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Holomorphic imbeddings of symmetric domains

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The purpose of this paper is to determine all (equivariant) holomorphic imbeddings of a symmetric domain D into another symmetric domain D'; a part of results has been announced in [3] without proofs.

In the case D' is of type $(\mathrm{III})_p$ or of type $(\mathrm{I})_{p,q}$, this problem was solved completely (and partially in the case D' is of type $(\mathrm{II})_p$) by Satake in his paper [4]. Our methods are similar to those adopted in [4], but depend further on general properties of Lie algebras. Our results are essentially applicable to any cases.

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be the Lie algebras of the groups of all analytic automorphisms of D and D' respectively. Then the problem is equivalent to that of finding all monomorphisms of $\mathfrak{g}$ into $\mathfrak{g}'$ satisfying a condition called (H_1) in [4]. Therefore we shall consider a slightly generalized problem as was done in [4], that is to determine all *homomorphisms* of a Lie algebra *of hermitian type* into another satisfying (H_1). A more precise exposition of our problem will be given in § 1.

We shall make reductions of the problem in § 2. Namely, if we find all *regular subalgebras* (see 2.3) of $\mathfrak{g}'$, and if we determine all pairs $(\mathfrak{g}, \rho)$ of a Lie algebra of hermitian type and a homomorphism of $\mathfrak{g}$ into a regular subalgebra $\mathfrak{g}_1'$ of $\mathfrak{g}'$ satisfying a certain condition (H_2) stronger than (H_1), we shall get all solutions; moreover, we shall be able to assume that both $\mathfrak{g}$ and $\mathfrak{g}'$ are simple. The determination of regular subalgebras of each non-compact simple Lie algebra of hermitian type will be done in § 4. For a simple Lie algebra $\mathfrak{g}'$ of type $(\mathrm{I})_{p,q}$, $(\mathrm{II})_p$, or $(\mathrm{III})_p$, all pairs $(\mathfrak{g}, \rho)$ of a non-compact simple Lie algebra and a homomorphism into $\mathfrak{g}'$ satisfying (H_2) can be determined by combining the results tabulated in [4] and our results in § 4; for the remaining cases $\mathfrak{g}' = (\mathrm{IV})_p$, (EIII), or (EVII), they are determined in § 5. Our § 3 is devoted to the preparations to § 5. In the Appendix, we shall refer to some results, supplementary to those given by Satake in [5], about the correspondence of boundary components by holomorphic imbeddings of symmetric domains.

Throughout this paper, the usual symbols $(\mathrm{I})_p$, $(\mathrm{II})_p$, $(\mathrm{III})_p$, $(\mathrm{IV})_p$, (EIII), and (EVII) for the irreducible symmetric domains will be also used to denote the corresponding simple Lie algebras, and the symbols $(A_l), (B_l), \cdots$ for the complex simple Lie algebras to denote their compact real forms.

On account of the well known isomorphisms between simple Lie algebras, one may restrict the parameters as follows:

$$(\mathrm{I})_{p,q}:\ 1 \leqq p \leqq q\,, \qquad (\mathrm{II})_p:\ p \geqq 5\,.$$
$$(\mathrm{III})_p:\ p \geqq 2\,, \qquad (\mathrm{IV})_p:\ p \geqq 5\,.$$

§ 1. Exposition of the problem.

1.1. Let D and D' be symmetric domains. A *holomorphic imbedding* of D into D' is a holomorphic isometry φ of D into D' such that $\varphi(D)$ is totally geodesic in D'. Let G (resp. G') be the connected component of the identity of the group of all analytic automorphisms. Two holomorphic imbeddings φ_1 and φ_2 of D into D' are said to be *equivalent* if there is an element g of G' such that $\varphi_2 = g \circ \varphi_1$. (For convenience, we shall not say equivalent when g is an analytic automorphism not contained in G'.) The problem is precisely to find out, for a given symmetric domain D', all equivalence classes (defined naturally) of pairs (D, φ) of a symmetric domain D and a holomorphic imbedding φ of D into D'. Hence, if o and o' are origins fixed in D and D' respectively, we may assume that

$$\varphi(o) = o'\,. \tag{1}$$

If $\mathfrak{g}$ is the Lie algebra of G, one has the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

corresponding to o, where $\mathfrak{k}$ is the (maximal compact) subalgebra of $\mathfrak{g}$, corresponding to the isotropy subgroup at o. The vector space $\mathfrak{p}$ is identified with the tangent space to D at o, and has a complex structure J which agrees with that of D. Similarly we have $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$ and J'. For a holomorphic imbedding φ of D into D' satisfying (1), we have a monomorphism ρ of $\mathfrak{g}$ into $\mathfrak{g}'$ such that

$$\rho(\mathfrak{k}) \subset \mathfrak{k}'\,, \qquad \rho(\mathfrak{p}) \subset \mathfrak{p}' \tag{2}$$

and

$$\rho \circ J = J' \circ \rho \quad \text{on} \quad \mathfrak{p}\,. \tag{3}$$

Conversely, for a monomorphism ρ saitisfying (2) and (3), there corresponds a holomorphic imbedding satisfying (1).

On the other hand, there is the uniquely determined element H_0 (resp. H_0') in the center of $\mathfrak{k}$ (resp. $\mathfrak{k}'$) such that

$$J = ad(H_0) \quad \text{on} \quad \mathfrak{p}\,, \qquad J' = ad(H_0') \quad \text{on} \quad \mathfrak{p}'\,. \tag{4}$$

The inner automorphism $\exp \pi ad(H_0)$ (resp. $\exp \pi ad(H_0'))$ of $\mathfrak{g}$ (resp. $\mathfrak{g}'$) gives rise to the involution corresponding to the Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ (resp. $\mathfrak{g}'=\mathfrak{k}'+\mathfrak{p}'$). Thus we have a condition

$$(H_1) \qquad \rho \circ ad(H_0) = ad(H_0') \circ \rho$$

which is equivalent to (2) and (3) taken together.

Clearly the problem is equivalent to determine all equivalence classes under the group of all inner automorphisms of $\mathfrak{g}'$ of monomorphisms ρ of $\mathfrak{g}$ into $\mathfrak{g}'$ satisfying (H_1). We shall investigate the problem in this manner, but actually in a slightly generalized form similar to that treated in [4].

1.2. A semi-simple Lie algebra over $\boldsymbol{R}$ is *of hermitian type* if all the non-compact simple components of it correspond to symmetric domains (i. e. the center of a maximal compact subalgebra of each non-compact simple component has the dimension 1). Let $\mathfrak{g}$ be a Lie algebra of hermitian type, and $\mathfrak{k}$ a maximal compact subalgebra of $\mathfrak{g}$. Let further

$$\mathfrak{g}=\mathfrak{g}_0+\sum_{i=1}^{d}\mathfrak{g}_i \tag{5}$$

be a decomposition of $\mathfrak{g}$ into the direct sum of ideals, where $\mathfrak{g}_0$ is compact and $\mathfrak{g}_i$ $(1 \leqq i \leqq d)$ are simple and non-compact, and put

$$\mathfrak{k}_i=\mathfrak{g}_i \cap \mathfrak{k} \qquad (1 \leqq i \leqq d).$$

Hence $\mathfrak{k}=\mathfrak{g}_0+\sum_{i=1}^{d}\mathfrak{k}_i$. To each pair $(\mathfrak{g}_i, \mathfrak{k}_i)$ $(1 \leqq i \leqq d)$, there corresponds a (simply connected) symmetric space D_i and an element H_{0i} of the center of $\mathfrak{k}_i$ which determines a complex structure of D_i so as to make it a symmetric domain. Actually, H_{0i} is unique within the sign $\pm$. For simplicity, we shall say that the element of $\mathfrak{g}$ defined by

$$H_0=\sum_{i=1}^{d}H_{0i} \tag{6}$$

is a *complex structure of the pair* $(\mathfrak{g}, \mathfrak{k})$. If $\mathfrak{g}$ is compact, clearly $H_0=0$. It is easy to see that a complex structure H_0 of $(\mathfrak{g}, \mathfrak{k})$ is contained in the center of $\mathfrak{k}$ and the inner automorphism $\exp \pi ad(H_0)$ of $\mathfrak{g}$ is the Cartan involution of $(\mathfrak{g}, \mathfrak{k})$. Let $\mathfrak{g}'$ be another Lie algebra of hermitian type, $\mathfrak{k}'$ its maximal compact subalgebra, and H_0' a complex structure of $(\mathfrak{g}', \mathfrak{k}')$. The condition (H_1) may be stated for any *homomorphism* ρ of $\mathfrak{g}$ into $\mathfrak{g}'$ and complex structures H_0 and H_0'. Then our problem will be as follows:

For a given Lie algebra $\mathfrak{g}'$ of hermitian type, determine all equivalence classes of pairs $(\mathfrak{g}, \rho)$ of a Lie algebra $\mathfrak{g}$ of hermitian type and a homomorphism ρ of $\mathfrak{g}$ into $\mathfrak{g}'$ satisfying (H_1) (for given H_0 and H_0' w.r.t. fixed maximal compact subalgebras $\mathfrak{k}$ and $\mathfrak{k}'$).

Since $\exp \pi\, ad(H_0)$ (resp. $\exp \pi\, ad(H_0')$) is the Cartan involution attached to

$\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ (resp. $\mathfrak{g}'=\mathfrak{k}'+\mathfrak{p}'$), a homomorphism satisfying (H_1) also satisfies (2).

1.3. For a semi-simple Lie algebra $\mathfrak{g}$ and a maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, $\mathrm{Int}(\mathfrak{g})$ and K will denote the group of all inner automorphisms of $\mathfrak{g}$ and its subgroup corresponding to $\mathfrak{k}$. We shall say briefly that two subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of $\mathfrak{g}$ are (k)-conjugate if they are conjugate in $\mathfrak{g}$ under an element of K (i. e. there is an element $k\in K$ such that $\mathfrak{g}_2=k(\mathfrak{g}_1)$). Let ρ_1 and ρ_2 be two homomorphisms of a semi-simple Lie algebra $\mathfrak{g}$ into another $\mathfrak{g}'$. We shall say that they are *(k)-equivalent* if there is an element k of K' such that $\rho_2=k\circ\rho_1$.

LEMMA 1. *Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a semi-simple Lie algebra over $\boldsymbol{R}$ with a fixed Cartan decomposition, and $\mathfrak{g}_1=\mathfrak{k}_1+\mathfrak{p}_1$, $\mathfrak{g}_2=\mathfrak{k}_2+\mathfrak{p}_2$ be Cartan decompositions of two semi-simple subalgebras of $\mathfrak{g}$ such that $\mathfrak{k}_i\subset\mathfrak{k}$ and $\mathfrak{p}_i\subset\mathfrak{p}$ $(i=1,2)$. Then $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are conjugate in $\mathfrak{g}$ if and only if they are (k)-conjugate.*

PROOF. The *only if* part should be shown. Since $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are conjugate in $\mathfrak{g}$, there is an element s_0 in $\mathrm{Int}(\mathfrak{g})$ such that $\mathfrak{g}_2=s_0(\mathfrak{g}_1)$. Put $\mathfrak{k}_2'=s_0(\mathfrak{k}_1)$ and $\mathfrak{p}_2'=s_0(\mathfrak{p}_1)$. By the conjugacy of Cartan decompositions, we have an element t in $\mathrm{Int}(g_2)$ such that $t(\mathfrak{k}_2')=\mathfrak{k}_2$ and $t(\mathfrak{p}_2)=\mathfrak{p}_2$. Let s_1 be the element in the subgroup of $\mathrm{Int}(\mathfrak{g})$ corresponding to $\mathfrak{g}_2$ such that its restriction on $\mathfrak{g}_2$ is t, and put $s=s_1s_0$. Then,

$$s(\mathfrak{g}_1)=\mathfrak{g}_2\,,\qquad s(\mathfrak{k}_1)=\mathfrak{k}_2\,,\qquad s(\mathfrak{p}_1)=\mathfrak{p}_2\,. \tag{7}$$

On the other hand, one knows that

$$s=kv\,, \tag{8}$$

where k is a uniquely determined element in K and

$$v=\exp ad(X)\,,\qquad X\in\mathfrak{p}\,.$$

By (7) and (8), we have

$$v(\mathfrak{k}_1)\subset\mathfrak{k}\,,\qquad v(\mathfrak{p}_1)\subset\mathfrak{p}\,. \tag{9}$$

Putting $v_1=\cosh ad(X)$ and $v_2=\sinh ad(X)$, we have

$$\begin{aligned} v_1(\mathfrak{k}_1)&\subset\mathfrak{k}\,, & v_1(\mathfrak{p}_1)&\subset\mathfrak{p}\,,\\ v_2(\mathfrak{k}_1)&\subset\mathfrak{p}\,, & v_2(\mathfrak{p}_1)&\subset\mathfrak{k}\,, \end{aligned} \tag{10}$$

and $v=v_1+v_2$. Since $v_2(\mathfrak{k}_1)=v_2(\mathfrak{p}_1)=0$ by (9) and (10), the subalgebra $\mathfrak{g}_1=\mathfrak{k}_1+\mathfrak{p}_1$ is contained in the kernel of semi-simple linear transformation $v_2=\sinh ad(X)$ whose eigen-values are altogether real, hence also in the kernel of $ad(X)$. Thus the inner automorphism v acts on $\mathfrak{g}_1$ as the identity, and we see that

$$s(\mathfrak{g}_1)=k(\mathfrak{g}_1)\,, \qquad \text{q. e. d.}$$

PROPOSITION 1. *Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ and $\mathfrak{g}'=\mathfrak{k}'+\mathfrak{p}'$ be Lie algebras of hermitian types with fixed Cartan decompositions. Suppose that two homomorphisms ρ_1*

and ρ_2 satisfy (H_1) for same complex structures H_0 of $(\mathfrak{g}, \mathfrak{k})$ and H_0' of $(\mathfrak{g}', \mathfrak{k}_1)$. Then ρ_1 and ρ_2 are equivalent (in $\mathfrak{g}'$) if and only if they are (k)-equivalent.

PROOF. Let $s = kv$ ($k \in K'$, $v \in \exp ad(\mathfrak{p}')$) be an element of $\mathrm{Int}(\mathfrak{g}')$ such that $\rho_2 = s \circ \rho_1$. Clearly, $s(\rho_1(\mathfrak{k})) = \rho_2(\mathfrak{k}) \subset \mathfrak{k}'$. Then, we can prove by the same way as in the proof of Lemma 1 that v acts on $\rho_1(\mathfrak{g})$ as the identity, and we have $\rho_2 = k \circ \rho_1$. q. e. d.

It is easy to see that the condition (H_1) is invariant under (k)-equivalence, hence our problem is equivalent *to determining all (k)-equivalence classes of* $(\mathfrak{g}, \rho)$.

1.4. Let $\mathfrak{g}$ be a Lie algebra of hermitian type, and $\mathfrak{k}$ a (fixed) maximal compact subalgebra. It is known that there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{k}$. All such Cartan subalgebras are mutually (k)-conjugate, and can be regarded as those of $\mathfrak{k}$. We denote by $\mathfrak{g}_C$ and $\mathfrak{q}_C$ the complexifications of $\mathfrak{g}$ and a vector subspace $\mathfrak{q}$ of it. One knows that the root system $\mathfrak{r}$ of $\mathfrak{g}_C$ relative to $\mathfrak{h}_C$ is the disjoint union of two subsets $\mathfrak{u}$ and $\mathfrak{v}$ consisting of so-called *compact roots* and *non-compact roots* respectively, i. e.

$$\mathfrak{k}_C = \mathfrak{h}_C + \sum_{\alpha \in \mathfrak{u}} \mathfrak{g}_\alpha , \qquad \mathfrak{p}_C = \sum_{\alpha \in \mathfrak{v}} \mathfrak{g}_\alpha , \tag{11}$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_C \mid ad(H)X = \alpha(H)X \text{ for every } H \in \mathfrak{h}_C\}$ (see e. g. [**2**]). Moreover, one knows that $\alpha(H_0) = 0$ or $\pm\sqrt{-1}$ according as $\alpha \in \mathfrak{u}$ or $\alpha \in \mathfrak{v}$. Put

$$\mathfrak{v}_\pm = \{\alpha \in \mathfrak{r} \mid \alpha(H_0) = \pm\sqrt{-1}\} , \tag{12}$$

and take an order of $\mathfrak{r}$ so as to make all the roots of $\mathfrak{v}_+$ positive. Let further Π be the fundamental root system w. r. t. this order, and

$$\Pi = \Pi_0 \cup \Pi_1 \cup \cdots \cup \Pi_d \tag{13}$$

the decomposition of Π into a union of *ideals*; where Π_0 contains only compact roots, and Π_i $(1 \leqq i \leqq d)$ has a connected Dynkin diagram in which there is one and only one non-compact positive root. The decomposition (13) corresponds actually to (5). In fact, each Π_i $(0 \leqq i \leqq d)$ is a fundamental system of roots of the ideal $\mathfrak{g}_{iC}$ of $\mathfrak{g}_C$ relative to $\mathfrak{h}_{iC} = \mathfrak{h}_C \cap \mathfrak{g}_{iC}$.

If $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_l\}$, we have a basis $H_1, \cdots, H_l$ of the vector space $\sqrt{-1}\mathfrak{h}$ over $\boldsymbol{R}$ such that

$$\alpha_i(H_j) = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} . \tag{14}$$

In other words by the usual identification, $H_j = 2\alpha_j / \langle \alpha_j, \alpha_j \rangle$. In the following §§, we shall also put $H_\alpha = 2\alpha / \langle \alpha, \alpha \rangle$ for every root $\alpha \in \mathfrak{r}$, and regard it as an element of $\sqrt{-1}\mathfrak{h}$.

Since $\sqrt{-1}^{-1} H_0 \in \sqrt{-1}\mathfrak{h}$, we can express it as a linear combination of $H_1, \cdots, H_l$ with real coefficients. We shall give here such expressions of

$\sqrt{-1}^{-1}H_0$ for all the simple Lie algebras of hermitian types. The element α_1 in Π will always be the unique non-compact simple root, hence $\alpha_1(\sqrt{-1}^{-1}H_0)=1$ and $\alpha_i(\sqrt{-1}^{-1}H_0)=0$ $(i\geqq 2)$. On the other hand, putting

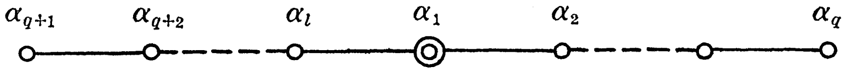
$$a_{ij}=\alpha_i(H_j)\,, \qquad \frac{1}{\sqrt{-1}}H_0=\sum_{j=1}^{l} r_j H_j\,,$$

we have

$$\alpha_i\Big(\frac{1}{\sqrt{-1}}H_0\Big)=\sum_{j=1}^{l} a_{ij} r_j \qquad (1\leqq i\leqq l)\,.$$

Since the Cartan matrix (a_{ij}) is non-singular, this equation in r_j can be solved. Now we have the following list:

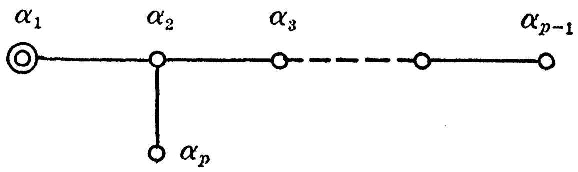
$(\mathrm{I})_{p,q}$ $(1\leqq p\leqq q)$: $[\mathfrak{k},\mathfrak{k}]\cong(A_{p-1})\times(A_{q-1})$.

I: 

$(l=p+q-1)$.

$$\frac{1}{\sqrt{-1}}H_0=\frac{1}{p+q}(qH_{q+1}+2qH_{q+2}+\cdots+(p-1)qH_l+pqH_1+p(q-1)H_2+\cdots+pH_q)\,.$$

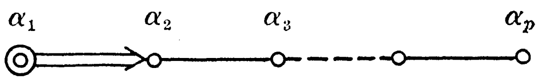
$(\mathrm{II})_p$ $(p\geqq 5)$: $[\mathfrak{k},\mathfrak{k}]\cong(A_{p-1})$.

II: 

.

$$\frac{1}{\sqrt{-1}}H_0=\frac{1}{4}(pH_1+2(p-2)H_2+2(p-3)H_3+\cdots+2H_{p-1}+(p-2)H_p)\,.$$

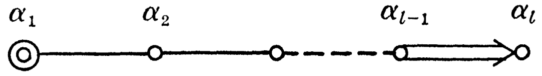
$(\mathrm{III})_p$ $(p\geqq 2)$: $[\mathfrak{k},\mathfrak{k}]\cong(A_{p-1})$.

