J. Math. Soc. Japan
Vol. 20, Nos. 1-2, 1968

On the relation for two-dimensional theta constants of level three

Dedicated to Professor Iyanaga on the occation of his 60th birthday

By Hisasi MORIKAWA

(Received June 18, 1967)
(Revised Sept. 18, 1967)

Let $\{Q_{11}, Q_{12}, Q_{22}\}$ be a system of indeterminates and denote

$$\vartheta_{\mathbf{a}}(Q)=\sum_{\mathbf{m}\in\mathbf{Z}^2} Q\left(\mathbf{m}+\frac{\mathbf{a}}{3}, \mathbf{m}+\frac{\mathbf{a}}{3}\right) \qquad (\mathbf{a}=(a_1, a_2)\,;\ a_1, a_2=0, 1, -1),$$

where $Q\left(\mathbf{m}+\frac{\mathbf{a}}{3}, \mathbf{m}+\frac{\mathbf{a}}{3}\right)$ means $Q_{11}^{\left(m_1+\frac{a_1}{3}\right)^2} Q_{12}^{2\left(m_1+\frac{a_1}{3}\right)\left(m_2+\frac{a_2}{3}\right)} Q_{22}^{\left(m_2+\frac{a_2}{3}\right)^2}$. In the present note we shall give an explicit defining equation for the projective scheme $\operatorname{Proj}\mathbf{Z}[\vartheta_{(0,0)}(Q), \vartheta_{(1,0)}(Q), \vartheta_{(0,1)}(Q), \vartheta_{(1,1)}(Q), \vartheta_{(1,-1)}(Q)]$. The defining equation $\Delta(X_{(0,0)}, X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)})=0$ is a rather simple equation of degree ten. From this equation we can conclude the following important result:

Let ζ be a primitive cubic root of unity and Γ_0 a transformation group on $\mathbf{Q}(\zeta, \vartheta_{(0,0)}(Q), \vartheta_{(1,0)}(Q), \vartheta_{(0,1)}(Q), \vartheta_{(1,1)}(Q), \vartheta_{(1,-1)}(Q))$ consisting of all the elements

$$(\alpha, \beta)\,;\ \vartheta_{\mathbf{a}}(Q)\rightarrow \zeta^{\mathbf{a}\beta^t\alpha^t\mathbf{a}}\vartheta_{\mathbf{a}\alpha}(Q) \qquad (\mathbf{a}\in GF(3)^2),$$

where α, β are 2×2-matrices with coefficients in $GF(3)$ such that $\det\alpha'\neq 0$ and $\beta^t\alpha=\alpha^t\beta$. Then the invariant subfield of $\mathbf{Q}(\zeta, \vartheta_{(1,0)}(Q)/\vartheta_{(0,0)}(Q), \vartheta_{(0,1)}(Q)/\vartheta_{(0,0)}(Q), \vartheta_{(1,1)}(Q)/\vartheta_{(0,0)}(Q), \vartheta_{(1,-1)}(Q)/\vartheta_{(0,0)}(Q))$ with respect to the group Γ_0 of automorphisms is the rational function field $\mathbf{Q}(\zeta, \sum_{\mathbf{a}\neq(0,0)}\vartheta_{\mathbf{a}}(Q)^3/\vartheta_{(0,0)}(Q)^3, \sum_{\mathbf{a}\neq(0,0)}\vartheta_{\mathbf{a}}(Q)^6/\vartheta_{(0,0)}(Q)^6, \vartheta_{(1,0)}(Q)\vartheta_{(0,1)}(Q)\vartheta_{(1,1)}(Q)\vartheta_{(1,-1)}(Q)/\vartheta_{(0,0)}(Q)^4)$.

§ 1. Canonical systems of theta constants on abstract abelian varieties.

1.1. Let $\mathbf{A}$ be an abelian variety defined over an algebraically closed field of characteristic p, where p is a prime number or zero. Let ξ be an algebraic equivalent class on $\mathbf{A}$ and X be a divisor in ξ. We denote by $\mathfrak{g}_X$ the group of all the points a in A such that $X_a\sim X$[1]. Since $\mathfrak{g}_X$ depends only the class ξ, we may denote $\mathfrak{g}_\xi$ instead of $\mathfrak{g}_X$. If $\mathfrak{g}_\xi$ is a finite group, the divisor class ξ (the divisor X) is called non-degenerate. For any prime number l

1) $X_a\sim X$ means that X is linearly equivalent to X.

coprime to p we choose a system $\tau^{(l)}={}^t(\tau_1^{(l)},\cdots,\tau_{2r}^{(l)})$ of l-adic coordinates on A and an isomorphism $lg^{(l)}$ of the group of roots of unity of l-power degree onto the additive residue group $\mathbf{Q}_l/\mathbf{Z}_l$. Denote by $E_l(\xi)$ the skew symmetric l-adic integral matrix associated with a divisor class ξ with respect to $\tau^{(l)}$ and $lg^{(l)}$ and denote by $\mathfrak{g}_{\xi,l}$ the l-Sylow subgroup of $\mathfrak{g}_\xi$. Then, if ξ is non-degenerate, the group $\mathfrak{g}_{\xi,l}$ coinsides with the set of all the points of l-power order such that $E_l(\xi)\tau^{(l)}(a)\equiv 0 \mod 1$.

Putting

$$a=a_p+\sum_l a_l\,,\quad b=b_p+\sum_l b_l \qquad (a_l,\,b_l\in\mathfrak{g}_{\xi,l})$$

and

$$e_\xi(a,\,b)=\prod_l lg^{(l)-1}({}^t\tau^{(l)}(a_l)E_l(\xi)\tau^{(l)}(b_l))\,,$$

we have a function $e_\xi(\ ,\ )$ on $\mathfrak{g}_\xi\times\mathfrak{g}_\xi$ such that

$$\begin{aligned} &e_\xi(a,\,b+c)=e_\xi(a,\,b)e_\xi(a,\,c)\,,\\ &e_\xi(a+b,\,c)=e_\xi(a,\,c)e_\xi(b,\,c)\,, \qquad\qquad (a,\,b\in\mathfrak{g}_\xi)\\ &e_\xi(a,\,a)=1\,,\qquad e_\xi(a,\,b)e_\xi(b,\,a)=1\,. \end{aligned}$$

Moreover, if ξ is non-degenerate and the order of $\mathfrak{g}_\xi$ is coprime to p then the function $e_\xi(\ ,\ )$ is non-degenerate, i. e. $e_\xi(a,\,b)=1$ for every b in $\mathfrak{g}_\xi$ implies $a=0$.

Since $E_l(\xi)$ is skew symmetric, there exists a direct sum decomposition $\mathfrak{g}_{\xi,l}=\mathfrak{g}_l\oplus\hat{\mathfrak{g}}_l$ such that

$${}^t\tau^{(l)}(a)E_l(\xi)\tau^{(l)}(b)\equiv{}^t\tau^{(l)}(\hat{a})E_l(\xi)\tau^{(l)}(\hat{b})\equiv 0 \mod 1 \qquad (a,\,b\in\mathfrak{g}_l\,;\ \hat{a},\,\hat{b}\in\hat{\mathfrak{g}}_l)\,.$$

If ξ is non-degenerate and the order of $\mathfrak{g}_\xi$ is coprime to p, we have a direct sum decomposition

$$\mathfrak{g}_\xi=\mathfrak{g}\oplus\hat{\mathfrak{g}} \qquad (\mathfrak{g}=\sum_l\mathfrak{g}_l,\ \hat{\mathfrak{g}}=\sum_l\hat{\mathfrak{g}}_l)$$

such that $e_\xi(\mathfrak{g},\,\mathfrak{g})=e_\xi(\hat{\mathfrak{g}},\,\hat{\mathfrak{g}})=1$ and $|\mathfrak{g}|=|\hat{\mathfrak{g}}|=\sqrt{|\mathfrak{g}|_\xi}$, where $|\mathfrak{g}|$, $|\hat{\mathfrak{g}}|$, $|\mathfrak{g}_\xi|$ are the orders of $\mathfrak{g}$, $\hat{\mathfrak{g}}$, $\mathfrak{g}_\xi$, respectively. We call the subgroup $\mathfrak{g}$ a standard subgroup with respect to ξ and call the direct sum decomposition $\mathfrak{g}_\xi=\mathfrak{g}\oplus\hat{\mathfrak{g}}$ *the standard direct sum decomposition with respect to* ξ.

