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On the prime ideal theorem

Dedicated to Professor S. Iyanaga on his 60th birthday

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Let K be an algebraic number field. Let $\pi_K(x)$ be the number of the prime ideals $\mathfrak{p}$ with $N(\mathfrak{p}) \leqq x$, then we have the following asymptotic formula;

$$\pi_K(x) = \int_2^x \frac{dt}{\log t} + O(x \exp(-c(\log x)^{1/2})) \tag{1}$$

(Landau [2], Satz 191). As a special case, if K is the rational number field, we have the formula;

$$\pi(x) = \sum_{p \leqq x} 1 = \int_2^x \frac{dt}{\log t} + O(x \exp(-c(\log x)^{1/2})), \tag{2}$$

which was first proved, in 1899, by de la Vallée Poussin. Since then, the remainder term of (2) has been improved by many authors; to obtain these improvements, the method of trigonometrical sums is very important. (Cf. Prachar [3], Titchmarsh [4].)

The purpose of this paper is to improve the remainder term of (1). Our main result is stated as follows;

MAIN THEOREM. *Let K be an algebraic number field. Let $\pi_K(x)$ be the number of the prime ideals whose norms are less than x. Then we have*

$$\pi_K(x) = \int_2^x \frac{dt}{\log t} + O\left(x \exp\left(-c \frac{(\log x)^{3/5}}{(\log \log x)^{1/5}}\right)\right).$$

We shall begin by proving the following theorem concerning trigonometrical sums, which will be regarded as a generalization of the theorem of Vinogradov [5] and will be fundamental for the proof of Main Theorem;

THEOREM I. *Let $\mathfrak{a}$ be an ideal of K. Let $L(\mathfrak{a})$ be the set of the principal ideals divisible by $\mathfrak{a}$. Let t be a large number, and A and B two real numbers such that*

$$\exp((\log t)^{2/3}) \leqq A < B \leqq 2A < 2t^{6n/5},$$

where n is the degree of K. We define a trigonometrical sum $S(t; A, B)$ as follows;

$$S(t; A, B) = \sum_{\substack{\mathfrak{b} \in L(\mathfrak{a}) \\ A \leqq N\mathfrak{b} < B}} \exp(2\pi i t \log N(\mathfrak{b})).$$

Then we have

$$|S(t\,;A,B)|\leqq c_1A^{1-c_2/\alpha^2},$$

where $\alpha=n\log t/\log A$, *and* c_1 *and* c_2 *are positive constants depending only on* K *and* $\mathfrak{a}$.

In the first half of § 1, the estimation of $S(t\,;A,B)$ will be reduced to that of the sum of the following type;

$$T=\sum_{x,y=1}^{Y}\exp\,(2\pi i\sum_{s=1}^{k}B_s x^s y^s)$$

(Lemmas 1-3). In Lemma 6, the estimation of T will be given in the analogous way to the theorem of [**5**], by making use of Lemma 4 concerning the property of B_s and Vinogradov's mean value theorem, which will be quoted from Hua [**1**] as Lemma 5. After these lemmas, the proof of Theorem I will be easily obtained.

In § 2, we shall estimate, by making use of Theorem I, the Dedekind zeta function $\zeta_K(s)$ (Lemmas 7-9). Then the estimation of $\zeta'_K(s)/\zeta_K(s)$ will be obtained (Lemmas 10 and 11). From these lemmas and Lemma 12 on the sum $\vartheta(x)$, our Main Theorem will be easily proved.

We now explain the notations which will be used in the following. Let $K^{(q)}$ $(q=1,\cdots,r_1)$ be the real conjugates of K, and $K^{(p)}$, $K^{(p+r_2)}=\overline{K^{(p)}}$ $(p=r_1+1,\cdots,r_1+r_2)$ the complex conjugates of K, so that $n=r_1+2r_2$. The conjugates of the number μ of K in $K^{(i)}$ are denoted by $\mu^{(i)}$ $(i=1,\cdots,n)$. We write $N(\mu)=\mu^{(1)}\cdots\mu^{(n)}$. If $|\xi^{(i)}|\leqq a$ (or $\geqq a$) $(i=1,\cdots,n)$, we write $|\xi|\leqq a$ (or $\geqq a$). A small Roman letter c means positive constant which depends only on K. It does not always mean the same constant at each time it appears. If X and Y are two numbers such that $|X|\leqq cY$, then we write $X=O(Y)$ or $X\ll Y$.

§ 1. Proof of Theorem I.

LEMMA 1. *Let* $\varepsilon_1,\cdots,\varepsilon_r$ $(r=r_1+r_2-1)$ *be the fundamental units of* K. *Let* (α_{ik}) *be the inverse of the square matrix* $(\log|\varepsilon_k^{(i)}|)_{1\leqq i,k\leqq r}$ *of degree* r. *Let* $\mathfrak{M}$ *be the set of the integers* μ *of* $\mathfrak{a}$ *such that*

$$A\leqq|N(\mu)|<B,\qquad 0\leqq t_k(\mu)<1\qquad(k=1,\cdots,r),$$

where

$$t_k(\mu)=\sum_{i=1}^{r}\alpha_{ik}\Big(\log|\mu^{(i)}|-\frac{1}{n}\log|N(\mu)|\Big)\qquad(k=1,\cdots,r).$$

Then

$$S(t\,;A,B)=\frac{1}{w}\sum_{\mu\in\mathfrak{M}}\exp\,(2\pi it\log|N(\mu)|),$$

where w is the number of the roots of unity in K.