II: 

$$\frac{1}{\sqrt{-1}}H_0=\frac{1}{2}(pH_1+(p-1)H_2+\cdots+2H_{p-1}+H_p)\,.$$

$(\mathrm{IV})_p$ $(p\geqq 5)$:

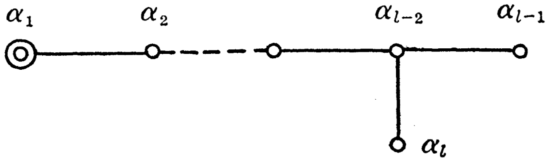
$p=$ odd: $[\mathfrak{k},\mathfrak{k}]\cong(B_{l-1})$, $l=\dfrac{p+1}{2}$.

II: 

.

$$\frac{1}{\sqrt{-1}}H_0=\frac{1}{2}(2H_1+2H_2+\cdots+2H_{l-1}+H_l)\,.$$

$p=$ even: $[\mathfrak{k},\mathfrak{k}]\cong(D_{l-1})$, $l=\dfrac{p}{2}+1$.

Π: 

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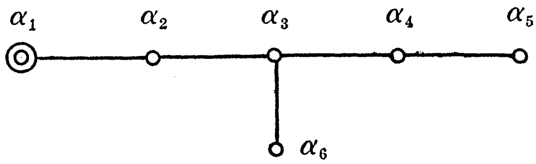
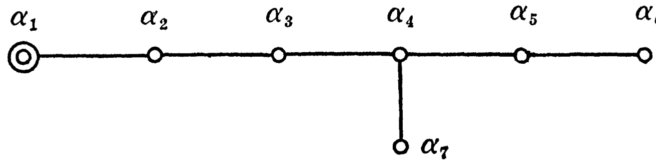
$$\frac{1}{\sqrt{-1}}H_0=\frac{1}{2}(2H_1+\cdots+2H_{l-2}+H_{l-1}+H_l)\,.$$

(EIII): $[\mathfrak{k},\mathfrak{k}]\cong(D_5)$.

Π: 

$$\frac{1}{\sqrt{-1}}H_0=\frac{1}{3}(4H_1+5H_2+6H_3+4H_4+2H_5+3H_6)\,.$$

(EVII): $[\mathfrak{k},\mathfrak{k}]\cong(E_6)$.

Π: 

$$\frac{1}{\sqrt{-1}}H_0=\frac{1}{2}(3H_1+4H_2+5H_3+6H_4+4H_5+2H_6+3H_7)\,.$$

§ 2. Reduction of the problem.

2.1. Let the notations $\mathfrak{g}, \mathfrak{g}', \cdots, H_0, H_0'$ be as before. The following condition (H_2) about a homomorphism ρ of $\mathfrak{g}$ into $\mathfrak{g}'$, which is stronger than (H_1), will play an important rôle for the reduction of our problem:

$$(H_2) \qquad \rho(H_0)=H_0'\,.$$

Now that all the Cartan subalgebras of $\mathfrak{g}'$ in $\mathfrak{k}'$ are mutually (k)-conjugate, any homomorphism satisfying (H_1) must be (k)-equivalent to a homomorphism ρ such that

$$(15) \qquad \rho(\mathfrak{h})\subset\mathfrak{h}'\,.$$

Hence, in the following, we shall always assume without mentioning specifically that a homomorphism ρ satisfying (H_1) or (H_2) also satisfies (15).

2.2. Let $\mathfrak{g}_C$ be a complex semi-simple Lie algebra, $\mathfrak{g}_u$ a compact real form of $\mathfrak{g}_C$. The usual complex conjugation acts on $\mathfrak{g}_C$ by

$$Z=X+\sqrt{-1}\,Y\to\bar{Z}=X-\sqrt{-1}\,Y \qquad (X,\,Y\in\mathfrak{g}_u)\,.$$

It is well known that, twisting this action in the following manner, we have all (sets of real points of) real forms of $\mathfrak{g}_C$: if τ_u is an involutive automorphism of $\mathfrak{g}_C$ which preserves $\mathfrak{g}_u$ and if we put

$$f_a = \begin{cases} 1 & (a = id.) \\ \tau_u & (a \neq id.) \end{cases} \quad \text{for} \quad a \in \mathrm{Aut}_{\boldsymbol{R}}(\boldsymbol{C}), \tag{16}$$

we can define another action of $\mathrm{Aut}_{\boldsymbol{R}}(\boldsymbol{C})$ on $\mathfrak{g}_C$ by

$$[a](X) = f_a(\bar{X}). \tag{17}$$

The set of '$\boldsymbol{R}$-rational points' $\mathfrak{g}$ of $\mathfrak{g}_C$ corresponding to this action is usually called a *real form* of $\mathfrak{g}_C$, and $\sigma = [a]$ $(a \neq id.)$ a *conjugation* of $\mathfrak{g}_C$ w.r.t. $\mathfrak{g}$. Let $\mathfrak{g}'_C$ be another complex semi-simple Lie algebra with a real form $\mathfrak{g}'$, σ' the conjugation of $\mathfrak{g}'_C$ w.r.t. $\mathfrak{g}'$. A homomorphism ρ of $\mathfrak{g}_C$ into $\mathfrak{g}'_C$ is defined over $\boldsymbol{R}$, or in other words ρ induces a homomorphism of $\mathfrak{g}$ into $\mathfrak{g}'$, if and only if

$$\rho \circ \sigma = \sigma' \circ \rho. \tag{18}$$

Suppose that the compact real form $\mathfrak{g}'_u$ of $\mathfrak{g}'_C$ is taken as $\rho(\mathfrak{g}_u) \subset \mathfrak{g}'_u$. Then, in view of the relations (16) and (17), the condition (18) is equivalent to

$$\rho \circ \tau_u = \tau'_u \circ \rho. \tag{19}$$

Now, let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (resp. $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$) be a Lie algebra of hermitian type with a Cartan decomposition, H_0 (resp. H'_0) a complex structure of $(\mathfrak{g}, \mathfrak{k})$ (resp. $(\mathfrak{g}', \mathfrak{k}')$). Taking the compact duals of $\mathfrak{g}$ and $\mathfrak{g}'$ for $\mathfrak{g}_u$ and $\mathfrak{g}'_u$ respectively, i.e.

$$\mathfrak{g}_u = \mathfrak{k} + \sqrt{-1}\,\mathfrak{p}, \qquad \mathfrak{g}'_u = \mathfrak{k}' + \sqrt{-1}\,\mathfrak{p}',$$

we can see that the conjugations of $\mathfrak{g}_C$ and $\mathfrak{g}'_C$ w.r.t. $\mathfrak{g}$ and $\mathfrak{g}'$ are defined as above by

$$\tau_u = \exp \pi\, ad_C(H_0) \quad \text{and} \quad \tau'_u = \exp \pi\, ad_C(H'_0) \tag{20}$$

respectively, where ad_C denotes the adjoint representations of the complex Lie algebras $\mathfrak{g}_C$ and $\mathfrak{g}'_C$. If ρ is a homorphism of $\mathfrak{g}_C$ into $\mathfrak{g}'_C$ such that $\rho(\mathfrak{g}_u) \subset \mathfrak{g}'_u$ and that

$$\rho \circ ad_C(H_0) = ad_C(H'_0) \circ \rho, \tag{21}$$

it satifies the condition (19), hence also (18). Then the image of $\mathfrak{g}$ by ρ is in $\mathfrak{g}'$, and according to (21) the restriction of ρ to $\mathfrak{g}$ satisfies (H_1). Conversely, if ρ is a homomorphism of $\mathfrak{g}$ into $\mathfrak{g}'$ satisfying (H_1), its extension to that of $\mathfrak{g}_C$ into $\mathfrak{g}'_C$ carries $\mathfrak{g}_u$ into $\mathfrak{g}'_u$ and has the property (21). Thus we have proved the following

PROPOSITION 2. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be of hermitian type. Then a homomorphism ρ of $\mathfrak{g}_C$ into $\mathfrak{g}'_C$ induces one of $\mathfrak{g}$ into $\mathfrak{g}'$ satisfying (H_1) (resp. (H_2)) if and only if $\rho(\mathfrak{g}_u) \subset \mathfrak{g}'_u$ and $\rho \circ ad_C(H_0) = ad_C(H'_0) \circ \rho$* (resp. $\rho(H_0) = H'_0$), *where* $\mathfrak{g}_u$ (resp. $\mathfrak{g}'_u$) *denotes the compact dual of* $\mathfrak{g}$ (resp. $\mathfrak{g}'$).

2.3. In this paragraph, we shall define a certain type of subalgebras of a Lie algebra of hermitian type.

Let the notations be as in 1.4. A subset Δ of the root system $\mathfrak{r}$ is called by Dynkin a Π-system if it has the following properties:

(i) If $\alpha = \Delta$ and $\beta \in \Delta$, $\alpha-\beta$ is not in $\mathfrak{r}$.

(ii) Δ is a linearly independent system in $\sqrt{-1}\mathfrak{h}$.

If Δ is a Π-system, the subset $\mathfrak{g}_C(\Delta)$ of $\mathfrak{g}_C$ defined by

$$\mathfrak{g}_C(\Delta) = \sum_{\alpha\in\Delta} \boldsymbol{C}H_\alpha + \sum_{\alpha\in\mathfrak{r}(\Delta)} \mathfrak{g}_\alpha\,; \quad \mathfrak{r}(\Delta) = (\sum_{\alpha\in\Delta} \boldsymbol{Z}\alpha) \cap \mathfrak{r}\,, \tag{22}$$

is a *regular* semi-simple subalgebra of $\mathfrak{g}_C$; $\mathfrak{r}(\Delta)$ can be considered as the root system of $\mathfrak{g}_C(\Delta)$ relative to the Cartan subalgebra $\sum_{\alpha\in\Delta} \boldsymbol{C}H_\alpha$, in which the Π-system Δ forms a simple root system ([1], Chap. II).

A Π-system, however, will be needed in this paper, only when it has moreover the following property:

(iii) Each connected component of the Dynkin diagram of Δ contains at most one positive non-compact root besides compact roots.

Therefore we shall say that *a subset Δ of $\mathfrak{r}$ is a Π-system if it has the property* (iii) *in addition to* (i) *and* (ii).

PROPOSITION 3. *Let Δ be a Π-system in $\mathfrak{r}$. Then the regular subalgebra $\mathfrak{g}_C(\Delta)$ of $\mathfrak{g}_C$ is defined over $\boldsymbol{R}$ whose real form $\mathfrak{g}(\Delta) = \mathfrak{g}_C(\Delta) \cap \mathfrak{g}$ is a Lie algebra of hermitian type, and there is a unique complex structure of* $(\mathfrak{g}(\Delta), \mathfrak{k}(\Delta))$ *by which the injection of* $\mathfrak{g}(\Delta)$ *into* $\mathfrak{g}$ *satisfies* (H_1), *where* $\mathfrak{k}(\Delta) = \mathfrak{g}(\Delta) \cap \mathfrak{k}$.

PROOF. Let $\Delta = \Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_d$ be the decomposition of Δ into a disjoint union, where Δ_0 contains only compact roots, and $\Delta_i = \{\gamma, \beta_1, \cdots, \beta_t\}$ $(1 \leqq i \leqq d)$ has a connected Dynkin diagram, in which γ denotes the unique positive non-compact root. Then all the roots in $\mathfrak{r}(\Delta_i)$ $(\subset \mathfrak{r})$ are of the form

$$\pm\Big(\gamma + \sum_{j=1}^{t} c_j\beta_j\Big) \in \mathfrak{v}_\pm \quad \text{or} \quad \pm\sum_{j=1}^{t} d_j\beta_j \in \mathfrak{u}\,, \tag{23}$$

where c_j and d_j are non-negative integers. Define an element H_{0i} of $\mathfrak{g}_C(\Delta_i)$ by

$$\frac{1}{\sqrt{-1}}\gamma(H_{0i}) = 1\,, \qquad \frac{1}{\sqrt{-1}}\beta_j(H_{0i}) = 0 \qquad (1 \leqq j \leqq t)\,, \tag{24}$$

and an element $H_{0\Delta}$ of $\mathfrak{g}_C(\Delta)$ by

$$H_{0\Delta} = \sum_{i=1}^{d} H_{0i}\,. \tag{25}$$

In view of (12), (23), (24), and (25), we have

$$\alpha(H_{0\Delta}) = \alpha(H_0) \qquad \text{for} \quad \alpha \in \mathfrak{r}(\Delta)\,. \tag{26}$$

Now, let $\mathfrak{g}_u(\Delta)$ be a compact real form of $\mathfrak{g}_C(\Delta)$ contained in $\mathfrak{g}_u$, i.e.

$$\mathfrak{g}_u(\Delta) = \mathfrak{g}_C(\Delta) \cap \mathfrak{g}_u\,.$$

The equations (24) and (25) show that $H_{0\Delta}$ is an element of the center of the intersection $\mathfrak{k}(\Delta)$ of $\mathfrak{g}_u(\Delta)$ and $\mathfrak{k}$, and that the inner automorphism $\tau_{u\Delta}$ of $\mathfrak{g}_C(\Delta)$ defined by

$$\tau_{u\Delta} = \exp \pi ad_C(H_{0\Delta})$$

is involutive and keeps $\mathfrak{g}_u(\Delta)$ invariant. The real form $\mathfrak{g}(\Delta)$ of $\mathfrak{g}_C(\Delta)$ defined by $\tau_{u\Delta}$ is a Lie algebra of hermitian type, in which $\mathfrak{k}(\Delta)$ is a maximal compact subalgebra, and $H_{0\Delta}$ is a complex structure of $(\mathfrak{g}(\Delta), \mathfrak{k}(\Delta))$. Since the equation (26) means that the injection ι of $\mathfrak{g}_C(\Delta)$ into $\mathfrak{g}_C$ satisfies

$$\iota \circ ad_C(H_{0\Delta}) = ad_C(H_0) \circ \iota\,,$$

the proposition follows from Proposition 2, q. e. d.

A subalgebra of a Lie algebra $\mathfrak{g}$ of hermitian type will also be called in the following a *regular subalgebra* (of $\mathfrak{g}$), if it is a real form $\mathfrak{g}(\Delta)$ of a regular semi-simple subalgebra $\mathfrak{g}_C(\Delta)$ of $\mathfrak{g}_C$, as defined in Proposition 3 for a Π-system Δ in our sense. Two regular subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of $\mathfrak{g}$ (w. r. t. the fixed Cartan subalgebra) will be called *(k)-equivalent* if they are (k)-conjugate. (Note that our definition of a regular subalgebra depends on the choice of a Cartan subalgebra in $\mathfrak{k}$.)

2.4. In this paragraph, we shall give a condition of (k)-equivalence between regular subalgebras of a Lie algebra of hermitian type.

Let $\mathfrak{g}$ be a Lie algebra of hermitian type. Let further $\mathfrak{k}$, $\mathfrak{h}$, $\mathfrak{r}$, and Π be same things as in 1.4, K the subgroup of $\mathrm{Int}(\mathfrak{g})$ corresponding to $\mathfrak{k}$. The subgroup W_K of the Weyl group of $\mathfrak{g}_C$ generated by all 'reflections' associated to compact roots can be considered as the Weyl group of the reductive Lie algebra $\mathfrak{k}$: a reflection w_β (in the dual vector space of $\sqrt{-1}\,\mathfrak{h}$) associated to a root β is defined by

$$w_\beta(\alpha) = \alpha - \alpha(H_\beta)\beta \qquad (\alpha \in \mathfrak{r})\,. \tag{27}$$

One knows that W_K is isomorphic to the quotient group of the normalizer $N_K(\mathfrak{h})$ of $\mathfrak{h}$ in K by the centralizer $Z_K(\mathfrak{h})$ of $\mathfrak{h}$ in K. Now that an element of W_K carries a compact root to a compact root, positive non-compact root to a positive non-compact root respectively, every image of a Π-system by it must be also a Π-system.

Now we shall show the following

THEOREM 1. *Let $\mathfrak{g}$ be a Lie algebra of hermitian type, $\mathfrak{k}$ a maximal compact subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ in $\mathfrak{k}$. Take an order of the root system $\mathfrak{r}$ of $\mathfrak{g}_C$ relative to $\mathfrak{h}_C$ w.r.t. a complex structure of $(\mathfrak{g}, \mathfrak{k})$, and let Δ_1, Δ_2 be Π-systems in $\mathfrak{r}$ w.r.t. this order. Then the regular subalgebras $\mathfrak{g}(\Delta_1)$ and $\mathfrak{g}(\Delta_2)$ of $\mathfrak{g}$ are conjugate in $\mathfrak{g}$, if and only if there is an element w of the Weyl group W_K of $\mathfrak{k}$ such that $\Delta_2 = w(\Delta_1)$.*

By Lemma 1, it is easy to see that two regular subalgebras are mutually conjugate if and only if they are (k)-equivalent.

To begin with, we establish some lemmas.

LEMMA 2. *Two subalgebras* $\mathfrak{h}_1$ *and* $\mathfrak{h}_2$ *of* $\mathfrak{g}$ *contained in* $\mathfrak{h}$ *are* (k)*-conjugate, if and only if there is an element* k *of* $N_K(\mathfrak{h})$ *such that* $\mathfrak{h}_2 = k(\mathfrak{h}_1)$.

PROOF. Suppose that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are (k)-conjugate, and let k be an element of K such that $\mathfrak{h}_2 = k(\mathfrak{h}_1)$. Then we have another Cartan subalgebra $\mathfrak{h}^* = k(\mathfrak{h})$ of $\mathfrak{k}$, and $\mathfrak{h}_2$ is contained in $\mathfrak{h} \cap \mathfrak{h}^*$. The normalizer of $\mathfrak{h} \cap \mathfrak{h}^*$ in $\mathfrak{k}$ is a direct sum

$$\mathfrak{n}_{\mathfrak{k}}(\mathfrak{h} \cap \mathfrak{h}^*) = \mathfrak{a} + \mathfrak{k}_0 \tag{28}$$

of an abelian subalgebra $\mathfrak{a}$ and a compact semi-simple subalgebra $\mathfrak{k}_0$. The Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}^*$ are clearly in $\mathfrak{n}_{\mathfrak{k}}(\mathfrak{h} \cap \mathfrak{h}^*)$, hence there are Cartan subalgebras $\mathfrak{h}_0$ and $\mathfrak{h}_0^*$ of $\mathfrak{k}_0$ such that

$$\mathfrak{h} = \mathfrak{a} + \mathfrak{h}_0, \qquad \mathfrak{h}^* = \mathfrak{a} + \mathfrak{h}_0^*. \tag{29}$$

Since all Cartan subalgebras of a compact semi-simple Lie algebra are mutually conjugate, there is an element k_1 of a subgroup of $\mathrm{Int}(\mathfrak{g})$ corresponding to $\mathfrak{k}_0$ such that

$$k_1(\mathfrak{h}_0^*) = \mathfrak{h}_0, \qquad k_1 = id. \text{ on } \mathfrak{a}, \qquad k_1 \mid \mathfrak{k}_0 \in \mathrm{Int}(\mathfrak{k}_0). \tag{30}$$

Since the intersection $\mathfrak{h} \cap \mathfrak{h}^*$ is contained in $\mathfrak{a}$, the automorphism k_1 preserves each element of $\mathfrak{h}_2$. Therefore the composition $k_1 k$ is an element of $N_K(\mathfrak{h})$ (by (29) and (30)), which maps $\mathfrak{h}_1$ onto $\mathfrak{h}_2$, q. e. d.

LEMMA 3. *Let* $\mathfrak{g}$ *be a non-compact simple Lie algebra of hermitian type,* $\mathfrak{k}$ *a maximal compact subalgebra of* $\mathfrak{g}$, *and let* $\mathfrak{h}$ *be a Cartan subalgebra of* $\mathfrak{g}$ *contained in* $\mathfrak{k}$. *Take an order of the root system of* $\mathfrak{g}_C$ *relative to* $\mathfrak{h}_C$ *as above for a fixed complex structure of* $(\mathfrak{g}, \mathfrak{k})$. *Then for each fundamental system* $\{\beta_1, \beta_2, \cdots, \beta_{l-1}\}$ *of compact roots of* $\mathfrak{g}_C$, *there is a uniquely determined positive non-compact root* γ *such that* $\{\gamma, \beta_1, \cdots, \beta_{l-1}\}$ *becomes a fundamental root system of* $\mathfrak{g}_C$.

