)$ spanned by $\{X(s);\ s\leqq t\}$.

Now suppose that a Gaussian process $\{X_1(t, w)\}$ is equivalent to an additive process expressed in the form (5.1). Then by Theorem 2, it has a representation (2.8) where the equivalence operator F can be assumed to be a self-adjoint operator. This equivalence operator F is reduced by $L_t^2(X)$ for every $t\in T$ if and only if $\{X_1(t, w)\}$ is also an additive process, in fact,

$$(5.3)\qquad \begin{aligned}&\langle F[X(t)-X(s)], FX(u)\rangle\\ &\quad=\text{Covariance}[X_1(t, w)-X_1(s, w), X_1(u, w)],\ t\geqq s\geqq u,\end{aligned}$$

and $F^*F=F^2$ and F are reduced by $L_t^2(X)$ at the same time. If F is reduced by $L_t^2(X)$ for every $t\in T$, then it is reduced by $L^2(B)$, $M(X(0))$ and all $M(Y_{t_j})$'s by their definition (see T. Hida [4]). Determine real constants α, α_j's, m, m_j's and functions $g(v, u)$, $m'(u)$ by the equalities

$$(5.4)\qquad \begin{aligned}&FX(0)=\alpha X(0), \qquad FY_{t_j}=\alpha_j Y_{t_j},\\ &\mathfrak{f}[X(0)]=m, \qquad \mathfrak{f}[Y_{t_j}]=m_j,\\ &FB(t, w)=B(t, w)+\int_T\int_0^t g(v, u)dvdB(u, w),\\ &\mathfrak{f}[B(t, w)]=\int_0^t m'(u)du.\end{aligned}$$

Since $\{FB(t, w)\}$ is also an additive process, $g(v, u)\equiv 0$. Noting that $F-I$ is of Hilbert-Schmidt type and $\mathfrak{f}$ is a bounded linear functional, we have the following proposition.

PROROSITION 1. *A Gaussian additive process* $\{X_1(t, w)\}$ *is equivalent to the Gaussion additive process* $\{X(t, w)\}$ *expressed in the form* (5.1) *if and only if it has the following representation*

(5.5)
$$X_1(t, w) \underset{(L)}{=} \alpha X(0, w) + \int_0^t c(u) dB(u, w) + \sum_{t_j \leqq t} \alpha_j a_j Y_{t_j}(w)$$
$$+ m + \int_0^t c(u) m'(u) du + \sum_{t_j \leqq t} a_j m_j , \qquad t \in T ,$$

where α, α_j's, m, m_j's *are real constants such that*

(5.6)
$$\sum_{t_j \in T} (\alpha_j - 1)^2 < +\infty ,$$

(5.7)
$$\sum_{t_j \in T} m_j^2 < +\infty ,$$

α *and* α_j's *are non-vanishing, and* $m'(u)$ *is a real function satisfying* (3.21).

This proposition enables us to extend the Skorokhod [**6**]'s results on the equivalence of two Gaussian additive processes.

(2) **On the general case.**

Let $\{X(t, w)\}$ be a process with mean zero and $T = [0, +\infty)$ and put

(5.8)
$$N(X) = \bigcap_{t \in T} L_t^2(X) .$$

Then $\{X(t, w)\}$ has a representation

(5.9)
$$X(t, w) = \sum_i \int_0^t c_i(t, u) dB_i(u, w) + \sum_{t_j \leqq t} \sum_{q=1}^{N_j} b_j^q(t) Y_{t_j}^q(w)$$
$$+ \sum_k a_k(t) h_k(w) , \qquad t \in T ,$$

where $\{B_i(t, w)\}$'s are mutually independent Brownian motions and $Y_{t_j}^q(w)$'s are O. N. S. of $L^2(X)$ such that

(5.10)
$$L^2(X) = N(X) \oplus \{ \sum_i^{N_j} \oplus L^2(B_i) \} \oplus \{ \sum_{t_j \in T} \sum_{q=1}^{N_j} \oplus M(Y_{t_j}^q) \} ,$$

$h_k(w)$'s are C. O. N. S. of $N(X)$, and $c_i(t, u)$'s, $b_j^q(t)$'s and $a_k(t)$'s are real functions such that

(5.11)
$$\sum_i \int_0^t c_i(t, u)^2 du + \sum_{t_j \leqq t} \sum_{q=1}^{N_j} b_j^q(t)^2 + \sum_k a_k(t)^2 < +\infty ,$$

for every $t \in T$ (T. Hida [**4**]).

If we define an equivalence operator F and a bounded linear functional $\mathfrak{f}$ on $L^2(X)$ in the same manner as in (5.4), then we have the following proposition.

PROPOSITION 2. *A Gaussian process* $\{X_1(t, w)\}$ *is equivalent to the Gaussian process* $\{X(t, w)\}$ *expressed in the form* (5.9) *if it has a representation*

$$(5.12)\qquad X_1(t, w) \underset{(L)}{=} \sum_i \int_0^t c_i(t, u)\Big\{dB_i(u, w) + \int_T g_i(u, z)dB_i(z, w)du\Big\}$$
$$+ \sum_{t_j \leqq t} \sum_{q=1}^{N_j} \beta_j^q b_j^q(t) Y_{t_j}^q(w) + \sum_k \alpha_k a_k(t) h_k(w)$$
$$+ \sum_i \int_0^t c_i(t, u) m_i'(u) du + \sum_{t_j \leqq t} \sum_{q=1}^{N_j} b_j^q(t) m_j^b + \sum_k a_k(t) n_k, \qquad t \in T,$$

where β_j^q*'s and* α_k*'s are non-vanishing real constants and* $g_i(v, u)$*'s are real functions such that*

$$(5.13)\qquad \sum_i \int_T \int_T g_i(v, u)^2 dv du + \sum_j \sum_q (\beta_j^q - 1)^2 + \sum_k (\alpha_k - 1)^2 < +\infty,$$

and m_j^q*'s and* n_k*'s are real constants, and* $m_i'(u)$*'s are real functions such that*

$$(5.14)\qquad \sum_i \int_T m'(u)^2 du + \sum_j \sum_q (m_j^q)^2 + \sum_k (n_k)^2 < +\infty,$$

and the linear operators F_i; $i = 1, 2, \cdots$, *on* $L^2(B_i)$ *determined by*

$$(5.15)\qquad F_i B_i(t, w) = B_i(t, w) + \int_T \int_0^t g_i(v, u) dv dB_i(u, w), \qquad t \in T,$$

are invertible.

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