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On certain square-integrable irreducible unitary representations of some $\mathfrak{p}$-adic linear groups

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Introduction.

Let P be a $\mathfrak{p}$-adic number field. Denote by $\mathfrak{O}$, $\mathfrak{P}$ and $\mathfrak{O}^*$ the ring of integers, the maximal ideal of $\mathfrak{O}$ and the unit group respectively. We consider the subgroup G of $GL(n^{1)}, P)$ formed by matrices with determinant in $\mathfrak{O}^*$. $K = GL(n, \mathfrak{O})$ is a maximal compact subgroup of G. In this paper, we construct continuous irreducible unitary representations of K parametrized by certain characters (which must satisfy rather restrictive conditions) of compact Cartan subgroups of G. We then show that unitary representations of G induced by these irreducible representations of K are irreducible and square integrable.

This paper is divided into four sections and last two sections are divided into several subsections. In § 1, we prove results in the theory of induced representations of finite groups, which are basic in our argument. In § 2, using results of § 1, we show that continuous irreducible unitary representations of K, which are not reduced to representations of $GL(n, \mathfrak{O}/\mathfrak{P})$, are induced by certain irreducible representations of some subgroups of K (Theorem 1). We further show that there exists a rather large family of irreducible unitary representations of K which are monomial (Theorem 2). In § 3, we study unitary representations of G induced by irreducible unitary representations of K. After the study of general properties of such representations, we show that there exists a rather large family of irreducible unitary representations of K which induce square integrable irreducible unitary representations of G (Theorem 3). We also show that an analogue of Frobenius' formula for induced characters is valid. In § 4, we first study correspondence between compact Cartan subgroups of G and extensions of P of degree n. Then, in 4-4 and 4-5, we construct irreducible unitary representations of K parametrized by certain characters, which satisfy certain conditions ('strong regularity' in our terminology), of compact Cartan subgroups of G. We show that they induce square integrable irreducible unitary representations of G (Theorem 4). Theo-

1) n is a natural number $\geqq 2$.

rem 4 is in fact a refinement of a special case of Theorem 3. In 4-7, we study equivalences between constructed representations of G and influences of outer automorphisms of G on these representations. In the last 4-8, restrictions of constructed irreducible unitary representations of G to the subgroup $G'=SL(n, P)$ are studied.

Mautner first observed that there exist square integrable irreducible unitary representations of $PGL(2, P)$ which are induced by irreducible representations of certain maximal compact subgroup. This work was done during the last one year. Meanwhile, the author could get copies of J. A. Shalika's lectures given in "Seminar on Representations of Lie Groups" held at Princeton in 1966; "Representations of the two by two unimodular group over local fields; I and II".

In the first lecture, he constructed discrete series of irreducible unitary representations of $SL(2, P)$ by a different method and showed that most of them were induced by suitable irreducible unitary representations of maximal compact subgroups. In the second lecture, he constructed irreducible unitary representations of $SL(2, \mathfrak{O})$ by using induced representations. He pointed out the possibility of extending his results to wider classes of $\mathfrak{p}$-adic linear groups.

Most of the author's work were done independently of Shalika's work. But the author owes a certain part of 4-5 to Shalika's second lecture. A part of results in this paper was announced in [**6**].

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NOTATIONS:

For a ring R, we denote by $M(n_1, n_2; R)$ the set of n_1 by n_2 matrices with elements in R. We put $M(n, R)=M(n, n; R)$. When R is commutative, we denote by $\det x$ the determinant of an element x in $M(n, R)$.

§ 1. Preliminary results in the theory of induced representations.

Let G be a finite group and H be a subgroup of G. Let ν be a representation of H on a finite dimensional vector space V over the complex number field. We denote by V_ν the vector space of V-valued functions f on G which satisfy the following condition:

$$f(hg)=\nu(h)f(g) \qquad \text{for every } h\in H.$$

We define a representation μ of G on V_ν as follows:

$$(\mu(g)f)(g')=f(g'g) \qquad (f\in V_\nu;\ g, g'\in G).$$

We call μ the representation of G induced by the representation ν of H and we denote $\mu=\operatorname{Ind}_{H\uparrow G}\nu$. Now let H be a normal subgroup of G. We denote by

$\hat{H}$ the set of all one-dimensional representations of H.

For any $g \in G$ and $\chi \in \hat{H}$ we define $g \cdot \chi \in \hat{H}$ as follows:

$$g\chi(h) = \chi(g^{-1}hg) \qquad (h \in H).$$

Thus G operates naturally on $\hat{H}$. For every $\chi \in \hat{H}$, we put $I_\chi = \{g \in G; g \cdot \chi = \chi\}$. I_χ is a subgroup of G containing H. Let μ be an irreducible representation of G on a finite dimensional vector space W. For each $\chi \in \hat{H}$, we associate a subspace W_χ of W as follows:

$$W_\chi = \{w \in W; \mu(h)w = \chi(h)w \qquad \text{for every } h \in H\}.$$

LEMMA 1-1. *Suppose $W_{\chi_0} \neq \{0\}$ for at least one $\chi_0 \in \hat{H}$ (this is always the case when H is abelian). Let O be the G-orbit in $\hat{H}$ containing χ_0. Then we have $W = \sum_{\chi \in O} W_\chi$ and every W_χ is a non-zero I_χ-invariant subspace and μ induces naturally a representation μ_χ of I_χ on W_χ. Then μ_χ is an irreducible representation of I_χ and $\operatorname{Ind}_{I_\chi \uparrow G} \mu_\chi$ is equivalent to μ.*

PROOF. All these things are a special case of the theorem in Curtis-Reiner's book [1] p. 348. q. e. d.