PROOF. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ be the simple root system of $\mathfrak{g}_C$ w. r. t. the fixed order, in which α_1 is the simple non-compact root. Then $\{\alpha_2, \alpha_3, \cdots, \alpha_l\}$ becomes a fundamental system of compact roots, hence there is an element w of W_K such that $w(\{\alpha_i\}) = \{\beta_j\}$. Thus we have a positive non-compact root $\gamma = w(\alpha_1)$, which is required. Suppose that we get another positive non-compact root γ_1 such that $\{\gamma_1, \beta_1, \cdots, \beta_{l-1}\}$ becomes a fundamental root system. Then it is easy to see that the following equations should be held:

$$\gamma_1 = \gamma + \sum_{i=1}^{l-1} c_i \beta_i, \qquad \gamma = \gamma_1 + \sum_{j=1}^{l-1} d_j \beta_j,$$

where c_i, d_j are all non-negative integers. But since $\beta_1, \cdots, \beta_{l-1}$ are linearly independent, we can see at once

$$c_i = d_i = 0, \qquad \text{and hence}$$

$$\gamma_1 = \gamma, \qquad \text{q. e. d.}$$

PROOF OF THEOREM 1. Suppose that there is an element $w \in W_K$, such that $\Delta_2 = w(\Delta_1)$. Then we have an element k of $N_K(\mathfrak{h})$ such that

$$k\left(\frac{1}{\sqrt{-1}} H_\alpha\right) = \frac{1}{\sqrt{-1}} H_{w(\alpha)} \qquad (\alpha \in \mathfrak{r}). \tag{31}$$

Extending k linearly to $\mathfrak{g}_C$, we can see that $k(\mathfrak{g}_\alpha) = \mathfrak{g}_{w(\alpha)}$, where $\mathfrak{g}_\alpha$ (resp. $\mathfrak{g}_{w(\alpha)}$) is the eigen space of the adjoint representation belonging to a root α (resp. $w(\alpha)$). Thus we have

$$k(\mathfrak{g}_C(\Delta_1)) = \mathfrak{g}_C(\Delta_2). \tag{32}$$

Since k preserves $\mathfrak{g}$, (32) means that $\mathfrak{g}(\Delta_2) = k(\mathfrak{g}(\Delta_1))$. Conversely, let $\mathfrak{g}(\Delta_1)$ and $\mathfrak{g}(\Delta_2)$ be conjugate in $\mathfrak{g}$. Then they are (k)-equivalent, i.e. there is an element k of K such that $k(\mathfrak{g}(\Delta_1))$ is equal to $\mathfrak{g}(\Delta_2)$. Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be the Cartan subalgebras of $\mathfrak{g}(\Delta_1)$ and $\mathfrak{g}(\Delta_2)$ generated by $\{\sqrt{-1}^{-1}H_\alpha : \alpha \in \Delta_1\}$ and $\{\sqrt{-1}^{-1}H_\alpha : \alpha \in \Delta_2\}$ respectively; both of them are contained in $\mathfrak{h}$. Then $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are (k)-conjugate, since $k(\mathfrak{h}_1)$ is, as well as $\mathfrak{h}_2$, a Cartan subalgebra of $\mathfrak{g}(\Delta_2)$ contained in $\mathfrak{k}(\Delta_2)$. Hence, by Lemma 2, there is an element k_0 of $N_K(\mathfrak{h})$ such that $\mathfrak{h}_2 = k_0(\mathfrak{h}_1)$. Taking w_0 in W_K which corresponds to k_0 modulo $Z_K(\mathfrak{h})$, we have $k_0(H_\alpha) = H_{w_0(\alpha)}$ for all $\alpha \in \mathfrak{r}$; in particular,

$$w_0(\mathfrak{r}(\Delta_1)) = \mathfrak{r}(\Delta_2). \tag{33}$$

Therefore we have another fundamental system $w_0(\Delta_1)$ of $\mathfrak{r}(\Delta_2)$ besides Δ_2. Using Lemma 3, it is easy to see that there is an element w in the subgroup of W_K generated by w_β where β runs over all compact roots in Δ_2, such that $ww_0(\Delta_1) = \Delta_2$, q. e. d.

2.5. The following theorem corresponds to the Proposition 1 in [4], when the Lie algebra $\mathfrak{g}'$ in our problem is $(\mathrm{III})_p$. It will give a reduction of our problem.

THEOREM 2. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras of hermitian type, $\mathfrak{k}$ and $\mathfrak{k}'$ maximal compact subalgebras of $\mathfrak{g}$ and $\mathfrak{g}'$, and let H_0 and H_0' be complex structures of $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{g}', \mathfrak{k}')$ respectively. Let further ρ be a homomorphism of $\mathfrak{g}$ into $\mathfrak{g}'$ satisfying (H_1) w.r.t. H_0 and H_0'. Then there is a regular subalgebra $\mathfrak{g}''$ of $\mathfrak{g}'$ such that the image $\rho(\mathfrak{g})$ is contained in $\mathfrak{g}''$ and that $\rho: \mathfrak{g} \to \mathfrak{g}''$ satisfies (H_2) with respect to H_0 and the natural complex structure of $(\mathfrak{g}'', \mathfrak{k}'')$ defined by H_0'. In particular, $\mathfrak{g}''$ can be taken as a proper subalgebra if $\rho: \mathfrak{g} \to \mathfrak{g}'$ does not satisfies (H_2).*

PROOF. The condition (H_1) implies that the image $\rho(\mathfrak{g})$ is contained in the centralizer $\mathfrak{g}_1$ of $H_0'-\rho(H_0)$ in $\mathfrak{g}'$. Clearly $\mathfrak{g}_1=\mathfrak{g}'$, if and only if ρ satisfies (H_2). Let $\mathfrak{r}'$ be the root system of $\mathfrak{g}_C'$ relative to $\mathfrak{h}_C'$, and $\mathfrak{r}''$ the subset of $\mathfrak{r}'$ consisting of roots such as

$$\alpha'(H_0'-\rho(H_0))=0\,. \tag{34}$$

It is easily seen that $\mathfrak{r}''$ is a subsystem of $\mathfrak{r}'$, and

$$\mathfrak{g}_{1C}=\mathfrak{h}_C'+\sum_{\alpha'\in\mathfrak{r}''}\mathfrak{g}_{\alpha'}', \qquad \mathfrak{g}_1=\mathfrak{g}_{1C}\cap\mathfrak{g}\,,$$

where $\mathfrak{g}_{\alpha'}'$ is the eigen space of the adjoint representation of $\mathfrak{h}_C'$ belonging to $\alpha'\in\mathfrak{r}'$. Let $\mathfrak{g}''$ denote the semi-simple part of $\mathfrak{g}_1$, and put $H_0''=\rho(H_0')$. The image $\rho(\mathfrak{g})$ is semi-simple, so that $\rho(\mathfrak{g})\subset\mathfrak{g}''$. Since $\mathfrak{r}''$ can be considered as the root system of $\mathfrak{g}_C''$ relative to the Cartan subalgebra $\mathfrak{h}_C''=\mathfrak{g}_C''=\cap\mathfrak{h}_C'$, the subalgebra $\mathfrak{g}_u''=\mathfrak{g}_C''\cap\mathfrak{g}_u'$ is a compact real form of $\mathfrak{g}_C''$. The equation (34) implies that H_0'' is an element of the center of $\mathfrak{k}''=\mathfrak{g}_C''\cap\mathfrak{k}'=\mathfrak{g}''\cap\mathfrak{g}_u''$, and that the inner automorphism τ'' of $\mathfrak{g}''$ defined by

$$\tau''=\exp\pi ad(H_0'')$$

is the restriction of the Cartan involution τ' of $(\mathfrak{g}',\mathfrak{k}')$ to $\mathfrak{g}''$. Hence τ'' is a Cartan involution of $\mathfrak{g}''$, and $\mathfrak{k}''$ is the maximal compact subalgebra of $\mathfrak{g}''$ defined by τ'' which contains $\mathfrak{h}''=\mathfrak{h}_C''\cap\mathfrak{g}''$. Therefore the subalgebra $\mathfrak{g}''$ is of hermitian type, and H_0'' is the complex structure of $(\mathfrak{g}'',\mathfrak{k}'')$ by which the homomorphism $\rho:\mathfrak{g}\to\mathfrak{g}''$ and the injection: $\mathfrak{g}''\to\mathfrak{g}'$ satisfy (H_2) and (H_1) respectively. Moreover the equation (34) also implies that a compact (resp. non-compact) root of $\mathfrak{g}_C''$ relative to $\mathfrak{h}_C''$ is (the restriction to $\mathfrak{h}_C''$ of) a compact (resp. non-compact) root in $\mathfrak{r}'$. Hence a simple root system of $\mathfrak{r}''$ w.r.t. a suitable order (by which a root $\alpha'\in\mathfrak{r}''$ satisfying $\alpha'(H_0'')=\sqrt{-1}$ is positive) is a Π-system, q.e.d.

2.6. In virtue of Theorem 2, our study of the problem can be divided generally into two steps: the first is to determine all (k)-equivalence classes of regular subalgebras of a given Lie algebra $\mathfrak{g}'$ of hermitian type; the second is to find out, for a fixed regular subalgebra $\mathfrak{g}''$ of $\mathfrak{g}'$, all equivalence classes of pairs $(\mathfrak{g},\rho)$ of a Lie algebra of hermitian type and a homomorphism of $\mathfrak{g}$ into $\mathfrak{g}''$ satisfying (H_2).

Now, we make here a little more reductions. When we say that ρ is a homomorphism of $\mathfrak{g}$ into $\mathfrak{g}'$ satisfying (H_1), we always assume that H_0 and H_0', complex structures of $(\mathfrak{g},\mathfrak{k})$ and $(\mathfrak{g}',\mathfrak{k}')$ respectively, are given, where $\mathfrak{k}$ (resp. $\mathfrak{k}'$) is a maximal compact subalgebra of $\mathfrak{g}$ (resp. $\mathfrak{g}'$). In the first place,

We may consider only the case $\mathfrak{g}'$ is simple.

In fact, if $\mathfrak{g}'$ is a direct sum

$$\mathfrak{g}'=\sum_{i=1}^{d'}\mathfrak{g}_i'$$

of simple ideals $\mathfrak{g}'_i$, we have a projection ρ_i of ρ and H'_{0i} of H'_0 on $\mathfrak{g}'_i$; it is clear that H'_{0i} is a complex structure of $(\mathfrak{g}_i, \mathfrak{k}'_i)$ $(\mathfrak{k}'_i=\mathfrak{k}'\cap\mathfrak{g}'_i)$, and that the homomorphism ρ_i of $\mathfrak{g}$ into $\mathfrak{g}'$ satisfies (H_1). If ρ satisfies (H_2), each ρ_i satisfies also (H_2). It is easy to see that two homomorphisms $\rho^{(1)}$ and $\rho^{(2)}$ of $\mathfrak{g}$ into $\mathfrak{g}'$ satisfying (H_1) are (k)-equivalent, if and only if the projections $\rho_i^{(1)}$ and $\rho_i^{(2)}$ of them on $\mathfrak{g}'_i$ are (k)-equivalent in $\mathfrak{g}'_i$ for all i.

Secondly, suppose that $\rho:\mathfrak{g}\rightarrow\mathfrak{g}'$ satisfies the stronger condition (H_2), and that $\mathfrak{g}$ is a direct sum of simple ideals $\mathfrak{g}_i$ $(1\leqq i\leqq d)$. By the above remark, we may assume again that $\mathfrak{g}'$ is simple. Let $\rho_{(i)}$ be the restriction of ρ to $\mathfrak{g}_i$, and H_{0i} the projection of H_0 on $\mathfrak{g}_i$. The restriction $\rho_{(i)}$ is a homomorphism of $\mathfrak{g}_i$ into $\mathfrak{g}'$ satisfying (H_1) w. r. t. H_{0i} ana H'_0. Since $\mathfrak{g}_i$'s are simple, some of $\rho_{(i)}$'s are injective and others are trivial. Let $\rho_{(1)}, \cdots, \rho_{(e)}$ $(e\leqq d)$ be injective. Then

$$\rho_{(1)}\oplus\cdots\oplus\rho_{(e)}:\ \mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_e\rightarrow\mathfrak{g}'$$

is a homomorphism satisfying (H_2). If one of $\rho_{(i)}$'s satisfies (H_2), say $\rho_{(1)}(H_{01})=H'_0$, it is easy to see by injectivity that $H_{02}, \cdots, H_{0e}$ are equal to zero, i.e. $\mathfrak{g}_2, \cdots, \mathfrak{g}_e$ are compact. On the other hand, if non of $\rho_{(i)}$'s satisfies (H_2), we see from Theorem 2 that each $\rho_{(i)}$ maps $\mathfrak{g}_i$ into a proper regular subalgebra $\mathfrak{g}'_i$ of $\mathfrak{g}'$ satisfying (H_2). We can see that $[\mathfrak{g}'_i, \mathfrak{g}'_j]=0$ if $i\neq j$[1]. Even if $\mathfrak{g}'_i$ is not simple, we can again apply the above consideration. If $\mathfrak{g}$ is compact and if ρ satisfies (H_2), $\mathfrak{g}'$ must obviously be compact.

Thus our problem has been reduced to the following three:

a) *Determine all (k)-equivalence classes of regular subalgebras of each non-compact simple Lie algebra of hermitian type.*

b) *Find out, for each non-compact simple Lie algebra $\mathfrak{g}'$ of hermitian type, all (k)-equivalence classes of pairs $(\mathfrak{g}, \rho)$ of a simple Lie algebra of hermitian type and a homomorphism of $\mathfrak{g}$ into $\mathfrak{g}'$ satisfying (H_2).*

c) *Let $\mathfrak{g}'_1$ and $\mathfrak{g}'_2$ be non-compact simple regular subalgebras of simple $\mathfrak{g}'$, ρ_1 and ρ_2 representatives of (k)-equivalence classes of homomorphisms of a simple Lie algebra $\mathfrak{g}$ of hermitian type into $\mathfrak{g}'_1$ and $\mathfrak{g}'_2$ satisfying (H_2) respectively. Examine whether $\iota_1\circ\rho_1$ and $\iota_2\circ\rho_2$ are (k)-equivalent in $\mathfrak{g}'$, where ι_i is the injection of $\mathfrak{g}'_i$ into $\mathfrak{g}'$ $(i=1, 2)$.*

The solutions to the problem a) will be given in §4. The problem b) is already solved by Satake in [4] (if the results in it and our certain solutions for a) are combined) when $\mathfrak{g}'$ is $(\mathrm{I})_{p,q}$, $(\mathrm{II})_p$, or $(\mathrm{III})_p$; while for the remaining cases that $\mathfrak{g}'$ is $(\mathrm{IV})_p$, (EIII), or (EVII), it will be done in §5 of this paper. We shall never describe in this paper the solutions to *c*), because they would depend too much on the respective circumstances. But, for each case specified, it will be not difficult to carry it out by applying Theorem 3 in §3.

1) Proof of this property will be given elsewhere.

EXAMPLE. There is a regular subalgebra $\mathfrak{g}''$ of type $(\mathrm{IV})_{2l-2}$ in $\mathfrak{g}'=(\mathrm{IV})_{2l-1}$. Take this $\mathfrak{g}''$ for both of $\mathfrak{g}_1'$ and $\mathfrak{g}_2'$ in *c*). If $\mathfrak{g}=(\mathrm{IV})_{2l-2}$, there are two (k)-equivalence classes of homomorphisms of $\mathfrak{g}$ into $\mathfrak{g}''$ satisfying (H_2) (see 5.1). Let ρ_1 and ρ_2 be their representatives. Then we can see that $\iota\circ\rho_1$ and $\iota\circ\rho_2$ are equivalent in $\mathfrak{g}'$. On the other hand, there is also a regular subalgebra $\mathfrak{g}''$ of type $(\mathrm{IV})_8$ $(l=5)$ in $\mathfrak{g}'=(\mathrm{EIII})$. But, in this case, we can see that $\iota\circ\rho_1$ and $\iota\circ\rho_2$ are not equivalent in $\mathfrak{g}'$. Hence we have two classes of homomorphisms of $(\mathrm{IV})_8$ into (EIII) satisfying (H_1).

§ 3. Some properties of homomorphisms of Lie algebras.

In this section, we prove some propositions which will be needed in § 5. Throughout this section, ρ denotes always (the complexification of) a monomorphism of $\mathfrak{g}$ into $\mathfrak{g}'$ satisfying (H_1), where both $\mathfrak{g}$ and $\mathfrak{g}'$ are non-compact simple Lie algebras of hermitian types; hence we may assume moreover the property (15). Other notations are the same as those used before. Especially,

$$H_\alpha=\frac{2\alpha}{\langle\alpha,\alpha\rangle}\in\sqrt{-1}\,\mathfrak{h}\ (\alpha\in\mathfrak{r}),\qquad H_i=H_{\alpha_i};$$

the root α_1 means always the non-compact simple root. All these notations are used with prime for $\mathfrak{g}'$.

Recall that there are at most two sorts of the length of roots for each simple Lie algebra $\mathfrak{g}_C$. A root of the maximal (resp. minimal) length will be called simply a *longer* (resp. *shorter*) root. Let α (resp. β) be a longer (resp. shorter) root. Then, if a non-compact real form $\mathfrak{g}$ of $\mathfrak{g}_C$ is of hermitian type, one knows that

$$\frac{\langle\alpha,\alpha\rangle}{\langle\beta,\beta\rangle}=\frac{\langle H_\beta,H_\beta\rangle}{\langle H_\alpha,H_\alpha\rangle}=\begin{cases}2\ (\mathfrak{g}=(\mathrm{III})_p,\ (\mathrm{IV})_{2n+1}),\\ 1\ (\text{others}).\end{cases}$$

3.1. Let ϕ and ϕ' be the Killing forms of $\mathfrak{g}_C$ and $\mathfrak{g}_C'$ respectively. Since $\phi'(\rho(X),\rho(Y))$ $(X, Y\in\mathfrak{g}_C)$ is again an invariant non-degenerate bilinear form of simple Lie algebra $\mathfrak{g}_C$, there is a constant c_ρ depending only on ρ such that

$$\phi'(\rho(X),\rho(Y))=c_\rho\phi(X,Y).$$

Therefore we have

$$\langle\rho(X),\rho(Y)\rangle'=c_\rho\langle X,Y\rangle\qquad(X,Y\in\sqrt{-1}\,\mathfrak{h}).\tag{35}$$

Now, since $H_1',\cdots,H_{l'}'$ forms a basis of $\sqrt{-1}\,\mathfrak{h}'$, we can put

$$\rho(H_i)=\sum_{j=1}^{l'}\mu_j^{(i)}H_j'.\tag{36}$$

PROPOSITION 4. *All the coefficients $\mu_j^{(i)}$ of the equation* (36) *are rational integers. In particular, if ρ satisfies (H_2),*

$$\mu_1^{(i)}=0 \quad for \quad i\geqq 2.$$

PROOF. Let λ_j' be the dominant weight defined by $\lambda_j'(H_k')=\delta_{jk}$, the usual Kronecker's delta. Then we have

$$\lambda_j'\circ\rho(H_i)=\mu_j^{(i)}. \tag{37}$$

If Λ_j is the irreducible representation of $\mathfrak{g}_C'$ defined by λ_j', the linear form $\lambda_j'\circ\rho$ on $\mathfrak{h}_C$ is a weight of the representation $\Lambda_j\circ\rho$ of $\mathfrak{g}_C$ relative to $\mathfrak{h}_C$. Therefore the equation (37) implies that $\mu_j^{(i)}$'s must be rational integers. If $i\geqq 2$ (resp. $j\geqq 2$), α_i (resp. α_j') is a compact root; hence we have $\alpha_i(H_0)=0$ (resp. $\alpha_j'(H_0')=0$), or equivalent to saying $\langle H_i, \sqrt{-1}\,H_0\rangle=0$ (resp. $\langle H_j', \sqrt{-1}\,H_0'\rangle'=0$). If ρ satisfies (H_2), we have therefore

$$\begin{aligned}\mu_1^{(i)}&=\langle\rho(H_i), \sqrt{-1}\,H_0'\rangle'=\langle\rho(H_i), \sqrt{-1}\,\rho(H_0)\rangle'\\&=c_\rho\langle H_i, \sqrt{-1}\,H_0\rangle=0,\end{aligned}$$ q. e. d.