A divisor X on A is called *symmetric* if $(-\delta_{\mathrm{A}})^{-1}(X)=X$, where $-\delta_{\mathrm{A}}$ is the automorphism $x\to -x$. We denote by $l(\xi)$ the dimension $l(X)=\dim H^0(A,\mathcal{O}_{\mathrm{A}}(X))$ for a divisor X in the algebraic equivalence class ξ.

LEMMA 1.1. *Let X be a non-degenerate positive divisor on an abelian variety* **A** *defined over an algebraically closed field k. Then there exists the unique system of functions $\{\phi_a\,|\,a\in\mathfrak{g}_X\}$ on* **A** *defined over k such that*

$$(\phi_a)=X_{-a}-X\,,$$

$$\phi_0(x)=1\,,\quad \phi_{a+b}(x)=\phi_a(x+b)\phi_b(x)\qquad (a,\, b\in \mathfrak{g}_X)\,.$$

PROOF. By the definition of $\mathfrak{g}_X$ there exist function f_a $(a\in\mathfrak{g}_X)$ defined over k such that $(f_a)=X_{-a}-X$ $(a\in\mathfrak{g}_X)$. Then it follows that

$$\begin{aligned}(\text{divisor of } f_a(x+b)) &= X_{-(a+b)}-X_{-b}=(X_{-(a+b)}-X)-(X_{-b}-X)\\ &=(f_{a+b})-(f_b)=(f_{a+b}f_b^{-1})\qquad (a,\, b\in\mathfrak{g}_X)\,.\end{aligned}$$

This means that $f_{a+b}(x)=\gamma_{a,b}f_a(x+b)f_b(x)$ $(a,\, b\in\mathfrak{g}_X)$ holds with a non-zero element $\gamma_{a,b}$ in k and γ is a 2-cocycle of $\mathfrak{g}_X$ with coefficients in the multiplicative group k^* of k, i.e.

$$(\partial\gamma)(a,\, b,\, c)=\gamma_{b,c}\gamma_{a+b,c}^{-1}\gamma_{a,b+c}\gamma_{a,b}^{-1}=1\qquad (a,\, b,\, c\in\mathfrak{g}_X)\,.$$

Since the field k is algebraically closed and the 2-cohomology of finite group with coefficients in the multiplicative group of an algebraically closed field is always trivial[2], there exist elements β_a $(a\in\mathfrak{g}_X)$ in k^* such that $\gamma_{a,b}=\beta_{a+b}^{-1}\beta_a\beta_b$ $(a,\, b\in\mathfrak{g}_X)$. Putting $\phi_a=\beta_a f_a$ $(a\in\mathfrak{g}_X)$, we have functions ϕ_a $(a\in\mathfrak{g}_X)$ in Lemma 1.1.

PROPOSITION 1.2. *Let ξ be a non-degenerate algebraic equivalence class on an abelian variety* A *and $\mathfrak{g}_\xi=\mathfrak{g}\oplus\hat{\mathfrak{g}}$ be the standard direct sum decomposition with respect to ξ. If ξ contains a positive divisor and $l(\xi)$ is coprime to the characteristic p, then there exists a symmetric positive divisor X in ξ such that*

$$X_{\hat{a}}=X\qquad (\hat{a}\in\hat{\mathfrak{g}})\,.$$

Moreover, the symmetric positive divisor X is uniquely determined up to translations by 2*-division points on* A.

We call the symmetric positive divisor X *the standard divisor in ξ with respect to the standard group* $\mathfrak{g}$.

PROOF OF PROPOSITION 1.2. Let D be a positive divisor in ξ and $\{\phi_a|a\in\mathfrak{g}_\xi\}$ be the system of functions given, in Lemma 1.1. Putting

$$\psi_a(x)=\sum_{\hat{b}\in\hat{\mathfrak{g}}}e_\xi(a,\,\hat{b})^{-1}\phi_{a+\hat{b}}(x)\qquad (a\in\mathfrak{g})\,,$$

we observe that

$$D-(\psi_a)_\infty\succ 0$$

and

$$\begin{aligned}\psi_a(x+\hat{c}) &= \sum_{\hat{b}\in\hat{\mathfrak{g}}}e_\xi(a,\,\hat{b})^{-1}\phi_{a+\hat{b}}(x+\hat{c})\\ &= e_\xi(a,\,\hat{c})\sum_{\hat{b}\in\hat{\mathfrak{g}}}e_\xi(a,\,\hat{b}+\hat{c})^{-1}\phi_{a+\hat{b}+\hat{c}}(x)\phi_{\hat{c}}(x)^{-1}\\ &= e_\xi(a,\,\hat{c})\phi_{\hat{c}}(x)^{-1}\psi_a(x)\qquad (a\in\mathfrak{g},\ \hat{c}\in\hat{\mathfrak{g}})\,.\end{aligned}$$

Since $e_\xi(\ ,\ )$ is non-degenerate, at least one function ψ_{a_0} is not constant zero. Hence, putting $(\psi_{a_0})=V-D$ $(V\succ 0)$, we have

2) See [**5**].

$$(\text{divisor of } \psi_{a_0}(x+\hat{c})) = V_{-\hat{c}} - D_{-\hat{c}} = (\phi_{\hat{c}}^{-1}\psi_{a_0})$$
$$= D - D_{-\hat{c}} + V - D = V - D_{-\hat{c}}\,.$$

This means that $V \in \xi$ and $V_{\hat{a}} = V$ ($\hat{a} \in \hat{\mathfrak{g}}$). Let $\mathbf{B}$ be the quotient abelian variety $A/\hat{\mathfrak{g}}$ and π be the natural separable isogeny of A onto $\mathbf{B}$. Then by virtue of Proposition 38, 78° § IX [6] there exists a positive divisor U on $\mathbf{B}$ such that $V = \pi^{-1}(U)$. By virtue of *Riemann-Roch theorem on abelian varieties*[3] we have

$$\sqrt{|\mathfrak{g}_\xi|} = l(\xi) = l(V) = l(\pi^{-1}(U)) = \nu(\pi) l(U) = |\hat{\mathfrak{g}}|\, l(U)\,,$$

and thus $l(U) = 1$, where $\nu(\pi)$ means the degree of the separable isogeny π. Since $(-\delta_{\mathbf{B}})^{-1}(U) \equiv U$, there exists a point s such that $(-\delta_{\mathbf{B}})^{-1}(U) \sim U_{2s}$. From $l(U) = 1$ it follows $(-\delta_{\mathbf{B}})^{-1}(U) = U_{2s}$. This means $(-\delta_{\mathbf{B}})^{-1}(U_s) = U_s$. Putting $X = \pi^{-1}(U_s)$, we have a standard divisor $X^{(\mathfrak{g})}$ with respect to $\mathfrak{g}$.

Finally we shall prove the uniqueness of the standard divisor up to translations by 2-division points. Let π' be the isogeny of $\mathbf{B}$ onto $\mathbf{A}$ such that $\pi\pi' = \nu(\pi)\delta_{\mathbf{B}}$ and Y be any standard divisor in ξ with respect to $\mathfrak{g}$. We denote by W the symmetric positive divisor on $\mathbf{B}$ such that $Y^{(\mathfrak{g})} = \pi^{-1}(W)$. Then it is sufficient to prove $W = U_{s+c}$ with a 2-division point c on $\mathbf{B}$. Since $Y \equiv X$, it follows[4]

$$\nu(\pi)^2 W \equiv (\nu(\pi)\delta_{\mathbf{B}})^{-1}(W) = \pi'^{-1}(\pi^{-1}(W)) \equiv \pi'^{-1}(Y)$$
$$\equiv \pi'^{-1}(X) \equiv \pi'^{-1}\pi^{-1}(U_s)) \equiv (\nu(\pi)\delta_{\mathbf{B}})^{-1}(U_s) \equiv \nu(\pi)^2 U_s\,.$$

Since abelian variety has no torsion with respect to algebraic equivalence[5], we have $W \equiv U_s$. Therefore from $l(W) = l(U_s) = 1$ it follows $W_a = U_s$ with a point a. From $(-\delta_{\mathbf{B}})^{-1}(W) = W$ and $(-\delta_{\mathbf{B}})^{-1}(U_s) = Us$ we have

$$W_a = (-\delta_{\mathbf{B}})^{-1}(W_a) = (-\delta_{\mathbf{B}})^{-1}(W)_{-a} = W_{-a}\,, \qquad W = W_{2a}\,.$$

This implies $2a = 0$, because $l(W) = 1$.