Now let ν_χ be an irreducible representation of I_χ ($\chi \in \hat{H}$) on a finite dimensional vector space V such that $\nu_\chi(h) = \chi(h) \cdot 1$ for every $h \in H$.

LEMMA 1-2. *The representation $\operatorname{Ind}_{I_\chi \uparrow G} \nu_\chi$ is irreducible.*

Moreover, $\operatorname{Ind}_{I_{\chi_1} \uparrow G} \nu_{\chi_1}$ and $\operatorname{Ind}_{I_{\chi_2} \uparrow G} \nu_{\chi_2}$ are mutually equivalent if and only if there exists $g_0 \in G$ such that $\chi_1 = g_0 \cdot \chi_2$ and if ν_{χ_1} and $\nu_{\chi_2}^{g_0}$ are mutually equivalent representations of I_{χ_1}, where $\nu_{\chi_2}^{g_0}$ is defined as follows:

$$\nu_{\chi_2}^{g_0}(g) = \nu_{\chi_2}(g_0^{-1}gg_0) \qquad (g \in I_{\chi_1}).$$

PROOF. For any $x \in G$, we denote by I_χ^x the subgroup $xI_\chi x^{-1} \cap I_\chi$ of I_χ. We define the representations ν_χ^x and $\nu_\chi | I_\chi^x$ of I_χ^x on V as follows:

$$\nu_\chi^x(g) = \nu_\chi(x^{-1}gx) \quad \text{and} \quad \nu_\chi | I_\chi^x(g) = \nu_\chi(g) \qquad (g \in I_\chi^x).$$

Then to prove $\operatorname{Ind}_{I_\chi \uparrow G} \nu_\chi$ is irreducible, it is sufficient to show that ν_χ^x and $\nu_\chi | I_\chi^x$ are disjoint representations[2] of I_χ^x when $x \notin I_\chi$ (see Curtis-Reiner [1] p. 328). Suppose that these two representations were not disjoint for some $x \notin I_\chi$. Then there exists a non-zero linear transformation S of V such that $S\nu_\chi(x^{-1}gx) = \nu_\chi(g)S$ for every $g \in I_\chi^x$. Since $x \notin I_\chi$, there exists an element $h \in H$ such that $\chi(x^{-1}hx) \neq \chi(h)$. Since h, $x^{-1}hx \in H \subset I_\chi^x$, we have

$$S\nu_\chi(x^{-1}hx) = \nu_\chi(h)S \quad \text{and} \quad \chi(x^{-1}hx)S = \chi(h)S.$$

Since $\chi(x^{-1}hx) \neq \chi(h)$, we must have $S = 0$. This contradicts the assumption

2) Two representations of a finite group are said to be disjoint if they have no common irreducible components.

that S is a non-zero linear transformation. $\operatorname{Ind}_{I_{\chi_1}\uparrow G} \nu_{\chi_1}$ and $\operatorname{Ind}_{I_{\chi_2}\uparrow G} \nu_{\chi_2}$ are equivalent if and only if there exists $g_0 \in G$ such that representations ν_{χ_1} and $\nu_{\chi_2}^{g_0}: g \to \nu_{\chi_2}(g_0^{-1} g g_0)$ of the group $I_{\chi_1,\chi_2}^{g_0} = I_{\chi_1} \cap g_0 I_{\chi_2} g_0^{-1}$ are not disjoint (see Curtis-Reiner [1] p. 329).

Assume that they are not disjoint. The above argument yields that $\chi_1(h) = \chi_2(g_0^{-1} g g_0)$ for every $h \in H$. Then $\chi_1 = g_0 \chi_2$ and $I_{\chi_2} = g_0^{-1} I_{\chi_1} g_0$, hence ν_{χ_1} and $\nu_{\chi_2}^{g_0}$ are both irreducible representations of $I_{\chi_1,\chi_2}^{g_0} = I_{\chi_1}$. Since they are not disjoint, they must be equivalent. The converse is obvious. q. e. d.

§ 2. Irreducible unitary representations of a maximal compact subgroup of a $\mathfrak{p}$-adic general linear group.

Let P be the completion of an algebraic number field with respect to a discrete valuation. Let $\mathfrak{O}$, $\mathfrak{P}$ and π be the ring of integers of P, the prime ideal of $\mathfrak{O}$ and a generator of $\mathfrak{P}$. Put $\mathfrak{R}_l = \mathfrak{O}/\mathfrak{P}^l$ $(l = 1, 2, \cdots)$. Denote by q the number of elements of the residue class field $\mathfrak{R}_1 = \mathfrak{O}/\mathfrak{P}$. Then $\mathfrak{R}_l$ is a finite ring with q^l elements. Let p be the characteristic of $\mathfrak{R}_1$. Denote by φ_l the natural projection of $\mathfrak{O}$ onto $\mathfrak{R}_l$. We extend φ_l naturally to a ring homomorphism of $M(n, \mathfrak{O})$ onto $M(n, \mathfrak{R}_l)$ and denote this extension of φ_l by the same symbol.