PROOF. Let (μ) be the principal ideal generated by an integer μ such that $A \leqq |N(\mu)| < B$. Put $a_k = [t_k(\mu)]$ $(k=1, \cdots, r)$. Then, we see that $\mu_1 = \mu \prod_{k=1}^{r} \varepsilon_k^{-a_k}$ is contained in $\mathfrak{M}$ and that the number $\mu_1\varepsilon$, the product of μ_1 and any unit ε of K, is contained in $\mathfrak{M}$ if and only if ε is the root of unity. Hence, the number of the generators of (μ) which are contained in $\mathfrak{M}$ is equal to w. Since their norms have the same absolute value $|N(\mu)|$, we have

$$\exp(2\pi it \log N((\mu))) = \frac{1}{w} \sum_{\substack{\lambda \in \mathfrak{M} \\ (\lambda)=(\mu)}} \exp(2\pi it \log |N(\lambda)|),$$

whence the lemma follows.

Now let J be the subset of $\{1, \cdots, r_1\}$. In the n-dimensional space E_1 we define the set X_J;

$$X_J = \left\{ (x_1, x_2, \cdots, x_n); \sum_{i=1}^{n} x_i \gamma_i^{(q)} \begin{cases} >0 & \text{for } q \in J \\ <0 & \text{for } q \notin J,\ 1 \leqq q \leqq r_1 \\ \neq 0 & \text{for } r_1+1 \leqq q \leqq n \end{cases} \right\},$$

where $\gamma_1, \cdots, \gamma_n$ is the basis of $\mathfrak{a}$. Let f_J be the mapping from X_J into the n-dimensional space E_2, which is defined as follows;

$$f_J(x_1, \cdots, x_n) = (y_0, y_1, \cdots, y_r, \theta_1, \cdots, \theta_{r_2}),$$

where

$$y_0 = \sum_{q=1}^{n} \log \Big| \sum_{i=1}^{n} x_i \gamma_i^{(q)} \Big|,$$

$$y_k = \sum_{q=1}^{r} \alpha_{qk} \Big(\log \Big| \sum_{i=1}^{n} x_i \gamma_i^{(q)} \Big| - \frac{1}{n} y_0 \Big) \qquad (k=1, \cdots, r),$$

$$\theta_p = \arg \sum_{i=1}^{n} x_i \gamma_i^{(p+r_1)} \qquad (p=1, \cdots, r_2).$$

It is easily seen that f_J is one to one mapping. Let V be the subset of E_2 as follows;

$$V = \left\{ (y_0, y_1, \cdots, y_r, \theta_1, \cdots \theta_{r_2}); \begin{array}{ll} \log A \leqq y_0 < \log B, & \\ 0 \leqq y_k < 1 & (k=1, \cdots, r) \\ 0 \leqq \theta_p < 2\pi & (p=1, \cdots, r_2) \end{array} \right\}.$$

Then the integer $\mu = \sum_{i=1}^{n} m_i \gamma_i$ of $\mathfrak{a}$ belongs to $\mathfrak{M}$ if and only if $(m_1, \cdots, m_n) \in f_J^{-1}(V)$ for some J. Hence, putting

$$F(\nu) = \exp(2\pi it \log |N(\nu)|)$$

for any number $\nu (\neq 0) \in K$, we have

$$S(t\,;\,A,\,B)=\frac{1}{w}\sum_{J}\sum_{(m_1,\cdots,m_n)\in f_J^{-1}(V)}F(\sum_{i=1}^{n}m_i\gamma_i)\,, \tag{3}$$

where J runs through all subsets of $\{1, 2, \cdots, r_1\}$ and $(m_1, \cdots, m_n)$ runs through all n-tuples of rational integers such that $f_J(m_1, \cdots, m_n)\in V$.

LEMMA 2. *We define for rational integers* $k_1, \cdots, k_n$ *the cube* Q *in* E_1;

$$Q=Q(k_1, \cdots, k_n)=\{(x_1, \cdots, x_n)\,;\; k_qM\leqq x_q<(k_q+1)M \quad (q=1, \cdots, n)\}$$

where

$$M=[A^{6/7n}]\,. \tag{4}$$

Then, the inner sum of (3) *is written as follows;*

$$\sum_{(m_1,\cdots,m_n)\in f_J^{-1}(V)}F(\sum_{i=1}^{n}m_i\gamma_i)=\sum_{Q\subset f_J^{-1}(V)}\sum_{(m_1,\cdots,m_n)\in Q}F(\sum_{i=1}^{n}m_i\gamma_i)+O(A^{1-1/7n})\,. \tag{5}$$

In the right-hand side, Q *runs through all cubes contained in* $f_J^{-1}(V)$.

PROOF. Let δ be the maximum of the diameters of the $f_J(Q)$ having points with V in common. We have, then,

$$\delta\ll MA^{-1/n}\,. \tag{6}$$

In fact, let $(x_1^0, \cdots, x_n^0)$ be a point of $Q\cap f_J^{-1}(V)$ and $f_J(x_1^0, \cdots, x_n^0)=(y_0^0, y_1^0, \cdots, y_r^0, \cdots)$. By the definitions of y_0 and y_k we have

$$\log|\sum_{i=1}^{n}x_i^0\gamma_i^{(q)}|=\frac{1}{n}y_0^0+\sum_{k=1}^{r}y_k^0\log|\varepsilon_k^{(q)}|=\frac{1}{n}\log A+O(1)\quad (q=1, \cdots, r+1)\,.$$

Hence, if $(x_1, \cdots, x_n)$ is a point in the same Q, then

$$\log|\sum_{i=1}^{n}x_i\gamma_i^{(q)}|=\log|\sum_{i=1}^{n}(x_i^0+x_i-x_i^0)\gamma_i^{(q)}|=\log|\sum_{i=1}^{n}x_i^0\gamma_i^{(q)}+O(M)|$$

$$=\log|\sum_{i=1}^{n}x_i^0\gamma_i^{(q)}|+O(MA^{-1/n})\qquad (q=1, \cdots, n)$$

and

$$\arg\sum_{i=1}^{n}x_i\gamma_i^{(p)}=\arg\sum_{i=1}^{n}x_i^0\gamma_i^{(p)}+O(MA^{-1/n})\qquad (p=r_1+1, \cdots, r_1+r_2)\,,$$

which proves the assertion (6). Since we may take A large, δ is small.