THEOREM 1.3. *Let ξ be a non-degenerate algebraic equivalence class on an abelian variety* A *and $\mathfrak{g}_\xi = \mathfrak{g} \oplus \hat{\mathfrak{g}}$ be the standard direct sum decomposition with respect to ξ. If ξ contains a positive divisor and $l(\xi)$ is coprime to both* 2 *and the characteristic p, then there exist functions $\{\varphi_a \mid a \in \mathfrak{g}\}$ on* A *such that the poles $(\varphi_a)_\infty$ $(a \in \mathfrak{g})$ are the same divisor contained in ξ and*

$$\begin{aligned} &\varphi_0(x) = 1\,, \quad \varphi_a(-x) = \varphi_{-a}(x)\,, \\ &\varphi_{a+b}(x) = \varphi_a(x+b)\varphi_b(x)\,, \qquad\qquad (a, b \in \mathfrak{g}\,;\ \hat{b} \in \hat{\mathfrak{g}})\,. \\ &\varphi_a(x+\hat{b}) = e_\xi(a, \hat{b})\varphi_a(x) \end{aligned}$$

3) See [3].
4) See [3].
5) See [4].

Moreover, the functions φ_a $(a \in \mathfrak{g})$ *are uniquely determined up to simultaneous translations by* 2-*division points on* **A**.

PROOF. Let π be the natural separable isogeny of **A** onto the quotient abelian variety $\mathbf{B} = A/\hat{\mathfrak{g}}$. Then from the proof of Proposition 1.2 there exists a symmetric positive divisor U on **B** such that $l(U)=1$ and $X = \pi^{-1}(U)$ is a divisor in ξ. Let ϕ_a $(a \in \mathfrak{g})$ be functions on **A** in Lemma 1.1, i.e.

$$(\phi_a) = X_{-a} - X\,, \quad \phi_0(x) = 1\,, \qquad (a, b \in \mathfrak{g})\,.$$
$$\phi_{a+b}(x) = \phi_a(x+b)\phi_b(x)$$

Since $X_{\hat{b}} = X$ $(\hat{b} \in \hat{\mathfrak{g}})$, there exist non-zero constants $\chi(a, \hat{b})$, $\chi(\hat{b})$ $(a \in \mathfrak{g},\ \hat{b} \in \hat{\mathfrak{g}})$ such that

$$\phi_a(x+\hat{b}) = \chi(a, \hat{b})\phi_a(x)\,, \qquad \phi_{\hat{b}}(x) = \chi(\hat{b})\,,$$
$$\chi(a, \hat{b}+\hat{c}) = \chi(a, \hat{b})\chi(a, \hat{c})\,,$$
$$\chi(a+b, \hat{c}) = \chi(a, \hat{c})\chi(b, \hat{c})\,, \qquad (a, b \in \mathfrak{g}\,;\ \hat{b}, \hat{c} \in \hat{\mathfrak{g}})\,,$$
$$\chi(\hat{b}+\hat{c}) = \chi(\hat{b})\chi(\hat{c})\,.$$

Let n be the degree of the isogeny of π and α the isogeny of B onto A such that $n\delta_{\mathbf{B}} = \pi \circ \alpha$. Let F_a $(na=0)$ be the functions satisfying $(F_a) = (n\delta_{\mathbf{B}})^{-1}(U_{-a}) - (n\delta_{\mathbf{B}})^{-1}(U)$. Then by virtue of the definition of $e_{U,n}(a, b)$ in § IX [**6**] we have

$$F_a(x+b) = e_{U,n}(a, b)F_a(x) \qquad (na = nb = 0)\,.$$

Since $n\delta_{\mathbf{B}} = \pi \circ \alpha$, it follows that

$$\phi_{\alpha a}(\alpha x) = \gamma_a F_{na}(x)\,,$$
$$\phi_{\alpha a}(\alpha x + \alpha b) = \gamma_a F_{na}(x+b) = e_{U,n}(na, b)\phi_{\alpha a}(\alpha x)$$
$$(n^2 a = 0,\ \alpha a \in \mathfrak{g}\,;\ \alpha b \in \hat{\mathfrak{g}},\ \gamma_a \neq 0)\,.$$

Since $n^2 U \equiv (n\delta_{\mathbf{B}})^{-1}(U)$ and

$$e_{U,n}(s, t) = \prod_l l_g^{(l)^{-1}}({}^t\tau^{(l)}(s_l) n E_l(U) \tau^{(l)}(t_l)) \qquad (ns = nt = 0)\,,$$

it follows

$$\begin{aligned}
\chi(\alpha a, \alpha b) &= e_{U,n}(na, b) \\
&= \prod_l l_g^{(l)^{-1}}({}^t\tau^{(l)}(na_l) n E_l(U) \tau^{(l)}(b_l)) \\
&= \prod_l l_g^{(l)^{-1}}({}^t\tau^{(l)}(a_l) E_l(n^2 U) \tau^{(l)}(b_l)) \\
&= \prod_l l_g^{(l)^{-1}}({}^t\tau^{(l)}(a_l) E_l((n\delta_{\mathbf{B}})^{-1}(U) \tau^{(l)}(b_l)) \\
&= \prod_l l_g^{(l)^{-1}}({}^t\tau^{(l)}(\alpha a_l) E_l(\pi^{-1}(U)) \tau^{(l)}(\alpha b_l)) \\
&= e_{\pi^{-1}(U)}(\alpha a, \alpha b) = e_\xi(\alpha a, \alpha b)
\end{aligned}$$
$$(a = \sum_l a_l,\ b = \sum_l b_l,\ n^2 a = 0,\ \alpha a \in \mathfrak{g},\ \alpha b \in \hat{\mathfrak{g}})\,.$$

Namely we have

$$\chi(a, \hat{b}) = e_{\xi}(a, \hat{b}) \qquad (a \in \mathfrak{g},\ \hat{b} \in \hat{\mathfrak{g}}).$$

Since X is symmetric, it follows $(-\delta_{\mathbf{A}})^{-1}(X_{-a}) = X_a$ $(a \in \mathfrak{g})$. Therefore there exist non-zero constants ρ_a $(a \in \mathfrak{g})$ such that

$$\phi_a(-x) = \rho_a \phi_{-a}(x) \quad \text{and} \quad \rho_a \rho_b = \rho_{a+b} \qquad (a, b \in \mathfrak{g}).$$

Since the order of g is odd, we have a unique element $\frac{1}{2}a$ in $\mathfrak{g}$ such that $\frac{1}{2}a + \frac{1}{2}a = a$ $(a \in \mathfrak{g})$. Putting

$$\varphi_a(x) = \rho_{\frac{1}{2}a}^{-1} \phi_a(x) \qquad (a \in \mathfrak{g})$$

we have

$$\begin{aligned} \varphi_a(-x) &= \rho_{\frac{1}{2}a}^{-1} \phi_a(-x) = \rho_{\frac{1}{2}a}^{-1} \rho_a \phi_{-a}(x) \\ &= \rho_{\frac{1}{2}a}^{-1} \phi_{-a}(x) = \rho_{-\frac{1}{2}a}^{-1} \phi_{-a}(x) = \varphi_{-a}(x) \qquad (a \in \mathfrak{g}). \end{aligned}$$

By virtue of Proposition 1.2 the freedom of the choice of the standard divisor X in ξ is only the translations by 2-division points on $\mathbf{A}$. Hence the functions $\{\varphi_a \mid a \in \mathfrak{g}\}$ are uniquely determined up to simultaneous translations by 2-division points on $\mathbf{A}$. This completes the proof of Theorem 1.3.

DEFINITION 1.4. We shall call the system of functions $\{\varphi_a \mid a \in \mathfrak{g}\}$ in Theorem 1.3, *the standard system of functions with respect to a standard direct sum decomposition* $\mathfrak{g}_{\xi} = \mathfrak{g} \oplus \hat{\mathfrak{g}}$.

The standard system is uniquely determined up to simultaneous translations by 2-division points.

PROPOSITION 1.5. *The standard system of funcitons* $\{\varphi_a \mid a \in \mathfrak{g}\}$ *with respect to a standard direct sum decomposition* $\mathfrak{g}_{\xi} = \mathfrak{g} \oplus \hat{\mathfrak{g}}$ *forms a linear base of the linear system* $\mathcal{L}(X)$ *where* X *is the pole divisor* $(\varphi_a)_{\infty}$.

PROOF. From Theorem 1.3 the functions φ_a $(a \in \mathfrak{g})$ belong to $\mathcal{L}(X)$ and are linearly independent. On the other hand $l(\xi) = l(X) = \sqrt{|\mathfrak{g}_{\xi}|} = |\mathfrak{g}|$, hence φ_a $(a \in \mathfrak{g})$ form a linear base of $\mathcal{L}(X)$.