For two natural numbers m_1, m_2 such that $m_1 \leqq m_2$, there exists a ring homomorphism $\varphi_{m_1}^{m_2}$ of $\mathfrak{R}_{m_2}$ onto $\mathfrak{R}_{m_1}$ such that $\varphi_{m_1} = \varphi_{m_1}^{m_2} \varphi_{m_2}$. We extend $\varphi_{m_1}^{m_2}$ naturally to a homomorphism of $M(n, \mathfrak{R}_{m_2})$ onto $M(n, \mathfrak{R}_{m_1})$ and denote this extension of $\varphi_{m_1}^{m_2}$ by the same symbol.

Put

$$M(n, 1; \mathfrak{R}_m) \text{ (resp. } M(n, 1; \mathfrak{O})) = (\mathfrak{R}_m)^n \text{ (resp. } (\mathfrak{O})^n).$$

Every element $x \in M(n, \mathfrak{R}_m)$ (resp. $M(n, \mathfrak{O})$) operates naturally on $(\mathfrak{R}_m)^n$ (resp. $(\mathfrak{O})^n$). We call n elements $v_1, \cdots, v_n \in (\mathfrak{R}_m)^n$ (resp. $(\mathfrak{O})^n$) form a base when every element of $(\mathfrak{R}_m)^n$ (resp. $(\mathfrak{O})^n$) can be uniquely expressed as $\mathfrak{R}_m$-linear (resp. $\mathfrak{O}$-linear) combination of $v_1, \cdots, v_n$. Put

$$e_i = \begin{pmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} i-1 \\ 1 \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} n-i \end{pmatrix} \in (\mathfrak{R}_m)^n \text{ (resp. } (\mathfrak{O})^n) \ (1 \leqq i \leqq n),$$

then $e_1, \cdots, e_n$ form a base of $(\mathfrak{R}_m)^n$ (resp. $(\mathfrak{O})^n$). We call an element x of $M(n, \mathfrak{R}_m)$ (resp. $M(n, \mathfrak{O})$) *quasi-regular* if the minimal polynomial of $\varphi_1^m(x)$ (resp. $\varphi_1(x)$) over $\mathfrak{R}_1$ is of degree n.

LEMMA 2-1. *Let x be a quasi-regular element of $M(n, \mathfrak{R}_m)$ (resp. $M(n, \mathfrak{O})$). There exists an element v of $(\mathfrak{R}_m)^n$ (resp. $(\mathfrak{O})^n$) such that $v, xv, \cdots, x^{n-1}v$ form a base of $(\mathfrak{R}_m)^n$ (resp. $(\mathfrak{O})^n$).*

PROOF. Put $\bar{x}=\varphi_1^m(x)$ (resp. $\varphi_1(x)$). Since the minimal polynomial of $\bar{x}$ over $\mathfrak{R}_1$ is of degree n, $\bar{x}$ is a cyclic $\mathfrak{R}_1$-linear transformation of $(\mathfrak{R}_1)^n$. Hence there exists a $\bar{v}\in(\mathfrak{R}_1)^n$ such that $\bar{v}, \bar{x}\bar{v}, \cdots, \bar{x}^{n-1}\bar{v}$ form a $\mathfrak{R}_1$-base of $(\mathfrak{R}_1)^n$. Take a $v\in(\mathfrak{R}_m)^n$ (resp. $(\mathfrak{O})^n$) such that $\varphi_1^m(v)$ (resp. $\varphi_1(v))=\bar{v}$. Since $e_1, \cdots, e_n$ form a base of $(\mathfrak{R}_m)^n$ (resp. $(\mathfrak{O})^n$), there exists $Y=(\alpha_{ik})\in M(n, \mathfrak{R}_m)$ (resp. $M(n, \mathfrak{O})$) such that $x^{i-1}v=\sum_{k=1}^{n}\alpha_{ik}e_k$ $(1\leqq i\leqq n)$. Since $\bar{v}, \bar{x}\bar{v}, \cdots, \bar{x}^{n-1}\bar{v}$ is a base of $(\mathfrak{R}_1)^n$, it is obvious that $\varphi_1(Y)\in GL(n, \mathfrak{R}_1)$. Hence $Y\in GL(n, \mathfrak{R}_m)$ (resp. $GL(n, \mathfrak{O})$). It is proved that $v, xv, \cdots, x^{n-1}v$ form a base. q. e. d.

COROLLARY 1. *Every element of $M(n, \mathfrak{R}_m)$ (resp. $M(n, \mathfrak{O})$) which commutes with x can be expressed as an $\mathfrak{R}_m$-linear ($\mathfrak{O}$-linear) combination of $1, x, x^2, \cdots, x^{n-1}$.*

PROOF. We assume that $tx=xt$, where $t\in M(n, \mathfrak{R}_m)$ (resp. $M(n, \mathfrak{O})$). Since $v, xv, \cdots$ and $x^{n-1}v$ form a base, we can put $tv=\sum_{i=1}^{n}a_ix^{i-1}v$, where $a_i\in\mathfrak{R}_m$ (resp. $\mathfrak{O}$) $(i=1, \cdots, n)$. Let $w=\sum_{i=1}^{n}b_ix^{i-1}v$ ($b_i\in\mathfrak{R}_m$ (resp. $\mathfrak{O}$)) be any element of $(\mathfrak{R}_m)^n$ (resp. $(\mathfrak{O})^n$). We have $tw=\sum_{i=1}^{n}b_ix^{i-1}tv=\sum_{i=1}^{n}b_ix^{i-1}(\sum_{j=1}^{n}a_jx^{j-1}v)=(\sum_{j=1}^{n}a_jx^{j-1})\sum_{i=1}^{n}b_ix^{i-1}v$ $=(\sum_{j=1}^{n}a_jx^{j-1})w$. Hence we have $t=\sum_{j=1}^{n}a_jx^{j-1}$. q. e. d.