Now we shall define the sets V_1 and V_2 as follows;

$$V_1=\left\{(y_0, y_1, \cdots, y_r, \theta_1, \cdots, \theta_{r_2})\,;\quad \begin{array}{l}\log A-\delta\leqq y_0\leqq\log B+\delta\\ -\delta\leqq y_k\leqq 1+\delta\quad (k=1, \cdots, r)\\ 0\leqq\theta_p\leqq 2\pi\quad (p=1, \cdots, r_2)\end{array}\right\}\,,$$

$$V_2=\left\{(y_0, y_1, \cdots, y_r, \theta_1, \cdots, \theta_{r_2});\quad \begin{array}{l}\log A+\delta \leqq y_0 \leqq \log B-\delta \\ \delta \leqq y_k \leqq 1-\delta \quad (k=1, \cdots, r) \\ \delta \leqq \theta_p \leqq 2\pi-\delta \quad (p=1, \cdots, r_2)\end{array}\right\}.$$

If $\log B-\delta < \log A+\delta$, then we put $V_2=\phi$ (empty set). It is easily seen that

$$f_J^{-1}(V_1) \supset \bigcup_{Q \cap f_J^{-1}(V) \neq \phi} Q \supset \bigcup_{Q \subset f_J^{-1}(V)} Q \supset f_J^{-1}(V_2).$$

Hence

(7)
$$\left|\sum_{(m_1, \cdots, m_n) \in f_J^{-1}(V)} - \sum_{Q \subset f_J^{-1}(V)} \sum_{(m_1, \cdots, m_n) \in Q} F(\sum_{i=1}^{n} m_i \gamma_i)\right| \leqq \text{the volume of } f_J^{-1}(V_1-V_2).$$

Since

$$\left|\frac{\partial(x_1, \cdots, x_n)}{\partial(y_0, y_1, \cdots, y_r, \theta_1, \cdots, \theta_{r_2})}\right| = c_0 e^{y_0},$$

where c_0 is a constant, the volume of $f_J^{-1}(V_1-V_2)$ is

$$\begin{aligned}\int_{f_J^{-1}(V_1-V_2)} dx_1 \cdots dx_n &= c_0 \int_{V_1-V_2} e^{y_0} dy_0 dy_1 \cdots dy_r d\theta_1 \cdots d\theta_{r_2} \\ &= c_0(Be^{\delta}-Ae^{-\delta})(1+2\delta)^r(2\pi)^{r_2} \\ &\quad -c_0(Be^{-\delta}-Ae^{\delta})(1-2\delta)^r(2\pi-2\delta)^{r_2} \ll \delta A.\end{aligned}$$

Hence, by (6) and (7), we obtain the proof.

Now we shall consider the inner sum in the right-hand side of (5);

$$\sum_{(m_1, \cdots, m_n) \in Q} F(\sum_{i=1}^{n} m_i \gamma_i),$$

where $Q=Q(k_1, \cdots, k_n) \subset f_J^{-1}(V)$. This sum is rewritten as follows;

$$S(\lambda)=\sum_{m_1, \cdots, m_n=0}^{M-1} F(\lambda+\sum_{i=1}^{n} m_i \gamma_i),$$

where

$$\lambda = M \sum_{i=1}^{n} k_i \gamma_i \in \mathfrak{M}.$$

The following inequality is obvious;

(8)
$$cA^{1/n} \leqq |\lambda| \leqq cA^{1/n}.$$

Now we put

$$L=(\log A)^{2/3},$$
$$d=([LA^{-1/n}Mk_1], \cdots, [LA^{-1/n}Mk_n]),$$

(9)
$$m_{1i}=[LA^{-1/n}Mk_i]/d \qquad (i=1, \cdots, n).$$

d is not zero; if $d=0$, then $LA^{-1/n}\lambda$ would be $O(1)$, which contradicts to (8). We may assume that $m_{11}\neq 0$. We put

$$d_1=|m_{11}|, \qquad d_i=(d_{i-1}, |m_{1i}|) \qquad (i=2, \cdots, n)$$

and take the rational integers l_i such that

$$l_i m_{1i}\equiv -d_i \pmod{d_{i-1}}, \qquad |l_i|\leqq d_{i-1} \quad (i=2, \cdots, n). \tag{10}$$

Moreover we put

$$m_{ij}=\begin{cases} l_i m_{1j}/d_{i-1} & (1\leqq j<i\leqq n), \\ (d_i+l_i m_{1i})/d_{i-1} & (2\leqq i=j\leqq n), \\ 0 & (2\leqq i<j\leqq n), \end{cases} \tag{11}$$

$$\alpha_i=\sum_{k=1}^{n} m_{ik}\gamma_k \qquad (i=1, 2, \cdots, n).$$

The m_{ik} are rational integers and

$$\det(m_{ik})=m_{11}\prod_{i=2}^{n}\frac{d_i}{d_{i-1}}=\pm d_n=\pm 1.$$

Hence, $\alpha_1, \cdots, \alpha_n$ is the basis of $\mathfrak{a}$ and, by (9), (10) and (11), we see that

$$\alpha_i\ll\sum_{k=1}^{n}|m_{ik}|\ll\sum_{k=1}^{n}|m_{1k}|\ll L \qquad (i=1, \cdots, n). \tag{12}$$

If we write

$$\gamma_i=\sum_{k=1}^{n} c_{ik}\alpha_k \qquad (k=1, \cdots, n),$$

where the c_{ik} are rational integers, then $\sum_{i=1}^{n} m_i\gamma_i=\sum_{k=1}^{n} a_k\alpha_k$, where

$$a_k=\sum_{i=1}^{n} c_{ik}m_i \qquad (k=1, \cdots, n).$$

Since the matrix (c_{ik}) is the inverse of (m_{ik}) and $m_{ik}\ll L$, it is easily seen that $c_{ik}\ll L^{n-1}$. Hence

$$a_k\ll ML^{n-1} \qquad (k=1, \cdots, n). \tag{13}$$

If $a_2, \cdots, a_n$ are fixed, then a_1 ranges over a set of some consecutive integers. Therefore, we can write

$$S(\lambda)=\sum_{a_2}\cdots\sum_{a_n}\sum_{a_1=a}^{b} F(\lambda+\sum_{k=1}^{n} a_k\alpha_k),$$

where a and b are rational integers depending on $a_2, \cdots, a_n$.

