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On some results on the Picard numbers of certain algebraic surfaces

To Professor Iyanaga for his 60th birthday

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§ 0. Introduction.

The Picard number of an algebraic variety is closely related to the arithmetical properties of the algebraic variety. The well-known Lefschetz-Hodge theorem asserts that, in the case of algebraic surfaces, 2-cycles on an algebraic surface are algebraic if and only if the periods of all the holomorphic 2-forms are zero (cf. Lefschetz [**5**], Kodaira-Spencer [**4**]). However, the determination of values of the periods on algebraic surfaces are extremely difficult. In this paper we examine some properties of the periods of holomorphic 2-forms on the algebraic surface $S=S_n(a^{(1)}, a^{(2)})$ in the three dimensional projective space $P_3(C)$ defined by

$$\prod_{j=1}^{n}(x_3-a_j^{(1)}x_2)=\prod_{j=1}^{n}(x_1-a_j^{(2)}x_0)\,, \tag{0.1}$$

where (x_0, x_1, x_2, x_3) are homogenous co-ordinates of $\boldsymbol{P}_3(\boldsymbol{C})$, and study the Picard number of this surface.

We shall summarize our results briefly. The first three sections are preliminaries. We calculate the Picard number in the final section. In the first section we show the following properties of our surface S defined by (0.1):

Let C_i be the (plane) algebraic curve defined by

$$u_2^n=\prod_{j=1}^{n}(u_1-a_j^{(i)}u_0) \qquad (i=1, 2)\,, \tag{0.2}$$

where (u_0, u_1, u_2) are homogenous co-ordinates of projective plane $\boldsymbol{P}_2(\boldsymbol{C})$, and let $G_n=\{\sigma_n^i : i=1, 2, \cdots, n\}$ be the automorphism group of C_i defined by

$$\sigma_n(u_0, u_1, u_2)=(u_0, u_1, \zeta_n u_2)\,, \qquad \zeta_n=\exp\left(\frac{2\pi\sqrt{-1}}{n}\right).$$

Then we prove that S is birationally equivalent to the quotient surface $(C_1\times C_2)/G_n$ (Lemma 1.1).

Let $\rho(S)$ be the Picard number of S and let $\rho^{(G_n)}(C_1\times C_2)$ be the number of homologically independent algebraic curves on $C_1\times C_2$ whose homology classes are invariant under the operations of G_n. Then we obtain from Lemma 1.1

easily the following

LEMMA 1.2.

(0.3) $$\rho(S)=\rho^{(G_n)}(C_1\times C_2)+n^2-2n+2 .$$

In the second section we determine certain Betti bases Γ_j^k ($k=1, \cdots, n-1$, $j=1, \cdots, n-2$) of the algebraic curve C_i in such a way that

(0.4) $$\begin{cases}(\sigma_n)_*(\Gamma_j^k)=\Gamma_j^{k+1} , \\ (\sigma_n)_*(\Gamma_j^{n-1})=-\sum_{k=1}^{n-1}\Gamma_j^k ,\end{cases} \qquad (k=1, \cdots , n-2)$$

holds, where $(\sigma_n)_*$ denotes the operation of the automorphism σ_n on the 1-cycles on C_i. In the third section we examine some properties of periods of holomorphic 1-forms on the plane algebraic curve C: $y^n=\prod_{j=1}^{n}(x-a_j)$ (where (x, y) are affine co-ordinates). Let $\omega^{r,\nu}=x^r y^{\nu-(n-1)}dx$ ($0\leqq r+\nu\leqq n-3$) be holomorphic 1-forms on C, and let J, J_1, J_2 be the Jacobian varieties of the algebraic curves C, C_1, C_2. First we give 'period matrices' X^ν, $X_\nu^{(1)}$, $X_\nu^{(2)}$ whose entries are the periods $\int_{\Gamma_j^1} x^r y^{\nu-(n-1)}dx$ for J, J_1, J_2, and give period relations of these 'period matrices' and a formula (concerning matrix representations of homomorphisms between J_1 and J_2) expressed by 'periods'. These results are restatements of G. Shimura [7] under somewhat weaker conditions. We give *power-series expansions of the holomorphic functions* $W_j^{\nu,r}(a)=\int_{\Gamma_j^1} x^r y^{\nu-(n-1)}dx$ at the point $a_0=(\zeta_n, \zeta_n^2, \cdots, \zeta_n^{n-1}, 1)$ and, as an application of the power-series expansions, we obtain the following results:

THEOREM 3.1. *Let* $\eta_j^{\nu,r}(a)=W_j^{\nu,r}(a)/W_{n-2}^{\nu,r}(a)$ ($j=1, \cdots, n-3$) *and define a holomorphic mapping* $H^{r,\nu}(a)$ *from a neighborhood of the point* a_0 *into* $\boldsymbol{C}^{n-3}$ *by*

$$H^{r,\nu}(a)=(\eta_1^{\nu,r}(a), \cdots , \eta_{n-3}^{\nu,r}(a)) .$$

Then the rank of the Jacobian matrix of the mapping $H^{\nu,r}(a)$ *at the point* a_0 *is equal to* $n-3$.

In Section 4, we calculate the Picard number $\rho(S)$ of our surface S. From Theorem 3.1 and Lemma 1.1 we obtain easily

THEOREM 4.2. *If* n *is a prime number, for general values of the parameters* $(a^{(1)}, a^{(2)})$, *we have*

$$\rho(S_n(a^{(1)}, a^{(2)}))=n^2-2n+2 ,$$

and for general values of the parameters (a)

$$\rho(S_n(a, a))=n^2-2n+2+(n-1) .$$

Secondly we calculate the Picard number of the quartic and quintic surfaces $S_4(a^{(1)}, a^{(2)})$ and $S_5(a^{(1)}, a^{(2)})$. For the quartic surface $S_4(a^{(1)}, a^{(2)})$, putting

$\tau(a)=\int_{\Gamma_1^1(a)} y^{-2}dx \Big/ \int_{\Gamma_2^1(a)} y^{-2}dx$, we obtain

THEOREM 4.4.

(I) *If* $\tau(a^{(1)}), \tau(a^{(2)}) \in \boldsymbol{Q}(\sqrt{-1})$, *then* $\rho(S_4(a^{(1)}, a^{(2)}))=20$.

(II) *If* $\tau(a^{(1)}), \tau(a^{(2)}) \notin \boldsymbol{Q}(\sqrt{-1})$, *and if there is a relation*

$$(0.4)\qquad \tau(a^{(2)})=\frac{m_{11}\tau(a^{(1)})+m_{12}}{m_{21}\tau(a^{(1)})+m_{22}} \text{ for a matrix } \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \in M_2(\boldsymbol{Q}),$$

then $\rho(S_4(a^{(1)}, a^{(2)}))=19$.

(III) *If there is no relation* (0.4) *between them, then* $\rho(S(a^{(1)}, a^{(2)}))=18$.

For the quintic surface $S_5(a^{(1)}, a^{(2)})$, we obtain the following partial results: Put

$$\tau_1(a)=\int_{\Gamma_1^1(a)} u_2^{-2}du_1 \Big/ \int_{\Gamma_3^1(a)} u_2^{-2}du_1,$$

$$\tau_2(a)=\int_{\Gamma_2^1(a)} u_2^{-2}du_1 \Big/ \int_{\Gamma_3^1(a)} u_2^{-2}du_1.$$

Then we obtain

THEOREM 4.6. *For any* $a^{(1)}, a^{(2)}$

$$37 \geqq \rho(S_5(a^{(1)}, a^{(2)})) \geqq 17.$$

If $\tau_1(a^{(1)}), \tau_2(a^{(2)}) \in \boldsymbol{Q}(\zeta_5)$,

$$\rho(S_5(a^{(1)}, a^{(2)}))=37.$$

Finally we calculate the Picard number $\rho(S_n')$ of a surface S_n' of the Fermat type $x_0^n+x_1^n+x_2^n+x_3^n=0$. Let l_n be the number of (i, j) (mod n) which satisfy the simultaneous congruences (0.5) for some ν, p, p':

$$(0.5)\qquad i\equiv \nu p,\quad j\equiv \nu p' \pmod n \quad \begin{array}{l} \nu=2, \cdots, n-2 \\ p=1, \cdots, \nu-1,\ p'=1, \cdots, n-(\nu-1). \end{array}$$

Then we obtain

THEOREM 4.7.

$$\rho(S_n')=n^2-2n+2+(n-1)(n^2-l_n)$$
$$\geqq n^2-2n+2+(n-1)(2n-5).$$

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§1. Some elementary properties of S.

1. First we shall make a simply observation on the algebraic surfaces with a certain type of fibering to which our surfaces belong. Let $\tilde{k}_0$ be a field of any characteristic, $\tilde{S}_0$ an algebraic surface and $(\tilde{x}, \tilde{y})$ a generic point of $\hat{S}_0$ over $\tilde{k}_0$ such that $\dim \tilde{k}_0(\tilde{x}) = 1$ and $\tilde{k}_0(\tilde{x}, \tilde{y})$ is a regular extension of $\tilde{k}_0(\tilde{x})$. We denote by $\tilde{D}$ an algebraic curve which is a locus of $(\tilde{x})$ over $\tilde{k}_0$ and define a rational map $\tilde{\omega}$ from $\tilde{S}_0$ to $\tilde{D}$ by $\tilde{\omega}(\tilde{x}, \tilde{y}) = \tilde{x}$. Moreover we denote by $\tilde{C}(\tilde{x})$ the "fibre at $\tilde{x}$" which is a locus of $(\tilde{y})$ over $\tilde{k}_0(\tilde{x})$.

Now we make the following

ASSUMPTION. There is an algebraic curve $\tilde{F}$ defined over $\tilde{k}_0$ which is biregularly equivalent to $\tilde{C}(\tilde{x})$ over the universal domain. We may assume that a biregular map $\tilde{T}$ between $\tilde{F}$ and $\tilde{C}(\tilde{x})$ is defined over an algebraic extension $\tilde{k}_0(\tilde{z})$ of $\tilde{k}_0(\tilde{x})$. Moreover we assume that $\tilde{k}_0(\tilde{z})$ is a regular Galois extension of $\tilde{k}_0(\tilde{z})$ and $(\tilde{z})$ and $(\tilde{y})$ are linearly disjoint over $\tilde{k}_0(\tilde{x})$. Let $\tilde{E}$ be a locus of $(\tilde{z})$ over $\tilde{k}_0$. Define a rational map $\tilde{r}$ from $\tilde{E}$ to $\tilde{D}$ by $\tilde{r}(\tilde{z}) = (\tilde{x})$ and a rational map $\tilde{R}$ from the product $\tilde{E} \times \tilde{F}$ to $\hat{S}$ by

$$\begin{cases} (\tilde{x}) = \tilde{R}((t), (\tilde{z})) = \tilde{r}(\tilde{z}), \\ (\tilde{y}) = \tilde{R}((t), (\tilde{z})) = \tilde{T}(t), \end{cases}$$

where t is a generic point of F over k_0. Then we obtain easily the following commutative diagram:

$$\begin{array}{ccc} \tilde{S} & \xleftarrow{\tilde{R}} & \tilde{E} \times \tilde{F} \\ \downarrow & & \downarrow Pr_{\tilde{F}} \\ \tilde{D} & \xleftarrow{r} & \tilde{E} \end{array} .$$

It can be easily proved that the product $\tilde{E} \times \tilde{F}$ is a Galois extension of $\tilde{S}$ whose Galois group is isomorphic to $\tilde{G}_n$. In fact, let $\alpha_\sigma = (T^\sigma)^{-1} \cdot T$ for every $\sigma \in G$. Put $\alpha_\sigma = (T^\sigma)^{-1} \cdot T$. Then the α_σ's form a group G' of automorphisms of F, and the action of σ on $E \times F$ is given by $\sigma(t, z) = (\alpha_\sigma(t), \sigma(z))$. By a suitable choice of $h(z)$, we can assume that $T^\sigma \neq T$ unless $\sigma = 1$. Then G' is isomorphic to G.