COROLLARY 2. *Let x be a quasi-regular element of $M(n, \mathfrak{R}_m)$ and let X be an element of $M(n, \mathfrak{O})$ such that $\varphi_m(X)=x$. We denote by C_x (resp. C_X) the centralizer of x (resp. X) in $GL(n, \mathfrak{R}_m)$ (resp. $GL(n, \mathfrak{O})$). Then C_X and C_x are abelian groups and we have $C_x=\varphi_m(C_X)$.*

PROOF. Obvious from Cor. 1. q. e. d.

COROLLARY 3. *Put* $\det(t\cdot 1-x)=t^n+\sum_{i=1}^{n}C_it^{n-i}$ ($C_i\in\mathfrak{R}_m$ (resp. $\mathfrak{O}$)), *where t is an indeterminate. We have*

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & & & & 0 & 1 \\ -C_n & \cdots & \cdots & \cdots & \cdots & (-C_1) \end{pmatrix}=YxY^{-1}\,[3].$$

PROOF. From the definition of Y, we have

$$\begin{pmatrix} v \\ xv \\ \vdots \\ x^{n-1}v \end{pmatrix}=Y\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

3) Y is given in the proof of Lemma 2-1.

in matrix notations. Hence

$$\begin{pmatrix} xv \\ x^2v \\ \vdots \\ x^nv \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & & & & 1 \\ -C_n & \cdots & & & (-C_1) \end{pmatrix} Y \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = Yx \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

(We have $x^n + \sum_{i=1}^{n} C_i x^{n-i} = 0$ from Hamilton-Cayley theorem.) Hence

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & & & & 1 \\ -C_n & \cdots & & & (-C_1) \end{pmatrix} Y = Yx .$$

Since $Y \in GL(n, \mathfrak{R}_m)$ (resp. $GL(n, \mathfrak{O})$), the Corollary 3 is proved. q. e. d.

COROLLARY 4. *Let X_1 and X_2 be two quasi-regular elements of $M(n, \mathfrak{O})$ whose characteristic polynomials are identical. Then X_1 and X_2 are conjugate in $GL(n, \mathfrak{O})$.*

PROOF. Obvious from Cor. 3. q. e. d.

Let G be the subgroup of $GL(n, P)$ formed by all elements with determinant in $\mathfrak{O}^*$[4]. Put $K = GL(n, \mathfrak{O})$. K is a maximal compact subgroup of $GL(n, P)$. We denote by K_m ($m = 1, 2, \cdots$) the invariant subgroup of K formed by all integral matrices congruent to the identity modulo $\mathfrak{P}^m$. Then $K_1 \supset K_2 \supset \cdots$ form a neighborhood basis of the identity in K. Take a character χ of the additive group P which is trivial on $\mathfrak{O}$ and is not trivial on $\pi^{-1}\mathfrak{O}$. Let r be a natural number $\geqq 2$ and put $s = \left[\frac{r}{2}\right]$. For every $x \in M(n, \mathfrak{R}_s)$, we define the function χ_x^r on K_{r-s} as follows:

$$\chi_x^r(k) = \chi(\pi^{-r} \operatorname{trace} x(k-1)) \qquad (k \in K_{r-s}{}^{5)}) .$$

LEMMA 2-2. *Let notations be as above.*

(i) *χ_x^r is a one-dimensional representation of K_{r-s} which is trivial on K_r.*

(ii) *For every $k \in K$, we have $k \cdot \chi_x^r = \chi^r_{\varphi_s(k) x \varphi_s(k)^{-1}}$ (for the definition of $k \cdot \chi_x^r$, see § 1).*

(iii) *The mapping: $x \to \chi_x^r$ defines an isomorphism of the additve group $M(n, \mathfrak{R}_s)$ onto the multiplicative group of all one-dimensional representations of K_{r-s} which are trivial on K_r.*

PROOF. (i) Let k_1, k_2 be two elements of K_{r-s}. We put $k_1 = 1 + \pi^{r-s} y_1$ and $k_2 = 1 + \pi^{r-s} y_2$ ($y_1, y_2 \in M(n, \mathfrak{O})$). Then we have

4) We denote by $\mathfrak{O}^*$ the unit group of P.