LEMMA 3. *Put*

$$\eta=\eta(a_2, \cdots, a_n)=\frac{1}{\alpha_1}(\lambda+\sum_{i=2}^{n} a_i\alpha_i), \tag{14}$$

$$B_s=B_s(m)=\frac{(-1)^{s-1}}{s}t\sum_{i=1}^{n}(\eta^{(i)}+m)^{-s}\quad (s=1,\cdots,k;\ |m|\leqq cML^{n-1}),$$

$$T_m=\sum_{x,y=1}^{Y}\exp\left(2\pi i\sum_{s=1}^{k}B_s x^s y^s\right),$$

where

$$Y=[A^{5/12n}],\qquad k=[6\alpha]+1. \tag{15}$$

Then we have

$$S(\lambda)\ll M^n(Y^{-2}\max_{|m|\leqq cML^{n-1}}|T_m|+A^{-1/7n})+Y^2(ML^{n-1})^{n-1}. \tag{16}$$

PROOF. We have

$$S(\lambda)=\sum_{a_2}\cdots\sum_{a_n}F(\alpha_1)\sum_{a_1=a}^{b}F(\eta+a_1)\ll\sum_{a_2}\cdots\sum_{a_n}\Big|\sum_{a_1}F(\eta+a_1)\Big|, \tag{17}$$

$$\begin{aligned}\sum_{a_1=a}^{b}F(\eta+a_1)&=Y^{-2}\sum_{x,y=1}^{Y}\sum_{m=a-xy}^{b-xy}F(\eta+m+xy)\\&=Y^{-2}\sum_{x,y=1}^{Y}\Big\{\sum_{m=a}^{b}F(\eta+m+xy)+O(xy)\Big\}\\&\ll Y^{-2}\sum_{m=a}^{b}\Big|\sum_{x,y=1}^{Y}F\Big(1+\frac{xy}{\eta+m}\Big)\Big|+Y^2.\end{aligned} \tag{18}$$

By (14), $|\alpha_1(\eta+m)|=|\lambda+O(ML^n)|\geqq cA^{1/n}$, so that

$$\left|\frac{xy}{\eta+m}\right|\leqq c|\alpha_1|A^{-1/n}Y^2\leqq cLA^{-1/6n}.$$

Hence, we have the following expansions;

$$t\log\Big|N\Big(1+\frac{xy}{\eta+m}\Big)\Big|=\sum_{s=1}^{k}B_s x^s y^s+O(t(cLA^{-1/6n})^{k+1}),$$

$$\begin{aligned}\sum_{x,y=1}^{Y}F\Big(1+\frac{xy}{\eta+m}\Big)&=T_m+O(Y^2t(cLA^{-1/6n})^{k+1})\\&=T_m+O(Y^2A^{-1/7n}).\end{aligned} \tag{19}$$

Since

$$\sum_{a_2}\cdots\sum_{a_n}\sum_{a_1}1=M^n,\qquad \sum_{a_2}\cdots\sum_{a_n}1\ll(ML^{n-1})^{n-1},$$

the lemma follows from (17), (18) and (19).

LEMMA 4. *If s is even, then*

$$\frac{t}{s}(cA^{-1/n})^s\leqq|B_s|\leqq\frac{t}{s}(cLA^{-1/n})^s. \tag{20}$$

PROOF. By (14) and (9),

$$\alpha_1(\eta+m)=\lambda+O(ML^n)=\lambda(1+O(ML^nA^{-1/n})),$$

$$\alpha_1=\frac{1}{d}(LA^{-1/n}\lambda+O(1))=\frac{1}{d}LA^{-1/n}\lambda(1+O(L^{-1}))\,.$$

Hence

$$s\arg(\eta^{(p)}+m)=s\arg(1+O(L^{-1}))\ll\alpha/L\ll(\log A)^{-1/6}$$
$$(p=r_1+1,\cdots,r_1+r_2)\,,$$

since $\alpha\leqq c(\log A)^{1/2}$, and we may assume that

$$\cos\left(s\arg(\eta^{(p)}+m)\right)\geqq\frac{1}{\sqrt{2}}\qquad(p=r_1+1,\cdots,r_1+r_2)\,.$$

Therefore, for even s,

$$\frac{t}{s}\cdot\frac{1}{\sqrt{2}}\sum_{i=1}^{n}|\eta^{(i)}+m|^{-s}\leqq|B_s|\leqq\frac{t}{s}\sum_{i=1}^{n}|\eta^{(i)}+m|^{-s}\,.$$

Since

$$|\eta+m|^{-1}\geqq c|\alpha_1|A^{-1/n}\,,\qquad|\eta+m|^{-1}\leqq c|\alpha_1|A^{-1/n}\leqq cLA^{-1/n}$$

and

$$\max(|\alpha_1^{(1)}|,\cdots,|\alpha_1^{(n)}|)\geqq1\,,$$

the lemma is proved.