2. Let $\boldsymbol{P}_3(\boldsymbol{C})$ be a three dimensional projective space with homogenous[1),2)] co-ordinates (x_0, x_1, x_2, x_3). We define a non-singlar algebraic surface in $\boldsymbol{P}_3(\boldsymbol{C})$ by

$$\prod_{i=1}^{n} (x_3 - a_i^{(1)} x_2) = \prod_{i=1}^{n} (x_1 - a_i^{(2)} x_0), \tag{1.1}$$

where two systems of parameters $(a^{(1)}) = (a_1^{(1)}, \cdots, a_n^{(1)})$ and $(a^{(2)}) = (a_1^{(2)}, \cdots, a_n^{(2)})$ satisfy the inequalities

$$\prod_{i \neq j} (a_i^{(1)} - a_j^{(1)}) \neq 0, \qquad \prod_{i \neq j} (a_i^{(2)} - a_j^{(2)}) \neq 0. \tag{1.2}$$

We denote by S the (non-singular) algebraic surface defined by (1.1). Let C_1

1) These facts were pointed out by Professor G. Shimura.

2) The arguments of n° 6 are not necessarily needed below but it will clarify our situations.

and C_2 be two plane curves defined, respectively, by the equations

$$(1.3)\qquad \begin{cases} (u_2^{(1)})^n = \prod_{i=1}^{n} (u_1^{(1)} - a_i^{(1)} u_0^{(1)}) , \\ (u_2^{(2)})^n = \prod_{i=1}^{n} (u_1^{(2)} - a_i^{(2)} u_0^{(2)}) , \end{cases}$$

where $u^{(1)}=(u_0^{(1)}, u_1^{(1)}, u_2^{(2)})$ and $u^{(2)}=(u_0^{(2)}, u_1^{(2)}, u_2^{(2)})$ are two systems of homogeneous co-ordinates and two systems of parameters $(a^{(1)})$ and $(a^{(2)})$ satisfy the inequalities (1.2). Let $\sigma_n^{(1)}$ and $\sigma_n^{(2)}$ be, respectively, the biregular automorphisms of C_1 and C_2 defined by

$$\begin{cases} \sigma_n^{(1)} \cdot (u_0^{(1)}, u_1^{(1)}, u_2^{(1)}) = (u_0^{(1)}, u_1^{(1)}, \zeta_n u_2^{(1)}) , \\ \sigma_n^{(2)} \cdot (u_0^{(2)}, u_1^{(2)}, u_2^{(2)}) = (u_0^{(2)}, u_1^{(2)}, \zeta_n u_2^{(2)}) , \end{cases}$$

where ζ_n is a primitive n-th root of unity. Define a group $G_n^{(k)}$ of automorphisms of C_k by

$$G_n^{(k)} = \{(\sigma_n^{(k)})^i ;\ i=1, 2, \cdots, n\} .$$

Let $P^{(k)}$ $(k=1, 2)$ be points of C_k and let $C_1 \times C_2$ denote the product of C_1 and C_2. Define a biregular automorphism σ_n of $C_1 \times C_2$ by

$$(1.4)\qquad \sigma_n(P^{(1)} \times P^{(2)}) = \sigma_n^{(1)}(P^{(1)}) \times \sigma_n^{(2)}(P^{(2)}) ,$$

and denote by G_n the group of automorphisms of $C_1 \times C_2$ generated by σ_n. Let $(C_1 \times C_2)/G_n$ be the quotient surface of the product $C_1 \times C_2$ by G_n. Consider points $P_i^{(1)}$ on C_1 and $P_i^{(2)}$ on C_2, respectively, with homogenous co-ordinates $(1, a_i^{(1)}, 0)$ and $(1, a_i^{(2)}, 0)$ $(i=1, 2, \cdots, n)$. Then it can be easily verified that the quotient surface $C_1 \times C_2/G_n$ has n^2 singular points corresponding to $P_i^{(1)} \times P_j^{(2)}$ $(i, j=1, 2, \cdots, n)$. First we obtain[3)]

LEMMA 1.1. *The surface S is birationally equivalent to the quotient surface $(C_1 \times C_2)/G_n$.*

PROOF. (i) Let $\boldsymbol{P}_2^{(k)}(\boldsymbol{C})$, $k=1, 2$, be two projective planes with homogenous co-ordinates $(y_0^{(k)}, y_1^{(k)}, y_2^{(k)})$ and let $\boldsymbol{P}_1(\boldsymbol{C}) = (\boldsymbol{C}^{(1)} \cup \boldsymbol{C}^{(2)})$ be a projective line with affine co-ordinate $t^{(k)}$ in $\boldsymbol{C}^{(k)}$ $(k=1, 2)$ such that $t^{(1)} \cdot t^{(2)} = 1$.

Let $W^{(k)}$ be the product variety $\boldsymbol{P}_2^{(k)}(\boldsymbol{C}) \times \boldsymbol{C}_k$ and form the variety $W = W^{(1)} \cup W^{(2)}$ by means of the transformation law

$$(1.5)\qquad t_1 t_2 = 1 , \quad y_0^{(1)} = y_0^{(2)} , \quad y_1^{(2)} t_1 = y_1^{(1)} , \quad y_2^{(2)} t_1 = y_2^{(1)} .$$

We define a non-singular algebraic surface S_1 in W by

$$(1.6)\qquad \begin{cases} \prod_{i=1}^{n} (y_2^{(1)} - a_i^{(1)} y_1^{(1)}) = (y_0^{(1)})^n \cdot \prod_{i=1}^{n} (t_1 - a_i^{(2)}) \quad (\text{in } W_1) \\ \prod_{i=1}^{n} (y_2^{(2)} - a_i^{(1)} y_1^{(2)}) = (y_0^{(2)})^n \cdot \prod_{i=1}^{n} (1 - a_i^{(2)} t_2) \quad (\text{in } W_2) \end{cases}$$

3) The proof of Lemma 1.1 is quite easy. But, for the calculations done below, the precise relation between S and $(C_1 \times C_2)/G_n$ are needed.

and a regular map Ψ_1 from S_1 onto S by

$$(1.7)\qquad \begin{cases} X_3Y_0^{(1)}-X_0Y_2^{(1)}=0\,, \quad X_2Y_0^{(1)}-X_0Y_1^{(1)}=0\,, \quad X_1-t_1X_0=0\,, \\ \qquad\qquad X_3Y_1^{(1)}-X_2Y_2^{(1)}=0 \quad \text{in} \quad W_1\,, \\ X_3Y_0^{(2)}-X_1Y_2^{(2)}=0\,, \quad X_2Y_0^{(2)}-X_1Y_1^{(2)}=0\,, \quad t_2X_1-X_0=0\,, \\ \qquad\qquad X_3Y_2^{(2)}-X_2Y_2^{(2)}=0 \quad \text{in} \quad W_2\,. \end{cases}$$

Let $\theta_i^{(1)}$ $(i=1, 2, \cdots, n)$ denote, respectively, the rational curves on S_1 defined by

$$y_0^{(k)} : y_1^{(k)} : y_2^{(k)} = 0 : 1 : a_i^{(1)}\,, \qquad k=1, 2\,,$$

and by Q_i $(i=1, 2, \cdots, n)$ the points on S with the homogenous co-ordinates

$$x_0 : x_1 : x_2 : x_3 = 0 : 0 : 1 : a_i^{(1)}\,.$$

Then it can be easily verified that $\Psi_1(\theta_i^{(1)})=Q_i$ $(i=1, 2, \cdots, n)$ and Ψ_1 induces a biregular map between $S_1-\bigcup_{i=1}^{n}\theta_i^{(1)}$ and $S-\bigcup_{i=1}^{n}Q_i$.

(ii) Let (ω_1, ω_2) be a homogeneous co-ordinate on a projective line $\boldsymbol{P}_1(\boldsymbol{C})$. Form a variety $X=W\times\boldsymbol{P}_1(\boldsymbol{C})=(W_1\times\boldsymbol{P}_1(\boldsymbol{C})\cup W_2\times\boldsymbol{P}_1(\boldsymbol{C}))$ and define an algebraic surface S_0 in X by

$$(1.8)\qquad \begin{cases} \prod_{i=1}^{n}(y_2^{(1)}-a_i^{(1)}y_1^{(1)})=(y_0^{(1)})^n\cdot\prod_{i=1}^{n}(t_1-a_i^{(2)})\,, \\ w_1y_1^{(1)}-w_2y_2^{(1)}=0\,, \end{cases} \quad \text{in } W_1\times\boldsymbol{P}_1(\boldsymbol{C})$$
$$\begin{cases} \prod_{i=1}^{n}(y_2^{(2)}-a_i^{(2)}y_1^{(2)})=(y_0^{(2)})^n\cdot\prod_{i=1}^{n}(1-a_i^{(2)}t_2)\,, \\ w_1y_1^{(2)}-w_2y_2^{(2)}=0\,, \end{cases} \quad \text{in } W_2\times\boldsymbol{P}_1(\boldsymbol{C})\,.$$

Denote by R_i $(i=1, 2, \cdots, n)$ the points on S_1 with co-ordinates

$$t_1=a_i^{(2)}\,, \qquad y_0^{(1)} : y_1^{(1)} : y_2^{(1)} = 1 : 0 : 0\,.$$

Denote by $\theta_i^{(0)}$ $(i=1, 2, \cdots, n)$ the rational curve $R_i\times\boldsymbol{P}_1(\boldsymbol{C})$ on S_0. Let Pr_1 be the projection from X $(=W\times\boldsymbol{P}_1(\boldsymbol{C}))$ onto the variety W and denote the restriction of Pr_1 to S_0 by the same symbol Pr_1. Then it can be easily verified that $Pr_1(\theta_i^{(0)})=R_i$ $(i=1, 2, \cdots, n)$ and that Pr_1 induces a biregular map between $S_0-\bigcup_i\theta_i^{(0)}$ and $S-\bigcup_i R_i$.

(iii) Define a rational map Ψ_0 from $C_1\times C_2$ into S_0 by[4)]

$$(1.9)\qquad \begin{cases} t_1u_0^{(2)}-u_1^{(2)}=0\,, \\ y_1^{(1)}u_0^{(2)}u_2^{(1)}-y_0^{(1)}u_2^{(2)}u_0^{(1)}=0\,, \\ y_2^{(1)}u_0^{(2)}u_2^{(1)}-y_0^{(1)}u_2^{(2)}u_1^{(1)}=0\,, \\ w_1u_2^{(1)}-w_2u_1^{(1)}=0\,, \end{cases} \quad (\text{from } C_1\times C_2 \text{ into } S_0\cap W_1)\,,$$

4) Cf. Kodaira [3], p. 584.