5) Take $X \in M(n, \mathfrak{O})$ such that $\varphi_s(X) = x$. Then $\chi(\pi^{-r} \operatorname{trace} X(k-1))$ ($k \in K_{r-s}$) depends only on x. We put $\chi(\pi^{-r} \operatorname{trace} X(k-1)) = \chi(\pi^{-r} \operatorname{trace} x(k-1))$. In the following we use such notations frequently without further references.

$$\begin{aligned}\chi_x^r(k_1k_2) &= \chi(\pi^{-r}\operatorname{trace} x\{1+\pi^{r-s}(y_1+y_2)+\pi^{2(r-s)}y_1y_2-1\}) \\ &= \chi(\pi^{-s}\operatorname{trace} xy_1)\chi(\pi^{-s}\operatorname{trace} xy_2)\chi(\pi^{r-2s}\operatorname{trace} xy_1y_2) \\ &= \chi_x^r(k_1)\chi_x^r(k_2) \qquad (\chi(\pi^{r-2s}\operatorname{trace} xy_1y_2)=1)\,.\end{aligned}$$

Hence χ_x^r is a one-dimensional representation of K_{r-s}. When $k\in K_r$, we have $\pi^{-r}\operatorname{trace} x(k-1)\in\mathfrak{O}$ and $\chi_x^r(k)=1$. Therefore χ_x^r is trivial on K_r.

(ii) By the definition of $k\cdot\chi_x^r$, we have, for every $h\in K_{r-s}$,

$$\begin{aligned}k\cdot\chi_x^r(h) &= \chi_x^r(k^{-1}hk)=\chi(\pi^{-r}\operatorname{trace} x(k^{-1}hk-1)) \\ &= \chi(\pi^{-r}\operatorname{trace}\{\varphi_s(k)x\varphi_s(k)^{-1}\}(h-1)) \\ &= \chi^r_{\varphi_s(k)x\varphi_s(k)^{-1}}(h)\,.\end{aligned}$$

(iii) It is obvious that the mapping: $x\to\chi_x^r$ defines a homomorphism of the additive group $M(n,\mathfrak{R}_s)$ into the character group of K_{r-s}/K_r. When $\chi_x^r\equiv 1$, we have $\chi(\pi^{-s}\operatorname{trace} xy)=1$ for every $y\in M(n,\mathfrak{O})$ and we have $x=0$. Therefore, this homomorphism is injective. Since $M(n,\mathfrak{R}_s)$ and K_{r-s}/K_r are finite abelian groups of the same order, this homomorphism is surjective. q. e. d.

Let ν be a non-trivial continuous irreducible unitary representation of K on a Hilbert space V. Then V is finite dimensional and there exists a natural number $r=r(\nu)$ such that ν is trivial on K_r and is not trivial on K_{r-1}. Then ν can be identified with a representation of the finite group $K/K_r\cong GL(n,\mathfrak{R}_r)$. We assume that $r\geqq 2$ and put $s=\left[\frac{r}{2}\right]$. Then K_{r-s}/K_r is a normal abelian subgroup of K/K_r. For every $x\in M(n,\mathfrak{R}_s)$, we define the subspace V_x of V as follows:

$$V_x=\{v\in V\,;\ \nu(k)v=\chi_x^r(k)v^{6)} \qquad \text{for every } k\in K_{r-s}\}\,.$$

We denote by O_ν the set of $x\in M(n,\mathfrak{R}_s)$ such that $V_x\neq\{0\}$. We define the adjoint transformation of an element of $GL(n,\mathfrak{R}_l)$ on $M(n,\mathfrak{R}_l)$ as follows ($l=1,2,\cdots$):

$$Ad\ k\cdot x=kxk^{-1}\quad (k\in GL(n,\mathfrak{R}_l),\quad x\in M(n,\mathfrak{R}_l))\,.$$

THEOREM 1. *Let ν be a continuous irreducible unitary representation of K which is not trivial on K_1. Put $r=r(\nu)$ and $s=\left[\frac{r}{2}\right]$ (we have $r\geqq 2$). Let notations be as above.*

(i) *$GL(n,\mathfrak{R}_s)$ acts on O_ν transitively by the adjoint transformation and for every $x\in O_\nu$ we have $\varphi_1^s(x)\neq 0$.*

(ii) *For every $x\in M(n,\mathfrak{R}_s)$, we denote by I_x the centralizer of χ_x^r in K (i.e. the set of $k\in K$ such that $k\cdot\chi_x^r=\chi_x^r$). I_x is given as follows:*

$$I_x=\{k\in K\,;\ \varphi_s(k)x\varphi_s(k)^{-1}=x\}\,.$$

6) χ_x^r is defined in Lemma 2-2.

(iii) *For every* $x \in O_\nu$, V_x *is an* I_x*-invariant subspace of* V. *We denote by* ν_x *the representation of* I_x *on* V_x *defined as follows:*

$$\nu_x(k) = \nu(k)|_{V_x} \qquad (k \in I_x).$$

Then ν_x *is an irreducible representation of* I_x *which coincides with* $\chi_x^r \cdot 1$ *on* K_{r-s}, *and* $\operatorname{Ind}_{I_x \uparrow K} \nu_x$ *is equivalent with* ν.

(iv) *Let* μ *be an irreducible unitary representation of* I_x (x *is any element in* $M(n, \mathfrak{R}_s)$ *such that* $\varphi_1^s(x) \neq 0$) *which coincides with* $\chi_x^r \cdot 1$ *on* K_{r-s}. *Then* $\nu = \operatorname{Ind}_{I_x \uparrow K} \mu$ *is a continuous irreducible unitary representation of* K *such that* $r(\nu) = r$ *and* $x \in O_\nu$.