THEOREM 1.6. *Let* ξ *be non-degenerate algebraic equivalent class on an abelian variety* $\mathbf{A}$ *such that* ξ *contains a positive divisor and* $l(\xi)$ *is coprime to 2 and the chracterstic* p. *Let* $\{\varphi_a \mid a \in \mathfrak{g}\}$ *be the standard system of functions on* $\mathbf{A}$ *with respect to a standard direct sum decomposition* $\mathfrak{g}_{\xi} = \mathfrak{g} \oplus \hat{\mathfrak{g}}$ *and* $(\varphi_a(0))_{a \in \mathfrak{g}}$ *be the image of the origin of* $\mathbf{A}$ *in the projective space by the* map: $x \to (\varphi_a(x))_{a \in \mathfrak{g}}$. *Then it follows*

$$\operatorname{rank}\,(\varphi_{a+b}(0)\varphi_{-a+b}(0))_{\mathfrak{g} \times \mathfrak{g}} \leqq 2^{\dim \mathbf{A}}, \tag{1}$$

where $(\varphi_{a+b}(0)\varphi_{-a+b}(0))_{\mathfrak{g} \times \mathfrak{g}}$ *means the* $|\mathfrak{g}| \times |\mathfrak{g}|$- *matrix of which* (a, b)- *component is* $\varphi_{a+b}(0)\varphi_{-a+b}(0)$ $(a, b \in \mathfrak{g})$.

PROOF. Let π be the natural separable isogeny of $\mathbf{A}$ onto the quotient abelian variety $\mathbf{B}=\mathbf{A}/\hat{\mathfrak{g}}$ and U the symmetric positive divisor on $\mathbf{B}$ such that $(\varphi_a)_\infty=\pi^{-1}(U)$ $(a\in\mathfrak{g})$. Since

$$\varphi_a(x+\hat{b})\varphi_{-a}(x+\hat{b})=e_\xi(a,\hat{b})e_\xi(-a,\hat{b})\varphi_a(x)\varphi_{-a}(x)=\varphi_a(x)\varphi_{-a}(x)\qquad(a\in\mathfrak{g},\ \hat{b}\in\hat{\mathfrak{g}}),$$

there exists the unique system of functions $\{h_a\,|\,a\in\mathfrak{g}\}$ on $\mathbf{B}$ such that $\varphi_a(x)\varphi_{-a}(x)=h_a(\pi x)$ and $(h_a)_\infty=2U$. By virtue of $l(2U)=2^{\dim \mathbf{A}}l(U)=2^{\dim \mathbf{A}}$ we observe that the number of linearly independent h_a $(a\in\mathfrak{g})$ is at most $2^{\dim \mathbf{A}}$. Hence it follows

$$\begin{aligned}\operatorname{rank}(\varphi_{a+b}(x)\varphi_{-a+b}(x))_{\mathfrak{g}\times\mathfrak{g}}&=\operatorname{rank}(\varphi_a(x+b)\varphi_{-a}(x+b))_{\mathfrak{g}\times\mathfrak{g}}\\&=\operatorname{rank}(h_a(\pi x+\pi b))_{\mathfrak{g}\times\mathfrak{g}}\leqq 2^{\dim \mathbf{A}}.\end{aligned}$$

Specializing x to the origin on $\mathbf{A}$, we complete the proof of Theorem 1.6.

DEFINITION 1.7. In the notation in Theorem 1.6 the system of homogeneous coordinates $(\varphi_a(0))_{a\in\mathfrak{g}}$ is called *the canonical system of theta constants with respect to a standard direct sum* $\mathfrak{g}_\xi=\mathfrak{g}\oplus\hat{\mathfrak{g}}$. If $l(\xi)=n^{\dim \mathbf{A}}$ and $\mathfrak{g}_\xi$ is the group of all the n-division points, we call $(\varphi_a(0))_{a\in\mathfrak{g}}$ *the canonical system of theta constants of level* n.

From (1) follow many equalities for theta constants $(\varphi_a(0))_{a\in\mathfrak{g}}$. In the next paragraph we shall show that two dimensional theta constants of level three satisfy a unique explicitly expressed irreducible equation of degree ten.

§ 2. Canonical systems of two-dimensional theta constants of level three.

2.1. Let $\{X_{(0,0)}, X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)}\}$ be a system of indeterminates and put

$$\begin{aligned}X_{(-1,0)}&=X_{(1,0)}, & X_{(0,-1)}&=X_{(0,1)},\\ X_{(1,1)}&=X_{(-1,-1)}, & X_{(1,-1)}&=X_{(-1,1)}.\end{aligned}$$

Regarding 0, 1, -1 as the elements of *the prime field* $GF(3)$ of characteristic three, we may consider the suffix $\mathbf{a}$ of $X_{\mathbf{a}}$ as a vector in $GF(3)$. A 2×2-matrix α with coefficients in $GF(3)$ operate on $\{X_{\mathbf{a}}\}$ as follows $X_{\mathbf{a}}\to X_{\mathbf{a}\alpha}$. From the definition it follows that

$$X_{\mathbf{a}}=X_{-\mathbf{a}}\qquad(\mathbf{a}\in GF(3)^2).$$

Let $\Gamma_0=\Gamma_0(GF(3))$ be the group of all the 4×4-matrices

$$\underbrace{\begin{pmatrix}\alpha & \beta\\ 0 & {}^t\alpha^{-1}\end{pmatrix}}_{2\quad 2}\begin{matrix}\}2\\ \}2\end{matrix}$$

with coefficients in $GF(3)$ such that $\alpha{}^t\beta=\beta{}^t\alpha$ and let $\bar{\Gamma}_0=\bar{\Gamma}_0(GF(3))$ be the quotient group $\Gamma_0/\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right\}$. Let ζ be a fixed primitive cubic root of unity. Since $\mathbf{a}\beta^t\alpha^t\mathbf{a}=(-\mathbf{a})\beta^t\alpha(-\mathbf{a})$, we may make Γ_0 operate on $X_{\mathbf{a}}$ ($\mathbf{a}\in GF(3)^2$) as follows;

$$X_{\mathbf{a}}^{\begin{pmatrix}\alpha & \beta\\ 0 & {}^t\alpha^{-1}\end{pmatrix}}=\zeta^{\mathbf{a}\beta^t\alpha^t\mathbf{a}}X_{\mathbf{a}\alpha},$$

where $\zeta^0=1$, $\zeta^1=\zeta$, $\zeta^{-1}=\zeta^2$ for the elements 0, 1, -1 in $GF(3)$.

LEMMA 2.1. *Let $\bar{N}$ and Γ_{00} be the subgroups in Γ_0 $\left\{\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\middle|\,{}^t\beta=\beta\right\}$ and $\left\{\begin{pmatrix}\alpha & 0\\ 0 & {}^t\alpha^{-1}\end{pmatrix}\middle|\det\alpha\neq 0\right\}\Big/\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right\}$, respectively. Then it follows that*

1° *Γ_0 is a split extension $\Gamma_{00}\bar{N}$ of $\bar{N}$ by the subgroup Γ_{00},*

2° *Γ_{00} operates faithfully on $\{X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)}\}$ as the symmetric group of four elements,*

3° *the index of the commutator group (Γ_0, Γ_0) in Γ_0 is two and an element σ in Γ_{00} belongs to (Γ_0, Γ_0) if and only if σ induces an even permutation on $\{X_{(1,0)}, X_{(0,1)}, X_{(1,1)} X_{(1,-1)}\}$,*

4° *if $\beta=\begin{pmatrix}\beta_{11} & \beta_{12}\\ \beta_{12} & \beta_{22}\end{pmatrix}$ then $\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}$ operates on $X_{\mathbf{a}}$ as follows*

$$X_{(0,0)}^{\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}}=X_{(0,0)}, \qquad X_{(1,0)}^{\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}}=\zeta^{\beta_{11}}X_{(1,0)}, \qquad X_{(0,1)}^{\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}}=\zeta^{\beta_{22}}X_{(0,1)},$$

$$X_{(1,1)}^{\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}}=\zeta^{\beta_{11}+2\beta_{12}+\beta_{22}}X_{(1,1)}, \qquad X_{(1,-1)}^{\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}}=\zeta^{\beta_{11}-2\beta_{12}+\beta_{22}}X_{(1,-1)}.$$

PROOF. From the definitions 1° and 4° follow immediately. The group Γ_{00} operates faithfully on $\{X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)}\}$ as a permutation group and the order of Γ_{00} is 24, hence Γ_{00} operates on $\{X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)}\}$ as the symmetric group. Next we shall show that the commutator $(\Gamma_{00}, \bar{N})$ generates $\bar{N}$. From the relations

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}-\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}-\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},$$

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}{}^t\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}-\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$$

follow the commutator relations:

$$\left(\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\right)=\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

$$\left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\left(\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and thus the commutator $(\Gamma_{00}, \bar{N})$ generates $\bar{N}$. Therefore we conclude that the index of the commutator group (Γ_0, Γ_0) in Γ_0 is two and an element σ in Γ_{00} does not belong to (Γ_0, Γ_0) if and only if σ induces an odd permutation on $\{X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)}\}$. This completes the proof of Lemma 2.1.