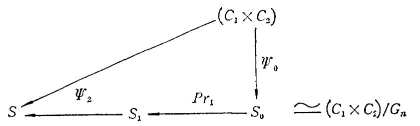
$$\begin{cases} t_2 u_1^{(2)} - u_0^{(2)} = 0\,, \\ y_2^{(2)} u_0^{(2)} u_2^{(1)} - y_0^{(2)} u_2^{(2)} u_0^{(1)} = 0\,, \\ y_2^{(2)} u_1^{(2)} u_2^{(1)} - y_0^{(2)} u_2^{(2)} u_1^{(1)} = 0\,, \\ w_1 u_2^{(1)} - w_2 u_2^{(1)} = 0\,. \end{cases} \qquad (\text{from } C_1 \times C_2 \text{ into } S_0 \cap W_2)\,,$$

Denote by Ξ_{ij} the rational curve on S_0 defined by the following equations:

$$t_1 = a_i^{(2)}\,, \qquad y_2^{(1)} - a_i^{(1)} y_i^{(1)} = 0\,, \qquad w_1 : w_2 = a_j^{(1)} : 1\,.$$

Then Ψ_0 is a regular map from $C_1 \times C_2 - \bigcup_{i,j=1}^{n} P_i^{(1)} \times P_j^{(2)}$ onto $S_0 - \Xi_{ij}$, and for points $P_1 \times P_2$, $P_1' \times P_2'$ on $C_1 \times C_2 - \bigcup_{i,j=1}^{n} P_i^{(1)} \times P_j^{(2)}$, $\Psi_0(P_1 \times P_2) = \Psi_0(P_1' \times P_2')$ holds if and only if: $(\sigma_n)(P_1 \times P_2) = P_1' \times P_2'$. Thus we see that S_0 is a non-singular model of the quotient surface $C_1 \times C_2 / G_n$.

From (i), (ii), (iii), we obtain the following diagram,



This proves Lemma 1.1.

3. We denote by $\rho(S)$, $\rho(S_0)$, $\rho(S_1)$ the Picard numbers of S, S_0, S_1, respectively. We define $\rho^{(G_n)}(C_1 \times C_2)$ to be the number of homologically independent curves which are invariant under the actions of G_n, where we consider homology with coefficients in $\boldsymbol{Q}$. Then we have

LEMMA 1.2.

$$\rho(S) = \rho^{(G_n)}(C_1 \times C_2) + n^2 - 2n\,.$$

PROOF. (i) As was shown in the proof of Lemma 1.1, the surface S is obtained from the surface S_0 by contracting $2n$ exceptional curves (of the first kind) on S. Hence we obtain

$$\rho(S) = \rho(S_0) - 2n\,. \tag{1.10}$$

(ii) Define local co-ordinates $\xi^{(k)}$ of C_k $(k=1, 2)$ at the points $P_j^{(k)}$ $(k=1, 2,$ $j=1, 2, \cdots, n)$ by

$$\xi^{(k)} = u_2^{(k)} / u_0^{(k)} \qquad (k=1, 2)\,.$$

Define local co-ordinates $(z_1^{(k)}, z_2^{(k)})$ of $S_0 \cap W_k$ $(k=1, 2)$ in a neighborhood of Ξ_{ij} by

$$z_1^{(2)} = \prod_{j=1}^{n} (t_1 - a_j^{(2)})\,, \qquad z_2^{(1)} = y_1^{(1)} / y_0^{(1)}\,,$$

$$z_1^{(1)} = \prod_{j=1}^{n} (w_1 / w_2 - a_i^{(1)})\,, \qquad z_2^{(2)} = y_0^{(1)} / y_1^{(1)}\,.$$

Then the rational map Ψ_0 is expressed in terms of these local co-ordinates in the following form:

$$\text{(1.11)}\quad \begin{cases} z_1^{(1)}=(\xi^{(1)})^n\,, \quad z_2^{(1)}=\xi^{(1)}/\xi^{(2)}\,, & \text{if} \quad \xi^{(2)}\neq 0\,,\\ z_1^{(2)}=(\xi^{(2)})^n\,, \quad z_n^{(2)}=\xi^{(2)}/\xi^{(1)}\,, & \text{if} \quad \xi^{(1)}\neq 0\,,\end{cases}$$

(cf. Kodaira [3]). We obtain

$$\text{(1.12)}\quad \begin{cases} d\log\xi^{(1)}=(1/n)d\log z_1^{(1)}\,, \quad d\log\xi^{(2)}=(1/n)d\log z_1^{(1)}-d\log z_2^{(1)}\,,\\ d\log\xi^{(2)}=(1/n)d\log z_1^{(2)}\,, \quad d\log\xi^{(1)}=(1/n)d\log z_1^{(2)}-d\log z_2^{(2)}\,.\end{cases}$$

We define a G_n-invariant holomorphic function H at $P_i^{(1)}\times P_j^{(2)}$ and $f\cdot d\xi^{(1)}+g\cdot d\xi^{(2)}$ a G_n-invariant holomorphic 1-form at $P_i^{(1)}\times P_j^{(2)}$. We write

$$H(\xi^{(1)},\xi^{(2)})=\sum_{k\geqq 0}^{\infty}\sum_{p+q=kn}a_{p,q}^{(k)}(\xi^{(1)})^p\cdot(\xi^{(2)})^q\,,$$

$$\text{(1.13)}\quad f\cdot d\xi^{(1)}+g\cdot d\xi^{(2)}=\sum_{k_1\geqq 1}^{\infty}\Big\{\sum_{\substack{p_1+q_1=k_1n\\ p_1\geqq 1}}b_{p_1,q_1}^{(k_1)}(\xi^{(1)})^{p_1}(\xi^{(2)})^{q_1}\Big\}d\log\xi^{(1)}$$

$$+\sum_{k_2\geqq 1}^{\infty}\Big\{\sum_{\substack{p_2+q_2=k_2n\\ q_2\geqq 1}}c_{p_2,q_2}^{(k_2)}(\xi^{(1)})^{p_2}(\xi^{(2)})^{q_2}\Big\}d\log\xi^{(2)}\,.$$

Then the holomorphic function $(\Psi_0)_*(H)$ and the holomorphic 1-form $(\Psi_0)_*(f\cdot d\xi^{(1)}+g\cdot d\xi^{(2)})$ in a neighborhood of Ξ_{ij} are given by

$$(\Psi_0)_*(H)=\sum_{k=0}^{\infty}\Big\{\sum_{q=0}^{kn}a_{p,q}^{(k)}(\xi^{(2)})^q\Big\}(\xi^{(1)})^k \quad (\text{in } W_1)\,,$$

$$=\sum_{k=0}^{\infty}\Big\{\sum_{q=0}^{kn}a_{p,q}^{(k)}(\xi^{(1)})^q\Big\}(\xi^{(2)})^k \quad (\text{in } W_2)\,,$$

$$(\Psi_0)_*(f\cdot d\xi^{(1)}+g\cdot d\xi^{(2)})=\Big\{\sum_{k_1\geqq 1}^{\infty}\sum_{p_1\geqq 1}^{k_1n}b_{p_1,k_1n-p}^{(k_1)}(z_2^{(1)})^{k_1n-p_1}(z_1)^{p_1}\Big\}\Big(\frac{1}{n}d\log z_1^{(1)}\Big)$$

$$+\Big\{\sum_{k_2\geqq 1}^{\infty}\sum_{q_2\geqq 1}^{k_2n}c_{k_2n-q_2,k_2}^{(k_2)}(z_2^{(1)})^{q_2}(z_1^{(1)})^{k_2}\Big\}\Big\{\frac{1}{n}d\log z_1^{(1)}-d\log z_2^{(1)}\Big\} \quad (\text{in } W_1)$$

$$=\Big\{\sum_{k_1\geqq 1}^{\infty}\sum_{p_1=1}^{k_1n}b_{p_1,k_1n-p}^{(k_1)}(z_2^{(2)})^{p_1}(z_1^{(2)})^{k_1}\Big\}\Big\{\frac{1}{n}d\log z_1^{(2)}-d\log z_2^{(2)}\Big\}$$

$$+\Big\{\sum_{k_2\geqq 1}^{\infty}\sum_{q_2=1}^{k_2n}c_{k_2n-q_2,q_2}^{(k_2)}(z_1^{(2)})^{q_2}(z_2^{(2)})^{k_2}\Big\}\Big\{\frac{1}{n}d\log z_1^{(2)}\Big\} \quad (\text{in } W_2)\,.$$

(iii) Let $\{\mathcal{D}_1,\cdots,\mathcal{D}_{\rho G_n(C_1\times C_2)}\}$ be a base of G_n-invariant algebraic curves on $C_1\times C_2$. We may assume that the divisors $\mathcal{D}_k$ $(k=1,2,\cdots)$ are defined by the quotients of G_n-invariant holomorphic functions $H_1^{(k)}(\xi^{(1)},\xi^{(2)})$ and $H_2^{(k)}(\xi^{(1)},\xi^{(2)})$ at the point $P_i^{(1)}\times P_j^{(2)}$. Then we obtain the divisor $(\Psi_0)_*(\mathcal{D}_k)$ which is defined at the points on Ξ_{ij} by

$$(\Psi_0)_*(H_1^{(k)})/(\Psi_0)_*(H_2^{(k)})\,.$$

Let $\mathcal{E}$ be a divisor on S, and put

$$\mathcal{E}=\mathcal{E}_1+\sum m_{ij}\cdot \Xi_{ij},$$

where $\mathcal{E}_1$ does not contain Ξ_{ij} as components. Let $\mathcal{E}_1'$ be the divisor on $C_1\times C_2$ induced from $\mathcal{E}_1$ by the rational map Ψ_0. By definition we can find integers $m, m_1, \cdots, m_{\rho(C_1\times C_2)}^{(\rho n)}$ such that

$$m\mathcal{E}_1'-\sum_k m_k\mathcal{D}_k\sim 0\,, \tag{1.14}$$

where the symbol $\sim$ indicates homology with coefficients in $\boldsymbol{Q}$. Then there is a G_n-invariant meromorphic closed 1-form $\tilde{h}$ such that $m\mathcal{E}_1'-\sum_k m_k\mathcal{D}_k$ are logarithmic divisors of $\tilde{h}$[5] (cf. Hodge-Atiyah [2]). Let $(\Psi_0)_*(\tilde{h})$ be a closed 1-form on S which can be induced by the process of (ii). Then we verify that the logarithmic divisor of $(\Psi_0)_*(\tilde{h})$ is expressed in the form

$$(m\mathcal{E}-\sum_k m_k(\Psi_0)_*(\mathcal{D}_k))+\sum_{i,j} m_{i,j}'\Xi_{i,j}\sim 0\,.$$

On the other hand we can easily verify that $(\Psi_0)_*(\mathcal{D}_k)$ and Ξ_{ij} are homologically independent. q. e. d.

§ 2. Some homological properties.

4. Let C be an affine algebraic curve defined by

$$y^n=\prod_{i=1}^{n}(x-a_i)\,, \tag{2.1}$$

where (x, y) are affine co-ordinates of the two-dimensional affine space $\boldsymbol{C}^2$. Take a projective line $\boldsymbol{P}_1(\boldsymbol{C})=\boldsymbol{C}_1\cup\boldsymbol{C}_2$ and let ν_k be an affine co-ordinate of the affine line $\boldsymbol{C}_k$ $(k=1, 2)$, where $\nu_1\nu_2=1$. We let π denote the projection from C onto $\boldsymbol{C}_1$ defined by

$$\nu_1=\pi(x, y)=x\,. \tag{2.2}$$

Let $\tilde{C}$ be a non-singular complete algebraic curve whose (rational) function field is isomorphic to that of C and denote the canonical map from $\tilde{C}$ onto the completion of C by $\tilde{\mu}$. Denote the extension to the completion of C of the projection π also by π and put $\tilde{\pi}=\pi\cdot\tilde{\mu}$. Then $\tilde{\pi}$ is a projection from $\tilde{C}$ onto $\boldsymbol{P}_1(\boldsymbol{C})$. The ramification points of the projection $\tilde{\pi}$ are given by

$$\nu_1=a_j \qquad (j=1, 2, \cdots, n)\,.$$

We denote by D_j the point $\nu_1=a_j$. Take a point P_0 with a co-ordinate x_0 in $\boldsymbol{C}_1$ which is different from D_j $(j=1, 2, \cdots, n)$. We draw oriented[6] C^∞-arcs γ_j

5) This means that the residue of $\tilde{h}$ around $\mathcal{E}_1'$, $\mathcal{D}_k$ are equal to m, m_k.