PROOF. Theorem 1 follows from Lemma 1-1, Lemma 1-2, Lemma 2-2 and the definition of $r(\nu)$. (Although K is an infinite group, here we consider only representations of K which are trivial on a certain normal subgroup of finite index in K. So results in § 1 are applicable.) q. e. d.

When there exists an element of O_ν which is quasi-regular, any element of O_ν is quasi-regular, since $GL(n, \mathfrak{R}_s)$ acts on O_ν transitively. In this case, we say that O_ν is quasi-regular.

LEMMA 2-3. *Let* x *be a quasi-regular element of* $M(n, \mathfrak{R}_s)$. *Take* $X \in M(n, \mathfrak{O})$ *such that* $\varphi_s(X) = x$. *Then we have* $I_x = C_X \cdot K_s$, *where we denote by* C_X *the subgroup of* K *formed by elements which commute with* X.

PROOF. Take $k \in I_x$. Then, by definition, $\varphi_s(k) x \varphi_s(k)^{-1} = x$. By Corollary 2 to Lemma 2-1, there exists $c \in C_X$ such that $\varphi_s(c) = \varphi_s(k)$. Put $h = c^{-1}k$. Then we have $h \in K_s$ and $k = ch$. Therefore we have $I_x \subset C_X \cdot K_s$. Since the inverse inclusion relation is obvious, we have $I_x = C_X \cdot K_s$. q. e. d.

COROLLARY. *We assume that* r *is even. Then we have* $s = r - s = \frac{r}{2}$. *Let* μ *be an irreducible representation of* I_x *which coincides with* $\chi_x^r \cdot 1$ *on* K_{r-s}. *Then* μ *is a one-dimensional representation of* I_x *and there exists a character* ξ *of* C_X *which coincides with* χ_x^r *on* $C_X \cap K_{r-s}$ *such that*

$$\mu(ak) = \xi(a) \chi_x^r(k) \qquad (a \in C_X,\ k \in K_{r-s}).$$

PROOF. By Lemma 2-3, we have $I_x = C_X \cdot K_{\frac{r}{2}}$. Since μ is an irreducible representation which coincides with $\chi_x^r \cdot 1$ on $K_{\frac{r}{2}}$ the restriction of μ to C_X is still irreducible. Corollary 2 to Lemma 2-1 shows that C_X is abelian, hence μ must be 1-dimensional. The restriction of μ to C_X defines a character ξ of C_X. Obviously, ξ coincides with χ_x^r on $C_X \cap K_{\frac{r}{2}}$ and we have

$$\mu(ak) = \xi(a) \chi_x^r(k) \qquad (a \in C_X,\ k \in K_{\frac{r}{2}}). \qquad \text{q. e. d.}$$

Let x be a quasi-regular element of $M(n, \mathfrak{R}_s)$. We take $X \in M(n, \mathfrak{O})$ such that $\varphi_s(X) = x$. For every character ξ of C_X which coincides with χ_x^{2s} on K_s, we put $\mu_{\xi,x}(ak)=\xi(a)\chi_x^{2s}(k)$ $(a\in C_X,\ k\in K_s)$. Then $\mu_{\xi,x}$ defines a one-dimensional representation of $I_x=C_X\cdot K_s$ which coincides with χ_x^{2s} on K_s.

THEOREM 2. $\nu=\operatorname{Ind}_{I_x\uparrow K}\mu_{\xi,x}$ *is a continuous irreducible unitary representation of* K *such that* $r(\nu)=2s$ *and* $x\in O_\nu$. *Conversely every continuous irreducible unitary representation* ν *of* K *such that* $r(\nu)$ *is even and that* O_ν *is quasi-regular can be constructed in this manner.*

PROOF. This theorem follows from Theorem 1 and Corollary to Lemma 2-3.

§ 3. Unitary representations of G induced by irreducible unitary representations of K.

3-1. Let G be the subgroup of $GL(n;P)$ formed by all elements with determinant in $\mathfrak{O}^*$. In this section we denote by $\hat{K}$ the set of equivalence classes of continuous irreducible unitary representations of K. We normalize Haar measure dg on G as follows:

$$\int_K dg=1\,.$$

For every $\nu\in\hat{K}$, we denote by V_ν a representation space of ν. V_ν is a finite dimensional Hilbert space. We denote by $\mathcal{H}_\nu$ the set of V_ν-valued functions on G which satisfy following two conditions:

1. $f(kg)=\nu(k)f(g)$ for every $k\in K$ and every $g\in G$.
2. $\int_G(f(g),\ f(g))dg<\infty$.

(We denote by $(\,,)$ the inner product in V_ν). We define inner product $[\,,]$ in $\mathcal{H}_\nu$ as follows:

$$[f,\ h]=\int_G(f(g),\ h(g))dg \qquad (f,\ h\in\mathcal{H}_\nu)\,.$$

Then $\mathcal{H}_\nu$ becomes a Hilbert space. We define the representation U_ν of G on $\mathcal{H}_\nu$ as follows:

$$(U_\nu(g)f)(g')=f(g'g) \qquad (g,\ g'\in G,\ f\in\mathcal{H}_\nu)\,.$$

Then U_ν is a continuous unitary representation of G. We denote by $U_\nu|K$ the representation of K on $\mathcal{H}_\nu$ obtained by restricting U_ν to K. For every $\mu\in\hat{K}$, we denote by $i(U_\nu|K,\ \mu)$ the multiplicity of μ in $U_\nu|K$.