3.2. A root α is said to be *strongly orthogonal to* another root β, if $\alpha\pm\beta$ are not roots. A subset S of the root system $\mathfrak{r}$ of $\mathfrak{g}_C$ relative to $\mathfrak{h}_C$ is called a *strongly orthogonal system* of roots, if any root in S is strongly orthogonal to all the other roots in S. Each connected component of a strongly orthogonal system contains only one root, so it is a Π-system (in our sense). One knows that there is a strongly orthogonal system Δ_0 consisting of positive non-compact roots $\gamma_1, \gamma_2, \cdots, \gamma_r$, where r is the dimension of a maximal abelian subalgebra contained in $\mathfrak{p}$ of $\mathfrak{g}$. The dimension r is called the *rank* of the pair $(\mathfrak{g}, \mathfrak{k})$. The regular subalgebra $\mathfrak{g}(\Delta_0)$ is isomorphic to a direct sum of r-copies of 3-dimensional Lie algebras of type $(\mathrm{I})_{1,1}$; hence, in view of Proposition 3, we have a monomorphism

$$\kappa:\ \underbrace{(\mathrm{I})_{1,1}+\cdots+(\mathrm{I})_{1,1}}_{r\text{-copies}}\longrightarrow\mathfrak{g}$$

satisfying (H_1), called a *Hermann map* (see [**5**]).

One knows that all the roots in Δ_0 are longer roots; γ_1 may be considered as the non-compact simple root α_1, and γ_r the highest root γ. We denote by $\gamma_j', r', \Delta_0', \cdots$ the similar things belonging to $\mathfrak{g}'$.

In the rest of this paper, we shall denote by E_α a basis of $\mathfrak{g}_\alpha$ for each root α of $\mathfrak{g}_C$ (and similarly by $E_{\alpha'}'$ for α' of $\mathfrak{g}_C'$) such that

$$\bar{E}_\alpha=-E_{-\alpha}, \qquad [E_\alpha, E_{-\alpha}]=H_\alpha, \tag{38}$$

where - denotes the conjugation of $\mathfrak{g}_C$ w. r. t. the compact dual $\mathfrak{g}_u$ of $\mathfrak{g}$ (see e. g. [**2**], pp. 219-221). Then we get a basis of $\mathfrak{g}_u$:

$$\sqrt{-1}\,H_i\ (1\leqq i\leqq l), \qquad E_\alpha-E_{-\alpha}, \sqrt{-1}\,(E_\alpha+E_{-\alpha}) \qquad (\alpha\in\mathfrak{r}).$$

Since $\mathfrak{g}_u$ is the compact dual of $\mathfrak{g}$, a basis of $\mathfrak{p}$ is given by $\sqrt{-1}\,(E_\gamma-E_{-\gamma})$, $E_\gamma+E_{-\gamma}$, where γ runs over all positive non-compact roots.

PROPOSITION 5. *The homomorphism ρ satisfying (H_1) can be modified by an element of K' preserving the condition* (15) *in such a way that there is a subset $S(\gamma)$ of Δ_0' for every $\gamma \in \Delta_0$ such that*

$$\rho(H_\gamma)=\sum_{\gamma'\in S(\gamma)} H'_{\gamma'}, \qquad \rho(E_\gamma)=\sum_{\gamma'\in S(\gamma)} E'_{\gamma'}, \qquad \rho(E_{-\gamma})=\sum_{\gamma'\in S(\gamma)} E'_{-\gamma'}. \tag{39}$$

PROOF. (Cf. Satake [**5**], Proposition 3.) Put $X_\gamma=E_\gamma+E_{-\gamma}$, $X'_{\gamma'}=E'_{\gamma'}+E_{-\gamma'}$. Then one knows that $\{X_\gamma : \gamma\in\Delta_0\}$ (resp. $\{X'_{\gamma'} : \gamma'\in\Delta_0'\}$) spans a maximal abelian subalgebra $\mathfrak{a}$ (resp. $\mathfrak{a}'$) in $\mathfrak{p}$ (resp. $\mathfrak{p}'$). Let $\mathfrak{b}$ (resp. $\mathfrak{b}'$) be the centralizer of $\mathfrak{a}$ (resp. $\mathfrak{a}'$) in $\mathfrak{h}$ (resp. $\mathfrak{h}'$). Then we get Cartan subalgebras $\mathfrak{h}_0=\mathfrak{b}+\mathfrak{a}$ and $\mathfrak{h}_0'=\mathfrak{b}'+\mathfrak{a}'$ of $\mathfrak{g}$ and $\mathfrak{g}'$ respectively, and direct sum decompositions

$$\mathfrak{h}=\mathfrak{b}+[ad(H_0)\mathfrak{a},\mathfrak{a}]=\mathfrak{b}+\mathfrak{a}_1,$$

$$\mathfrak{h}'=\mathfrak{b}'+[ad(H_0)\mathfrak{a}',\mathfrak{a}']=\mathfrak{b}'+\mathfrak{a}_1',$$

to the mutually orthogonal subspaces of $\mathfrak{h}$ and $\mathfrak{h}'$ respectively w.r.t. the Killing form. Since all the maximal abelian subalgebras in $\mathfrak{p}'$ are (k)-conjugate to $\mathfrak{a}'$, there is an element k_1 of K' such that $k_1\circ\rho(\mathfrak{a})\subset\mathfrak{a}'$. As ρ satisfies (H_1), it is easy to see that $k_1\circ\rho(\mathfrak{a}_1)\subset\mathfrak{a}_1'$. Let $\mathfrak{h}_0''=\mathfrak{b}''+\mathfrak{a}'$ be another Cartan subalgebra of $\mathfrak{g}'$ which is conjugate to $\mathfrak{h}_0'$ and contains $k_1\circ\rho(\mathfrak{h}_0)$; hence clearly $k_1\circ\rho(\mathfrak{b})\subset\mathfrak{b}''$. One knows ([**6**], § 1, Proposition 5) that there is an element k_2 of K' such that

$$k_2(X)=X \qquad \text{for all} \quad X\in\mathfrak{a},$$

and

$$k_2(\mathfrak{b}'')=\mathfrak{b}'.$$

Clearly $(k_2k_1)\circ\rho(\mathfrak{h})\subset\mathfrak{h}'$ and $(k_2k_1)\circ\rho(\mathfrak{a})\subset\mathfrak{a}'$. Therefore we may assume that $\rho(\mathfrak{h})\subset\mathfrak{h}'$ and $\rho(\mathfrak{a})\subset\mathfrak{a}'$. Now, let γ be a root in Δ_0. Then we can write

$$\rho(X_\gamma)=\sum_{\gamma'\in\Delta_0'}\kappa_{\gamma'}X'_{\gamma'}, \qquad \kappa_{\gamma'}\in\boldsymbol{R}.$$

Since ρ satisfies (H_1), $\sqrt{-1}$-eigenspace of $ad(H_0)\,|\,\mathfrak{p}_C$ is carried by ρ to $\sqrt{-1}$-eigenspace of $ad(H_0')\,|\,\mathfrak{p}_C'$; hence we have

$$\rho(E_\gamma)=\sum_{\gamma'\in\Delta_0'}\kappa_{\gamma'}E'_{\gamma'}, \qquad \rho(E_{-\gamma})=\sum_{\gamma'\in\Delta_0'}\kappa_{\gamma'}E'_{-\gamma'}.$$

The relation $[E_\gamma, E_{-\gamma}]=H_\gamma$ implies that $\rho(H_\gamma)=\sum|\kappa_{\gamma'}|^2H'_{\gamma'}$, and hence the relation $[H_\gamma, E_\gamma]=2E_\gamma$ implies $|\kappa_{\gamma'}|^2=1$ if $\kappa_{\gamma'}\neq 0$. Putting $S(\gamma)=\{\gamma'\in\Delta_0'\,|\,\kappa_{\gamma'}\neq 0\}$, we have therefore

$$\rho(H_\gamma)=\sum_{\gamma'\in S(\gamma)}H'_{\gamma'}, \qquad \rho(E_{\pm\gamma})=\sum_{\gamma'\in S(\gamma)}\kappa_{\gamma'}E'_{\pm\gamma'}, \qquad \kappa_{\gamma'}=\pm 1.$$

Since $S(\gamma)$ forms a linearly independent system in $\sqrt{-1}\mathfrak{h}'$, we can find an element H' of $\sqrt{-1}\mathfrak{h}'$ such that $\gamma'(H')=\pi(|\kappa_{\gamma'}|-\kappa_{\gamma'})/2$ for all $\gamma'\in S(\gamma)$. Then the inner automorphism $h=\exp ad(\sqrt{-1}H')$ of $\mathfrak{g}'$ is contained in K' and keeps every element of $\mathfrak{h}'$ invariant; the homomorphism $h\circ\rho$ is what we are looking for, q. e. d.

Hence, in the following, we assume that ρ satisfies also (39) for all $\gamma\in\Delta_0$.

3.3. For the non-compact simple root $\gamma_1=\alpha_1$, we shall denote in the following by $S(1)$ the subset $S(\gamma_1)$ of Δ_0' given in Proposition 5. By the equation (35), $S(\gamma_i)$ and $S(\gamma_j)$ are disjoint if $i\neq j$. On the other hand, every element γ in Δ_0 is transposed to α_1 by a suitable element of W_K, hence it is easy to see that the number of elements of $S(\gamma)$ is equal to that of $S(1)$ which will be denoted in the following by m_ρ. The number m_ρ depends only on the (k)-equivalence class containing ρ. Clearly,

(40) $$m_\rho r\leqq r'.$$

PROPOSITION 6. i) *The positive integer m_ρ is equal to the coefficient $\mu_1^{(1)}$ in the equation* (36).

ii) *Put*

(41) $$\delta=\langle H_1, H_1\rangle, \qquad \delta'=\langle H_1', H_1'\rangle'.$$

Then $c_\rho=m_\rho\ \delta'/\delta$, or in other words,

(42) $$(1/\delta')\langle\rho(X), \rho(Y)\rangle'=(m_\rho/\delta)\langle X, Y\rangle \qquad (X, Y\in\sqrt{-1}\,\mathfrak{h}).$$

iii) *Let $H_0=\sqrt{-1}\sum a_iH_i$ and $H_0'=\sqrt{-1}\sum a_j'H_j'$ be the expression as in* 1.4 *of the given complex structures. Suppose ρ satisfies the stronger condition* (H_2). *Then $m_\rho=a_1'/a_1$.*

PROOF. Every positive non-compact root γ' is of the form

$$\gamma'=\alpha_1'+\sum_{j=2}^{l'}c_j\alpha_j'.$$

If γ' is in Δ_0', it is a longer root; accordingly, we see that

$$H_{\gamma'}'=H_1'+\sum_{j=2}^{l'}d_jH_j'.$$

Therefore the property i) follows from Proposition 5. The property iii) follows from i) easily. For ii), we see from (35) and (39) that

$$\begin{aligned}\langle\rho(H_1), \rho(H_1)\rangle' &= c_\rho\langle H_1, H_1\rangle\\ &=\sum_{\gamma'\in S(1)}\langle H_{\gamma'}', H_{\gamma'}'\rangle'=m_\rho\langle H_1', H_1'\rangle',\end{aligned}$$

hence if follows that $c_\rho=m_\rho\delta'/\delta$, q. e. d.

3.4. PROPOSITION 7. *Suppose m_ρ is equal to* 1, *and let α be a longer root of $\mathfrak{g}_C$. Then there is a root α' of $\mathfrak{g}_C'$ such that $\rho(H_\alpha)=H_{\alpha'}'$, $\rho(E_\alpha)=\kappa_{\alpha'}E_{\alpha'}'$, $\rho(E_{-\alpha})=\bar{\kappa}_{\alpha'}E_{-\alpha'}'$, $|\kappa_{\alpha'}|=1$. The root α' is compact or positive non-compact according as the root α is.*

PROOF. Put $\rho(E_\alpha)=\sum\kappa_{\alpha'}E_{\alpha'}'$. Since $[H_\alpha, E_\alpha]=2E_\alpha$, we have $\alpha'(\rho(H_\alpha))=2$

if $\kappa_{\alpha'}\neq 0$, and so

(43) $$\langle H', \rho(H_\alpha)\rangle' = \langle H', H'\rangle' \quad \text{where} \quad H' = H'_{\alpha'} .$$

On the other hand, from the Schwartz inequality, it follows that

(44) $$\langle H', \rho(H_\alpha)\rangle'^2 \leqq \langle H', H'\rangle' \cdot \langle \rho(H_\alpha), \rho(H_\alpha)\rangle' ,$$

where the equality holds if and only if $\rho(H_\alpha)$ is a scalar multiple of H'. If $m_\rho=1$ and α is a longer root, it follows from (43), (44), and (42) that $(1/\delta')\langle H', H'\rangle' \leqq (1/\delta)\langle H_\alpha, H_\alpha\rangle = 1$. As α'_1 is a longer root,

$$\frac{1}{\delta'}\langle H', H'\rangle' = \frac{\langle \alpha'_1, \alpha'_1\rangle'}{\langle \alpha', \alpha'\rangle'} = 1 .$$

Therefore the equality should be hold. Hence, again by (43), we have $\rho(H_\alpha) = H'_{\alpha'}$, and so $\rho(E_\alpha) = \kappa_{\alpha'}E'_{\alpha'}$. Now that ρ satisfies (H_1), it must also satisfy $\rho(\mathfrak{g}_u)\subset \mathfrak{g}'_u$, i.e. $\rho(\bar{X}) = \overline{\rho(X)}$ $(X\in \mathfrak{g}_C)$, where — denotes both of the conjugations of $\mathfrak{g}_C$ and $\mathfrak{g}'_C$ w.r.t. $\mathfrak{g}_u$ and $\mathfrak{g}'_u$ respectively. Thus, from (38), we see that

$$\rho(E_{-\alpha}) = \bar{\kappa}_{\alpha'}E'_{-\alpha'} , \qquad |\kappa_{\alpha'}| = 1 .$$

The remaining part of the proposition is clear, q.e.d.

COROLLARY. *If $\mathfrak{g}$ is not of type $(\mathrm{III})_p$ or $(\mathrm{IV})_{2l'-1}$ and if $m_\rho=1$, then the image $\rho(\mathfrak{g})$ is a regular subalgebra of $\mathfrak{g}'$. Let β'_i be the root corresponding to the simple root α_i. Then ρ is (k)-equivalent to ρ_0 defined by*

(45) $$\rho_0(H_i) = H'_{\beta_i'} , \qquad \rho_0(E_{\pm\alpha_i}) = E'_{\pm\beta_i'} .$$

PROOF. The first part is a direct consequence from Proposition 7. In view of the proof of the proposition, we may put

$$\rho(E_{\alpha_i}) = e^{\sqrt{-1}\theta_i}E'_{\beta_i'} , \qquad 0\leqq \theta_i < 2\pi \ (1\leqq i\leqq l) .$$

Since $\beta'_1, \cdots, \beta'_l$ are linealy independent, there is an element H' of $\sqrt{-1}\,\mathfrak{h}'$ such that $\beta'_i(H') = -\theta_i$. Then we have

$$\rho_0 = \exp ad(\sqrt{-1}\,H')\circ\rho , \qquad \text{q.e.d.}$$

3.5. Two homomorphisms ρ_1 and ρ_2 of a complex semi-simple Lie algebra $\mathfrak{g}_C$ into another $\mathfrak{g}'_C$ are said to be *L-equivalent*, if $\Lambda\circ\rho_1$ and $\Lambda\circ\rho_2$ are equivalent representations of $\mathfrak{g}_C$ for any representation Λ of $\mathfrak{g}'_C$. It is known that, if $\mathfrak{g}'_C$ is of type (A), (B), or (C), two L-equivalent homomorphisms of $\mathfrak{g}_C$ into $\mathfrak{g}'_C$ are equivalent in $\mathfrak{g}'_C$ (Dynkin [1], Chap. I).

PROPOSITION 8. *Let $\mathfrak{g}'$ be of type $(\mathrm{I})_{p,q}$, $(\mathrm{II})_p$, $(\mathrm{III})_p$, or $(\mathrm{IV})_{2l-1}$. Let ρ_1 and ρ_2 be two homomorphisms of a non-compact simple Lie algebra $\mathfrak{g}$ of hermitian type into $\mathfrak{g}'$ satisfying (H_2). Then, if ρ_1 and ρ_2 coincide on the fixed Cartan subalgebra of $\mathfrak{g}$, ρ_1 and ρ_2 are (k)-equivalent.*

PROOF. We denote the complexifications of ρ_1 and ρ_2 by the same notations respectively. By the assumption, we have $\rho_1|\mathfrak{h}_C = \rho_2|\mathfrak{h}_C$. Then ρ_1 and ρ_2 are

L-equivalent homomorphisms of $\mathfrak{g}_C$ into $\mathfrak{g}'_C$ ([1], Chap. I, Theorem 1.1), so they are equivalent if $\mathfrak{g}'_C \neq (\mathrm{II})_p$. Suppose $\mathfrak{g}' \neq (\mathrm{II})_p$. Then there is an inner automorphism $s \in \mathrm{Int}(\mathfrak{g}'_C)$ such that $\rho_2 = s \circ \rho_1$. Since $\rho_i(\mathfrak{g}_u) \subset \mathfrak{g}'_u$ $(i=1, 2)$, we may assume that s is in $\mathrm{Int}(\mathfrak{g}_u)$ (cf. Lemma 1; $\mathfrak{g}_u$ and $\mathfrak{g}'_u$ are maximal compact subalgebras of semi-simple Lie algebras $\mathfrak{g}_C$ and $\mathfrak{g}'_C$ considered over $\boldsymbol{R}$). As ρ_1 and ρ_2 satisfy (H_2), it follows that $s(H'_0) = H'_0$. In fact,

$$H'_0 = \rho_2(H_0) = s \circ \rho_1(H_0) = s(H'_0) .$$

Therefore it follows from Proposition 2 that $s(\mathfrak{g}') = \mathfrak{g}'$; so the restriction of s to $\mathfrak{g}'$ gives an inner automorphism of $\mathfrak{g}'$. Hence it follows from Proposition 1 that ρ_1 and ρ_2 are (k)-equivalent in $\mathfrak{g}'$. Now, let $\mathfrak{g}'$ be of type $(\mathrm{II})_p$. It is known that there is an automorphism a of $\mathfrak{g}'_C$ such as $\rho_2 = a \circ \rho_1$ ([1], Chap. I, § 1, NO. 5). As above, it is easy to see that a can be taken as an automorphism of $\mathfrak{g}'_u$, and that $a(H'_0) = H'_0$. Therefore a induces an automorphism on $\mathfrak{g}$. On the other hand, it is well known that a mod $\mathrm{Int}(\mathfrak{g}'_C)$ corresponds to an automorphism ω of the Dynkin diagram. Now that $a(H'_0) = H'_0$, ω must keep α'_1 invariant. Hence ω is the identity: in fact, an automorphism of the Dynkin diagram of $\mathfrak{g}'_C$ is either the identity or that giving the permutation between α'_1 and α'_p. Therefore a must be inner, q. e. d.

REMARK. This proposition is also proved in [4] for the case $\mathfrak{g}' = (\mathrm{I})_{p,q}$, $(\mathrm{II})_p$, and $(\mathrm{III})_p$ (p. 439 Lemma 2).

The following proposition seems to hold good more generally, but the author does not know a general proof (for the case $m_\rho = 1$, see the corollary to Proposition 7).

PROPOSITION 9. *Let* $\mathfrak{g}'$ *be of type* $(\mathrm{IV})_p$, (EIII), *or* (EVII), *and* ρ *a homomorphism of* $\mathfrak{g}$ *into* $\mathfrak{g}'$ *satisfying* (H_2), *where* $\mathfrak{g}$ *is a non-compact simple Lie algebra of hermitian type. Then, modifying by* (k)*-equivalence,* ρ *can be determined by the following equations:*

$$\rho(H_i) = \sum_{\alpha' \in S(i)} H'_{\alpha'} , \qquad \rho(E_{\pm\alpha_i}) = \sum_{\alpha' \in S(i)} E'_{\pm\alpha'} , \tag{46}$$

where $S(1)$ *is a subset of* Δ'_0, $S(i)$ $(i \geqq 2)$ *a strongly orthogonal system of compact roots, which is uniquely determined up to* $W_{K'}$*-equivalence for each fixed* $S(1)$.