6) We define the direction from P_0 to D_j as positive orientation.

from P_0 to D_j $(j=1, 2, \cdots, n)$ in such a way that any two of them have no common point other than P_0. Define a subset Σ of $\boldsymbol{P}_1(\boldsymbol{C})$ by

$$\Sigma=\boldsymbol{P}_1(\boldsymbol{C})-\bigcup_j \gamma_j .$$

Now we determine branches $y^{(i)}(x)$ $(i=1, 2, \cdots, n)$ of the algebraic function y of x defined by (2.1) at the point P_0 in such a way that

$$y^{(j)}(x)=\zeta_n^{i-j} y^{(i)}(x) . \tag{2.3}$$

To a point P on γ_j we continue analytically $y^{(i)}(x)$ along γ_j and to a point P on $\Sigma \cap \boldsymbol{C}_1^{(1)}$ we continue analytically $y^{(i)}(x)$ along any arc γ starting at P_0 and passing between γ_n and γ_1. (See fig. 1.) Now we define mappings $\tilde{\lambda}_j^i$ $(j=1, \cdots, n)$ from the oriented arcs γ_j $(j=1, \cdots, n)$ into the algebraic curve $\tilde{C}$ by

$$\tilde{\lambda}_j^i(x)=\tilde{\mu}^{-1}(x, y^{(i)}(x)) \qquad (i \bmod n) . \tag{2.4}$$

We also define a cross-section $\tilde{\lambda}^i$ from the intersection $\Sigma \cap \boldsymbol{C}_1$ into $\tilde{C}$ by

$$\tilde{\lambda}^i(x)=\tilde{\mu}^{-1}(x, y^{(i)}(x)) . \tag{2.5}$$

The cross-section $\tilde{\lambda}^i$ can be easily extended to the subset Σ. We denote this (extended) cross-section by the same symbol $\tilde{\lambda}^i$. We define an oriented 1-cell $\tilde{\gamma}_j^i$ (on the algebraic curve $\tilde{C}$) to be the image $\tilde{\lambda}_j^i$ (γ_j). We define an oriented 2-cell $\tilde{\Sigma}^i$ to be the image of Σ by $\tilde{\lambda}^i$. Moreover put $\tilde{D}_0^i=\tilde{\lambda}^i(P_0)$ and $\tilde{D}_j^i=\tilde{\lambda}^i(D_j)$. Thus we obtain a cell decomposition K of $\tilde{C}$ consisting of $(\tilde{\Sigma}^i, \tilde{\gamma}_j^i, \tilde{P}_0^i, \tilde{D}_j^i)$. Take C^{∞}-arcs δ_j $(j=1, 2, \cdots, n)$ which issue and end at P_0 and surround, respectively, the arcs γ_j in a positive direction (see fig. 2). Then the results of analytic

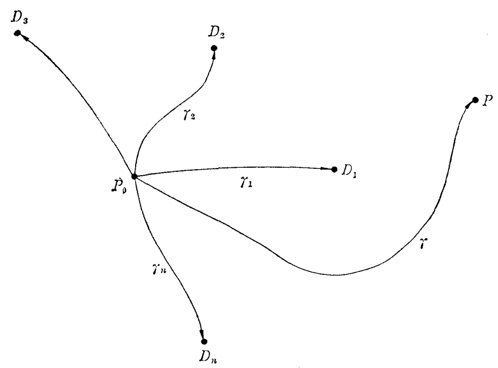


Fig. 1.

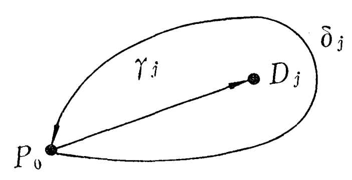


Fig. 2.

continuation of $y^{(i)}$ along δ_j are as follows:

$$y^{(i)} \rightarrow \zeta_n \cdot y^{(i)} \qquad (j=1, 2, \cdots, n) . \tag{2.6}$$

5. Now we obtain easily boundary operators on the cell-decomposition K of $\tilde{C}$. First, for the 1-cells $\tilde{\gamma}_j^i$, we obtain

$$\partial(\tilde{\gamma}_j^i)=\tilde{D}_0^i-\tilde{D}_j \qquad (j=1, 2, \cdots, n) , \tag{2.7}$$

and, for the 2-cells $\tilde{\Sigma}^i$, from the results of $(2.6)_j$,

$$(2.8)_j \qquad \partial(\tilde{\Sigma}^i)=\sum_{j=1}^{n}\{\tilde{\gamma}_j^{i+j-1}-\tilde{\gamma}_j^{i+j}\} \qquad \left(\begin{matrix} i+j,\ i+j-1 \pmod n \\ j=1,2,\cdots,n \end{matrix}\right).$$

Define 1-cycles Γ_j^i on $\tilde{C}$ by

$$(2.9)^{7)} \qquad \Gamma_j^i=\gamma_j^i-\gamma_j^{i+1}+\gamma_{j+1}^{i+1}-\gamma_{j+1}^i \qquad \left(\begin{matrix} i,\ i+1 \pmod n \\ j=1,\cdots,n \end{matrix}\right).$$

Then the operation $(\sigma_n)_*$ of the automorphism σ_n on these 1-cycles are given by

$$(2.10) \qquad (\sigma_n)_*(\Gamma_j^i)=\Gamma_j^{i+1} \qquad \left(\begin{matrix} i,\ i+1 \pmod n \\ j=1,2,\cdots,n \end{matrix}\right).$$

From the boundary relation (2.8), we obtain

LEMMA 2.1.[8] *The 1-cycles*

$$(2.11) \qquad \Gamma_j^i \qquad \left(\begin{matrix} i=1,\cdots,n-1 \\ j=1,\cdots,n-2 \end{matrix}\right),$$

constitute a Betti-base of 1-cycles on $\tilde{C}$.

6. Now, using the previous result (2.10) and Lemma 2.1, we examine some homological properties of our surface S_1 from the point of view taken up by Lefschetz. Let O_i $(i=1,2,\cdots,n)$ be the points on $\boldsymbol{P}_1(\boldsymbol{C})$ with the affine coordinates $t_1=a_i^{(2)}$, $\Pi=\boldsymbol{P}_1(\boldsymbol{C})-\bigcup_{i=1}^{n}O_i$ and let $\pi_1(\Pi)$ be the fundamental group of the domain Π. Take a point O_0 on $\boldsymbol{P}_1(\boldsymbol{C})$ which is different from O_i $(i=1,2,\cdots,n)$ and let δ_i $(i=1,2,\cdots,n)$ be C^∞-arcs which surround the point O_i in the positive sense and have no common points other than O_0 (see figure 3).

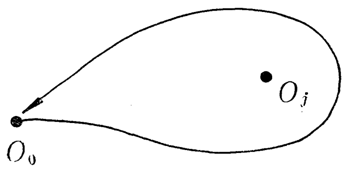


Fig. 3.

Denote the homotopy class which is represented by the arc δ_j $(j=1,2,\cdots,n)$ by $\tilde{\delta}_j$. Let $\tilde{\omega}_1$ be the regular map from the algebraic surface S_1 onto $\boldsymbol{P}_1(\boldsymbol{C})$ defined by

$$\tilde{\omega}_1(t_k,(y_0^{(k)},y_1^{(k)},y_2^{(k)}))=t_k \qquad (k=1,2).$$

Then the singular fibres with respect to this map $\tilde{\omega}_1$ are $\tilde{\omega}^{-1}(O_j)$, $j=1,2,\cdots,n$ (cf. Kodaira [3]). Now we denote the operation of the fundamental group $\pi_1(\Pi)$ on the one dimensional homology group $H_1(\tilde{\omega}^{-1}(O_0),Z)$ of the (general) fibre $\tilde{\omega}^{-1}(O_0)$ by $\tilde{\chi}$. Let ι_0 be the isomorphism from $\tilde{C}_1$ onto $\tilde{\omega}^{-1}(O_0)$ and let

7) If $j=n+1$, we understand that $\Gamma_j^i=\gamma_n^i-\gamma_n^{i+1}+\gamma_1^{i+1}-\gamma_1^i$.

8) In the case of n=prime odd, these cycles are used in Lefschetz [5].

$(\iota_0)_*$ denote the operation of ι_0 on the chain groups of $\tilde{C}_1$. Concerning this operation, we obtain easily the following

THEOREM 2.1.

$$\tilde{\chi}(\tilde{\delta}_j)_*(\Gamma_j^i)=(\iota_0)^*(\Gamma_j^{i-1}) \qquad \begin{pmatrix} i,\ i+1\ (\mathrm{mod}\ n) \\ j=1,2,\cdots,n \end{pmatrix}.$$

Proof is obvious.

§ 3. Some results on periods[9].

7. It is known that any rational 1-form on the algebraic curve C defined by (2.1) is expressed as a linear combination (with coefficients in $\boldsymbol{C}$) of 1-forms of the following type:

$$\omega^{(\beta_j),\nu,r}=\prod_{j=1}^{n}(x-a_j)^{\beta_j}y^{-\nu}\Phi_r(x)dx\,,$$

where β_j and ν are positive integers and $\Phi_r(x)$ is a polynomial of degree r which does not vanish at $x=a_j$ $(j=1,\cdots,n)$ (cf. Lefschetz [**5**]). It can be easily verified that $\omega^{(\beta_j),\nu,r}$ is of the first kind if and only if

$$\left.\begin{aligned} &\beta_j-\frac{\nu}{n}>-1 \\ &\nu-\sum_{j=1}^{n}\beta_j\geqq n \end{aligned}\right. \qquad (j=1,\cdots,n)$$

Put $r_\nu=\nu-1$, $\beta_j=0$. Let a be a complex number different from a_j's $(j=1,2,\cdots,n)$, and define the holomorphic 1-form $\omega^{\nu,r}$ by

$$\omega^{\nu,r}=\prod_j (x-a)^r y^{-\nu}dx \qquad \begin{pmatrix} \nu=1,\cdots,n-1 \\ r=1,\cdots,r_\nu \end{pmatrix}, \tag{3.1}$$

so $\omega^{\nu,r}$ constitute the basis of holomorphic 1-forms on C. And we see that

$$r_\nu+r_{n-\nu}=n-2\,. \tag{3.2}$$

Denote the period of $\omega^{r,\nu}$ over the 1-cycle Γ_j^i by $\tau_{i,j}^{\nu,r}$. Then, from (2.10), we obtain

$$\tau_{i+k,j}^{\nu,r}=\zeta_n^{k\nu}\cdot\tau_{i,j}^{\nu,r}\,, \qquad (i,\ i+k(\mathrm{mod}\ n))\,. \tag{3.3}$$

For the sake of simplicity, put $\tau_j^{\nu,r}=\tau_{i,j}^{\nu,r}$. Define r_ν-vectors $\mathfrak{x}_j^\nu$ by

$${}^t\mathfrak{x}_j^\nu=(\tau_j^{\nu,1},\tau_j^{\nu,2},\cdots,\tau_j^{\nu,r_\nu})\,,$$

and matrices X_ν $(1\leqq\nu\leqq[n/2])$ by

$$X_\nu=\begin{bmatrix} \mathfrak{x}_1^\nu, & \mathfrak{x}_2^\nu, & \cdots, & \mathfrak{x}_{n-2}^\nu \\ \bar{\mathfrak{x}}_1^{n-\nu}, & \bar{\mathfrak{x}}_2^{n-\nu}, & \cdots, & \bar{\mathfrak{x}}_{n-2}^{n-\nu} \end{bmatrix}. \tag{3.4}$$

9) In § 3, we mainly follow the notations of Shimura [7].