LEMMA 3-1. *We assume that* $i=i(U_\nu|K,\ \nu)<\infty$. *Then* U_ν *decomposes into direct sum of at most* i *irreducible representations.*

PROOF. We define the linear operator P_ν on $\mathcal{H}_\nu$ as follows:

$$P_\nu f=\dim V_\nu\int_K\overline{\chi_\nu(k)}U_\nu(k)fdk \qquad (f\in\mathcal{H}_\nu)\,,$$

where χ_ν is the character of ν. P_ν is a projection operator. We put

$$W_\nu = \{f \in \mathcal{H}_\nu ;\ P_\nu f = f\}.$$

By the assumption we have $\dim W_\nu = i \dim V_\nu$. Let X be any closed non-zero G-invariant subspace of $\mathcal{H}_\nu$. Let f be any non-zero element of X. Since f is a continuous function on G, there exists $g_0 \in G$ such that $f(g_0) \neq 0$. Then $P_\nu(U_\nu(g_0)f)(e) = f(g_0) \neq 0$. $P_\nu(U(g_0)f)$ is a continuous non-zero function on G and belongs to $W_\nu \cap X$. Thus $W_\nu \cap X \neq \{0\}$. Since X is G-invariant, we have

$$\dim (W_\nu \cap X) \geqq \dim V_\nu .$$

Let $X_1, X_2, \cdots, X_j$ be mutually orthogonal non-zero G-invariant subspaces of $\mathcal{H}_\nu$. Then we have

$$i \dim V_\nu = \dim W_\nu \geqq \sum_{l=1}^{j} \dim (W_\nu \cap X_l) \geqq j \dim V_\nu .$$

Hence we have $i \geqq j$. q. e. d.

COROLLARY. *When $i(U_\nu | K, \nu) = 1$, U_ν is an irreducible unitary representation of G.*

We say that a continuous irreducible unitary representation U of G on a Hilbert space $\mathcal{H}$ is square-integrable if there exists a non-zero element v of $\mathcal{H}$ such that $(U(g)v, v)$ is a square integrable function on G. When U is square integrable, it is known that there exists positive number d which depends only on the equivalence class of U and the normalization of Haar-measure dg of G such that

$$\int_G (U(g)u_1, v_1)\overline{(U(g)u_2, v_2)}dg = \frac{1}{d}(u_1, u_2)\overline{(v_1, v_2)}$$

for every $u_1, u_2, v_1, v_2 \in \mathcal{H}$ (see Godement [2] and Harish-Chandra [3] (ii)). The above relation is called Schur's orthogonality relation and d is called the formal degree of U.

LEMMA 3-2. *We assume that U_ν is irreducible. Then U_ν is a square-integrable representation of G and the formal degree of U_ν is equal to* $\dim V_\nu$.

PROOF. Take $v \in V_\nu$ such that $(v, v) = 1$. We define $f \in \mathcal{H}_\nu$ as follows:

$$f(g) = 0 \quad \text{when} \quad g \notin K$$

and

$$f(g) = \nu(g)v \quad \text{when} \quad g \in K .$$

Then we have $[f, f] = 1$ and

$$\int_G |[U_\nu(g)f, f]|^2 dg = \int_K \left| \int_K (\nu(k_1 k)v,\ \nu(k_1)v) dk_1 \right|^2 dk$$

$$= \int_K |\nu(k)v, v)|^2 dk = \frac{1}{\dim V_\nu} .$$

q. e. d.

LEMMA 3-3. *We assume that U_ν is irreducible and that for every $\mu \in \hat{K}$, $i(U_\nu | K, \mu) < \infty$. For every Schwartz-Bruhat function f on G, we define a linear* *operator* $U_\nu(f)$ *on* $\mathcal{H}_\nu$ *as follows:*

$$U_\nu(f)=\int_G f(g)U_\nu(g)dg\,.$$

Then $U_\nu(f)$ *is of trace-class and we have*

$$\text{Trace } U_\nu(f)=\int_G\Big(\int_K f(gkg^{-1}) \text{ trace } \nu(k)dk\Big)dg\,.$$

PROOF. For every $\mu\in\hat{K}$, we define a projection operator P_μ on $\mathcal{H}_\nu$ as follows:

$$P_\mu=(\dim V_\mu)\int_K \overline{\text{trace }\mu(k)}U_\nu(k)dk\,.$$

We put $\mathcal{H}_\nu(\mu)=P_\mu(\mathcal{H}_\nu)$. Then we have $\mathcal{H}_\nu=\bigoplus_{\mu\in\hat{K}}\mathcal{H}_\nu(\mu)$ (direct sum). By the assumption, $\mathcal{H}_\nu(\mu)$ is finite dimensional. There exists a complete orthonormal base $\{e_i\,;\ i=1, 2, \cdots\}$ of $\mathcal{H}_\nu$ such that every e_i belongs to some $\mathcal{H}_\nu(\mu)$ $(\mu\in\hat{K})$. Let f be a Schwartz-Bruhat function on G. Since f is a uniformly locally constant function on G with compact support, the set of $\mu\in\hat{K}$ such that $U_\nu(f)(\mathcal{H}_\nu(\mu))\neq\{0\}$ is a finite set. Hence the set of $(i, j)\in \boldsymbol{N}\times\boldsymbol{N}$ such that $[U_\nu(f)e_i, e_j]=[e_i, U_\nu(\check{f})e_j]\neq 0$ is a finite set, where we denote by $\boldsymbol{N}$ the set of natural numbers and put $\check{f}(g)=\overline{f(g^{-1})}$. Therefore we have $\sum_{i,j}|[U_\nu(f)e_i, e_j]|<\infty$. Hence $U_\nu(f)$ is of trace-class and we have

$$\text{Trace } U_\nu(f)=\sum_{i=1}^{\infty}[U_\nu(f)e_i, e_i]$$

(see Harish-Chandra [3] (i)).