Denote by $s_3(X)$, $s_6(X)$, $s_9(X)$, $t_4(X)$ the symmetric functions in the four variable $X_{(1,0)}$, $X_{(0,1)}$, $X_{(1,1)}$, $X_{(1,-1)}$

$$s_3(X) = X^3_{(1,0)} + X^3_{(0,1)} + X^3_{(1,1)} + X^3_{(1,-1)},$$
$$s_6(X) = X^6_{(1,0)} + X^6_{(0,1)} + X^6_{(1,1)} + X^6_{(1,-1)},$$
$$s_9(X) = X^9_{(1,0)} + X^9_{(0,1)} + X^9_{(1,1)} + X^9_{(1,-1)},$$
$$t_4(X) = X_{(1,0)} X_{(0,1)} X_{(1,1)} X_{(1,-1)}.$$

LEMMA 2.2. *Let k be a field of characteristic p such that k contains a primitive cubic root ζ of unity and p does not divide six. Then $k[X_{(0,0)}, s_3(X), s_6(X), s_9(X), t_4(X)]$ is the subalgebra $k[X_{(0,0)}, X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)}]^{\Gamma_0}$ consisting of all the the invariant elements with respect to the group Γ_0 of automorphisms and $k(X^{-3}_{(0,0)} s_3(X), X^{-6}_{(0,0)} s_6(X), X^{-9}_{(0,0)} s_9(X), X^{-4}_{(0,0)} t_4(X))$ is the subfield $k(X^{-1}_{(0,0)} X_{(1,0)}, X^{-1}_{(0,0)} X_{(0,1)}, X^{-1}_{(0,0)} X_{(1,1)}, X^{-1}_{(0,0)} X_{(1,-1)})^{\Gamma_0}$ consisting of all the invariant elements in $k(X^{-1}_{(0,0)} X_{(1,0)}, X^{-1}_{(0,0)} X_{(0,1)}, X^{-1}_{(0,0)} X_{(1,1)}, X^{-1}_{(0,0)} X_{(1,-1)})$ with respect to Γ_0.*

PROOF. From 4° in Lemma 2.1 it follows that a monomial in $X_{\mathbf{a}}$ ($\mathbf{a} \in GF(3)^2$) is invariant by $\bar{N}$ if and only if it is a product of $X_{(0,0)}$, $X^3_{(1,0)}$, $X^3_{(0,1)}$, $X^3_{(1,1)}$, $X^3_{(1,-1)}$, $X_{(1,0)} X_{(0,1)} X_{(1,1)} X_{(1,-1)}$. This shows that $k[X_{(0,0)}, X^3_{(1,0)}, X^3_{(0,1)}, X^3_{(1,1)}, X^3_{(1,-1)}, t_4(X)]$ is the subring $k[X_{(0,0)}, X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)}]^{\bar{N}}$ consisting of all the invariant with respect to $\bar{N}$. Since Γ_{00} is the symmetric group on $\{X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)}\}$, Γ_{00} is also regarded as the symmetric group on $\{X^3_{(1,0)}, X^3_{(0,1)}, X^3_{(1,1)}, X^3_{(1,-1)}\}$. On the other hand the characteristic p of k is zero or a prime number not less than four. Hence we have

$$\begin{aligned} k[X_{(0,0)}, X_{(1,0)}, \cdots, X_{(1,-1)}]^{\Gamma_0} &= k[X_{(0,0)}, t_4(X), X^3_{(1,0)}, \cdots, X^3_{(1,-1)}]^{\Gamma_{00}} \\ &= k[X_{(0,0)}, t_4(X), s_3(X), s_6(X), s_9(X)]. \end{aligned}$$

We shall prove the second part of Lemma 2.2. Denote

$$\operatorname{sign}\sigma=\begin{cases}1 & \text{if } \sigma\in(\bar{\Gamma}_0,\bar{\Gamma}_0)\\ -1 & \text{if } \sigma\notin(\bar{\Gamma}_0,\bar{\Gamma}_0),\end{cases}$$

Any element in $k(X_{(0,0)}^{-1}X_{(1,0)},\cdots,X_{(0,0)}^{-1}X_{(1,-1)})^{\bar{\Gamma}_0}$ is a quotient $f(X)/g(X)$ of homogeneous forms $f(X)$ and $g(X)$ of the same degree such that $f(X)^\sigma=\chi(\sigma)f(X)$ and $g(X)^\sigma=\chi(\sigma)g(X)$ $(\sigma\in\bar{\Gamma}_0)$ with a character χ of degree 1 of $\bar{\Gamma}_0$. By virtue of 3° in Lemma 2.1 the character χ must be sign σ or the trivial character. Hence it follows that $f(X)^\sigma=f(X)$ and $g(X)^\sigma=g(X)$ for σ in N. This shows that

$$k(X_{(0,0)}^{-1}X_{(1,0)},\cdots,X_{(0,0)}^{-1}X_{(1,-1)})^{\bar{\Gamma}_0}$$

is the invariant subfield in the field $k(X_{(0,0)}^3X_{(1,0)}^3,\cdots,X_{(0,0)}^{-3}X_{(1,-1)}^3,X_{(0,0)}^{-4}t_4(X))$ with respect to the group $\bar{\Gamma}_{00}$ of automorphisms. $\bar{\Gamma}_{00}$ is regarded as the symmetric group on $\{X_{(0,0)}^{-3}X_{(1,0)}^3,X_{(0,0)}^{-3}X_{(0,1)}^3,X_{(0,0)}^{-3}X_{(1,1)}^3,X_{(0,0)}^{-3}X_{(1,-1)}^3\}$. Hence we have

$$\begin{aligned}&k(X_{(0,0)}^{-1}X_{(1,0)},\cdots,X_{(0,0)}^{-1}X_{(1,-1)})^{\bar{\Gamma}_0}\\ &=k(X_{(0,0)}^{-3}X_{(1,0)}^3,\cdots,X_{(0,0)}^{-3}X_{(1,-1)}^3,X_{(0,0)}^{-4}t_4(X))^{\bar{\Gamma}_{00}}\\ &=k(X_{(0,0)}^{-4}t_4(X),X_{(0,0)}^{-3}s_3(X),X_{(0,0)}^{-6}s_6(X),X_{(0,0)}^{-9}s_9(X)).\end{aligned}$$

This completes the proof of Lemma 2.2.

2.2. Denote by $\Delta(X)$ the polynomial of degree ten defined by

$$\Delta(X)=\det\begin{pmatrix}X_{(0,0)}^2 & X_{(1,0)}^2 & X_{(0,1)}^2 & X_{(1,1)}^2 & X_{(1,-1)}^2\\ X_{(1,0)}^2 & X_{(0,0)}X_{(1,0)} & X_{(1,1)}X_{(1,-1)} & X_{(0,1)}X_{(1,-1)} & X_{(0,1)}X_{(1,1)}\\ X_{(0,1)}^2 & X_{(1,1)}X_{(1,-1)} & X_{(0,0)}X_{(0,1)} & X_{(1,0)}X_{(1,-1)} & X_{(1,0)}X_{(1,1)}\\ X_{(1,1)}^2 & X_{(0,1)}X_{(1,-1)} & X_{(1,0)}X_{(1,-1)} & X_{(0,0)}X_{(1,1)} & X_{(1,0)}X_{(0,1)}\\ X_{(1,-1)}^2 & X_{(0,1)}X_{(1,1)} & X_{(1,0)}X_{(1,1)} & X_{(1,0)}X_{(0,1)} & X_{(0,0)}X_{(1,-1)}\end{pmatrix}$$

and by $\mathbf{M}_{2,3}$ the projective scheme

$$\operatorname{Proj}\mathbf{Z}[X_{(0,0)},X_{(1,0)},X_{(0,1)},X_{(1,1)},X_{(1,-1)}]/(\Delta(X))$$

corresponding to the homogeneous ideal $(\Delta(X))$. From the direct calculation we observe that $\bar{\Gamma}_0$ leaves the polynomial $\Delta(X)$ invariant. Hence we may consider $\bar{\Gamma}_0$ as a group of automorphisms of the projective scheme $\mathbf{M}_{2,3}$.