Then, from the classical bilinear equality and inequality, we obtain the following 'Periods relations' of X_ν:

(3.5) $$\sqrt{-1}\bar{X}_\nu T_\nu^{-1}{}^t X_\nu = \begin{bmatrix} -\mathfrak{A}_\nu & 0 \\ 0 & \mathfrak{B}_\nu \end{bmatrix}$$

where $\mathfrak{A}_\nu \in M_{r_\nu}(\boldsymbol{C})$ and $\mathfrak{B}_\nu \in M_{r_{n-\nu}}(\boldsymbol{C})$ are positive definite Hermitian matrices and T_ν^{-1} is a skew-Hermitian matrix. Moreover T_ν^{-1} is expressed by means of certain polynomials $t_{ij}(\zeta)$ of one variable ζ with coefficients in $\boldsymbol{Q}$ in the following form:

$$T_\nu^{-1} = [t_{ij}(\zeta_n^\nu)] \quad \text{(cf. Shimura [7])}.$$

Denote by J, J_1, J_2 the Jacobian varieties of $\tilde{C}$, $\tilde{C}_1$, $\tilde{C}_2$ respectively, and by $\theta(\sigma_n)$ the automorphism of J which corresponds to the automorphism σ_n of C. By (3.3) the matrix representations $\Phi(\sigma_n)$ of $\theta(\sigma_n)$ by means of the holomorphic 1-forms $\omega^{r,\nu}$ is expressed in the following form:

(3.6) $$\Phi(\sigma_n) = \begin{bmatrix} \Phi_1(\zeta_n) & & & & \\ & \ddots & & & \\ & & \Phi_q(\zeta_n) & & \\ & & & \ddots & \\ & & & & \Phi_{[n/2]}(\zeta_n^{[n/2]}) \end{bmatrix},$$

where

$$\Phi_\nu(\zeta_n^\nu) = \begin{bmatrix} \zeta_n^\nu E_{r_\nu} & 0 \\ 0 & \bar{\zeta}_n^\nu E_{r_{n-\nu}} \end{bmatrix}.$$

Denotes, as usual, by $\mathrm{Hom}(J_1, J_2)$ the module of all homomorphisms of J_1 into J_2 and by $\mathcal{E}(J)$ the endomorphism ring of J. Put $\mathrm{Hom}_0(J_1, J_2) = \mathrm{Hom}(H_1, J_2) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$, $\mathcal{E}_0(J) = \mathcal{E}(J) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$. Let λ be an arbitrary element of $\mathrm{Hom}_0(J_1, J_2)$. We denote by Λ the matrix which represents λ. If the homomorphism λ satisfies

(3.7) $$\lambda \circ \theta(\sigma_n) = \theta(\sigma_n) \circ \lambda,$$

then the matrix Λ must be expressed in the following form:

(3.8) $$\Lambda = \begin{bmatrix} \Lambda_1' & & & \\ & \Lambda_2' & & \\ & & \ddots & \\ & & & \Lambda_{[n/2]}' \end{bmatrix},$$

where

$$\Lambda_\nu' = \begin{bmatrix} \Sigma_\nu & 0 \\ 0 & \Delta_\nu \end{bmatrix}, \quad \Sigma_\nu \in M_{r_\nu}(\boldsymbol{C}), \quad \Delta_\nu \in M_{r_{n-\nu}}(\boldsymbol{C}).$$

Put

$$\Lambda_\nu = \begin{bmatrix} \Sigma_\nu & 0 \\ 0 & \bar{\Delta}_\nu \end{bmatrix}.$$

Then, from (3.3) and (3.7), we obtain

(3.9) $$\Lambda_\nu X_\nu^{(1)} = X_\nu^{(2)}{}^t U_\nu, \quad U_\nu = (u_{ij}(\zeta_n^\nu))_{i,j=1,\cdots,n-2}, \quad (\nu = 1, \cdots, [n/2]),$$

where $u_{ij}(t)$ are polynomials of one variable t with coefficients in $\boldsymbol{Q}$ and $X_\nu^{(k)}$ $(k=1, 2)$ are the periods matrices of C_k $(k=1, 2)$. Conversely, if a matrix Λ of the form (3.8) satisfies the equation (3.9), then the matrix Λ represents an element of $\mathrm{Hom}_0(J_1, J_2)$. Thus, in order to determine $\mathrm{Hom}_0(J_1, J_2)$, it suffices to find matrices Λ_ν (or matrices U_ν) satisfying the condition stated above.

8. To clarify the dependence of our objects on the parameters (a), we denote the curve $\tilde{C}$, the rational 1-form ω on $\tilde{C}$, the 2-cycles γ_j^i on $C \cdots$, respectively, by $\tilde{C}(a)$, $\omega(a)$, $\gamma_j^i(a), \cdots$. Define a point (a_0) in $\boldsymbol{C}^n$ by $(a_0)=(\zeta_n, \zeta_n^2, \cdots, \zeta_n^{n-1}, 1)$ and, in some neighborhood $\mathfrak{A}$ of (a_0), define holomorphic 1-forms $\omega^{\nu,r}(a)$ on $\tilde{C}(a)$ (which depends holomorphically on the parameters (a) in $\mathfrak{A}$) by

$$\omega^{\nu,r}(a)=x^r y^{-\nu} dx\,, \qquad \nu=1, \cdots, n-1\,, \quad r=1, \cdots, r_\nu\,. \tag{3.10}$$

Define a holomorphic function $W_j^{\nu,r}(a)$ in $\mathfrak{A}$ by

$$W_j^{\nu,r}(a)=\int_{\Gamma_j^1(a)} \omega^{\nu,r}(a)\,, \tag{3.11}$$

where $\Gamma_j^1(a)$ is a 1-cycle on $\tilde{C}(a)$ defined in (2.9). Write the power series expansion of $W_j^{\nu,r}(a)$ at the point $a_0=(\zeta_n, \zeta_n^2, \cdots, \zeta_n^{n-1}, 1)$ in the following form:

$$W_j^{\nu,r}(a)=\sum_{i_1,\cdots,i_n=0}^{\infty} c_{i_1,\cdots,i_n}^{(j,\nu,r)}(a_1-\zeta_n)^{i_1} \cdots (a_{n-1}-\zeta_n^{n-1})^{i_{n-1}}(a_n-1)^{i_n}\,. \tag{3.12}$$

Now we examine certain relations between the coefficients $c_{i_1,\cdots,i_n}^{(j,\nu,r)}$.

(i) First we note that there is a biregular morphism τ_n of the algebraic curve $\tilde{C}(a_0): y^n=x^n-1$ such that $\tau_n(x, y)=(\zeta_n x, y)$ and that the operation $(\tau_n)_*$ of τ_n on the 1-cycles of the algebraic curve $\tilde{C}$ is expressed in the following form:

$$(\tau_n)_*(\Gamma_j^i(a_0))=\Gamma_{j+1}^i(a_0) \qquad (j,\ j+1 \ (\mathrm{mod}\ n))\,.$$

By a small deformation of Γ_j^i, we obtain 1-cycles δ_j^i such that the ramification points of $\tilde{C}$ do not lie on δ_j^i and such that

$$(\tau_n)_*(\delta_j^i(a_0))=\delta_{j+1}^i(a) \qquad (j,\ j+1 \ (\mathrm{mod}\ n))\,. \tag{3.13}$$

Secondly, we note that, by exchange of integration and differentiation, we obtain easily

$$c_{i_1\cdots i_n}^{(i,\nu,r)}=(1/i_1! \cdots i_n!)\cdot d_{i_1\cdots i_n}\cdot \int_{\delta_j^1(a_0)} x^r(x^n-1)^{\nu/n} \prod_{j=1}^n (x-\zeta_n^j)^{-i_j} dx \tag{3.14}$$

where

$$d_{i_1\cdots i_n}=(-1)^{i_1+\cdots+i_n} \prod_{j=1}^n (\nu/n) \cdots \{(\nu/n)-i_j+1)\}\,.$$

Hence, from (2.10) and (3.11), we obtain

$$c_{i_1\cdots i_n}^{(j,\nu,r)}=\zeta_n^{(j-1)\{(r+1)-\sum_{k=1}^n i_k\}} c_{i_j,i_{j+1},\cdots,i_{j+n-1}}^{(1,\nu,r)}\,. \tag{3.15}$$

(ii) Define definite integrals $B(r, \nu, i_1, \cdots, i_n)$ by

$$B(r, \nu, i_1, \cdots, i_n)=\int_{\Gamma_1^1(a_0)} x^r(x^n-1)^{\nu/n}\prod_{j=1}^{n}(x-\zeta_n^j)^{-i_j}dx\,. \tag{3.16}$$

For any integer $t\geqq 0$, we put

$$A_{i,t}=\sum_{(k)=(k_0,\cdots,k_{n-1})}\binom{i}{k_0}\cdot\binom{i-k_0}{k_1}\cdots\binom{i-k_0-k_1-\cdots-k_{n-1}}{k_n}, \tag{3.17}$$

where $(k)=(k_0, \cdots, k_{n-1})$ satisfies

$$\begin{cases} k_0\geqq 0, \cdots, k_{n-1}\geqq 0\,, \\ k_0+k_1+\cdots+k_{n-1}=i\,, \\ k_1+2k_2+\cdots+(n-1)k_{n-1}=t\,. \end{cases}$$

Moreover, for any integers $s\geqq 0$, we set

$$B_s^{(i_1,\cdots,i_n)}=\sum_{\substack{t_1+\cdots+t_n=s\\ 0\leqq t_1,\cdots,0\leqq t_n}} A_{i_1,t_1}, \cdots, A_{i_n,t_n}\cdot\zeta_n^{\sum_{j=1}^{n}-j(i_j+t_j)}\,. \tag{3.18}$$

Then, from the identity $(1/x)-\zeta_n^j=(1/x^n-1)\cdot\sum_{k=0}^{n-1}\zeta_n^{-j(k+1)}\cdot x^k$, we obtain easily

$$B(r, \nu, i_1, \cdots, i_n)=\sum_{s=0}^{(n-1)(i_1+\cdots+i_n)} B_s^{(i_1,\cdots,i_n)}\int_{\Gamma_1^1(a_0)} x^{r+s}(x^n-1)^{(\nu/n)-(i_1+\cdots+i_n)}dx\,. \tag{3.19}$$

Put

$$B_0(p, \nu, l)=\int_{\Gamma_1^1(a_0)} x^p\cdot(x^n-1)^{(\nu/n)-l}dx\,.$$

Now we shall examine two recursion formulas for the definite integrals $B_0(p, \nu, l)$. We obtain the following

PROPOSITION 3.1.

$$(p+1)\cdot B_0(p, \nu, l)+(\nu-ln)\cdot B_0(p+n, \nu, l+1)=0\,, \tag{3.20}$$

$$\{(p+1)+(\nu-ln)\}B(p+n, \nu, l+1)-(p+1)B(p, \nu, l+1)=0\,. \tag{3.21}$$

PROOF. For the rational function $x^{p+1}(x^n-1)^{(\nu/n)-l}$ on $\tilde{C}$, we obtain

$$\begin{cases} d(x^{p+1}(x^n-1)^{(\nu/n)-l})=(p+1)x^p(x^n-1)^{(\nu/n)-l}dx+n(q/n-l)x^{p+n}(x^n-1)dx \\ \qquad =(p+(\nu-ln))x^{p+n}(x^n-1)^{(\nu/n)-(l+1)}dx-(p+1)x^p(x^n-1)^{(\nu/n)-(l-1)}dx\,. \end{cases} \tag{3.22}$$

From the first equality of (3.22) we obtain

$$(p+1)B_0(p, \nu, l)+(q-nl)B_0(p+n, \nu, l+1)=0 \tag{3.23}$$

and, from the second equality of (3.22), we obtain

$$(p+1)+(\nu-ln)B_0(p+n, \nu, l+1)-(p+1)B_0(p, \nu, l+1)=0\,, \tag{3.24}$$

q. e. d.