Now we quote from [**1**] Vinogradov's mean value theorem;

LEMMA 5. *Let P and q be positive rational integers and $g, g_1,\cdots, g_a$ arbitrary rational integers. If $q>6a^2+\frac{a}{4}(a+1)$, then the number of the $2q$-tuples $(m_1,\cdots,m_{2q})$ of rational integers such that*

$$m_1^s+m_2^s+\cdots+m_q^s-m_{q+1}^s-\cdots-m_{2q}^s=g_s\qquad(s=1,\cdots,a)\,,$$
$$g<m_j\leqq g+P\qquad(j=1,\cdots,2q)$$

does not exceed

$$N(P;a,q)=(5q)^{30aq}(\log P)^{12a}P^{2q-\frac{a}{2}(a+1)+\frac{a}{2}(a+1)\left(1-\frac{1}{a}\right)^{6a}}\tag{21}$$

(cf. [**1**], p. 37).

LEMMA 6.

$$T_m\ll Y^{2-c/\alpha^2}\,.$$

PROOF. We put

$$b=\left[6k^2+\frac{k}{2}(k+1)+1\right],\qquad l=2b\,.$$

Applying Hölder's inequality twice, we have

$$|T_m|^l\leqq Y^{l-1}\sum_{x=1}^{Y}\sum_{y_1,\cdots,y_l=1}^{Y}\exp\left\{2\pi i\sum_{s=1}^{k}B_sx^s\left(\sum_{u=1}^{b}y_u^s-\sum_{v=b+1}^{2b}y_v^s\right)\right\}$$

and

$$|T_m|^{l^2} \leqq Y^{l(l-1)}\Big(\sum_{y_1,\cdots,y_l=1}^{Y} 1\Big)^{l-1}$$

$$\times \sum_{y_1,\cdots,y_l=1}^{Y} \Big| \sum_{x=1}^{Y} \exp\Big\{2\pi i \sum_{s=1}^{k} B_s x^s \Big(\sum_{u=1}^{b} y_u^s - \sum_{v=b+1}^{2b} y_v^s\Big)\Big\}\Big|^l$$

$$\leqq Y^{2l(l-1)} \sum_{y_1,\cdots,y_l=1}^{Y}$$

$$\times \sum_{x_1,\cdots,x_l=1}^{Y} \exp\Big\{2\pi i \sum_{s=1}^{k} B_s\Big(\sum_{u=1}^{b} x_u^s - \sum_{v=b+1}^{2b} x_v^s\Big)\Big(\sum_{u=1}^{b} y_u^s - \sum_{v=b+1}^{2b} y_v^s\Big)\Big\}.$$

Hence

$$|T_m|^{l^2} \leqq Y^{2l(l-1)} \sum_{z_1,\cdots,z_k} N_1(z_1, \cdots, z_k)$$

$$\times \Big| \sum_{y_1,\cdots,y_l=1}^{Y} \exp\Big\{2\pi i \sum_{s=1}^{k} B_s z_s \Big(\sum_{u=1}^{b} y_u^s - \sum_{v=b+1}^{2b} y_v^s\Big)\Big\}\Big|,$$

where $z_1, \cdots, z_k$ run through all rational integers such that

$$|z_s| < bY^s \qquad (s=1, \cdots, k)$$

and $N_1(z_1, \cdots, z_k)$ is the number of the l-tuples $(x_1, \cdots, x_l)$ of rational integers such that

$$x_1^s + \cdots + x_b^s - x_{b+1}^s - \cdots - x_{2b}^s = z_s \qquad (s=1, \cdots, k),$$

$$1 \leqq x_j \leqq Y \qquad (j=1, \cdots, l).$$

By Lemma 5, we have

$$|T_m|^{l^2} \leqq Y^{2l(l-1)} N(Y; k, b)$$

$$\times \sum_{z_1,\cdots,z_k} \Big| \sum_{y_1,\cdots,y_l=1}^{Y} \exp\Big\{2\pi i \sum_{s=1}^{k} B_s z_s \Big(\sum_{u=1}^{b} y_u^s - \sum_{v=b+1}^{2b} y_v^s\Big)\Big\}\Big|.$$

By Schwarz's inequality,

$$|T_m|^{2l^2} \leqq Y^{4l(l-1)} N(Y; k, b)^2 \sum_{z_1,\cdots,z_k} 1$$

$$\times \sum_{z_1,\cdots,z_k} \sum_{y_1,\cdots,y_{2l}=1}^{Y} \exp\Big\{2\pi i \sum_{s=1}^{k} B_s z_s \Big(\sum_{u=1}^{l} y_u^s - \sum_{v=l+1}^{2l} y_v^s\Big)\Big\}$$

$$\leqq Y^{4l(l-1)} N(Y; k, b)^2 l^k Y^{\frac{k}{2}(k+1)}$$

$$\times \sum_{d_1,\cdots,d_k} N_2(d_1, \cdots, d_k) \Big| \sum_{z_1,\cdots,z_k} \exp\Big(2\pi i \sum_{s=1}^{k} B_s d_s z_s\Big)\Big|,$$

where $d_1, \cdots, d_k$ run through all rational integers such that

$$|d_s| < lY^s \qquad (s=1, \cdots, k)$$

and $N_2(d_1, \cdots, d_k)$ is the number of the $2l$-tuples $(y_1, \cdots, y_{2l})$ of rational integers such that

$$y_1^s + \cdots + y_l^s - y_{l+1}^s - \cdots - y_{2l}^s = d_s \qquad (s=1, \cdots, k),$$
$$1 \leqq y_j \leqq Y \qquad (j=1, \cdots, 2l).$$

Hence, by Lemma 5,

$$|T_m|^{2l^2} \leqq Y^{4l(l-1)+\frac{k}{2}(k+1)} l^k N(Y; k, b)^2 N(Y; k, l)$$
$$\times \sum_{d_1,\cdots,d_k} \Big| \sum_{z_1,\cdots,z_k} \exp\Big(2\pi i \sum_{s=1}^{k} B_s d_s z_s\Big)\Big|.$$