THEOREM 2.3. *Let k be a field of characteristic p such that k contains a primitic cubic root ζ of unity and p does not divide six. Let $x_{(0,0)}, x_{(1,0)}, x_{(0,1)}, x_{(1,1)}, x_{(1,-1)}$ be the images of $X_{(0,0)}, X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)}$ in the residue ring $k[X_{(0,0)},X_{(1,0)},X_{(0,1)},X_{(1,1)},X_{(1,-1)}]/(\Delta(X))$. Then the rational function field $k(x_{(0,0)}^{-3}s_3(x),x_{(0,0)}^{9}s_6(x),x_{(0,0)}^{-4}t_4(x))$ in three variables is the invariant subfield in $k(x_{(1,0)}/x_{(0,0)},x_{(0,1)}/x_{(0,0)},x_{(1,1)}/x_{(0,0)},x_{(1,-1)}/x_{(0,0)})$ with respect to the group $\bar{\Gamma}_0$ of* *automorphisms where the operation is defined by*

$$x_{\mathbf{a}}^{\begin{pmatrix}\alpha & \beta \\ 0 & {}^t\alpha^{-1}\end{pmatrix}}=\zeta^{\mathbf{a}\beta{}^t\alpha{}^t\mathbf{a}}x_{\mathbf{a}\alpha}\,,\qquad \left(\begin{matrix}\mathbf{a}\in GF(3)^2 \\ \alpha,\,\beta\in(GF(3))_{2\times 2}\,,\end{matrix}\quad \det\alpha\neq 0\,,\quad \beta{}^t\alpha=\alpha{}^t\beta\right).$$

Moreover $x_{(0,0)}^{-9}s_9(x)$ *is expressed by* $x_{(0,0)}^{-3}s_3(x)$, $x_{(0,0)}^{-6}s_6(x)$ *and* $x_{(0,0)}^{-4}t_4(x)$ *as follows*

$$\begin{aligned}(2)\qquad x_{(0,0)}^{-9}s_9(x)&=x_{(0,0)}^{-10}t_4(x)\Big(\frac{1}{3}s_3(x)^2-s_6(x)\Big)\\&\quad+x_{(0,0)}^{-9}\Big(\frac{4}{3}s_3(x)s_6(x)-\frac{1}{3}s_3(x)^3\Big)+3x_{(0,0)}^{-8}t_4(x)^2\\&\quad+\frac{1}{3}x_{(0,0)}^{-7}t_4(x)s_3(x)+\frac{1}{6}x_{(0,0)}^{-6}(s_6(x)-s_3(x)^2)\\&\quad+\frac{1}{3}x_{(0,0)}^{-4}t_4(x)\,.\end{aligned}$$

PROOF. Since the characteristic p of k does not divide six, by the direct calculation it follows

$$\begin{aligned}\Delta(X)&=X_{(0,0)}^6X_{(1,0)}X_{(0,1)}X_{(1,1)}X_{(1,-1)}-X_{(0,0)}^4\sum_{\mathbf{a}\neq\mathbf{b}\neq(0,0)}X_{\mathbf{a}}^3X_{\mathbf{b}}^3\\&\quad+X_{(0,0)}^3X_{(1,0)}X_{(0,1)}X_{(1,1)}X_{(1,-1)}\sum_{\mathbf{a}\neq(0,0)}X_{\mathbf{a}}^3\\&\quad+9X_{(0,0)}^2X_{(1,0)}^2X_{(0,1)}^2X_{(1,1)}^2X_{(1,-1)}^2\\&\quad-6X_{(0,0)}\sum_{\mathbf{a}\neq\mathbf{b}\neq\mathbf{c}\neq(0,0)}X_{\mathbf{a}}^3X_{\mathbf{b}}^3X_{\mathbf{c}}^3+X_{(0,0)}\sum_{\mathbf{a}\neq\mathbf{b}\neq(0,0)}X_{\mathbf{a}}^6X_{\mathbf{b}}^3\\&\quad-2X_{(1,0)}X_{(0,1)}X_{(1,1)}X_{(1,-1)}\Big(\sum_{\mathbf{a}\neq(0,0)}X_{\mathbf{a}}^6-\sum_{\mathbf{a}\neq\mathbf{b}\neq(0,0)}X_{\mathbf{a}}^3X_{\mathbf{b}}^3\Big)\\&=-3X_{(0,0)}s_9(X)+9X_{(0,0)}^2t_4(X)^2+4X_{(0,0)}s_3(X)s_6(X)\\&\quad-X_{(0,0)}s_3(X)^3-3t_4(X)s_6(X)+X_{(0,0)}^3t_4(X)s_3(X)\\&\quad+X_{(0,0)}^6t_4(X)+s_3(X)^2t_4(X)-\frac{1}{2}X_{(0,0)}^4s_3(X)^2+\frac{1}{2}X_{(0,0)}^4s_6(X)\,.\end{aligned}$$

This shows that $\Delta(X)$ is absolutely irreducible as a polynomial in the five variable $X_{(0,0)}$, $s_3(X)$, $s_6(X)$, $s_9(X)$, $t_4(X)$, because $\Delta(X)$ contains the only one monomial $X_{(0,0)}s_9(X)$ containing the variable $s_9(X)$ and $X_{(0,0)}$ does not divide $\Delta(X)$. Hence the residue ring $k[X_{(0,0)}^{-3}s_3(X), X_{(0,0)}^{-6}s_6(X), X_{(0,0)}^{-9}s_9(X), X_{(0,0)}^{-4}t_4(X)]/(X_{(0,0)}^{-10}\Delta(X))$ is an integral domain and its quotient field $k(x_{(0,0)}^{-3}s_3(x), x_{(0,0)}^{-6}s_6(x), x_{(0,0)}^{-9}s_9(x), x_{(0,0)}^{-4}t_4(x))$ is a rational function field in three variables $x_{(0,0)}^{-3}s_3(x)$, $x_{(0,0)}^{-6}s_6(x)$, $x_{(0,0)}^{-4}t_4(x)$. We shall next show that $\Delta(X)$ is an absolutely irreducible polynomial in $X_{(0,0)}, X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)}$. Since $k[X_{(0,0)},\cdots,X_{(1,-1)}]$ is a unique factorization domain, there exists a unique factorization $\Delta(X)=P_1(X)\cdots P_r(X)$. Since the group $\bar{\Gamma}_0$ leaves $\Delta(X)$ invariant, $P_i(X)^\sigma$ $(1\leqq i\leqq r\,;\,\sigma\in\bar{\Gamma}_0)$ are also irreducible factors of $\Delta(X)$. This shows that there exists a subgroup H in $\bar{\Gamma}_0$ and an irreducible element $P(X)$ such that $\Delta(X)=\prod P(X)^{\sigma}$ where σ runs over a system of representatives of right cosets modulo H and $P(X)^{\sigma}$ ($\sigma \bmod H$) are coprime to each other. The degree of $\Delta(X)$ is ten and the index $[\bar{\Gamma}_0 : H]$ is a divisor of the order $2^3 3^4$ of $\bar{\Gamma}_0$. Hence the only possibility of the index $[\bar{\Gamma}_0 : H]$ is one or two. Assume for a moment that $[\bar{\Gamma}_0 : H]=2$. Then by virtue of Lemma 2.1 H coincides with the commutator group $(\bar{\Gamma}_0, \bar{\Gamma}_0)$ and $P(X)^{\sigma}=\chi(\sigma)P(X)$ ($\sigma\in(\bar{\Gamma}_0, \bar{\Gamma}_0)$) with a charactor χ. On the other hand $\bar{\Gamma}_0$ and $(\bar{\Gamma}_0, \bar{\Gamma}_0)$ are the extensions of the normal subgroup $\bar{N}$ by the symmetric group and the alternating group on $\{X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)}\}$, respectively. Hence $P(X)^{2^2 3^4}$ is an alternating function in $X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)}$ and thus $P(X)^{2^3 3^4}$ is a symmetric function in $X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)}$. This means that $P(X)^{2^3 3^4}$ is an invariant for $\bar{\Gamma}_0$. On the other hand $P(X)$ is an absolutely irreducible element in $k[X_{(0,0)}, s_3(X), s_6(X), s_9(X), t_4(X)]$ which is the invariant subring with respect to $\bar{\Gamma}_0$. This shows that $\Delta(X)^{2^2 3^4}=cP(X)^{2^2 3^4}$ with a constant c and $\Delta(X)=c'P(X)^2$. This is a contradiction. Therefore $\Delta(X)$ is absolutely irreducible in $k[X_{(0,0)}, \cdots, X_{(1,-1)}]$ and $k[x_{(0,0)}, \cdots, x_{(1,-1)}]=k[X_{(0,0)}, \cdots, X_{(1,-1)}]/(\Delta(X))$ is an integral domain. Finaly we shall prove that $k(x_{(0,0)}^{-3}s_3(x), x_{(0,0)}^{-6}s_6(x), x_{(0,0)}^{-4}t_4(x))$ is the invariant subfield in $k(x_{(0,0)}^{-1}x_{(1,0)}, x_{(0,0)}^{-1}x_{(0,1)}, x_{(0,0)}^{-1}x_{(1,1)}, x_{(0,0)}^{-1}x_{(1,-1)})$ with respect to $\bar{\Gamma}_0$. Let $f(x)$ be an invariant element in $k(x_{(0,0)}^{-1}x_{(1,0)}, \cdots, x_{(0,0)}^{-1}x_{(1,-1)})$ with respect to $\bar{\Gamma}_0$. Then there exists a pair $(g(X), h(X))$ of homogeneous elements of the same degree in $k[X_{(0,0)}, X_{(1,0)}, \cdots, X_{(1,-1)}]$ such that $f(x)=g(x)/h(x)$, $h(X)^{\sigma}=h(X)$ ($\sigma\in\bar{\Gamma}_0$) and $g(X)^{\sigma}-g(X)=l_{\sigma}(X)\Delta(X)$ with homogeneous elements $l_{\sigma}(X)$ whose degree is $\deg g(X)-10$. The system $(l_{\sigma}(X))_{\sigma\in\bar{\Gamma}_0}$ may be regarded as a 1-cocycle of $\bar{\Gamma}_0$ with coefficients in the additive group $k[X_{(0,0)}, \cdots, X_{(1,-1)}]$. Since the order $2^3 3^4$ of $\bar{\Gamma}_0$ is coprime to the characteristic p of k, the 1-cohomology group of $\bar{\Gamma}_0$ is trivial[6]. Therefore there exists an element $a(X)$ in $k[X_{(0,0)}, \cdots, X_{(1,-1)}]$ such that $l_{\sigma}(X)=a(X)^{\sigma}-a(X)$ ($\sigma\in\bar{\Gamma}_0$). Putting $g_1(X)=g(X)-a(X)\Delta(X)$, we have $f(x)=g_1(x)/h(x)$ and $g_1(X)^{\sigma}=g_1(X)$, $h(X)^{\sigma}=h(X)$ ($\sigma\in\bar{\Gamma}_0$). This shows that the invariant subfield in $k(x_{(0,0)}^{-1}x_{(1,0)}, \cdots, x_{(0,0)}^{-1}x_{(1,-1)})$ with respect to $\bar{\Gamma}_0$ is the quotient field of the integral domain A, where