From this proposition, we easily obtain

(3.25)
$$n^{l-1}(-\nu)(n-\nu)\cdots\{(l-2)n-\nu\}\{(l-1)n-\nu\}B(p,\nu,l)$$
$$=(p+1-n)(p+1-2n)\cdots(p+1-ln)B(p-ln,\nu,0)\,,$$

(3.26)
$$(p+1-n)(p+1-2n)\cdots(p+1-kn)B(p-kn,\nu,q)$$
$$=\{p+\nu+(1-q)n\}\{p+\nu+(2-q)n\}\cdots\{p+\nu+(k-q)n\}B(p,\nu,q)\,.$$

Let $p=p_0+tn$, $0\leqq p_0<n$. From (3.25) and (3.26) we obtain

(3.27)
$$(-\nu)(n-\nu)\cdots\{(l-1)n-\nu\}B(p,\nu,l)=\{p+1-(l-t+1)n\}$$
$$\cdots\{(p+1)-ln\}\{(p+\nu+n)\cdots(p+\nu)+(l-t)\}B(p_0,\nu,0)\,.$$

9. Now we fix the index (ν, r) and define holomorphic functions $W^{\nu,r}_{i,j}(a)$ in the neighborhood $\mathfrak{A}$ of (a_0) by

$$W^{\nu,r}_{i,j}(a)=\partial W^{\nu,r}_{j}(a)/\partial a_i\,.$$

Moreover we define a matrix $\Lambda^{\nu,r}(a)$ by

(3.28)
$$\Lambda^{\nu,r}(a)=\begin{bmatrix} W^{\nu,r}_1(a) & \cdots & W^{\nu,r}_n(a)\\ W^{\nu,r}_{1,1}(a) & \cdots & W^{\nu,r}_{1,n}(a)\\ W^{\nu,r}_{n,1}(a) & \cdots & W^{\nu,r}_{n,n}(a)\end{bmatrix}$$

and a cyclic matrix W by

(3.29)
$$W=\begin{bmatrix} W^{\nu,r}_{1,1}(a_0) & W^{\nu,r}_{n,1}(a_0) & \cdots & W^{\nu,r}_{2,1}(a_0)\\ W^{\nu,r}_{2,1}(a_0) & W^{\nu,r}_{1,1}(a_0) & \cdots & W^{\nu,r}_{3,1}(a_0)\\ \vdots & \vdots & & \vdots\\ W^{\nu,r}_{n,1}(a_0) & W^{\nu,r}_{n-1,1}(a_0) & \cdots & W^{\nu,r}_{1,1}(a_0)\end{bmatrix}.$$

We put

(3.30)
$$\begin{cases} L=\begin{bmatrix} 1 & 1 & \cdots & 1\\ 1 & \zeta_n & \cdots & \zeta_n^{n-1}\\ 1 & \zeta_n^2 & \cdots & \zeta_n^{2(n-1)}\\ \vdots & \vdots & & \vdots\\ 1 & \zeta_n^{n-1} & \cdots & \zeta_n^{(n-1)(n-1)}\end{bmatrix}\\[2ex] M=\begin{bmatrix} \zeta_n W^{\nu,r}_1(a) & 0\\ 0 & -\dfrac{\nu}{n}E_n\end{bmatrix}\\[2ex] N=\begin{bmatrix} 1 & & & 0\\ & \zeta_n^r & & \\ & & \ddots & \\ 0 & & & \zeta_n^{r(n-1)}\end{bmatrix}\end{cases}$$

and

(3.31)
$$B=\begin{bmatrix} B_0(r+n-1,\nu,1) & & & \\ & \zeta_n^{-1}B_0(r+n-2,\nu,1) & & \\ & & \ddots & \\ & & & \zeta_n^{n-1}B_0(r,\nu,1) \end{bmatrix}.$$

Then, from the formula (3.15), we obtain

(3.32)
$$\Lambda^{\nu,r}(a)=M\begin{bmatrix} 1 & \zeta_n & \cdots & \zeta_n^{n-1} \\ & & W & \end{bmatrix}N.$$

From the recursion formula (3.27), we obtain

(3.33)
$$\begin{bmatrix} 1 & 0 \\ 0 & L^{-1} \end{bmatrix}\cdot\begin{bmatrix} 1 & [1\ \zeta_n \cdots \zeta_n^{n-1}] \\ 0 & W \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 \\ 0 & L \end{bmatrix}=\begin{bmatrix} 1 & [o \cdots o\ n] \\ 0 & B \end{bmatrix}.$$

10. Define a holomorphic map $H^{r,\nu}$ from the neighborhood $\mathfrak{A}$ of (a_0) into the complex Euclid space C^{n-3} in the following way:

(3.34)
$$H^{r,\nu}(a_1,\cdots,a_n)=(\eta_1^{r,\nu}(a),\eta_2^{r,\nu}(a),\cdots,\eta_{n-3}^{r,\nu}(a)),$$

where

$$\eta_1^{r,\nu}(a)=W_1^{r,\nu}(a)/W_{n-2}^{r,\nu}(a),\cdots,\eta_{n-3}^{r,\nu}(a)=W_{n-3}^{r,\nu}(a)/W_{n-2}^{r,\nu}(a).$$

Denote by $J_0(H^{r,\nu}(a))$ the Jacobian matrix of the mapping $H^{r,\nu}(a)$ at the point a_0. Now we obtain

THEOREM 3.1. *The rank of* $J_0(H^{r,\nu}(a))$ *is equal to* $n-3$.

PROOF. (i) Let the indices $(i_1,\cdots,i_{n-2})$ and $(j_1,\cdots,j_{n-3})$ be subsets of $(1,2,\cdots,n)$. Define a matrix $W^{r,\nu}_{\binom{i_1,\cdots,i_{n-2}}{j_1,\cdots,j_{n-3}}}$ by

(3.35)
$$W^{r,\nu}_{\binom{i_1,\cdots,i_{n-2}}{j_1,\cdots,j_{n-3}}}=\begin{bmatrix} W_{i_1}^{r,\nu}(a_0) & \cdots & W_{i_{n-2}}^{r,\nu}(a_0) \\ W_{j_1,i_1}^{r,\nu}(a_0) & \cdots & W_{j_1,i_{n-2}}^{r,\nu}(a_0) \\ \vdots & & \vdots \\ W_{j_{n-3},i_1}^{r,\nu}(a_0) & \cdots & W_{j_{n-3},i_{n-2}}^{r,\nu}(a_0) \end{bmatrix}.$$

Then, from the transformation laws (3.32) and (3.33), we obtain

(3.36)
$$n\zeta_n^{-\sum_{j=1}^{n-1} i_j-1}\cdot\prod_{j=1}^{n-3}B_0(r+n-i_j,\nu,1)=\sum_{\substack{(i_1,\cdots,i_{n-2})\subset(1,2,\cdots,n)\\(j_1,\cdots,j_{n-3})\subset(1,2,\cdots,n)}}\det W^{r,\nu}_{\binom{i_1,\cdots,i_{n-2}}{j_1,\cdots,j_{n-3}}}c_{\binom{i_1,\cdots,i_{n-2}}{j_1,\cdots,j_{n-3}}}$$

where $c_{\binom{i_1,\cdots,i_{n-2}}{j_1,\cdots,j_{n-3}}}$ are constants.

(ii) On the other hand, we can verify that $x^{r+n-i}(x^n-1)^{\frac{\nu}{n}-1}dx$ is not a derived form, if $(r+n-i\neq n-1,-\nu)$. From (3.36) and the recursion formula (3.27), we can easily verify that

(3.37)
$$B_0(r+n-i,\nu,1)\neq 0,\qquad \text{if}\quad r+n-i\neq n-1.$$

(iii) From (i) and (ii), we conclude that, for some indices $\binom{i_1,\cdots,i_{n-2}}{j_1,\cdots,j_{n-3}}$.

(3.38) $$\det W^{r,\nu}_{\binom{i_1,\cdots,i_{n-2}}{j_1,\cdots,j_{n-3}}}\neq 0\,.$$

On the other hand, from the boundary relation (2.8) in § 2, we conclude that Γ^1_j $(j=1,\cdots,n)$ are expressed as a linear combination of Γ^1_j $(j=1,2,\cdots,n-2)$. Hence, for suitable indices $(j_1,\cdots,j_{n-3})$ we obtain,

(3.39) $$\det W^{r,\nu}_{\binom{1,2,\cdots,n-2}{j_1,\cdots,j_{n-3}}}\neq 0\,.$$

(iv) Let $(j_1,\cdots,j_{n-3})$ be the indices for which the inequality (3.39) holds and let $J_0(H^{r,\nu}(a))_{j_1,\cdots,j_{n-3}}$ be the sub-matrix of $J_0(H^{r,\nu}(a))$ defined by

(3.40) $$J_0(H^{r,\nu}(a))_{j_1,\cdots,j_{n-3}}=\begin{bmatrix}\partial\eta^{r,\nu}_2(a)/\partial a_{j_1} & \cdots & \partial\eta^{r,\nu}_2(a)/\partial a_{j_{n-3}}\\ \partial\eta^{r,\nu}_{n-3}(a)/\partial a_{j_1} & \cdots & \partial\eta^{r,\nu}_{n-3}(a)/\partial a_{j_{n-3}}\end{bmatrix}(a)=(a_0)\,.$$

Then we obtain

(3.41) $$\det J_0(H^{r,\nu}(a))=(W^{r,\nu}_{n-2}(a))^{-(n-2)}\cdot\det W^{r,\nu}_{\binom{1,2,\cdots,n-2}{j_1,\cdots,j_{n-2}}}\,.$$

Thus we prove the assertion of Theorem 3.1.

§ 4. Picard numbers.

We examine the module $\mathrm{Hom}_0^{(G_n)}(J(a^{(1)}),J(a^{(2)}))$ and the endomorphism ring $\mathcal{E}_0^{(G_n)}(J(a))$, where $J(a^{(k)})$ and $J(a)$ are the Jacobian varieties of the algebraic curves $C^{(k)}: y^n=\prod_{j=1}^{n}(x-a_j^{(k)})$ and $C: y^n=\prod_{j=1}^{n}(x-a_j)$, respectively.

11. Define an $(n-3)$-vector $Z(a)$, which depends on the parameter (a) by

(4.1) $${}^tZ(a^{(k)})=(\eta_1^{\nu_0,0}(a^{(k)}),\eta_2^{\nu_0,0}(a^{(k)}),\cdots,\eta_{n-3}^{\nu_0,0}(a^{(k)}))\qquad(k=1,2)\,,$$

and let $\mathfrak{B}^{(k)}$ $(k=1,2)$ be a small neighborhood of the point $Z(a_0^{(k)})$ $(k=1,2)$ in the complex space $\boldsymbol{C}^{n-3}$.