PROOF (Verifications of some properties in it are reserved to § 5). In view of Proposition 5, we can modify ρ so as to satisfy

$$\rho(H_1) = \sum_{\alpha' \in S(1)} H'_{\alpha'} , \qquad \rho(E_{\pm\alpha_1}) = \sum_{\alpha' \in S(1)} E'_{\pm\alpha'} .$$

For a fixed $S(1)$ in every case, we shall get in § 5 uniquely determined $W_{K'}$-class of strongly orthogonal system $S(i)$ of compact roots for each index i such that $\rho(H_i) = \sum_{\alpha' \in S(i)} H'_{\alpha'}$. Put $\rho(E_{\alpha_i}) = \sum \kappa^{(i)}_{\alpha'} E'_{\alpha'}$. Then, in view of (38), it is easy to see that $\rho(E_{-\alpha_i}) = \sum \overline{\kappa^{(i)}_{\alpha'}} E'_{-\alpha'}$. From the equation

$$\alpha_i(H_j)\rho(E_{\alpha_i})=[\rho(H_j),\ \rho(E_{\alpha_i})]=\sum\kappa^{(i)}_{\alpha'}\alpha'(\rho(H_j))E'_{\alpha'}\ ,$$

it follows that, if $\kappa^{(i)}_{\alpha'}\neq 0$,

$$\alpha'(\rho(H_i))=2\ , \qquad \alpha'(\rho(H_j))=\alpha_i(H_j)\ . \tag{47}$$

We shall see in § 5 for every case that *the set of compact roots satisfying* (47) *coincides with* $S(i)$. Then, from the relation $[E_{\alpha_i},\ E_{-\alpha_i}]=H_i$, we have

$$\rho(E_{\alpha_i})=\sum_{\alpha'\in S(i)}\kappa_{\alpha'}E'_{\alpha'}\ , \qquad \rho(E_{-\alpha_i})=\sum_{\alpha'\in S(i)}\overline{\kappa_{\alpha'}}E'_{-\alpha'}\ ,$$

$$\kappa_{\alpha'}=e^{\sqrt{-1}\theta_{\alpha'}}\ , \qquad 0\leqq\theta_{\alpha'}<2\pi\ .$$

It will be easily seen (in § 5) that the *union of* $S(i)$ $(1\leqq i\leqq l)$ *forms a linearly independent system in* $\sqrt{-1}\mathfrak{h}'$. Therefore there is an element H' of $\sqrt{-1}\mathfrak{h}'$ such as $\alpha'(H')=-\theta_{\alpha'}$ $(\alpha'\in S(i),\ 1\leqq i\leqq l)$. The inner automorphism $h=\exp ad(\sqrt{-1}H')$ of $\mathfrak{g}'$ is contained in K'; if it is extended linearly to the automorphism of $\mathfrak{g}'_C$, it keeps every element of $\sqrt{-1}\mathfrak{h}$ invariant, and $h(E'_{\alpha'})=e^{-\sqrt{-1}\theta_{\alpha'}}E'_{\alpha'}$. Therefore the homomorphism $h\circ\rho$, instead of ρ, satisfies clearly all the relations in (46), q. e. d.

THEOREM 3. *Let* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *be non-compact simple Lie algebras of hermitian types. Let* ρ_1 *and* ρ_2 *be homomorphisms of* $\mathfrak{g}$ *into* $\mathfrak{g}'$ *satisfying* (H_2) *and* (15). *Let further* Θ_i *be the restriction of* ρ_i *onto the fixed Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$. *Then* ρ_1 *and* ρ_2 *are* (k)-*equivalent if and only if there is an element* k *of* $N_{K'}(\mathfrak{h}')$ *such as* $\Theta_2=k\circ\Theta_1$.

PROOF. Firstly, suppose there is $k\in N_{K'}(\mathfrak{h}')$ such that $\Theta_2=k\circ\Theta_1$. Then, from Proposition 8 and 9, it follows easily that ρ_2 and $k\circ\rho_1$ are (k)-equivalent, so that ρ_1 and ρ_2 are. Conversely, let ρ_1 and ρ_2 be (k)-equivalent: $\rho_2=k_1\circ\rho_1$, $k_1\in K'$. Putting $\mathfrak{h}^*=k_1(\mathfrak{h}')$, we can find an element k_2 of K', by the same way as the proof of Lemma 2, such that the restriction of it to $\rho_2(\mathfrak{h})=k_1\circ\rho_1(\mathfrak{h})$ is the identity and that $k_2(\mathfrak{h}^*)=\mathfrak{h}'$. Clearly the element $k=k_2k_1$ of K' normalizes the Cartan subalgebra $\mathfrak{h}'$, and we have $\Theta_2=k\circ\Theta_1$, q. e. d.

§ 4. Determination of regular subalgebras.

In this section, we shall determine all regular subalgebras of each non-compact simple Lie algebra $\mathfrak{g}$ of hermitian type. But actually, it is sufficient to find all proper maximal regular subalgebras (i. e. proper subalgebras which are maximal among regular subalgebras) of all simple $\mathfrak{g}$, because others are determined automatically. In fact, if a regular subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$ is contained in another regular subalgebra $\mathfrak{g}_2$ of $\mathfrak{g}$, we may consider by definition $\mathfrak{g}_1$ to be a regular subalgebra of $\mathfrak{g}_2$. Especially, the injection ι_1 of $\mathfrak{g}_1$ into $\mathfrak{g}$ satisfies (H_2) if and only if both $\iota_1:\mathfrak{g}_1\to\mathfrak{g}_2$ and $\iota_2:\mathfrak{g}_2\to\mathfrak{g}$ satisfy (H_2).

Moreover, we have only to determine all W_K-equivalence classes of maximal Π-systems because of Proposition 3 and Theorem 1.

We shall refer freely the following known facts:

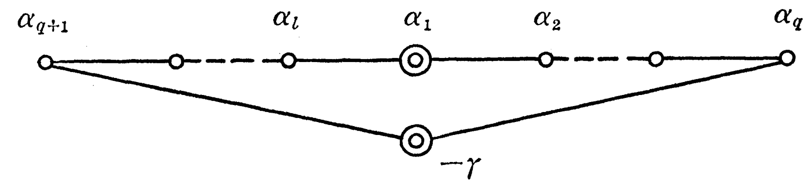
i) All positive non-compact roots of a same length are mutually permutable by translations of W_K.

ii) The highest root γ is a positive non-compact root, and has the maximal length.

iii) Let Δ_i be a system obtained from the extended fundamental system $\{\Pi, -\gamma\}$ (i.e. the union of the simple root system Π and the lowest root $-\gamma$) by omitting a compact simple root α_i, and by replacing all elements α in $\langle -\gamma \rangle$ by $-\alpha$, where $\langle -\gamma \rangle$ denotes the connected component which contains $-\gamma$. Then, except for the case $\mathfrak{g}=(\mathrm{I})_{p,q}$, Δ_i is a proper maximal Π-system in our sense (cf. [1], Chap. II, Theorem 5.5).

We shall say that Δ_i in iii) is obtained from the simple root system by an *elementary transformation*. A regular subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$ will be called *of class* (H_2) if the injection of $\mathfrak{g}_1$ into $\mathfrak{g}$ satisfies (H_2).

4.1. *The case* $\mathfrak{g}=(\mathrm{I})_{p,q}$ $(1 \leqq p \leqq q)$. In this case, the extended Dynkin diagram is given as follows:



; $l=p+q-1$

$\gamma=\alpha_1+\cdots+\alpha_l$.

All positive non-compact roots are in the W_K-orbit of α_1. If $p \geqq 2$, all compact roots are divided into two W_K-classes: one is the W_K-orbit of α_2, the other of α_l. If $p=1$, all compact roots are in the W_K-orbit of α_2. Let $\beta=\sum_{i=2}^{l} c_i\alpha_i$ be a compact root. If $\beta(H_1)$ is equal to -1, it is easily seen that $c_2=1$ or, when $p \geqq 2$, $c_l=1$. Hence β is really of the form

$$\beta=\alpha_2+\sum_{i=3}^{t}\alpha_i \ (t \leqq q) \quad \text{or, if} \quad p \geqq 2,$$

$$=\alpha_l+\sum_{i=s}^{l-1}\alpha_i \ (s \geqq q+1).$$

All such compact roots β are transposed to α_2 or α_l by the subgroup of W_K generated by $\{w_3, \cdots, w_q\}$ or $\{w_{q+1}, \cdots, w_l\}$ respectively, where w_i denotes the reflection w_{α_i} (defined by (27)); this subgroup keeps α_1 invariant. If

$$\beta(H_1)=\beta(H_2)=\cdots=\beta(H_{t-1})=0 \quad \text{and} \quad \beta(H_t)=-1,$$

it is easy to see by a similar argument that β is transposed to α_t by the subgroup of W_K generated by $\{w_{t+1}, \cdots, w_q\}$ which keeps $\alpha_1, \alpha_2, \cdots, \alpha_{t-1}$ invariant. Similarly, when $p \geqq 2$, a connected Dynkin diagram $\{\beta, \alpha_s, \alpha_{s+1}, \cdots, \alpha_l, \alpha_1\}$ of type (A) with a compact root β is equivalent under W_K to $\{\alpha_{s-1}, \alpha_s, \cdots, \alpha_l, \alpha_1\}$. Moreover, we can find an element of W_K which gives the following permutations:

$$\alpha_1 \longleftrightarrow \gamma, \qquad \alpha_2 \longleftrightarrow -\alpha_q, \qquad \alpha_l \longleftrightarrow -\alpha_{q+1}.$$

Under these considerations, we see that every proper maximal Π-system is equivalent to one of the following, those obtained by omitting two roots from the above diagram:

$(\mathrm{I})_{r,q}+(A_{p-r-1})$ $(1 \leqq r < p)$: $\{\alpha_{l-r+1}, \cdots, \alpha_{l-1}, \alpha_l, \alpha_1, \cdots, \alpha_q\}$
$\cup \{\alpha_{q+1}, \alpha_{q+2}, \cdots, \alpha_{l-r-1}\}$.

$(\mathrm{I})_{p,s}+(A_{q-s-1})$ $(p \leqq s < q)$: $\{\alpha_{q+1}, \cdots, \alpha_{l-1}, \alpha_l, \alpha_1, \cdots, \alpha_s\}$
$\cup \{\alpha_{s+2}, \alpha_{s+3}, \cdots, \alpha_q\}$.

$(\mathrm{I})'_{s,p}+(A_{q-s-1})$ $(1 \leqq s < p)$: $\{\alpha_s, \cdots, \alpha_2, \alpha_1, \alpha_l, \alpha_{l-1}, \cdots, \alpha_{q+1}\}$
$\cup \{\alpha_{s+2}, \alpha_{s+3}, \cdots, \alpha_q\}$.

$(\mathrm{I})_{r,s}+(\mathrm{I})_{p-r,q-s}$ $\left(\begin{matrix}1 \leqq r \leqq s \\ s-r \leqq q-p\end{matrix}\right)$: $\{\alpha_{l-r+1}, \cdots, \alpha_l, \alpha_1, \cdots, \alpha_s\}$
$\cup \{-\alpha_{l-r-1}, \cdots, -\alpha_{q+1}, \gamma, -\alpha_q, \cdots, -\alpha_{s+2}\}$.

$(\mathrm{I})'_{s,r}+(\mathrm{I})_{p-r,q-s}$ $(1 \leqq s < r < p)$: $\{\alpha_s, \cdots, \alpha_1, \alpha_l, \cdots, \alpha_{l-r+1}\}$
$\cup \{-\alpha_{l-r-1}, \cdots, -\alpha_{q+1}, \gamma, -\alpha_q, \cdots, -\alpha_{s+2}\}$.

$(A_{p-1})+(A_{q-1})$: $\{\alpha_{q+1}, \cdots, \alpha_{l-1}, \alpha_l\} \cup \{\alpha_2, \alpha_3, \cdots, \alpha_q\}$.

In this table, the following notations are used:

The elements in a pair of brace { } form a connected component of a Π-system of roots;

(A_0) denotes the empty set;

$(\mathrm{I})'_{s,r}$ means the regular subalgebra of type $(\mathrm{I})_{r,s}$, but not conjugate to that corresponding to the Π-system

$$\{\alpha_{l-s+1}, \cdots, \alpha_{l-1}, \alpha_l, \alpha_1, \alpha_2, \cdots, \alpha_r\}.$$

This will be denoted also by $(\mathrm{I})_{r,s}$ $(r > s)$.

Using the expressions of the complex structures as described in 1.4, we can find regular subalgebras of class (H_2) in the above table; we see at once that they must be of a form $\mathfrak{g}_1 = (\mathrm{I})_{r,s}+(\mathrm{I})_{p-r,q-s}$. The complex structures of $(\mathrm{I})_{r,s}$ and $(\mathrm{I})_{p-r,q-s}$ are

$$\frac{\sqrt{-1}}{r+s}[sH_{l-r+1}+2sH_{l-r+2}+\cdots+rsH_1+r(s-1)H_2+\cdots+rH_s],$$

$$\frac{\sqrt{-1}}{(p+q)-(r+s)}\left[-(q-s)H_{l-r-1}-2(q-s)H_{l-r-2}-\cdots+\right.$$
$$\left.+(p-r)(q-s)(H_1+H_2+\cdots+H_l)-\cdots-(p-r)H_{s+2}\right].$$

As $\mathfrak{g}_1$ is of class (H_2), we have

$$\frac{rs}{r+s}+\frac{(p-r)(q-s)}{(p+q)-(r+s)}=\frac{pq}{p+q},$$

and hence

$$(p-ar)(q-as)=0, \quad a=\frac{p+q}{r+s}.$$

Thus we have

$$r:s=p:q. \tag{48}$$

Conversely, it is easy to see that $\mathfrak{g}_1$ is of class (H_2), if (r, s) satisfies the relation (48). We have seen that

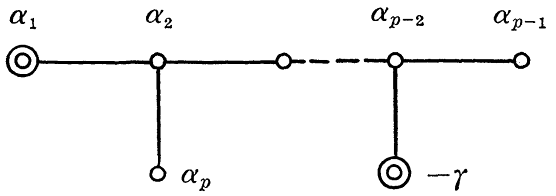
$$(\mathrm{I})_{r,s}+(\mathrm{I})_{p-r,q-s}:\quad r:s=p:q$$

are maximal regular subalgebras of class (H_2) of $(\mathrm{I})_{p,q}$.

Maximal regular subalgebras without compact factors not being of class (H_2) (i.e. non-compact regular subalgebras not of class (H_2) which is maximal w.r.t. this property) are (k)-equivalent to one of the following:

$$(\mathrm{I})_{p,q-1},\quad (\mathrm{I})_{p-1,q},\quad (\mathrm{I})_{r,s}+(\mathrm{I})_{p-r,q-s}(r:s\neq p:q).$$

4.2. *The case* $\mathfrak{g}=(\mathrm{II})_p$. In this case, the extended Dynkin diagram is given as follows:



$$;\quad \gamma=\alpha_1+2(\alpha_2+\cdots+\alpha_{p-2})+\alpha_{p-1}+\alpha_p.$$

All positive non-compact roots are in the W_K-orbit of α_1, all compact roots in that of α_2. By elementary transformations, we get some Π-systems representing (k)-equivalence classes of maximal regular subalgebras. Omitting $-\gamma$ and α_{p-1} from the above diagram, we get a Π-system corresponding to $(\mathrm{II})_{p-1}$, which is also proper maximal. If we omit α_1 and $-\gamma$ from the diagram, we get also a Π-system corresponding to (A_{p-1}) which is a maximal compact subalgebra, so we can see at once that there is no compact root α such that

$$\alpha(H_1)=\cdots=\alpha(H_{s-1})=\alpha(H_{s+1})=0, \qquad \alpha(H_s)=-1.$$

Hence, by arguments similar to those in 4.1, we get all maximal Π-systems such that positive non-compact roots in them have extreme positions in connected components of their Dynkin diagrams; it is easy to see that there is no regular subalgebra of type $(\mathrm{III})_r$, $(\mathrm{IV})_r$ $(r \geqq 7)$, (EIII), or (EVII). The remaining cases to examine are those that some simple factors of regular subalgebras are of type $(\mathrm{I})_{r,s}$. Let $\beta = \sum_{2\leqq i\leqq p} c_i\alpha_i$ be a compact root such that

$$\beta(H_i) = -1, \qquad \beta(H_2) = 0, \cdots, \beta(H_t) = 0, \tag{49}$$

where t is a positive integer smaller than $p-1$. The equations (49) are equivalent to the relation

$$c_p = c_2 = \cdots = c_{t+1} = 1.$$

Such a root β is transposed to $\alpha_2+\alpha_3+\cdots+\alpha_p$ by an element of the subgroup of W_K generated by $\{w_{t+2}, \cdots, w_{p-1}\}$; this subgroup keeps $\alpha_1, \alpha_2, \cdots, \alpha_t$ invariant. In the rest of this paragraph, we put

$$\beta = \alpha_2+\alpha_3+\cdots+\alpha_p.$$

Clearly $\beta(H_{p-1}) \neq 0$. Let $\beta_1 = \sum d_i\alpha_i$ be another compact root such that

$$\beta_1(H_1) = \cdots = \beta_1(H_{p-3}) = 0, \qquad \beta(H_{\beta_1}) = -1.$$

Then we see at once that $\beta_1 = -\alpha_{p-1}$. By a same manner, we have finally a maximal Π-system for each positive integer $r \leqq \left[\frac{p}{2}\right]$:

$$\{-\alpha_{p-r+2}, \cdots, -\alpha_{p-2}, -\alpha_{p-1}, \beta, \alpha_1, \cdots, \alpha_{p-r}\}$$

which corresponds to a regular subalgebra of type $(\mathrm{I})_{r,p-r}$. There is no root orthogonal to it.

Thus we have got all (k)-equivalence classes of maximal regular subalgebras; they are represented by the following Π-systems:

$(\mathrm{I})_{r,p-r}$ $\left(1 \leqq r \leqq \left[\frac{p}{2}\right]\right)$: $\{-\alpha_{p-r+2}, \cdots, -\alpha_{p-2}, -\alpha_{p-1}, \beta, \alpha_1, \alpha_2, \cdots, \alpha_{p-r}\}$.

$(\mathrm{II})_r+(\mathrm{II})_{p-r}$ $\left(\left[\frac{p}{2}\right] \leqq r \leqq p-2\right)$: $\left\{\begin{matrix}\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_{r-1} \\ \alpha_p\end{matrix}\right\}$

$$\cup \left\{\begin{matrix}\gamma, -\alpha_{p-2}, -\alpha_{p-3}, \cdots, -\alpha_{r+1} \\ -\alpha_{p-1}\end{matrix}\right\},$$

$(\mathrm{II})_{p-1}$: $\left\{\begin{matrix}\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_{p-2} \\ \alpha_p\end{matrix}\right\}$,

(A_{p-1}): $\{\alpha_p, \alpha_2, \alpha_3, \cdots, \alpha_{p-1}\}$,

$$((\mathrm{II})_2 = (\mathrm{I})_{1,1}+(A_1),\ (\mathrm{II})_3 = (\mathrm{I})_{1,3},\ (\mathrm{II})_4 = (\mathrm{IV})_6)$$

Since $H_\gamma = H_1+2(H_2+\cdots+H_{p-2})+H_{p-1}+H_p$, we see at once that the regular subalgebras $(\mathrm{II})_r+(\mathrm{II})_{p-r}$ $\left(\left[\frac{p}{2}\right] \leqq r \leqq p-2\right)$ are of class (H_2). Clearly, the regular subalgebra $(\mathrm{II})_{p-1}$ is not of class (H_2). The complex structure of a regular subalgebra $(\mathrm{I})_{r,p-r}$ is

$$\frac{1}{p}[-(p-r)H_{p-r+2}-2(p-r)H_{p-r+3}-\cdots+$$
$$+(r-1)(p-r)(H_2+\cdots+H_p)+r(p-r)H_1+\cdots+rH_{p-r}].$$

Hence, if $(\mathrm{I})_{r,p-r}$ is of class (H_2), we see that

$$\frac{r(p-r)}{p}=\frac{p}{4}, \quad \text{so that} \quad p=2r.$$

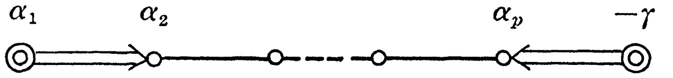
Conversely, if $p=2r$, the regular subalgebra $(\mathrm{I})_{r,r}$ is of class (H_2). Thus all (k)-equivalence classes of maximal regular subalgebras of class (H_2) are represented by

$$(\mathrm{I})_{r,r} \ (\text{if } p=2r), \qquad (\mathrm{II})_r+(\mathrm{II})_{p-r};$$

maximal non-compact regular subalgebras not of class (H_2) are (k)-equivalent to one of

$$(\mathrm{I})_{r,p-r} \left(r\neq\frac{p}{2}\right), \qquad (\mathrm{II})_r+(\mathrm{I})_{s,p-r-s} \left(s\neq\frac{p-r}{2}\right), \qquad (\mathrm{II})_{p-1}.$$

4.3. *The case* $\mathfrak{g}=(\mathrm{III})_p$. The extended Dynkin diagram of $\mathfrak{g}_C$ is



$; \quad \gamma=\alpha_1+2(\alpha_2+\cdots+\alpha_p).$

In this case, there are two sorts of the length of roots; all longer roots are non-compact. Hence there are two W_K-classes of positive non-compact roots represented by the longer root α_1 and by the shorter root $\gamma_1=\alpha_1+\alpha_2$ respectively. The orbit $W_K(\alpha_1)$ contains p elements and forms a strongly orthogonal system. On the other hand, all compact roots are mutually permutable by elements of W_K. Every compact root β satisfying $\alpha_1(H_\beta)=-2$ is equivalent to α_2 under an element of W_K which preserves α_1 fixed. Moreover, if a compact root β satisfies

$$\beta(H_1)=\beta(H_2)=\cdots=\beta(H_{t-1})=0, \qquad \beta(H_t)=-1,$$

it is transposed to α_{t+1} by an element of the subgroup of W_K generated by $\{w_{t+2},\cdots,w_p\}$; this subgroup preserves $\alpha_1,\alpha_2,\cdots,\alpha_t$ fixed. Thus we have seen that all maximal Π-systems containing longer roots are equivalent under W_K to one of those given by elementary transformations. Now, let $\alpha=\sum_{2\leq i\leq p} c_i\alpha_i$ be a compact root such as $\alpha(H_{\gamma_1})=-1$, where γ_1 is the positive non-compact root $\alpha_1+\alpha_2$. Since $H_{\gamma_1}=2H_1+H_2$, we can see at once that $c_3=1$. Hence α is of the form $\alpha_2+\sum_{3\leq i\leq t}\alpha_i$ or $\alpha_3+\sum_{4\leq i\leq t}\alpha_i$. Thus, by the same considerations as for the case of $(\mathrm{II})_p$, we have a representative

$$\{-\alpha_{p-r+3},\cdots,-\alpha_{p-1},-\alpha_p,\beta,\gamma_1,\alpha_3,\alpha_4,\cdots,\alpha_{p-r+1}\}$$

of an equivalence class of Π-systems which corresponds to a regular subalgebra of type $(\mathrm{I})_{r,p-r}$, where

$$\beta=\alpha_2+\alpha_3+\cdots+\alpha_p\,.$$

It is easy to see that there is no compact root α such as

$$\alpha(H_{\gamma_1})=0\,,\qquad \alpha(H_2)=\cdots=\alpha(H_{t-1})=\alpha(H_{t+1})=0\,,\qquad \alpha(H_t)=-1\,.$$

Hence there is no regular subalgebra of type $(\mathrm{II})_r$, $(\mathrm{IV})_r$, (EIII) or (EVIII).