Since

(22) $$\Big|\sum_{z_s} \exp(2\pi i B_s d_s z_s)\Big| \leqq \min(lY^s, |\sin(\pi B_s d_s)|^{-1}),$$

we have

$$|T_m|^{2l^2} \leqq Y^{4l(l-1)+\frac{k}{2}(k+1)} l^k N(Y; k, b)^2 N(Y; k, l) \cdot W,$$

where

(23) $$W = \prod_{s=1}^{k} \sum_{d_s} \min(lY^s, |\sin(\pi B_s d_s)|^{-1}).$$

We denote by J the set of *even* rational integers s such that $2\alpha \leqq s \leqq 4\alpha$. By (20), we have for $s \in J$

$$|B_s d_s| \leqq lY^s |B_s| \leqq ck^2 Y^2 t(cLA^{-1/n})^s \leqq L^{cL} A^{-\alpha/6n},$$

which implies that $|B_s d_s| \leqq 1/2$ for large A. Hence

$$|\sin(\pi B_s d_s)|^{-1} \leqq (2|B_s d_s|)^{-1} \qquad (s \in J)$$

and it follows from (20) and (23) that

$$W \leqq \prod_{\substack{s \notin J \\ 1 \leqq s \leqq k}} (2l^2 Y^{2s}) \prod_{\substack{s \in J \\ 1 \leqq s \leqq k}} \Big(lY^s + |B_s|^{-1} \sum_{m=1}^{lY^s} m^{-1}\Big)$$
$$\leqq (2l^2)^k Y^{k(k+1)} \prod_{s \in J} (Y^{-s} + t^{-1}(cA^{1/n})^s Y^{-2s} \log Y)$$
$$\leqq c^{k^2} Y^{k(k+1)} \prod_{s \in J} (Y^{-2\alpha} + Y^{-4\alpha/5} \log Y).$$

Since the number of the elements of J is $\geqq \alpha - 1/2$, it follows that

(24) $$W \leqq c^{k^2} Y^{k(k+1) - 2\alpha(2\alpha-1)/5} (\log Y)^k.$$

By (21), (22) and (24) we have

$$|T_m|^{2l^2} \leqq c^{k^2} (25bl)^{30lk} (\log Y)^{40k} Y^{4l^2 + \frac{3}{2}k(k+1)\left(1-\frac{1}{k}\right)^{6k} - \frac{2}{5}\alpha(2\alpha-1)}.$$

Now we see that

$$30kl \log(25bl) \leqq ck^3 \log k, \qquad 13k^2 \leqq l \leqq 14k^2, \qquad \Big(1-\frac{1}{k}\Big)^{6k} < \frac{1}{400},$$

$$\frac{3}{2}k(k+1)\Big(1-\frac{1}{k}\Big)^{6k} - \frac{2}{5}\alpha(2\alpha-1) \leqq \frac{3}{400}(6\alpha+1)(3\alpha+1) - \frac{2}{5}\alpha(2\alpha-1)$$
$$\leqq -\frac{\alpha^2}{12} \qquad (\alpha \geqq 5/6).$$

Therefore we have

$$T_m \ll (\log Y)^{c/k^3} Y^{2-\alpha^2/24l^2} \ll Y^{2-c/\alpha^2},$$

and the proof of the lemma is completed.

Now from (16) and Lemma 6 we obtain

$$S(\lambda) \ll M^n(A^{-c/\alpha^2}+A^{-1/7n})+Y^2(ML^{n-1})^{n-1} \ll M^n A^{-c/\alpha^2}. \tag{25}$$

Hence, by (3), (5), (25) and the fact that the number of the Q such that $Q \subset f_J^{-1}(V)$ is $O(AM^{-n})$, we have

$$S(t\,;\,A,\,B) \ll A^{1-c/\alpha^2}+A^{1-1/7n} \ll A^{1-c/\alpha^2}.$$

Thus the proof of Theorem I is completed.

§ 2. Proof of Main Theorem.

In this paragraph, we denote by $c_1, \cdots, c_8$ positive constants depending only on K.

LEMMA 7. *Let A and B be real numbers such that*

$$\exp\{(\log|t|)^{2/3}(\log\log|t|)^{1/3}\} \leqq A < B \leqq 2A \leqq 2|t|^{7n/6}, \tag{26}$$

where $|t| \geqq c$. We define the sum;

$$L(s\,;\,A,\,B) = \sum_{\substack{\mathfrak{b}\in L(\mathfrak{a})\\ A\leqq N\mathfrak{b}<B}} N(\mathfrak{b})^{-s},$$

where $s=\sigma+it$ is complex, $\sigma>0$. If $|t| \geqq c_1$ and

$$\sigma \geqq 1-c_2\Big(\frac{\log\log|t|}{\log|t|}\Big)^{2/3},$$

with suitably chosen constants c_1 and c_2, then we have

$$L(s\,;\,A,\,B) \ll 1.$$

PROOF. Let $H(x)$ be the number of the ideals $\mathfrak{b}$ in $L(\mathfrak{a})$ such that $N(\mathfrak{b}) \leqq x$. Then

$$L(s\,;\,A,\,B) = \sum_{A\leqq m<B} m^{-\sigma}\{H(m)-H(m-1)\}e^{-it\log m}.$$

By the partial summation,

$$|L(s\,;\,A,\,B)| \leqq A^{-\sigma}\max_{A\leqq C<B}\Big|\sum_{A\leqq m<C}\{H(m)-H(m-1)\}e^{-it\log m}\Big|$$

$$= A^{-\sigma}\max_{A\leqq C<B}\Big|\sum_{\substack{\mathfrak{b}\in L(\mathfrak{a})\\ A\leqq N\mathfrak{b}<C}}\exp(-it\log N(\mathfrak{b}))\Big| = A^{-\sigma}\max_{A\leqq C<B}\Big|S\Big(\frac{|t|}{2\pi}\,;\,A,\,C\Big)\Big|.$$

In view of (26), it follows from Theorem I that

$$L(s\,;\,A,\,B) \ll A^{1-\sigma-c/\alpha^2} \qquad (|t| \geqq c_1),$$

where $\alpha = n\log|t|/\log A$. Since $\alpha \leqq c(\log|t|/\log\log|t|)^{1/3}$, we have

$$L(s\,;A,B) \ll \exp\left\{\left(1-\sigma-c_2\left(\frac{\log\log|t|}{\log|t|}\right)^{2/3}\right)\log A\right\},$$

which gives the proof.