(i) Let λ be an element of $\mathrm{Hom}_0^{(G_n)}(J(a^{(1)}),J(a^{(2)}))$, Λ the matrix representation of λ (as was defined in § 3) and let $U=\begin{bmatrix}\mathrm{a} & \mathrm{b}\\ \mathrm{c} & \mathrm{d}\end{bmatrix}$ be the corresponding matrix appeared in (3.9), where $\mathrm{a}\in M_{n-3}(\boldsymbol{Q}(\zeta_n))$, $\mathrm{b}\in M_{n-3,1}(\boldsymbol{Q}(\zeta_n))$, $\mathrm{c}\in M_{1,n-3}(\boldsymbol{Q}(\zeta_n))$, $\mathrm{d}\in M_1(\boldsymbol{Q}(\zeta_n))$. Then, we obtain

(4.2) $$[{}^tZ(a^{(2)})\cdot\mathrm{c}+\mathrm{d}]\cdot Z(a^{(1)})=\mathrm{a}\cdot Z(a^{(2)})+\mathrm{b}$$

(cf. Shimura [7] § 4).

For an element λ of the endomorphism ring $\mathcal{E}_0(J(a))$, we obtain, by setting $(a^{(1)})=(a^{(2)})=(a)$, $C^{(1)}(a^{(1)})=C^{(2)}(a^{(2)})=C(a)$, a similar equation:

(4.2)′ $$[{}^tZ(a)\mathrm{c}+\mathrm{d}]Z(a)=\mathrm{a}Z(a)+\mathrm{b}\,,$$

where the matrices a, b, c, d have the same meaning as in (4.2). For any vectors $Z^{(1)}=(\eta_1^{(1)},\cdots,\eta_{n-3}^{(1)})$, $Z^{(2)}=(\eta_1^{(2)},\cdots,\eta_{n-3}^{(2)})$ and $Z=Z^{(1)}=Z^{(2)}$ in the neighborhoods $\mathfrak{B}_k$ $(k=1,2)$, we consider the following equations with respect to

$Z=\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}$:

$$[{}^tZ^{(2)}\mathrm{c}+\mathrm{d}]Z^{(1)}=\mathrm{a}Z^{(2)}+\mathrm{b}\,, \tag{4.3}$$

$$[{}^tZ\mathrm{c}+\mathrm{d}]Z=\mathrm{a}Z+\mathrm{b}\,. \tag{4.3$'$}$$

Then it can be easily verified that, for general values of $(Z^{(1)}, Z^{(2)})$ in $\mathfrak{B}^{(1)}\times\mathfrak{B}^{(2)}$, the equation (4.3) has no other solutions than $U=\begin{bmatrix} e & & \\ & \ddots & \\ & & e\end{bmatrix}$ $(e\in \boldsymbol{Q}(\zeta_n))$ (cf. Shimura [7] § 4).

(iii) On the other hand, since, by Theorem 3.1, the rank of the Jacobian matrix of $H^{r,\nu_0}(a_0)$ is equal to $n-3$, we conclude that a small neighborhood $\mathfrak{B}^k$ of $Z(a_0^{(k)})$ is fulfilled by the Image $H^{\nu_0,0}(\mathfrak{A})$, where $\mathfrak{A}$ is a certain neighborhood of the point a_0.

From (i), (ii) and the relation (4.3), we obtain

THEOREM 4.1. *If n is a prime number, for general values of the parameters $a^{(1)}$, $a^{(2)}$, we have*

$$\mathrm{Hom}_0^{(G_n)}(J(a^{(1)}), J(a^{(2)}))=0 \tag{4.4}$$

and, for general values of the parameters (a), we have

$$\mathcal{E}_0^{(G_n)}(J(a))\cong \boldsymbol{Q}(\zeta_n)\,. \tag{4.5}$$

As to the Picard number, we obtain in view of Lemma 1.2[10)]

THEOREM 4.2. *If n is a prime number, for general values of the parameters (a_1), (a_2) and (a), we have*

$$\rho(S_n(a^{(1)}, a^{(2)}))=n^2-2n+2 \tag{4.4$'$}$$

and

$$\rho(S_n(a, a))=n^2-2n+2+(n-1)\,. \tag{4.5$'$}$$

12. *Quartic and Quintic surfaces.* We examine some results concerning the maximum of the dimension of $\mathrm{Hom}_0^{(G_n)}(J(a^{(1)}), J(a^{(2)}))$ (and the maximum of the Picard number $\rho(S_n(a^{(1)}, a^{(2)}))$, in two cases: $n=4$ and $n=5$.

(i) $n=m=4$. In this case, our curves (and surfaces which are obtained as the quotient of product of our curves) are as follows:

$$\begin{cases} C(a): & y^4=\prod_{j=1}^{4}(x-a_j)\,, \\ S_4(a^{(1)}, a^{(2)}): & \prod_{j=1}^{4}(x_3-a_j^{(1)}x_2)=\prod_{j=2}^{4}(x_1-a_j^{(2)}x_0)\,. \end{cases}$$

For the curve $C(a)$, we have a base of holomorphic 1-forms consisting of the following forms:

$$\omega^{1,0}=dx/y^3\,,\qquad \omega^{1,1}=xdx/y^3\,,\qquad \omega^{2,0}=dx/y^2\,. \tag{4.6}$$

10) It is obvious that: $\dim_Q \mathrm{Hom}_0^{(G_n)}(J(a^{(1)}), J(a^{(2)}))+Q=\rho^{(G_n)}(C_1\times C_2)$.

We define rational 1-cycles $\tilde{\Gamma}_j^k(a)$ on the algebraic curve $C(a)$ ($k=1, 2, j=1, 2, 3$) by

(4.7)
$$\tilde{\Gamma}_1^k(a)=\frac{1}{2}(\Gamma_1^k(a)+\Gamma_2^k(a)), \qquad \tilde{\Gamma}_2^k(a)=\frac{1}{2}(\Gamma_2^k(a)+\Gamma_3^k(a))$$
$$\tilde{\Gamma}_3^k(a)=\frac{1}{2}(\Gamma_3^k(a)+\Gamma_1^k(a)), \qquad (k=1, 2).$$

Then the period matrix of the algebraic curve $C(a)$ with respect to bases of holomorphic 1-form: (4.6) and bases of 1-cycles: (4.7) are expressed in the following form:

(4.8)

	$\tilde{\Gamma}_1^1(a)$ $\tilde{\Gamma}_2^1(a)$ $\tilde{\Gamma}_1^2(a)$ $\tilde{\Gamma}_2^2(a)$	$\tilde{\Gamma}_3^1(a)$	$\tilde{\Gamma}_3^2(a)$
$\omega^{1,0}$ $\omega^{1,1}$	$\Omega(a)$	0	
$\omega^{2,0}$	0	$2\int_{\tilde{\Gamma}_3^1(a)}\omega^{2,0}$	$2\int_{\tilde{\Gamma}_3^2(a)}\omega^{2,0}$

where $\Omega(a)$ is a complex matrix of 2 rows and 4 columns.

Now we define two abelian varieties $J_1(a)$ and $J_2(a)$ by

(4.9)
$$J_1(a)=\boldsymbol{C}^2/\Omega(a),$$

(4.10)
$$J_2(a)=\boldsymbol{C}/\left(\int_{\Gamma_1^1(a)}\omega^{1,0}, \int_{\Gamma_2^1(a)}\omega^{2,0}\right).$$

Then, obviously, our Jacobian variety $J(a)$ is isogenous to the product $J_1(a)\times J_2(a)$. The automorphism $\theta'(\sigma_4)$ of the abelian variety $J_1(a)$ corresponding to the automorphism σ_4: $(x, y)\rightarrow(x, \zeta_4 y)$ of the algebraic curve $C(a)$ is expressed in the following form:

(4.11)
$$\theta'(\sigma_4)=\begin{bmatrix}\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{bmatrix}.$$

Thus the Jacobian variety $J_1(a)$ is isogenous to the product of two elliptic curves whose endomorphism ring is the Gaussian field $\boldsymbol{Q}(\sqrt{-1})$, (cf. Shimura [7], Prop. 14). Obviously $J_2(a)$ is an elliptic curve whose analytic invariant is:

$$\eta(a)=\int_{\Gamma_1^1(a)}y^{-2}dx\Big/\int_{\Gamma_1^2(a)}y^{-2}dx.$$

THEOREM 4.3.

(1) *If* $J(a^{(1)})$ *and* $J(a^{(2)})$ *are not isogenous, then* $\dim \mathrm{Hom}_0^{(G_n)}(J(a^{(1)}), J(a^{(2)}))=8$.

(2) *If* $J(a^{(1)})$ *and* $J(a^{(2)})$ *are isogenous, and are not isogenous to* $J(a_0)$, *then* $\dim \mathrm{Hom}_0^{(G_n)}(J(a^{(1)}), J(a^{(2)}))=9$.

(3) *If* $J(a^{(k)})$ $(k=1, 2)$ *are isogenous to* $J(a_0)$, *then* $\dim \mathrm{Hom}_0^{(G_n)}(J(a^{(1)}), J(a^{(2)}))=10$.

We indicate isogeny by the symbol $\sim$.

In view of Lemma 1.2, we obtain

THEOREM 4.4.

(I)′ *If* $J(a^{(1)}) \not\sim J(a^{(2)})$, *then* $\rho(S_4(a^{(1)}, a^{(2)}))=18$.

(II)′ *If* $J(a^{(1)}) \sim J(a^{(2)})$, $J(a^{(k)}) \not\sim J(a_0)$, *then* $\rho(S_4(a^{(1)}, a^{(2)}))=19$.

(III)′ *If* $J(a^{(k)}) \sim J(a_0)$, $k=1, 2$, *then* $\rho(S_4(a^{(1)}, a^{(2)}))=20$.

(ii) $n=5$. *In this case we have*

$$C(a): \quad y^5=\prod_{j=1}^{5}(x-a_j),$$

$$S_5(a_1, a_2): \quad \prod_{j=1}^{5}(x_3-a_j^{(1)}x_2)=\prod_{j=1}^{5}(x_1-a_j^{(2)}x_0).$$

For the sake of simplicity, we ask only the maximum of the dimension of the endomorphism ring $\mathcal{E}_0^{(G_n)}(J(a))$. Put

$$X^{(2)}(a)=[X_1^{(2)}(a), X_2^{(2)}(a)], \qquad X^{(3)}(a)=[X_2^{(3)}(a), X_2^{(3)}(a)],$$

and put

$$U^{(2)}=\begin{bmatrix} U_{1,1}^{(2)}, & U_{1,2}^{(2)} \\ U_{2,1}^{(2)}, & U_{2,2}^{(2)} \end{bmatrix},$$

where

$$\begin{cases} X_1^{(2)}(a),\ U_{1,1}^{(2)} \in M_1(\boldsymbol{C}), & X_2^{(2)}(a),\ U_{1,2}^{(2)} \in M_{1,2}(\boldsymbol{C}), \\ X_1^{(3)}(a),\ U_{2,1}^{(2)} \in M_2(\boldsymbol{C}), & X_2^{(3)}(a),\ U_{2,2}^{(2)} \in M_{2,2}(\boldsymbol{C}). \end{cases}$$

We define

$$U'^{(2)}=\begin{bmatrix} {}^tU_{1,1} & -{}^tU_{2,1} \\ {}^tU_{1,2} & -{}^tU_{2,2} \end{bmatrix},$$

$$Z^{(2)}(a)=X_1^{(2)}(a)^{-1}\cdot X_2^{(2)}(a), \qquad Z^{(3)}(z)=X_1^{(3)}(a)^{-1}\cdot X_2^{(3)}(a).$$

Then the equality (4.2) is reduced to the following:

$$\text{(4.12)} \qquad \begin{cases} [1, Z^2(a)]U'^{(2)}\cdot\begin{bmatrix} Z^3(a) \\ E_2 \end{bmatrix}=0 \\ [Z^3(a), E_2]U'^{(2)}\cdot{}^t[1, Z^2(a)]=0. \end{cases}$$

Now we define $P(a)$ to be the subalgebra of the full matrix algebra with coefficients in cyclotomis field $\boldsymbol{Q}(\zeta_5)$ $M_3(\boldsymbol{Q}(\zeta_5))$ consisting of matrices $U^{(2)}$ such that the corresponding matrices $U'^{(2)}$ satisfy the equality (4.11). From the inequality (3.5), we know that

$$\text{(4.13)} \qquad \det\begin{bmatrix} 1 & Z^{(2)}(a) \\ Z^{(3)}(a) & E_2 \end{bmatrix}\neq 0.$$

Hence, obviously, we obtain from (4.11) and (4.13)

$$\text{(4.14)} \qquad [P(a): \boldsymbol{Q}(\zeta_5)]\leqq 5.$$

Now we prove the existence of a parameter (a) for which the equality $[P(a):$

$Q(\zeta_5)] = 5$ holds.