We have seen that all (k)-equivalence classes of maximal regular subalgebras of $\mathfrak{g}$ are represented by the following:

$$(\mathrm{I})_{r,p-r}\ \left(1\leqq r\leqq\left[\frac{p}{2}\right]\right):\ \{-\alpha_{p-r+3},\cdots,-\alpha_{p-1},-\alpha_p,\beta,\gamma_1,\alpha_3,\alpha_4,\cdots,\alpha_{p-r+1}\}.$$

$$(\mathrm{III})_r+(\mathrm{III})_{p-r}\ \left(\left[\frac{p}{2}\right]\leqq r\leqq p-1\right):\ \{\alpha_1,\cdots,\alpha_r\}\cup\{\gamma,-\alpha_p,-\alpha_{p-1},-\alpha_{r+2}\}.$$

$$(A_{p-1}):\ \{\alpha_2,\alpha_3,\cdots,\alpha_p\}.$$

$$((\mathrm{III})_1=(\mathrm{I})_{1,1})$$

A maximal regular subalgebra of class (H_2) is (k)-equivalent to one of

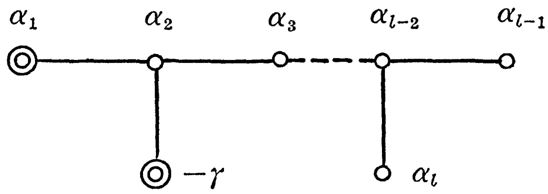
$$(\mathrm{I})_{r,r}\ (\text{if } p=2r)\,,\qquad (\mathrm{III})_r+(\mathrm{III})_{p-r}\,;$$

a maximal non-compact regular subalgebra not of class (H_2) to one of

$$(\mathrm{I})_{r,p-r}\ \left(r\neq\frac{p}{2}\right),\qquad (\mathrm{III})_{p-1}\,,\qquad (\mathrm{III})_r+(\mathrm{I})_{s,p-r-s}\ \left(s\neq\frac{p-r}{2}\right).$$

4.4. *The case* $\mathfrak{g}=(\mathrm{IV})_p$.

4.4.1. *The case* $p=2l-2$. The extended Dynkin diagram is as follows:



$$;\quad \gamma=\alpha_1+2(\alpha_2+\cdots\alpha_{l-2})+\alpha_{l-1}+\alpha_l\,.$$

All positive non-compact roots (resp. all compact roots) are in the W_K-orbit of α_1 (resp. α_2). Let α be a compact root. If α satisfies

$$\alpha(H_1)=\alpha(H_2)=\cdots=\alpha(H_{t-1})=0\,,\qquad \alpha(H_t)=-1\,,$$

where t is a positive integer smaller than $l-3$, then α is transposed to α_{t+1} by an element w of the subgroup of W_K generated by $w_{t+2},\cdots,w_l$, keeping $\alpha_1,\cdots,\alpha_t$ invariant. On the other hand, if α satisfies

$$\alpha(H_1)=\cdots=\alpha(H_{l-2})=0\,,\qquad \alpha(H_{l-1})=1\,,$$

then α is either α_{l-1} or α_l; there is no element of W_K that causes a permutation between α_{l-1} and α_l keeping $\alpha_1,\cdots,\alpha_{l-2}$ invariant. The root

$$\beta_1=\alpha_2+2(\alpha_3+\cdots+\alpha_{l-2})+\alpha_{l-1}+\alpha_l$$

is the only one compact root which satisfies

$$\beta_1(H_1) = -1\,, \qquad \beta_1(H_2) = 0\,;$$

the root

$$\beta_2 = \alpha_{l-2} + \alpha_{l-1} + \alpha_l$$

is the unique compact root such as

$$\beta_2(H_1) = \cdots = \beta_2(H_{l-4}) = \beta_2(H_{l-2}) = 0\,, \qquad \beta_2(H_{l-3}) = -1\,.$$

Thus we have the following table of (k)-equivalence classes of maximal regular subalgebras:

$(\mathrm{I})_{1,1} + (\mathrm{I})_{1,1} + (D_{l-2})$: $\{\alpha_1\} \cup \{\gamma\} \cup \left\{\begin{matrix}\alpha_3, \cdots, \alpha_{l-2}, \alpha_{l-1} \\ \alpha_l\end{matrix}\right\}$.

$(\mathrm{I})_{1,r} + (D_{l-r-1})$ $(2 \leqq r \leqq l-2)$: $\{\alpha_1, \cdots, \alpha_r\} \cup \left\{\begin{matrix}\alpha_{r+2}, \cdots, \alpha_{l-2}, \alpha_{l-1} \\ \alpha_l\end{matrix}\right\}$.

$(\mathrm{I})_{1,l-1}$: $\{\alpha_1, \cdots, \alpha_{l-2}, \alpha_{l-1}\}$

$(\mathrm{I})'_{1,l-1}$: $\{\alpha_1, \cdots, \alpha_{l-2}, \alpha_l\}$

$(\mathrm{I})_{2,2} + (A_{l-6})$: $\{\beta_1, \alpha_1, \alpha_2\} \cup \{\alpha_4, \alpha_5, \cdots, \alpha_{l-3}\}$.

$(\mathrm{IV})_{p-2}$: $\left\{\begin{matrix}\alpha_1, \alpha_2, \cdots, \alpha_{l-3}, \alpha_{l-2} \\ \beta_2\end{matrix}\right\}$.

(D_{l-1}): $\left\{\begin{matrix}\alpha_2, \alpha_3, \cdots, \alpha_{l-2}, \alpha_{l-1} \\ \alpha_l\end{matrix}\right\}$.

$$\left(\begin{matrix}(\mathrm{IV})_4 = (\mathrm{I})_{1,1},\ (D_1) = (A_1),\ (D_2) = (A_1) + (A_1) \\ (D_3) = (A_3)\,;\ (A_t) = \emptyset \quad \text{if} \quad t \leqq 0\end{matrix}\right)$$

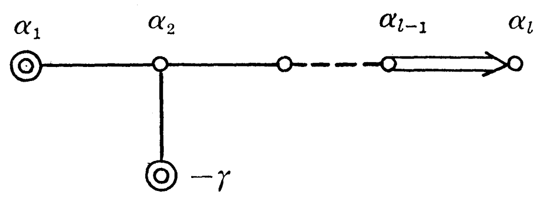
Among them, the regular subalgebras

$$(\mathrm{I})_{1,1} + (\mathrm{I})_{1,1} + (D_{l-2}),\ (\mathrm{I})_{2,2} + (A_{l-6}),\ (\mathrm{IV})_{p-2}$$

are of class (H_2); a maximal non-compact regular subalgebra not of class (H_2) is (k)-equivalent to one of

$$(\mathrm{I})_{1,l-1}, \qquad (\mathrm{I})'_{1,l-1}\,.$$

4.4.2. *The case $p = 2l-1$.* In this case, the diagram is



; $\gamma = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_l)$.

There is only one positive non-compact shorter root γ_1:

$$\gamma_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_{l-1} + \alpha_l\,.$$

The root γ_1 is orthogonal to every compact root; hence it is stable under the action of W_K. All the other positive non-compact roots are in the orbit $W_K(\alpha_1)$, and have the maximal length. All the compact roots are divided into two orbits under W_K; one is $W_K(\alpha_2)$ and another $W_K(\alpha_l)$. Putting

$$\beta_1=\alpha_2+2(\alpha_3+\cdots+\alpha_l),$$
$$\beta_2=\alpha_{l-1}+2\alpha_l,$$
$$\beta_3=\alpha_{l-1}+\alpha_l,$$

we get easily the following table of (k)-equivalence classes of maximal regular subalgebras:

$(\mathrm{I})_{1,1}+(\mathrm{I})_{1,1}+(B_{l-2})$: $\{\alpha_1\}\cup\{\gamma\}\cup\{\alpha_3,\alpha_4,\cdots,\alpha_l\}$.

$(\mathrm{I})_{1,r}+(B_{l-r-1})(2\leqq r\leqq l-2)$: $\{\alpha_1,\alpha_2,\cdots,\alpha_r\}\cup\{\alpha_{r+2},\alpha_{r+3},\cdots,\alpha_l\}$.

$(\mathrm{I})_{2,2}+(B_{l-3})$: $\{\beta_1,\alpha_1,\alpha_2\}\cup\{\alpha_4,\alpha_5,\cdots,\alpha_l\}$.

$(\mathrm{I})'_{1,1}+(B_{l-1})$: $\{\gamma_1\}\cup\{\alpha_2,\alpha_3,\cdots,\alpha_l\}$.

$(\mathrm{IV})_{p-1}$: $\left\{\begin{matrix}\alpha_1, \alpha_2, \cdots, \alpha_{l-2}, \alpha_{l-1}\\ \beta_2\end{matrix}\right\}$.

$(\mathrm{IV})_{p-2}+(A_1)$: $\{\alpha_1,\alpha_2,\cdots,\alpha_{l-2},\beta_3\}\cup\{\alpha_l\}$.

$$((\mathrm{IV})_4=(\mathrm{I})_{2,2},\ (\mathrm{IV})_3=(\mathrm{III})_2)$$

In them, the regular subalgebras

$$(\mathrm{I})_{1,1}+(\mathrm{I})_{1,1}+(B_{l-2}),\ (\mathrm{I})'_{1,1}+(B_{l-1}),\ (\mathrm{I})_{2,2}+(B_{l-3}),$$
$$(\mathrm{IV})_{p-1},\ (\mathrm{IV})_{p-2}+(A_1)$$

are of class (H_2). (Observe that $H_{\gamma_1}=H_1+H_\gamma=2\sqrt{-1}H_0$.) Every maximal non-compact regular subalgebra not of class (H_2) are (k)-equivalent to $(\mathrm{I})_{1,l-1}$.

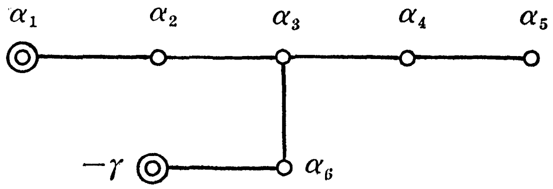
4.4.3. REMARKS. i) Let $p=2l-2$. There is a monomorphism ρ, unique up to within (k)-equivalence, of $(\mathrm{IV})_{p-1}$ into $(\mathrm{IV})_p$ satisfying (H_2) (see § 5, Theorem 4). It will be easily seen that the maximal regular subalgebra $(\mathrm{IV})_{p-2}$ of $(\mathrm{IV})_p$ is (k)-equivalent in $(\mathrm{IV})_p$ to the image $\rho((\mathrm{IV})_{p-2})$ of the maximal regular subalgebra $(\mathrm{IV})_{p-2}$ of $(\mathrm{IV})_{p-1}$. Hence the maximal *regular* subalgebra $(\mathrm{IV})_{p-2}$ of $(\mathrm{IV})_p$ is not a maximal subalgebra.

ii) $p=2l-1$. The regular subalgebras $(\mathrm{I})_{1,p-2}$ and $(\mathrm{I})'_{1,p-2}$ of $(\mathrm{IV})_{p-1}$ are (k)-equivalent in $(\mathrm{IV})_p$, if we consider $(\mathrm{IV})_{p-1}$ as the regular subalgebra of $(\mathrm{IV})_p$ corresponding to the Π-system given in the table in 4.4.2. In fact, $\beta_2=\alpha_{l-1}+2\alpha_l$ is transposed to α_{l-1} by w_l keeping $\alpha_1,\alpha_2,\cdots,\alpha_{l-2}$ invariant.

iii) In view of Proposition 5, we can see at once that the regular subalgebra $(\mathrm{I})'_{1,1}$ of $(\mathrm{IV})_{2l-1}$ is (k)-conjugate to the diagonal subalgebra (in a natural sense) of the regular subalgebra $(\mathrm{I})_{1,1}+(\mathrm{I})_{1,1}$ corresponding to the Π-system

$\{\alpha_1\} \cup \{\gamma\}$. In fact, $H_{\gamma_1} = H_1 + H_\gamma$.

4.5. *The case* $\mathfrak{g} = (\mathrm{EIII})$. The extended Dynkin diagram of $\mathfrak{g}_C$ is as follows:



$; \gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 .$

All positive non-compact roots (resp. all compact roots) are transposed to α_1 (resp. α_2) by elements of W_K. Putting

$$\beta_1 = \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \qquad \beta_2 = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,$$

we have the following table of (k)-equivalence classes of maximal regular subalgebras by the same considerations as above:

$(\mathrm{I})_{1,5} + (\mathrm{I})_{1,1}$: $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} \cup \{\gamma\}$; of class (H_2).

$(\mathrm{I})_{1,2} + (\mathrm{I})_{1,2} + (A_2)$: $\{\alpha_1, \alpha_2\} \cup \{\gamma, -\alpha_6\} \cup \{\alpha_4, \alpha_5\}$: of class (H_2).

$(\mathrm{I})_{2,4} + (A_1)$: $\{\beta_1, \alpha_1, \alpha_2, \alpha_3, \alpha_6\} \cup \{\alpha_5\}$; of class (H_2).

$(\mathrm{II})_5$: $\left\{\begin{matrix}\alpha_1, \alpha_2, \alpha_3, \alpha_4 \\ \beta_2 \end{matrix}\right\}$.

$(\mathrm{IV})_8$: $\left\{\begin{matrix}\alpha_1, \alpha_2, \alpha_3, \alpha_4 \\ \alpha_6 \end{matrix}\right\}$.

(D_5): $\left\{\begin{matrix}\alpha_2, \alpha_3, \alpha_4, \alpha_5 \\ \alpha_6 \end{matrix}\right\}$.

Maximal non-compact regular subalgebras not of class (H_2) are (k)-equivalent to

$$(\mathrm{I})_{1,5}, \qquad (\mathrm{I})_{1,4} + (\mathrm{I})_{1,1}, \qquad (\mathrm{II})_5, \qquad (\mathrm{IV})_8 .$$

REMARK. There are two (k)-equivalence classes of regular subalgebras of type $(\mathrm{I})_{1,4}$: one is

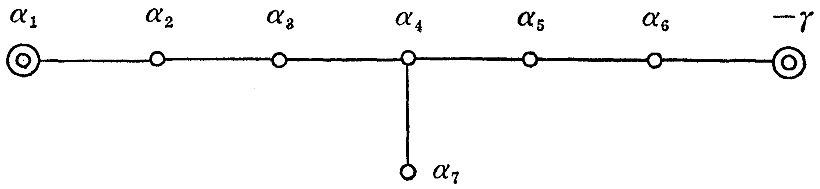
$(\mathrm{I})_{1,4}$: $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, and the other

$(\mathrm{I})'_{1,4}$: $\{\alpha_1, \alpha_2, \alpha_3, \alpha_6\}$.

Both of them are contained in the regular subalgebra $(\mathrm{IV})_8$.

4.6. *The case* $\mathfrak{g} = (\mathrm{EVII})$[2]. The extended Dynkin diagram is as follows:

2) The table in [3] given for this case was wrong: some II-systems in it are not maximal, which are omitted here.



;

$$\gamma=\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7 .$$

By the same considerations as above, we get the following list of (k)-equivalence classes of maximal regular subalgebras; here β_1, β_2, and β_3 denote the compact roots as follows:

$$\beta_1=\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+2\alpha_7 ,$$
$$\beta_2=\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7 ,$$
$$\beta_3=\alpha_4+\alpha_5+\alpha_6+\alpha_7 .$$

$(\mathrm{I})_{1,5}+(\mathrm{I})_{1,2}$: $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_7\} \cup \{\gamma, -\alpha_6\}$; of class (H_2).

$(\mathrm{I})_{1,3}+(\mathrm{I})_{1,3}+(A_1)$: $\{\alpha_1, \alpha_2, \alpha_3\} \cup \{\gamma, -\alpha_6, -\alpha_5\}$; of class (H_2).

$(\mathrm{I})_{2,6}$: $\{\beta_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$; of class (H_2).

$(\mathrm{I})_{3,3}+(A_2)$: $\{-\alpha_7, \beta_1, \alpha_1, \alpha_2, \alpha_3\} \cup \{\alpha_5, \alpha_6\}$; of class (H_2).

$(\mathrm{II})_6+(A_1)$: $\left\{\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & \alpha_7 \\ & \beta_2 & & & \end{matrix}\right\}$; of class (H_2).

$(\mathrm{IV})_{10}+(\mathrm{I})_{1,1}$: $\left\{\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & \alpha_5 \\ & & & \alpha_7 & \end{matrix}\right\}$; of class (H_2).

(EIII): $\left\{\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & \alpha_5 \\ & & \beta_3 & & \end{matrix}\right\}$.

(E_6): $\left\{\begin{matrix}\alpha_2, & \alpha_3, & \alpha_4, & \alpha_5, & \alpha_6 \\ & & \alpha_7 & & \end{matrix}\right\}$.

Maximal non-compact regular subalgebras not of class (H_2) are (k)-equivalent to one of the following:

$$(\mathrm{I})_{1,4}+(\mathrm{I})_{1,2},\ (\mathrm{I})_{1,5}+(\mathrm{I})_{1,1},\ (\mathrm{I})'_{1,5}+(\mathrm{I})_{1,1} ,$$
$$(\mathrm{I})_{1,6},\ (\mathrm{I})_{2,5},\ (\mathrm{IV})_{10},\ (\mathrm{EIII}) ,$$

where $(\mathrm{I})_{1,5}$ and $(\mathrm{I})'_{1,5}$ are not (k)-equivalent to each other:

$$(\mathrm{I})_{1,5}: \quad \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_7\} ,$$
$$(\mathrm{I})'_{1,5}: \quad \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} .$$

Both $(\mathrm{I})_{1,5}$ and $(\mathrm{I})'_{1,5}$ are contained in the regular subalgebra $(\mathrm{IV})_{10}$ in the above table.

§ 5. Determination of homomorphisms satisfying (H_2).