$$\begin{aligned} A &= k[X_{(0,0)}^{-1}X_{(1,0)}, \cdots, X_{(0,0)}^{-1}X_{(1,-1)}]/(X_{(0,0)}^{-10}\Delta(X)) \\ &= k[X_{(0,0)}^{-3}s_3(X), X_{(0,0)}^{-6}s_6(X), X_{(0,0)}^{-9}s_9(X), X_{(0,0)}^{-4}t_4(X)]/(X_{(0,0)}^{-10}\Delta(X)) \\ &= k[x_{(0,0)}^{-3}s_3(x), x_{(0,0)}^{-6}s_6(x), x_{(0,0)}^{-9}s_9(x), x_{(0,0)}^{-4}t_4(x)] \\ &= k[x_{(0,0)}^{-3}s_3(x), x_{(0,0)}^{-6}s_6(x), x_{(0,0)}^{-4}t_4(x)] . \end{aligned}$$

This completes the proof of Theorem 2.3.

COROLLARY 2.4. *Let k be a field whose characteristic p does not divide six.*

6) See [**5**].

Then the projective variety

$$\mathbf{M}_{2,3}\otimes_{\mathbf{Z}} k=\mathrm{Proj}\,(k[X_{(0,0)}, X_{(1,0)}, X_{(0,1)}, X_{(1,1)}, X_{(1,-1)}]/\Delta(X))$$

is asbolutely irreducible.

2.2. Let $\{Q_{11}, Q_{12}, Q_{22}, U_1, U_2\}$ be a system of indeterminates and denote briefly

$$U(\mathbf{m})=U_1^{m_1}U_2^{m_2}, \qquad Q(\mathbf{m}, \mathbf{n})=Q_{11}^{m_1n_1}Q_{12}^{(m_1n_2+m_2n_1)}Q_{22}^{m_2n_2}$$
$$(\mathbf{m}=(m_1, m_2), \mathbf{n}=(n_1, n_2)).$$

By a two dimensional formal theta function of level n we mean a formal series

$$\varphi(U)=\sum_{\mathbf{m}\in\mathbf{Z}^2}\lambda_{\mathbf{m}}U(\mathbf{m})^2,$$

such that

$$\varphi(Q(\mathbf{m})U)=Q(\mathbf{m}, \mathbf{m})^{-n}U(\mathbf{m})^{-2n}\varphi(U) \qquad (\mathbf{m}\in\mathbf{Z}^2),$$

where $(Q(\mathbf{m})U)(\mathbf{n})=Q(\mathbf{m}, \mathbf{n})U(\mathbf{n})$. Then two dimensional formal theta functions of level n form a vector space of dimension n^2 over the field of coefficients. We shall be concerned with the following formal series

$$\vartheta_{\mathbf{a}}(Q\,|\,U)=\sum_{\mathbf{m}\in\mathbf{Z}^2}Q\Big(\mathbf{m}+\frac{\mathbf{a}}{3}, \mathbf{m}+\frac{\mathbf{a}}{3}\Big)U\Big(\mathbf{m}+\frac{\mathbf{a}}{3}\Big)^2$$

$$\vartheta_{\mathbf{a}}(Q)=\sum_{\mathbf{m}\in\mathbf{Z}^2}Q\Big(\mathbf{m}+\frac{\mathbf{a}}{3}, \mathbf{m}+\frac{\mathbf{a}}{3}\Big) \qquad (\mathbf{a}=(a_1, a_2)\,;\ a_1, a_2=0, 1, -1).$$

These formal series satisfy the relations

$$\vartheta_{\mathbf{a}}(Q\,|\,Q(\mathbf{m})U)=Q(\mathbf{m}, \mathbf{m})^{-1}U(\mathbf{m})^{-2}\vartheta_{\mathbf{a}}(Q\,|\,U), \tag{3}$$

$$\vartheta_{-\mathbf{a}}(Q\,|\,U)=\vartheta_{\mathbf{a}}(Q\,|\,U^{-1}), \tag{4}$$

$$\vartheta_{\mathbf{a}}\Big(Q\,|\,Q\Big(\frac{\mathbf{b}}{3}\Big)U\Big)=Q\Big(\frac{\mathbf{b}}{3}, \frac{\mathbf{b}}{3}\Big)^{-1}U\Big(\frac{\mathbf{b}}{3}\Big)^{-2}\vartheta_{\mathbf{a}+\mathbf{b}}(Q\,|\,U), \tag{5}$$

$$\vartheta_{-\mathbf{a}}(Q)=\vartheta_{\mathbf{a}}(Q) \tag{6}$$

$$\begin{pmatrix} \mathbf{m}\in\mathbf{Z}^2 \\ \mathbf{a}=(a_1, a_2),\ \mathbf{b}=(b_1, b_2) \\ a_1, a_2, b_1, b_2=0, 1, -1 \end{pmatrix}$$

and the products $\vartheta_{\mathbf{a}}(Q\,|\,U)\vartheta_{-\mathbf{a}}(Q\,|\,U)$ are considered as formal theta functions of level 2. These relations (3)$\sim$(6) imply the inequality[7]

$$\mathrm{rank}\,(\vartheta_{\mathbf{a}+\mathbf{b}}(Q\,|\,U)\vartheta_{-\mathbf{a}+\mathbf{b}}(Q\,|\,U))\leqq 2^2=4,$$

where $(\vartheta_{\mathbf{a}+\mathbf{b}}(Q\,|\,U)\vartheta_{\mathbf{a}+\mathbf{b}}(Q\,|\,U))$ is the 9×9-matrix whose $(\mathbf{a}, \mathbf{b})$-component is $\vartheta_{\mathbf{a}+\mathbf{b}}(Q\,|\,U)\vartheta_{-\mathbf{a}+\mathbf{b}}(Q\,|\,U)$ and $(\mathbf{a}, \mathbf{b})$ run over the vector space $GF(3)^2$. Since $(\vartheta_{\mathbf{a}}(Q))$