Let

$$\mathfrak{H}^{(2)} = \{(Z_1', Z_2') ; |Z_1'|^2 + |Z_2'|^2 < 1\} . \tag{4.15}$$

Then, from Shimura [8], we know that, by means of suitable linear fractional functions L_1, L_2 with entries in $Q(\zeta_5)$, the generic point of $\mathfrak{H}^{(2)}$ is written in the form:

$$Z_1'(a) = L_1(Z_1(a)) , \qquad Z_2'(a) = L_2(a) . \tag{4.16}$$

Choosing $(Z_1'(a), Z_2'(a))$ so that the entries of $(Z_1'(a), Z_2'(a))$ belong to the cyclotomic field $Q(\zeta_5)$[11], we obtain the following results:

THEOREM 4.5. *For the algebraic cnrve $C(a)$ we have the inequalities*

$$20 \geqq \dim \mathcal{E}_0^{(G_n)}(J(a)) \geqq 4 . \tag{4.17}$$

Moreover, if $Z^{(1)}(a)$, $Z^{(2)}(a) \in Q(\zeta_5)$, then we have the equality

$$20 = \dim \mathcal{E}_0^{(G_n)}(J(a)) . \tag{4.18}$$

Hence, in view of Lemma 1.2, we obtain:

THEOREM 4.6. *For the Picard uumber $\rho(S_5(a^{(1)}, a^{(2)}))$ of the quintic surface $S_5(a^{(1)}, a^{(2)})$, we have*

$$37 \geqq \rho(S_5(a^{(1)}, a^{(2)})) \geqq 17 . \tag{4.17$'$}$$

Moreover, if $Z^{(1)}(a)$, $Z^{(2)}(a) \in Q(\zeta_5)$, the equality

$$\rho(S_5(a, a)) = 37 \tag{4.18$'$}$$

holds.

13. *Surfaces of Fermat type.* Now we consider a surface of Fermat type defined by an equation of the form: $x_0^n + x_1^n + x_2^n + x_3^n = 0$, where we assume that the order n is a prime number $\geqq 5$. We examine the dimension of the sub-endomorphism ring $\mathcal{E}_0^{(G_n)}(J(a^{(0)}))$ of the algebraic curve defined by the equation: $y^n = (x^n - 1)$; (i) From the recursion formula (3.15), we obtain

$$X^{(\nu)} = \begin{bmatrix} \tau_{(0)}^{(\nu,0)} & & 0 \\ & \ddots & \\ 0 & & \tau_{(0)}^{(\nu,\nu-1)} \end{bmatrix} \cdot \begin{bmatrix} 1 & \zeta_n & \cdots & \zeta_n^{n-3} \\ 1 & \zeta_n^2 & \cdots & \zeta_n^{2(n-3)} \\ \vdots & & & \\ 1 & \zeta_n^{(\nu-1)} & \cdots & \zeta_n^{(\nu-1)(n-3)} \end{bmatrix} . \tag{4.19}$$

We write

11) The existence of the parameters (a) such that $(z_1'(a), z_2'(a))$ are matrices with entries in $Q(\zeta_5)$ are assured by Theorem 4.1. Moreover, by Shimura [2], we know that the inverse function of $\tau_1(a)$, $\tau_2(a)$ are automorphic functions w, r, t certain arithmetic groups.

$$(4.20)\quad \begin{cases} (T^{(\nu)})^{-1}=\begin{bmatrix} T'^{(\nu)}_{1,1} & T'^{(\nu)}_{1,2} \\ T'^{(\nu)}_{2,1} & T'^{(\nu)}_{2,2} \end{bmatrix}, \\ U^{(\nu)}=\begin{bmatrix} U^{(\nu)}_{1,1}\,U & -{}^{(\nu)}_{1,2} \\ U^{(\nu)}_{2,1} & U^{(\nu)}_{2,2} \end{bmatrix}, \end{cases}$$

where

$$\begin{cases} T'^{(\nu)}_{1,1}, & U^{(\nu)}_{1,1}\in M_{\nu-1}(\boldsymbol{Q}(\zeta_n^\nu)) \\ T'^{(\nu)}_{1,2}, & U^{(\nu)}_{1,2}\in M_{\nu-1,n-\nu-1}(\boldsymbol{Q}(\zeta_n^\nu)) \\ T'^{(\nu)}_{2,1}, & U^{(\nu)}_{2,1}\in M_{\nu-1,n-\nu-1}(\boldsymbol{Q}(\zeta_n^\nu)) \\ T'^{(\nu)}_{2,2}, & U^{(\nu)}_{2,2}\in M_{\nu-1,n-\nu-1}(\boldsymbol{Q}(\zeta_n^\nu)) \end{cases}$$

and define a matrix $V^{(\nu)}$ ($\in M_{n-2}(\boldsymbol{Q}(\zeta_n^\nu))$) by

$$(4.21)\quad V^{(\nu)}=\begin{bmatrix} U^{(\nu)}_{1,1}, & -U^{(\nu)}_{1,2} \\ U^{(\nu)}_{2,1}, & -U^{(\nu)}_{2,2} \end{bmatrix}\begin{bmatrix} {}^tT^{(\nu)}_{1,1}, & {}^tT^{(\nu)}_{2,1} \\ -{}^tT^{(\nu)}_{1,2}, & -{}^tT^{(\nu)}_{2,2} \end{bmatrix}.$$

Then, the matrix $V^{(\nu)}$ has the following form:

$$(4.22)\quad V^{(\nu)}=(\nu_{ij}(\zeta_n^\nu))_{i,j=1,\cdots,n-2},$$

where $\nu_{ij}(t)$ is a polynomial of one variable t with coefficients in the rational number field $\boldsymbol{Q}$. Let

$$(4.23)\quad F^{(\nu)}=\begin{bmatrix} 1 & \zeta_n & \cdots & \zeta_n^{n-3} \\ \vdots & & & \\ 1 & \zeta_n^{\nu-1} & \cdots & \zeta_n^{(\nu-1)(n-3)} \end{bmatrix}.$$

Then, by the elimination of the matrix Λ_ν in (3.9) and replacing the matrix $U^{(\nu)}$ by the matrix $V^{(\nu)}$, we obtain the following period relations:

$$(4.24)\quad \begin{cases} F^{(\nu)}\cdot V^{(\nu)}\cdot {}^tF^{(n-\nu)}=0, \\ F^{(n-\nu)}\cdot V^{(\nu)}\cdot {}^tF^{\nu}=0, \end{cases}\quad \left(\nu=1,\cdots,\frac{n-1}{2}\right).$$

(ii) In what follows we assume that n is a prime number $\geqq 5$ and let P be the subalgebra of the full matrix algebra $M_{n-2}(\boldsymbol{Q}(\zeta_n))$ defined by

$$(4.25)\quad P=\Big\{V\in M_{n-2}(\boldsymbol{Q}(\zeta_n))\,;\ \text{where the conjugates } V^{(\nu)} \text{ of } V \text{ satisfy the equations (4.24) for } \nu=1,\cdots,\frac{n-1}{2}\Big\}.$$

Let $g_n^{(\nu)}$ be the automorphism of the cyclotomic field $\boldsymbol{Q}(\zeta_n^\nu)$ such that

$$g_n^{(\nu)}(\zeta_n^\nu)=\zeta_n$$

and let α_ν be the integer such that $\alpha_\nu\cdot\nu\equiv 1 \pmod n$. We denote by $g_n^{(\nu)}(F^{(\nu)})$ the matrix which is obtained by applying $g_n^{(\nu)}$ on the entries of the matrix $F^{(\nu)}$. Then the equation (4.24) is equivalent to

$$(4.26)\quad \begin{aligned} &g_n^\nu(F^\nu)\cdot V\cdot {}^tg_n(F^{(n-\nu)})=0, \\ &g_n^\nu(F^{n-\nu})\cdot V\cdot {}^tg_n^{(\nu)}(F^{(\nu)})=0. \end{aligned}$$

Define vectors $\mathfrak{x}_n^{(i)}$ by

$$^t\mathfrak{x}_n^{(i)} = (1, \zeta_n^i, \cdots, \zeta_n^{i(n-3)})$$

and denote by $\mathfrak{x}_n^{(i)} \otimes \mathfrak{x}_n^{(j)}$ the tensor product of $\mathfrak{x}_n^{(i)}$ and $\mathfrak{x}_n^{(j)}$. Consider the linear equation (whose indeterminates are the entries $\nu_{k,l}$ of the matrix V):

$$[\mathfrak{x}_n^{(i)} \otimes \mathfrak{x}_n^{(j)}](V) = \sum_{k,l=1}^{n-2} \zeta_n^{i(k-1)+j(l-1)} \cdot \nu_{k,l}^{(1)} = 0 .$$

Then the equation (4.26) is equivalent to

$$(4.27) \qquad [\mathfrak{x}_{(n)}^{(i_\nu)} \otimes \mathfrak{x}_{(n)}^{(j_\nu)}](V) = 0 , \qquad \begin{array}{l} i_\nu = \alpha_\nu, 2\alpha_\nu, \cdots, (\nu-1)\alpha_\nu , \\ j_\nu = \alpha_\nu, 2\alpha_\nu, \cdots, \alpha_\nu(n-\nu-1) , \\ (\nu = 1, \cdots, n-2) . \end{array}$$

Denote by l_n the number of linearly independent vectors $\mathfrak{x}_{(n)}^{(i_\nu)} \otimes \mathfrak{x}_{(n)}^{(j_\nu)}$, where i_ν and j_ν subordinate to the conditions in (4.27). Then we have

$$(4.28) \qquad [\dim P : \boldsymbol{Q}(\zeta_n)] = (n-2)^2 - l_n .$$

We note that, in the following six cases:

1) $i \equiv 0 \pmod n$, 2) $i \equiv 1 \pmod n$, 3) $j \equiv 0 \pmod n$,

4) $j \equiv 1 \pmod n$, 5) $i+j \equiv 0 \pmod n$, 6) $j-i \equiv 1 \pmod n$,

the congruences: $i_\nu \equiv i$, $j_\nu \equiv j \pmod n$ have no solutions i_ν, j_ν satisfying the conditions in (4.27). Hence we obtain,

$$[\dim P : \boldsymbol{Q}(\zeta_n)] \geqq 2n-3 .$$

For the surface S_n' of Fermat type defined by $x_0^n + x_1^n + x_2^n + x_3^n = 0$, where n is a prime number $\geqq 5$, we obtain $\rho(S_n) \geqq (n-1)(2n-5) + n^2 - 2n + 2$.

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