In this section, we determine all pairs $(\mathfrak{g}, \rho)$ of a simple Lie algebra $\mathfrak{g}$ of hermitian type and a monomorphism ρ of $\mathfrak{g}$ into a simple Lie algebra $\mathfrak{g}'$ of type $(\mathrm{IV})_p$, (EIII), and (EVII) satisfying the condition (H_2).

5.1. In the first place, we consider generally the case $\mathfrak{g}$ is isomorphic to $\mathfrak{g}'$. Identifying $\mathfrak{g}'$ with $\mathfrak{g}$ by the isomorphism, our problem becomes to determine all automorphisms ρ of $\mathfrak{g}$ such as $\rho(H_0)=H_0'$, where H_0 and H_0' are complex structures of $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{g}, \mathfrak{k}')$ respectively. Modyfying $\mathfrak{k}'$ by an inner automorphism of $\mathfrak{g}$, we may assume that $\rho(\mathfrak{k})=\mathfrak{k}'=\mathfrak{k}$; then H_0' becomes H_0 or $-H_0$ since $\mathfrak{g}$ is simple. Furthermore, we may also assume that $\rho(\mathfrak{h})=\mathfrak{h}$, i.e. that ρ induces an automorphism of $\mathfrak{h}$. Under these assumptions, it is easy to see the following facts (cf. Corollary to Proposition 7):

i) If $H_0=H_0'$, ρ is (k)-equivalent to an automorphism a of $\mathfrak{g}$ caused by an automorphism of the Dynkin diagram of $\mathfrak{g}_C$ keeping the non-compact simple root invariant;

ii) If $H_0'=-H_0$, ρ is (k)-equivalent to an automorphism $\bar{a}$ gvien by a composition of an automorphism of $\mathfrak{g}$, associated with the permutation of roots such as $\alpha\to-\alpha$ for any root α, and an automorphism a got in i) for the case $H_0=H_0'$.

Hence we have got the following *trivial* solutions:

A) $\mathfrak{g}=\mathfrak{g}'=(\mathrm{I})_{p,q}(p=q)$, $(\mathrm{II})_p$, $(\mathrm{III})_p$, $(\mathrm{IV})_p$ $(p=\text{odd})$, (EIII), (EVII).

$$\rho\sim id.,\ \overline{id.}\ \text{(according as } H_0'\sim H_0 \text{ or } -H_0).$$

B) $\mathfrak{g}=\mathfrak{g}'=(\mathrm{I})_{p,q}$, $(\mathrm{IV})_p$ $(p=\text{even})$.

$$\rho\sim\begin{cases} id.,\ \overline{id.}\ \text{(according as } H_0'\sim H_0 \text{ or } -H_0),\\ \rho_0,\ \bar{\rho}_0\ (\qquad\quad ,, \qquad\qquad\qquad\qquad). \end{cases}$$

In this table, ρ_0 denotes the outer automorphism caused by the automorphism of the Dynkin diagram different from the identity: in view of Theorem 3, it can be said that ρ_0 is determined (up to (k)-equivalence) by the automorphism Θ of $\mathfrak{h}_C$ such as

$$\Theta(H_1)=H_1,\ \Theta(H_{p\pm i})=H_{p\pm i+1}(0\leqq i\leqq p-2) \text{ for } (\mathrm{I})_{p,p};$$

$$\Theta(H_i)=H_i(1\leqq i\leqq l-2),\ \Theta(H_{l-1})=H_l,\ \Theta(H_l)=H_{l-1} \text{ for } (\mathrm{IV})_p.$$

In the following, we shall consider only the case $\mathfrak{g}$ is not isomorphic to $\mathfrak{g}'$.

5.2. *The case* $\mathfrak{g}'=(\mathrm{IV})_{p'}$. Let $(\mathfrak{g}, \rho)$ be a pair satisfying the above conditions, and $H_0=\sqrt{-1}\sum_{1\leqq i\leqq l} a_iH_i$ the fixed complex structure of $(\mathfrak{g}, \mathfrak{k})$. The complex structure of $(\mathfrak{g}', \mathfrak{k}')$ are expressed as

$$H_0' = \frac{\sqrt{-1}}{2}(2H_1' + \cdots + 2H_{l'-2}' + (2)H_{l'-1}' + H_{l'}'),\ (2) = \begin{cases} 2\ (p' = 2l'-1) \\ 1\ (p' = 2l'-2). \end{cases}$$

Hence, from Proposition 6 iii), we have $a_1 m_\rho = 1$. Since the number of elements of Δ_0' (see 3.2) is 2, m_ρ is equal to 1 or 2. Now that ρ is injective, the dimension of $\mathfrak{h}$ can not be larger than that of $\mathfrak{h}'$, and neither the number of elements of Δ_0 (i.e. the rank of $(\mathfrak{g}, \mathfrak{k})$) than that of Δ_0'.

In the first place, we treat the case $\mathfrak{g} = (\mathrm{IV})_p$. For convenience, p is admitted to be 1, 2, 3, or 4, and p' to be 2, 3, or 4 by the following identifications:

$$(\mathrm{IV})_1 = (\mathrm{I})_{1,1},\ (\mathrm{IV})_2 = (\mathrm{I})_{1,1} + (\mathrm{I})_{1,1},$$

$$(\mathrm{IV})_3 = (\mathrm{III})_2,\ (\mathrm{IV})_4 = (\mathrm{I})_{2,2}$$

(though $(\mathrm{IV})_2$ is not simple). If $\mathfrak{g} \neq (\mathrm{IV})_1$, we have $a_1 = a_1' = 1$, and hence $m_\rho = 1$. For $\mathfrak{g} = (\mathrm{IV})_1$, $a_1 = \frac{1}{2}$, so $m_\rho = 2$.

i) Let $p' = 2l'-2$, $p = p'-1$, and suppose that we have a monomorphism ρ satisfying (H_2). Clearly, $l = l'-1$. By Proposition 7 (and Proposition 5 for the case $p' = 2$), we can modify ρ by (k)-equivalence so as to make it to satisfy the following relations:

$$\rho(H_i) = H_i'(1 \leqq i \leqq l-1 = l'-2),\ \rho(H_l) = \sum_{j=1}^{l'} \mu_j H_j',$$

where μ_j's are integers and $\mu_1 = 0$ if $l \neq 1$. Proposition 6 ii) implies

$$\frac{1}{\delta'}\langle H_i', \rho(H_l)\rangle' = \begin{cases} 0 & (1 \leqq i \leqq l'-3) \\ -1 & (i = l-1 = l'-2), \end{cases}$$

$$\frac{1}{\delta'}\langle \rho(H_l), \rho(H_l)\rangle' = 2.$$

Hence we have

$$\mu_2 = \mu_3 = \cdots = \mu_{l'-2} = 0,$$

$$\frac{1}{2}(\mu_{l'-1} + \mu_{l'}) = 1,\ \mu_{l'-1}^2 + \mu_{l'}^2 = 2,$$

and so $\rho(H_l) = H_{l'-1}' + H_{l'}'$. Now we determine all compact roots satisfying the relation (47) in § 3. If $i \leqq l-1$, it is easy to see by induction that the root $\alpha' = \alpha_i'$ is the unique compact root such as

$$\alpha'(\rho(H_1)) = \cdots = \alpha'(\rho(H_{i-2})) = 0,$$

$$\alpha'(\rho(H_{i-1})) = -1,\ \alpha'(\rho(H_i)) = 2.$$

Let $\alpha' = \sum c_j \alpha_j$ be again a compact root satisfying now $\alpha'(\rho(H_i)) = 0$ $(i \leqq l-2)$. Then we can see at once that $c_j = 0$ $(j \leqq l'-2)$; hence α' must be equal to $\pm\alpha_{l'-1}'$ or $\pm\alpha_{l'}'$. If α' satisfies moreover

$$\alpha'(\rho(H_{l-1}))=-1, \qquad \alpha'(\rho(H_{l-1}))=2,$$

then we have $\alpha'=\alpha'_{l'-1}$ or $\alpha'_{l'}$. Thus, for the present case, we have proved Proposition 9 completely:

$$(50)\qquad \begin{aligned}&\rho(H'_i)=H'_i, \qquad \rho(E_{\pm\alpha_i})=E'_{\pm\alpha'_i}\ (1\leqq i\leqq l-1=l'-2),\\ &\rho(H_l)=H'_{l'-1}+H'_{l'}, \qquad \rho(E_{\pm\alpha_l})=E'_{\pm\alpha_{l'-1}}+E'_{\pm\alpha'_{l'}}.\end{aligned}$$

and all the other homomorphisms satisfying (H_2) are equivalent to ρ. (Observe that ρ is invariant under the automorphism ρ_0 of $\mathfrak{g}'$ given in 5.1.)

ii) Let $p'=2l'-1$, $p=p'-1$ (hence $l=l'$), and suppose that we have a monomorphism ρ satisfying (H_2). From Corollary to Proposition 7, it is easy to see that ρ can be modified by (k)-equivalence to satisfy

$$(51)\qquad \begin{aligned}&\rho(H_i)=H'_i, \qquad \rho(E_{\pm\alpha_i})=E'_{\pm\alpha'_i}\ (1\leqq i\leqq l-1=l'-1),\\ &\rho(H_l)=H'_{\beta'}=H'_{l'-1}+H'_{l'}, \qquad \rho(E_{\pm\alpha_l})=E'_{\pm\beta'},\end{aligned}$$

where $\beta'=\alpha'_{l'-1}+2\alpha'_{l'}$ is a compact root of the maximal length. Hence there is one and only one (k)-equivalence class of homomorphisms of $\mathfrak{g}=(\mathrm{IV})_{2l'-2}$ into $\mathfrak{g}'=(\mathrm{IV})_{2l'-1}$ satisfying (H_2). The image of $\mathfrak{g}$ under the homomorphism ρ defined by (51) is the regular subalgebra $(\mathrm{IV})_{p'-1}$ of $(\mathrm{IV})_{p'}$.

Combining the results of i) and ii), it can be easily seen that there is a unique (k)-equivalence class of homomorphism of $\mathfrak{g}=(\mathrm{IV})_p$ $(1\leqq p\leqq p'-1)$ into $\mathfrak{g}'=(\mathrm{IV})_{p'}$ $(p'\geqq 2)$ satisfying (H_2). Particularly, $\mathfrak{g}=(\mathrm{I})_{1,1}$ is imbedded in $\mathfrak{g}'=(\mathrm{IV})_{p'}$ by the monomorphism defined by

$$\rho(H_1)=H'_1+H'_{\gamma'}, \qquad \rho(E_{\pm\alpha_1})=E'_{\pm\alpha'_1}+E'_{\pm\gamma'},$$

where γ' is the highest root of $\mathfrak{g}'$; the image $\rho((\mathrm{I})_{1,1})$ is the diagonal subalgebra of the regular subalgebra $(\mathrm{I})_{1,1}+(\mathrm{I})_{1,1}$ of $\mathfrak{g}'$; if $p'=2l'-1$, there is a homomorphism ρ_1, (k)-equivalent to ρ, of $(\mathrm{I})_{1,1}$ into $(\mathrm{IV})_{p'}$ such that the image is the regular subalgebra $(\mathrm{I})'_{1,1}$ given in the table in 4.4.2:

$$\rho_1(H_1)=H'_1+H'_{\gamma'}=H'_{\gamma'_1}, \qquad \rho_1(E_{\pm\alpha_1})=E'_{\pm\gamma'_1}.$$

Now, let $\mathfrak{g}$ be of type $(\mathrm{I})_{p,q}$ $(p\leqq q)$, and suppose that we get a ρ. Clearly, p is at most 2 and $p+q\leqq l'+1$. By the expression of H_0, we have

$$m_\rho\frac{pq}{p+q}=1, \quad \text{i.e.} \quad (m_\rho p-1)(m_\rho q-1)=1.$$

For this equation, we get only two solutions of positive integers:

$$m_\rho=1,\ p=q=2; \qquad m_\rho=2,\ p=q=1.$$

But these two cases are already examined above as the cases $\mathfrak{g}=(\mathrm{IV})_4$ and $\mathfrak{g}=(\mathrm{IV})_1$ respectively.

In the second place, let $\mathfrak{g}$ be of type $(\mathrm{III})_p$ $(p\geqq 2)$. Since the rank of $(\mathfrak{g}, \mathfrak{k})$

is p, the only possible case for the existence of ρ is that $p=2$. But that is already examined above as the case $\mathfrak{g}=(\mathrm{IV})_3$.

For the remaining cases $\mathfrak{g}=(\mathrm{II})_p$ $(p\geqq 5)$, (EIII), and (EVII), it is easy to see from the relation $a_1 m_\rho=1$ that there is no homomorphism satisfying (H_2).

Thus we have proved the following

THEOREM 4. *Let* $(\mathfrak{g}, \rho)$ *be a pair of a simple Lie algebra of hermitian type and a homomorphism into* $\mathfrak{g}'=(\mathrm{IV})_{p'}$ *satisfying* (H_2) *w.r.t. a fixed complex structures of* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *respectively. Then* $\mathfrak{g}$ *is of type* $(\mathrm{IV})_p$ $(1\leqq p\leqq p'-1)$ *and* ρ *is equivalent to that derived inductively from* (50) *and* (51). *For the case* $p=1$, m_ρ *is equal to* 2, *and for others,* $m_\rho=1$.

5.3. *The case* $\mathfrak{g}'=(\mathrm{EIII})$. In this case, the rank of $(\mathfrak{g}', \mathfrak{k}')$ is equal to 2. Let $(\mathfrak{g}, \rho)$ be a pair what we are looking for and $H_0=\sqrt{-1}\sum_{1\leqq i\leqq l} a_i H_i$ the fixed complex structure of $(\mathfrak{g}, \mathfrak{k})$. Then Proposition 6 iii) implies $a_1 m_\rho=\frac{4}{3}$, where m_ρ is at most 2. Hence it is easy to see that $\mathfrak{g}$ can not be other than of type $(\mathrm{I})_{p,q}$. Firstly, let m_ρ be equal to 1. Then the image $\rho(\mathfrak{g})$ must be a regular subalgebra of $\mathfrak{g}'$. On the other hand, we have

$$\frac{pq}{p+q}=\frac{4}{3}, \quad \text{and so } (3p-4)(3q-4)=16\,.$$

This equation has a unique solution of positive integers: $p=2$, $q=4$. We have seen in 4.5 that there is certainly a regular subalgebra $(\mathrm{I})_{2,4}$ of class (H_2) represented by the Π-system $\{\beta_1', \alpha_1', \alpha_2', \alpha_3', \alpha_6'\}$, where $\beta_1'=\alpha_2'+2\alpha_3'+2\alpha_4'+\alpha_5'+\alpha_6'$. Therefore we can see that there is a monomorphism of $(\mathrm{I})_{2,4}$ into (EIII), unique up to (k)-equivalence: for instance, a representative ρ is defined by

$$(52)\qquad \begin{aligned} &\rho(H_i)=H_i', \qquad \rho(E_{\pm\alpha_i})=E'_{\pm\alpha'_i}\ (1\leqq i<5)\,,\\ &\rho(H_5)=H'_{\beta'}, \qquad \rho(E_{\pm\alpha_5})=E'_{\pm\beta'}\,. \end{aligned}$$

Secondly, let m_ρ be equal to 2. Then we have

$$\frac{2pq}{p+q}=\frac{4}{3}, \quad \text{and so } (3p-2)(3q-2)=4\,.$$

Hence we can see at once that $\mathfrak{g}$ must be of type $(\mathrm{I})_{1,2}$. By Proposition 5, the homomorphism ρ can be modified by (k)-equivalence so as to satisfy

$$\rho(H_1)=H_1'+H_{\gamma'}', \qquad \rho(H_2)=\sum_{2\leqq j\leqq 6}\mu_j H_j',$$

where γ' denotes the highest root of $\mathfrak{g}_C'$. By Proposition 6 ii), we have

$$(53)\qquad \begin{aligned} &\mu_2-\mu_6=2\,,\\ &\mu_2^2+\cdots+\mu_6^2-(\mu_2\mu_3+\mu_3\mu_4+\mu_3\mu_6+\mu_4\mu_5)=2\,. \end{aligned}$$

Since μ_j's are rational integers, it follows easily from (53) that

$$\mu_2 = 1, \qquad \mu_3 = \mu_4 = \mu_5 = 0, \qquad \mu_6 = -1.$$

Thus we have got a homomorphism ρ defined by

$$\begin{aligned} &\rho(H_1) = H_1' + H_{\gamma'}', \qquad \rho(E_{\pm\alpha_1}) = E'_{\pm\alpha'_1} + E'_{\pm\gamma'}, \\ &\rho(H_2) = H_2' - H_6' = H_2' + H'_{-\alpha_6}, \qquad \rho(E'_{\pm\alpha_2}) = E'_{\pm\alpha_2} + E'_{\mp\alpha_6}, \end{aligned} \tag{54}$$

It can be easily seen that ρ satisfies (H_2). Now, let $\alpha' = \sum c_j \alpha_j'$ be a compact root satisfying

$$\alpha'(\rho(H_1)) = -1, \qquad \alpha'(\rho(H_2)) = 2. \tag{55}$$

The first condition in (55) implies

$$\alpha'(H_1') = -1, \qquad \alpha'(H_{\gamma'}') = 0,$$

or

$$\alpha'(H_1') = 0, \qquad \alpha'(H_{\gamma'}') = -1,$$

and hence $(c_2, c_6) = (1, 0)$ or $(0, -1)$. Then, from the second condition of (55), it follows that $\alpha' = \alpha_2$ or $-\alpha_6$. Thus, for the present case, we have proved Proposition 9 completely: all homomorphisms of $(\mathrm{I})_{1,2}$ into (EIII) satisfying (H_2) are equivalent to ρ defined by (54).

We have proved the following

THEOREM 5. *Let $\mathfrak{g}'$ be of type* (EIII). *Then every pair $(\mathfrak{g}, \rho)$ of a simple Lie algebra of hermitian type and a homomorphism into $\mathfrak{g}'$ satisfying (H_2) is equivalent to one of the following:*

i) *$\mathfrak{g}$ is of type $(\mathrm{I})_{2,4}$, ρ is determined by* (52);
   $m_\rho = 1$, $\rho(\mathfrak{g}) =$ the regular subalgebra $(\mathrm{I})_{2,4}$ of (EIII).

ii) *$\mathfrak{g}$ is of type $(\mathrm{I})_{1,2}$, ρ is determined by* (54);
   $m_\rho = 2$, $\rho(\mathfrak{g}) =$ the diagonal subalgebra of the regular subalgebra
   $(\mathrm{I})_{1,2} + (\mathrm{I})_{1,2}$.

5.4. *The caes* $\mathfrak{g}' = (\mathrm{EVII})$. Since the rank of $(\mathfrak{g}', \mathfrak{k}')$ is 3, the integer m_ρ defined as before is at most 3; the coefficient a_1' in the expression of $\frac{1}{\sqrt{-1}} H_0'$ is equal to $\frac{3}{2}$.

Firstly, let $\mathfrak{g}$ be of type $(\mathrm{I})_{p,q}$. We have the following four solutions of the equation $m_\rho \frac{pq}{p+q} = \frac{3}{2}$:

i) $m_\rho = 1$, $p = 2$, $q = 6$.
ii) $m_\rho = 1$, $p = q = 3$.
iii) $m_\rho = 2$, $p = 1$, $q = 3$.
iv) $m_\rho = 3$, $p = q = 1$.