7) See [1].

is a specialization of $(\vartheta_{\mathbf{a}}(Q\,|\,U))$ over the replacement $U\left(\mathbf{m}+\frac{\mathbf{a}}{3}\right)$ by 1, it follows that

$$\text{(7)}\qquad \operatorname{rank}(\vartheta_{\mathbf{a}+\mathbf{b}}(Q)\vartheta_{-\mathbf{a}+\mathbf{b}}(Q)) \leqq 4 .$$

Since $\vartheta_{\mathbf{a}}(Q)=\vartheta_{-\mathbf{a}}(Q)$, we have

$$\begin{aligned}\vartheta_{\mathbf{a}+\mathbf{b}}(Q)\vartheta_{-\mathbf{a}+\mathbf{b}}(Q) &= \vartheta_{-\mathbf{a}+\mathbf{b}}(Q)\vartheta_{-(-\mathbf{a})+\mathbf{b}}(Q)\\ &= \vartheta_{-\mathbf{a}-\mathbf{b}}(Q)\vartheta_{-(-\mathbf{a})-\mathbf{b}}(Q) = \vartheta_{\mathbf{a}-\mathbf{b}}(Q)\vartheta_{-\mathbf{a}-\mathbf{b}}(Q) .\end{aligned}$$

This means that the inequality (7) is equivalent to

$$\text{(8)}\quad \det\begin{pmatrix} \vartheta_{(0,0)}(Q)^2 & \vartheta_{(1,0)}(Q)^2 & \vartheta_{(0,1)}(Q)^2 & \vartheta_{(1,1)}(Q)^2 & \vartheta_{(1,-1)}(Q)^2 \\ \vartheta_{(1,0)}(Q)^2 & \vartheta_{(0,0)}(Q)\vartheta_{(1,0)}(Q) & \vartheta_{(1,1)}(Q)\vartheta_{(1,-1)}(Q) & \vartheta_{(0,1)}(Q)\vartheta_{(1,-1)}(Q) & \vartheta_{(0,1)}(Q)\vartheta_{(1,1)}(Q) \\ \vartheta_{(0,1)}(Q)^2 & \vartheta_{(1,1)}(Q)\vartheta_{(1,-1)}(Q) & \vartheta_{(0,0)}(Q)\vartheta_{(0,1)}(Q) & \vartheta_{(1,0)}(Q)\vartheta_{(1,-1)}(Q) & \vartheta_{(1,0)}(Q)\vartheta_{(1,1)}(Q) \\ \vartheta_{(1,1)}(Q)^2 & \vartheta_{(0,1)}(Q)\vartheta_{(1,-1)}(Q) & \vartheta_{(1,0)}(Q)\vartheta_{(1,-1)}(Q) & \vartheta_{(0,0)}(Q)\vartheta_{(1,1)}(Q) & \vartheta_{(1,0)}(Q)\vartheta_{(0,1)}(Q) \\ \vartheta_{(1,-1)}(Q)^2 & \vartheta_{(0,1)}(Q)\vartheta_{(1,1)}(Q) & \vartheta_{(1,0)}(Q)\vartheta_{(1,1)}(Q) & \vartheta_{(1,0)}(Q)\vartheta_{(0,1)}(Q) & \vartheta_{(0,0)}(Q)\vartheta_{(1,-1)}(Q) \end{pmatrix} = 0 .$$

THEOREM 2.5. *Let k be a field of characteristic p such that k contains a primitive cubic root ζ of unity and p does not divide six. Then the projective variety defined by $\Delta(X_{(0,0)}, X_{(1,0)}, \cdots, X_{(1,-1)})=0$ is the projective locus of $(\vartheta_{(0,0)}(Q), \vartheta_{(1,0)}(Q), \vartheta_{(0,1)}(Q), \vartheta_{(1,1)}(Q), \vartheta_{(1,-1)}(Q))$ over k and the rational function field $k(\sum_{\mathbf{a}\neq(0,0)}\vartheta_{\mathbf{a}}(Q)^3/\vartheta_{(0,0)}(Q)^3, \sum_{\mathbf{a}\neq(0,0)}\vartheta_{\mathbf{a}}(Q)^6/\vartheta_{(0,0)}(Q)^6, \vartheta_{(1,0)}(Q)\vartheta_{(0,1)}(Q)\vartheta_{(1,1)}(Q)\vartheta_{(1,-1)}(Q)/\vartheta_{(0,0)}(Q)^4)$ is the invariant subfield in $k(\vartheta_{(1,0)}(Q)/\vartheta_{(0,0)}(Q), \vartheta_{(0,1)}(Q)/\vartheta_{(0,0)}(Q), \vartheta_{(1,1)}(Q)/\vartheta_{(0,0)}(Q), \vartheta_{(1,-1)}(Q)/\vartheta_{(0,0)}(Q))$ with respect to the group of automorphisms*

$$(\alpha, \beta):\ \vartheta_{\mathbf{a}}(Q)\to\zeta^{\mathbf{a}\beta^t\alpha^t\mathbf{a}}\vartheta_{\mathbf{a}\alpha}(Q)\qquad(\mathbf{a}\in GF(3)^2),$$

where α, β are 2×2-matrices with coefficients in $GF(3)$ such that $\det\alpha\neq 0$ and $\beta^t\alpha=\alpha^t\beta$. Moreover if $(\varphi_{\mathbf{a}}(0))$ is the canonical system of theta constants of level three on a two dimensional abstruct abelian variety, the point $(\varphi_{(0,0)}(0), \varphi_{(1,0)}(0), \varphi_{(0,1)}(0), \varphi_{(1,1)}(0), \varphi_{(1,-1)}(0))$ is a point on the variety $\Delta(X_{(0,0)}, X_{(1,0)}, \cdots, X_{(1,-1)})=0$.

PROOF. Since $\Delta(X_{(0,0)}, X_{(1,0)}, \cdots, X_{(1,-1)})$ is absolutely irreducible, for the first assertion it is sufficient to show that

$$\begin{aligned}\dim_k k(&\vartheta_{(1,0)}(Q)/\vartheta_{(0,0)}(Q), \vartheta_{(0,1)}(Q)/\vartheta_{(0,0)}(Q),\\ &\vartheta_{(1,1)}(Q)/\vartheta_{(0,0)}(Q), \vartheta_{(1,-1)}(Q)/\vartheta_{(0,0)}(Q))=3 .\end{aligned}$$

It is also sufficient to prove for the case $k=\mathbf{C}$. This is a well-known classical result. The second part is a direct consequence from Theorem 2.3. Let $(\varphi_{\mathbf{a}}(0))$ be the canonical system of theta constants of level three on a two dimensional abstruct abelian variety. Since

$$\varphi_{\mathbf{a}+\mathbf{b}}(0)\varphi_{-\mathbf{a}+\mathbf{b}}(0)=\varphi_{-\mathbf{a}+\mathbf{b}}(0)\varphi_{-(-\mathbf{a})+\mathbf{b}}(0)$$
$$=\varphi_{-\mathbf{a}-\mathbf{b}}(0)\varphi_{-(-\mathbf{a})-(-\mathbf{b})}(0)=\varphi_{\mathbf{a}-\mathbf{b}}(0)\varphi_{-\mathbf{a}-\mathbf{b}}(0)\,,$$

by virtue of Theorem 1.6 it follows

$$\Delta(\varphi_{(0,0)}(0),\ \varphi_{(1,0)}(0),\ \varphi_{(0,1)}(0),\ \varphi_{(1,1)}(0),\ \varphi_{(1,-1)}(0))=0\,.$$

This completes the proof of Theorem 2.5.

Mathematical Institute of
Nagoya University

References

[1] H. Morikawa, On the defining equations of abefian varieties, Nagoya Math. J., **30** (1967), 143-161.
[2] D. G. Northcott, An introduction to homological algebra, Cambridge, 1960.
[3] M. Nishi, The Frobenius theorem and the duality theorem on abelian variety, Mem. Coll. Sci. Univ. Kyoto Ser. A, **32** (1959), 333-350.
[4] J.-P. Serre, Quelques propriétés des variétés abéliennes en caractéristique *p*, Amer. J. Math., **80** (1958), 715-739.
[5] J.-P. Serre, Cohomologie Galoisienne, chapitre II, College de France, 1963.
[6] A. Weil, Variétés abéliennes et courbes algébriques, Paris, 1948.