LEMMA 8. *We define the function $\psi(s\,;\mathfrak{a})$ of a complex variable $s=\sigma+it$ as follows;*

$$\psi(s\,;\mathfrak{a}) = \sum_{\mathfrak{b}\in L(\mathfrak{a})} N(\mathfrak{b})^{-s} \qquad (\sigma>1).$$

Then $\psi(s\,;\mathfrak{a})$ is analytic for $\sigma>1-1/n$ except for a pole at $s=1$. If $|t|\geqq c_1$, and

$$\sigma \geqq 1-c_2\left(\frac{\log\log|t|}{\log|t|}\right)^{2/3},$$

then we have

$$\psi(s\,;\mathfrak{a}) \ll (\log|t|)^c.$$

PROOF. Let $\mathfrak{C}$ be the ideal class containing $\mathfrak{a}^{-1}$. Then $H(x)$ is equal to the number of ideals $\mathfrak{m}$ in $\mathfrak{C}$ such that $N(\mathfrak{m}) \leqq x/N(\mathfrak{a})$. Therefore

$$H(x) = \lambda\frac{x}{N(\mathfrak{a})} + O(x^{1-1/n}),$$

where λ is a constant depending only on K ([2], Satz 210). Let a and $b(>a)$ be rational integers. We have

$$\sum_{\substack{\mathfrak{b}\in L(\mathfrak{a})\\ a\leqq N\mathfrak{b}<b}} N(\mathfrak{b})^{-s} = \sum_{m=a}^{b-1} m^{-s}\{H(m)-H(m-1)\}$$

$$= s\int_a^{b-1}\frac{H(x)}{x^{s+1}}dx - \frac{H(a-1)}{a^s} - \frac{H(b-1)}{(b-1)^s}.$$

If $\sigma>1$, we can let $b\to\infty$. Therefore, putting $r(x)=H(x)-\lambda x/N(\mathfrak{a})$, we have

$$F(s) = \psi(s\,;\mathfrak{a}) - \sum_{\substack{\mathfrak{b}\in L(\mathfrak{a})\\ N\mathfrak{b}<a}} N(\mathfrak{b})^{-s} \tag{27}$$

$$= \frac{\lambda}{s-1}\frac{a^{1-s}}{N(\mathfrak{a})} + \frac{\lambda}{N(\mathfrak{a})a^s} + s\int_a^{\infty}\frac{r(x)}{x^{s+1}}dx - \frac{r(a-1)}{a^s}.$$

Since $r(x)\ll x^{1-1/n}$, the integral in the right-hand side of (27) converges uniformly for $\sigma\geqq\sigma_1>1-1/n$. Hence $F(s)$ is analytic for $\sigma>1-1/n$ except for a pole at $s=1$ and we obtain the first assertion of the lemma. Assume that $\sigma\geqq 1-1/7n$ and put $a=[X]$, where $X=|t|^{7n/6}$. Then

$$F(s) \ll |t|^{-1}X^{1-\sigma} + X^{-\sigma} + |t|X^{-\sigma+1-1/n} + X^{-\sigma+1-1/n} \ll 1. \tag{28}$$

Moreover put $X_0 = \exp\{(\log|t|)^{2/3}(\log\log|t|)^{1/3}\}$ and let m be the rational integer such that $2^{m-1}X_0 < X \leqq 2^m X_0$, then, by Lemma 7,

(29)
$$\sum_{\substack{\mathfrak{b}\in L(\mathfrak{a})\\ X_0\leqq N\mathfrak{b}<X}} N(\mathfrak{b})^{-s}=L(s\,;X_0,2X_0)+L(s\,;2X_0,2^2X_0)+\cdots$$
$$+L(s\,;2^{m-1}X_0,X)\ll\log|t|.$$

On the other hand, if $\sigma\geqq\sigma_0=1-c_2(\log\log|t|/\log|t|)^{2/3}$, then

(30)
$$\sum_{\substack{\mathfrak{b}\in L(\mathfrak{a})\\ N\mathfrak{b}<X_0}} N(\mathfrak{b})^{-s}\ll\frac{X_0^{1-\sigma_0}}{1-\sigma_0}\ll\left(\frac{\log|t|}{\log\log|t|}\right)^{2/3}\exp(c\log\log|t|)\ll(\log|t|)^c.$$

Collecting the results (28), (29) and (30), we have

$$\phi(s\,;\mathfrak{a})=\sum_{N\mathfrak{b}<X}N(\mathfrak{b})^{-s}+O(1)\ll(\log|t|)^c.$$

LEMMA 9. *Let $\zeta_K(s)$ be the Dedekind zeta function of K. If $|t|\geqq c_1$ and*

$$\sigma\geqq1-c_2\left(\frac{\log\log|t|}{\log|t|}\right)^{2/3},$$

then we have

$$\zeta_K(s)\ll(\log|t|)^{c_3},$$

where we may assume that $c_3>1$.