If $m_\rho = 1$, it follows from Corollary to Proposition 7 that the image $\rho(\mathfrak{g})$ must be (k)-conjugate to a regular subalgebra of $\mathfrak{g}'$ given in the table in 4.6. There are really regular subalgebras $(\mathrm{I})_{2,6}$ and $(\mathrm{I})_{3,3}$, of class (H_2), in it. Hence we get a homomorphism ρ of $\mathfrak{g}$ into $\mathfrak{g}'$ defined uniquely up to (k)-equivalence by

$$\rho(H_i)=H'_i\ (1\leqq i\leqq 6),\quad \rho(H_7)=H'_{\beta'}\ \text{for}\ \mathfrak{g}=(\mathrm{I})_{2,6},$$

and

$$\rho(H_i)=H'_i\ (1\leqq i\leqq 3),\quad \rho(H_4)=-H'_7,\quad \rho(H_5)=H'_{\beta'}\ \text{for}\ \mathfrak{g}=(\mathrm{I})_{3,3},$$

where β' denotes the compact root $\alpha'_2+2\alpha'_3+3\alpha'_4+2\alpha'_5+\alpha'_6+2\alpha'_7$. The automorphism ρ_0 of the regular subalgebra $(\mathrm{I})_{3,3}$ defined in 5.1 is induced by an inner automorphism of $\mathfrak{g}'$ corresponding to an element w of $W_{K'}$ such as

$$w:\ \alpha'_1\longleftrightarrow\alpha'_1,\qquad \alpha'_2\longleftrightarrow\beta',\qquad \alpha'_3\longleftrightarrow-\alpha'_7\,;$$

for instance, $w=w_7(w_0)^2w_3w_4w_5w_7w_4w_3w_0^{-1}$, where w_i $(i\neq 0)$ denotes the reflection associated with α'_i (cf. (27)) and $w_0=w_7w_4w_5w_6$. Hence $\rho_0\circ\rho$ is (k)-equivalent to ρ. For other homomorphisms satisfying (H_2), it is easy to see that they are (k)-equivalent to ρ. Now, for iv), it follows from Proposition 5 that all homomorphisms of $\mathfrak{g}=(\mathrm{I})_{1,1}$ into $\mathfrak{g}'$ satisfying (H_2) are (k)-equivalent to ρ determined by

$$\rho(H_1)=H'_1+H'_{\gamma'_1}+H'_{\gamma'},\qquad \rho(E_{\pm\alpha_1})=E'_{\pm\alpha'_1}+E'_{\pm\gamma'_1}+E'_{\pm\gamma'},$$

where γ' is the highest root, and γ'_1 is a positive non-compact root strongly orthogonal to both α'_1 and γ'. The image $\rho(\mathfrak{g})$ is the diagonal subalgebra of the regular subalgebra whose Π-system is $\Delta'_0=\{\alpha'_1,\gamma'_1,\gamma'\}$. Now, let $\mathfrak{g}$ be of type $(\mathrm{I})_{1,3}$ (the case iii)), and suppose we get a homomorphism of it into $\mathfrak{g}'$ satisfying (H_2). Modifying ρ by (k)-equivalence, we may put

$$\rho(H_1)=H'_1+H'_{\gamma'},\qquad \rho(H_2)=\sum_{j=2}^{7}\mu_jH'_j,\qquad \rho(H_3)=\sum_{j=2}^{7}\nu_jH'_j.$$

On the other hand, we have from Proposition 6 ii) the following relations:

$$\frac{1}{\delta'}\langle\rho(H_2),\ \rho(H_2)\rangle'=\frac{1}{\delta'}\langle\rho(H_3),\ \rho(H_3)\rangle'=2, \tag{56}$$

$$\frac{1}{\delta'}\langle\rho(H_1),\ \rho(H_2)\rangle'=\frac{1}{\delta'}\langle\rho(H_2),\ \rho(H_3)\rangle'=-1, \tag{57}$$

$$\frac{1}{\delta'}\langle\rho(H_1),\ \rho(H_3)\rangle'=0. \tag{58}$$

By (56), we see that the rational integers μ_j $(2\leqq j\leqq 7)$ and ν_j $(2\leqq j\leqq 7)$ are solutions of

$$x_2^2+x_3^2+\cdots+x_7^2-(x_2x_3+x_3x_4+x_4x_5+x_5x_6+x_4x_7)=2. \tag{59}$$

If $\{x_2, x_3, \cdots, x_7\}$ is an integral solution of (59), we put

$$N=\sum_{j=2}^{7}x_jH'_j=(x_2, x_3, \cdots, x_7)\,;$$

the vector N of $\sqrt{-1}\mathfrak{h}'$ will be called a (integral) solution of (59). The relation (57) implies $\mu_2-\mu_6=2$. Putting $x_2-x_6=2$. we get the following 9 integral solutions of (59):

$$N_1=(1,0,0,0,-1,0),\ N_2=(1,1,0,0,-1,0),\ N_3=(1,1,1,0,-1,0),$$
$$N_4=(1,1,1,0,-1,1),\ N_5=(1,0,0,-1,-1,0),\ N_6=(1,0,-1,-1,-1,0),$$
$$N_7=(1,0,-1,-1,-1,-1),\ N_8=(1,1,0,-1,-1,0),\ N_9=(2,2,2,1,0,1).$$

All N_i but N_9 are transposed to N_1 by elements of $N_{K'}(\mathfrak{h}')$ keeping H_1 invariant. In fact, denoting by k_i an element of the class in $N_{K'}(\mathfrak{h}')$ modulo $Z_{K'}(\mathfrak{h}')$ corresponding to the reflection $w_i\in W_{K'}$, we have the following diagram.

$$\begin{tikzcd}
N_4 \arrow[r, leftrightarrow, "k_7"] & N_3 \arrow[r, leftrightarrow, "k_4"] & N_2 \arrow[r, leftrightarrow, "k_3"] \arrow[d, leftrightarrow, "k_5"'] & N_1 \arrow[d, leftrightarrow, "k_5"] & & \\
& & N_8 \arrow[r, leftrightarrow, "k_3"'] & N_5 \arrow[r, leftrightarrow, "k_4"'] & N_6 \arrow[r, leftrightarrow, "k_7"'] & N_7
\end{tikzcd}$$

Therefore we may assume that $\rho(H_2)$ is equal to $N_1=H_2'-H_6'$ or N_9. By (57), we have $\nu_2=\nu_6$. On the other hand, the relation (58) implies that $\nu_3-\nu_5=2$ if $\rho(H_2)=N_1$, or that $-2\nu_2+\nu_6=2$ if $\rho(H_2)=N_9$. Under the relations $x_2=x_6$ and $x_3-x_5=2$, we have only one integral solution $(0,1,0,-1,0,0)$ of (59). On the other hand, there is no solution of (59) satisfying $x_2=x_6$ and $-2x_2+x_6=2$. Hence we get a (k)-equivalence class of homomorphisms of $\mathfrak{g}=(\mathrm{I})_{1,3}$ into $\mathfrak{g}'=(\mathrm{EVII})$ satisfying (H_2) represented by that defined by

$$(60)\qquad \begin{aligned}
&\rho(H_1)=H_1'+H_{\gamma'}', &&\rho(E_{\pm\alpha_1})=E'_{\pm\alpha'_1}+E'_{\pm\gamma'},\\
&\rho(H_2)=H_2'+H'_{-\alpha_6}, &&\rho(E_{\pm\alpha_2})=E'_{\pm\alpha'_2}+E'_{\mp\alpha'_6},\\
&\rho(H_3)=H_3'+H'_{-\alpha_5}, &&\rho(E_{\pm\alpha_3})=E'_{\pm\alpha'_3}+E'_{\mp\alpha'_5};
\end{aligned}$$

the sets $S(1)=\{\alpha_1',\gamma'\}$, $S(2)=\{\alpha_2',-\alpha_6'\}$, $S(3)=\{\alpha_3',-\alpha_5'\}$ are strongly orthogonal systems. Let α' be a compact root satisfying

$$\alpha'(\rho(H_1))=-1,\qquad \alpha'(\rho(H_2))=2.$$

Then, by the same way as before, we can see that α' is equal to α_2' or $-\alpha_6'$ and hence an element of $S(2)$. Let further $\beta'=\sum c_j\alpha_j'$ be a compact root satisfying

$$(61)\qquad \beta'(\rho(H_1))=0,\qquad \beta'(\rho(H_2))=-1,\qquad \beta'(\rho(H_3))=2.$$

The first condition in (61) implies

$$(c_2,c_6)=(0,0),\ (1,1),\ \text{or}\ (-1,-1),$$

and the second is equivalent to say

$$(62)\qquad \beta'(H_3')=-1\ (\text{resp. }0),\qquad \beta'(H_5')=0\ (\text{resp. }-1).$$

Hence, if $(c_2, c_6)=(0, 0)$, we see from the last condition that β' is equal to α'_3 or $-\alpha'_5$, i.e. an element of $S(3)$; for other cases (62) implies $2(c_3-c_5)=-1$, which can not happen. Therefore, by Proposition 9, we conclude that there is no other (k)-equivalence class than that represented by ρ. The image $\rho(\mathfrak{g})$ is clearly the diagonal subalgebra of the regular subalgebra $(\mathrm{I})_{1,3}+(\mathrm{I})_{1,3}$ of $\mathfrak{g}'$.

For the case $\mathfrak{g}$ is of type $(\mathrm{II})_p$ $(p\geqq 5)$, it is easy to see that p must be equal to 6, m_ρ to 1. Hence we get a unique (k)-equivalence class represented by a homomorphism ρ such that the image $\rho(\mathfrak{g})$ is the regular subalgebra $(\mathrm{II})_6$ of (EVII).

Now, let $\mathfrak{g}$ be of type $(\mathrm{III})_p$. The possible case is only that $m_\rho=1$ and $p=3$. Suppose we get a homomorphism ρ of $\mathfrak{g}=(\mathrm{III})_3$ into $\mathfrak{g}'=(\mathrm{EVII})$ satisfying (H_2). We may put

$$\rho(H_1)=H'_1, \qquad \rho(H_2)=\sum_{j=2}^{7}\mu_j H'_j, \qquad \rho(H_3)=\sum_{j=2}^{7}\nu_j H'_j.$$

From Proposition 6 ii), we have the following relations:

$$\frac{1}{\delta'}\langle\rho(H_2),\ \rho(H_2)\rangle'=\frac{1}{\delta'}\langle\rho(H_3),\ \rho(H_3)\rangle'=2\,, \tag{63}$$

$$\frac{1}{\delta'}\langle H'_1,\ \rho(H_2)\rangle'=\frac{1}{\delta'}\langle\rho(H_2),\ \rho(H_3)\rangle'=-1\,, \tag{64}$$

$$\frac{1}{\delta'}\langle H'_1,\ \rho(H_3)\rangle'=0\,. \tag{65}$$

The relation (63) implies that the rational integers μ_j $(2\leqq j\leqq 7)$ and ν_j $(2\leqq j \leqq 7)$ are solutions of (59). Then we can see that $\rho(H_2)$ should be equal to one of N_i $(1\leqq i\leqq 10)$ defined as follows:

$$\begin{aligned}
&N_1=(2, 2, 3, 2, 1, 2), \quad N_2=(2, 2, 3, 2, 1, 1), \quad N_3=(2, 2, 2, 2, 1, 1)\,,\\
&N_4=(2, 2, 2, 1, 1, 1), \quad N_5=(2, 2, 2, 1, 0, 1), \quad N_6=(2, 3, 3, 2, 1, 2)\,,\\
&N_7=(2, 3, 4, 2, 1, 2), \quad N_8=(2, 3, 4, 3, 1, 2), \quad N_9=(2, 3, 4, 3, 2, 2)\,,\\
&N_{10}=(2, 3, 3, 2, 1, 1)\,.
\end{aligned}$$

Taking $k_i\in N_{K'}(\mathfrak{h}')$ as above, we get easily the following diagram:

$$\begin{tikzcd}
N_5 \arrow[r, leftrightarrow, "k_6"] & N_4 \arrow[r, leftrightarrow, "k_5"] & N_3 \arrow[r, leftrightarrow, "k_4"] & N_2 \arrow[r, leftrightarrow, "k_7"] \arrow[d, leftrightarrow, "k_3"'] & N_1 \arrow[d, leftrightarrow, "k_3"] & & & \\
& & & N_{10} \arrow[r, leftrightarrow, "k_7"'] & N_6 \arrow[r, leftrightarrow, "k_4"'] & N_7 \arrow[r, leftrightarrow, "k_5"'] & N_8 \arrow[r, leftrightarrow, "k_6"'] & N_9
\end{tikzcd}$$

Hence in any case, we can modify ρ by (k)-equivalence keeping H'_1 invariant so as to satisfy

(66) $$\rho(H_2)=N_1=2H'_2+2H'_3+3H'_4+2H'_5+H'_6+2H'_7=H'_2+H'_{\beta'},$$

where

(67) $$\beta'=\alpha'_2+2\alpha'_3+3\alpha'_4+2\alpha'_5+2\alpha'_6+2\alpha'_7$$

is a compact root of $\mathfrak{g}'=$(EVII). Then the relations (64) and (65) with (66) implies

(68) $$\nu_2=0, \qquad \nu_2-(\nu_3/2)+(\nu_7/2)=-1.$$

There is only one solution $H'_3-H'_7$ of (59) satisfying (68). Thus we have a homomorphism satisfying (H_2) defined by

(69) $$\begin{aligned}&\rho(H_1)=H'_1, && \rho(E_{\pm\alpha_1})=E'_{\pm\alpha'_1},\\ &\rho(H_2)=H'_2+H'_{\beta'}, && \rho(E_{\pm\alpha_2})=E'_{\pm\alpha'_2}+E'_{\pm\beta'},\\ &\rho(H_3)=H'_3-H'_7, && \rho(E_{\pm\alpha_3})=E'_{\pm\alpha'_3}+E'_{\pm\alpha'_7};\end{aligned}$$

the sets $S(1)=\{\alpha'_1\}$, $S(2)=\{\alpha'_2, \beta'\}$, $S(3)=\{\alpha'_3, -\alpha'_7\}$ are strongly orthogonal systems. Let $\alpha'=\sum c_j\alpha'_j$ be a compact root satisfying

(70) $$\alpha'(\rho(H_1))=-1, \qquad \alpha'(\rho(H_2))=2,$$

Recall that β' given by (67), is contained in the Π-system corresponding to the regular subalgebra $(\mathrm{I})_{2,6}$ of $\mathfrak{g}'$ given in the table in 4.6. Then we see that β' is strongly orthogonal to all simple roots but α'_1 and α'_7, and that $\beta'(H_1)=-1$, $\beta'(H_7)=1$. Then it follows from the condition (70) that α' is equal to α'_2 or β'. By the same way, we can easily see that a compact root α' satisfying

$$\alpha'(\rho(H_1))=0, \qquad \alpha'(\rho(H_2))=-1, \qquad \alpha'(\rho(H_3))=0$$

is an element of $S(3)$. Therefore, for the present case, we have proved Proposition 9 completely, and so we see that there is a unique equivalence class of homomorphisms of $\mathfrak{g}=(\mathrm{III})_3$ into $\mathfrak{g}'$ satisfying (H_2) whose representative is given by the above ρ. The image of $\mathfrak{g}$ by ρ is clearly contained in the regular subalgebra $(\mathrm{I})_{3,3}$ of $\mathfrak{g}'$; we see at once that ρ is invariant by the automorphism ρ_0, defined in 5.1, of the regular subalgebra $(\mathrm{I})_{3,3}$.

There is no solution such that $\mathfrak{g}=(\mathrm{IV})_p$ $(p\geqq 5)$ or (EIII).

Thus we have proved the following

THEOREM 6. *Let* $\mathfrak{g}'$ *be of type* (EVII). *Then every pair* $(\mathfrak{g}, \rho)$ *of a simple Lie algebra of hermitian type and a homomorphism into* $\mathfrak{g}'$ *satisfying* (H_2) *is equivalent to one of those given as follows:*

i) $\mathfrak{g}=(\mathrm{I})_{2,6}$, $\rho(H_i)=H'_i$ $(1\leqq i\leqq 6)$, $\rho(H_7)=H'_{\beta'}$;
$m_\rho=1$, $\rho(\mathfrak{g})=$ *the regular subalgebra* $(\mathrm{I})_{2,6}$ *of* $\mathfrak{g}'$.

ii) $\mathfrak{g}=(\mathrm{I})_{3,3}$, $\rho(H_i)=H'_i$ $(1\leqq i\leqq 3)$, $\rho(H_4)=-H'_7$, $\rho(H_5)=H'_{\beta'}$;
$m_\rho=1$, $\rho(\mathfrak{g})=$ *the regular subalgebra* $(\mathrm{I})_{3,3}$ *of* $\mathfrak{g}'$.

iii) $\mathfrak{g}=(\mathrm{I})_{1,3}$, ρ *is determined by* (60);
$m_\rho=2$, $\rho(\mathfrak{g})=$ *the diagonal subalgebra of the regular subalgebra* $(\mathrm{I})_{1,3}+(\mathrm{I})_{1,3}$.

iv) $\mathfrak{g}=(\mathrm{I})_{1,1}$, $\rho(H_1)=H_1'+H_{r'-1}'+H_{r'}'$;
$m_\rho=3$, $\rho(\mathfrak{g})=$ *the diagonal subalgebra of the regular subalgebra* $(\mathrm{I})_{1,1}+(\mathrm{I})_{1,1}+(\mathrm{I})_{1,1}$.

v) $\mathfrak{g}=(\mathrm{II})_6$, ρ *is determined naturally;*
$m_\rho=1$, $\rho(\mathfrak{g})=$ *the regular subalgebra* $(\mathrm{II})_6$ *of* $\mathfrak{g}'$.

vi) $\mathfrak{g}=(\mathrm{III})_3$, ρ *is determined by* (69);
$m_\rho=1$, $\rho(\mathfrak{g})\subset$ *the regular subalgebra* $(\mathrm{I})_{3,3}$ *of* $\mathfrak{g}$.

Appendix

We give here some numerical results about correspondences of boundary components w.r.t. a holomorphic imbedding of a symmetric domain into another, supplementary to those given by Satake ([**5**]).

We recall at first the general theory given in [**5**]. Let φ be a holomorphic imbedding of symmetric domain D into another D' such as $\varphi(o)=o'$, where o and o' are fixed origin of D and D' respectively. Let further $\mathfrak{g}$ (resp. $\mathfrak{g}'$) be the Lie algebra of hermitian type corresponding to D (resp. D', ρ the homomorphism (and its complexification) of $\mathfrak{g}$ into $\mathfrak{g}'$ satisfying (H_1) which comes from φ. Suppose that D (resp. D') is imbedded in $\mathfrak{p}_+$ (resp. $\mathfrak{p}_+'$) by the Harish-Chandra imbedding, where

$$\mathfrak{p}_+=\sum_{\alpha\in\mathfrak{v}_+}\mathfrak{g}_\alpha \text{ (resp. } \mathfrak{p}_+'=\sum_{\alpha'\in\mathfrak{v}'_+}\mathfrak{g}_{\alpha'}').$$

Then φ is the restriction on D of the $\boldsymbol{C}$-linear map $\rho:\mathfrak{p}_+\rightarrow\mathfrak{p}_+'$. For each boundary component F of D, there is a uniquely determined boundary component F' of D' in which $\rho(F)$ is contained. If ρ is injective and F is proper boundary component (i.e. not D itself), F' is also proper. We may consider only the case both D and D' are irreducible; hence we assume so in the following. Let $\Delta_0=\{\gamma_i\}_{i\in R}$ be a maximal strongly orthogonal system of positive non-compact roots of $\mathfrak{g}_C$ ordered by $R=\{1,\cdots,r\}$, where r is the rank of $(\mathfrak{g},\mathfrak{k})$. For a number i in R, F_i denotes the i-th boundary component of D; that is defined as the boundary component of D containing the image under the Hermann map κ of the boundary component

$$\{1\}\underbrace{\times\cdots\times}_{i\text{-copies}}\{1\}\times U\underbrace{\times\cdots\times}_{(r-i)\text{-copies}}U$$

of U^r, where U is the unit disc (symmetric domain of type $(\mathrm{I})_{1,1}$) and hence $\{1\}$ is a boundary component of U. Every boundary component of D is equivalent under K to one of F_i's. Similarly Δ_0', γ_j', r', R', and F_j' are defined for

D'. Modifying ρ by (k)-equivalence, one can find $j \in R'$ for each $i \in R$ such that

$$\rho(F_i) \subset F'_j. \tag{71}$$

The number j is really a multiple of i:

$$j = mi. \tag{72}$$

Thus the correspondence of boundary components is considered to be determined by the multiplier m in (72).

We may take the non-compact simple root α_1 for γ_1. Then it can be seen that the properties (71) and (72) are equivalent to saying (modifying ρ by (k)-equivalence if necessary) that $\rho(H_1) = \sum_{1 \leqq t \leqq m} H'_{\gamma'_t}$. Therefore we see at once that *the number* m_ρ *defined in* 3.3 *is equal to* m *of* (72).

Let $\mathfrak{g}$ be a simple regular subalgebra of $\mathfrak{g}'$ corresponding to a Π-system Δ'. The injection of $\mathfrak{g}$ into $\mathfrak{g}'$ is denoted by ι. Let further α be the positive non-compact root in Δ'. Then, by Proposition 6 ii), we have

$$m_\iota = \frac{1}{\delta'} \langle H'_\alpha, H'_\alpha \rangle' = \frac{\langle \alpha'_1, \alpha'_1 \rangle'}{\langle \alpha, \alpha \rangle'}.$$

Hence m_ι *is equal to either* 1 *or* 2 *according as* α *is longer or shorter root of* $\mathfrak{g}'_C$.

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