PROOF. Let $\mathfrak{a}_1,\cdots,\mathfrak{a}_h$ be the representatives of ideal classes. Then it is easily seen that

$$\zeta_K(s)=\sum_{i=1}^{h}N(\mathfrak{a}_i)\phi(s\,;\mathfrak{a}_i)$$

for $\sigma>1-1/n$. Hence the lemma follows immediately from Lemma 8.

LEMMA 10. *Let $\zeta_K(s)\ll e^{\phi(t)}$ as $t\to\infty$ in the region*

$$1-\theta(t)\leqq\sigma\leqq2\qquad(t\geqq0),$$

where $\phi(t)$ and $1/\theta(t)$ are positive non-decreasing functions of t for $t\geqq0$, such that $\theta(t)\leqq1$, $\phi(t)\to\infty$ and $\phi(t)/\theta(t)=o(e^{\phi(t)})$ (o is Landau's symbol). Then $\zeta_K(s)$ has no zeros in the region

$$\sigma\geqq1-c_4\frac{\theta(2t+1)}{\phi(2t+1)}.$$

Moreover,

$$\frac{\zeta_K'}{\zeta_K}(s)\ll\frac{\phi(2t+3)}{\theta(2t+3)}$$

for $\sigma\geqq1-c_5\theta(2t+3)/\phi(2t+3)$, $|t|\geqq c$.

PROOF. The proof is easily obtained from Theorem 3.11, [4] by replacing $\zeta(s)$ by $\zeta_K(s)$.

In Lemma 10, we can take, by Lemma 9,

$$\theta(t)=c_2\left(\frac{\log\log(t+3)}{\log(t+3)}\right)^{2/3},\qquad\phi(t)=c_3\log\log(t+3).$$

Hence, putting $\psi(t)=(\log|t|)^{2/3}(\log\log|t|)^{1/3}$, we have the following

LEMMA 11. *$\zeta_K(s)$ has no zeros in the region*

$$\sigma \geqq 1 - \frac{c_6}{\psi(t)}, \qquad |t| \geqq c_1 .$$

If $\sigma \geqq 1 - c_7/\psi(t)$, $|t| \geqq 3c_1$, then we have

$$\frac{\zeta_K'}{\zeta_K}(s) \ll \psi(t) .$$

Now we put

$$l(t) = \begin{cases} \dfrac{c_8}{\psi(c_1)} & (|t| \leqq c_1) , \\ \dfrac{c_8}{\psi(t)} & (|t| \geqq c_1) , \end{cases}$$

where we take c_8 small. We define the function $K(s)$ as follows;

$$K(s) = -\frac{\zeta_K'}{\zeta_K}(s) - \sum_{\substack{\mathfrak{p}, m \\ m \geqq 2}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{ms}} \qquad (\sigma > 1/2) .$$

Then $K(s)$ is regular for $\sigma \geqq 1 - l(t)$ except at $s = 1$, and from Lemma 11 and the fact $\zeta_K(1+it) \neq 0$ for all t (cf. [2], Satz 188), it follows that

$$(31) \quad \begin{aligned} & K(s) = \frac{1}{s-1} + O(1) \qquad (1 - l(t) \leqq \sigma \leqq 2,\ |t| \leqq c_1) , \\ & K(s) \ll \psi(t) \qquad (1 - l(t) \leqq \sigma \leqq 2,\ |t| \geqq c_1) . \end{aligned}$$

LEMMA 12. *Put*

$$\vartheta(x) = \sum_{N\mathfrak{p} \leqq x} \log N(\mathfrak{p}) .$$

Then

$$\vartheta(x) = x + O(x e^{-cQ(x)}) ,$$

where $Q(x) = (\log x)^{3/5} (\log\log x)^{-1/5}$.

PROOF. We define

$$P(x) = \sum_{N\mathfrak{p} \leqq x} \log N(\mathfrak{p}) \log \frac{x}{N(\mathfrak{p})} ,$$

then

$$P(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} K(s) \frac{x^s}{s^2} ds = x + \frac{1}{2\pi i} \int_C K(s) \frac{x^s}{s^2} ds ,$$

where C is the curve defined by $\sigma = 1 - l(t)$ $(-\infty < t < \infty)$ (cf. [2], Satz 189). By (31) we have

$$\begin{aligned} \int_C K(s) \frac{x^s}{s^2} ds & \ll \int_0^{c_1} \frac{x^{1-c}}{1+t^2} dt + \int_{c_1}^{\infty} \frac{x^{1-l(t)}}{t^2} \psi(t) dt \\ & \ll x e^{-cQ(x)} + \left\{ \int_{c_1}^{e^{Q(x)}} + \int_{e^{Q(x)}}^{\infty} \right\} \frac{x^{1-l(t)}}{t^{3/2}} dt \end{aligned}$$

$$\ll xe^{-cQ(x)}+x\exp\{-l(e^{Q(x)})\log x\}+xe^{-\frac{1}{4}Q(x)}$$
$$\ll xe^{-cQ(x)}.$$

Hence

$$P(x)=x+O(xe^{-cQ(x)}).$$

From this the lemma follows in the same way as in the proof of [2], Satz 190.

Now, it is easily seen that

$$\pi_K(x)=\frac{\vartheta(x)}{\log x}+\int_2^x\frac{\vartheta(t)}{t\log^2 t}dt.$$

By Lemma 12,

$$\frac{\vartheta(x)}{\log x}=\frac{x}{\log x}+O(xe^{-cQ(x)}),$$

$$\int_2^x\frac{\vartheta(t)}{t\log^2 t}dt=\int_2^x\frac{t+O(te^{-cQ(t)})}{t\log^2 t}dt$$
$$=\int_2^x\frac{dt}{\log t}-\frac{x}{\log x}+O(xe^{-cQ(x)}).$$

Hence

$$\pi_K(x)=\int_2^x\frac{dt}{\log t}+O(xe^{-cQ(x)}).$$

Thus we complete the proof of Main Theorem.

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