From the assumption and Lemma 3-2, U_ν is a square-integrable irreducible unitary representation of G and the formal degree of U_ν is $\dim V_\nu$. In the following, we repeat the argument used in the proof of theorem 2 in Harish-Chandra [3], (ii).

Let e be any unit vector in $\mathcal{H}_\nu$. For every $x\in G$, we put $U_\nu^x(f)$ $U_\nu(x)^{-1}U_\nu(f)U_\nu(x)$. We have $[U_\nu^x(f)e, e]=[U_\nu(f)U_\nu(x)e, U_\nu(x)e]=\sum_{i=1}^{\infty}[U_\nu(f)U_\nu(x)e, e_i][e_i, U_\nu(x)e]=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}[U_\nu(f)e_j\,; e_i][U_\nu(x)e, e_j][e_i, U_\nu(x)e]$. The set of $(i, j)\in\boldsymbol{N}\times\boldsymbol{N}$ such that $[U_\nu(f)e_j, e_i][U_\nu(x)e, e_j][e_i, U_\nu(x)e]\neq 0$ is a finite set. Therefore we have, using Schur's orthogonality relation,

$$\begin{aligned}\int_G &[U_\nu^x(f)e, e]dx\\ &=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(U_\nu(f)e_j, e_i]\int_G [U_\nu(x)e, e_j]\overline{[U_\nu(x)e, e_i]}dx\\ &=\frac{1}{\dim V_\nu}\sum_{i=1}^{\infty}[U_\nu(f)e_i, e_i]=\frac{1}{\dim V_\nu}\text{Trace } U_\nu(f)\,.\end{aligned}$$

Let $\{v_i;\ i=1, \cdots, \dim V_\nu\}$ be an orthonormal base of V_ν. We define $f_i \in \mathcal{H}_\nu$ as follows:

$$f_i(g)=\begin{cases} 0, & \text{when } g \notin K. \\ \nu(g)v_i, & \text{when } g \in K. \end{cases}$$

Then f_i $(1 \leqq i \leqq \dim V_\nu)$ is a unit vector in $\mathcal{H}_\nu$. Hence

$$\begin{aligned}\text{Trace } U_\nu(f) &= \dim V_\nu \int_G [U_\nu^{(x)}(f)f_i, f_i]dx \\ &= \dim V_\nu \int_G dx \int_G f(g)[U_\nu(x^{-1}gx)f_i, f_i]dg \\ &= \dim V_\nu \int_G dx \int_G f(xgx^{-1})\Big(\int_G (f_i(hg), f_i(h))dh\Big)dg \\ &= \dim V_\nu \int_G dx \int_K f(xkx^{-1})\Big(\int_K (f_i(hk)v_i, f_i(h)v_i)dh\Big)dk \\ &= \dim V_\nu \int_G dx \int_K f(xkx^{-1})(\nu(k)v_i, v_i)dk\,.\end{aligned}$$

We have

$$\begin{aligned}\text{Trace } U_\nu(f) &= \sum_{i=1}^{\dim V_\nu} \int_G dx \int_K f(xkx^{-1})(\nu(k)v_i, v_i)dk \\ &= \int_G dx \int_K f(xkx^{-1}) \text{ trace } \nu(k)dk\,.\end{aligned}$$

q. e. d.

3-2. We define the subset H_+ of $\tilde{G}=GL(n, P)$ as follows:

$$H_+ = \left\{ h(m_1, m_2, \cdots, m_n) = \begin{pmatrix} \pi^{m_1} & & & \\ & \pi^{m_2} & & \\ & & \ddots & \\ & & & \pi^{m_n} \end{pmatrix};\ \begin{matrix} m_1, \cdots, m_n \in \boldsymbol{Z} \\ m_1 \geqq m_2 \geqq \cdots \geqq m_n \end{matrix} \right\}.$$

For every integer j, we define the subset H_+^j of H_+ as follows:

$$H_+^j = \{h(m_1, m_2, \cdots, m_n);\ m_1 + \cdots + m_n = j\}\,.$$

For every $h \in H_+$, we put $K^h = K \cap h^{-1}Kh$. We denote by ν^h the representation of K^h on V_ν defined as follows:

$$\nu^h(k) = \nu(hkh^{-1})\,.$$

It is well-known that

$$G = \bigcup_{h \in H_+^0} KhK \qquad \text{(disjoint union)}.$$

For every $h \in H_+^0$, we denote by $\mathcal{H}_\nu^h$ the subspace of $\mathcal{H}_\nu$ formed by functions which vanish outside open compact subset KhK of G. Then $\mathcal{H}_\nu^h$ is a closed subspace of $\mathcal{H}_\nu$ and we have

$$\mathcal{H}_\nu = \bigoplus_{h \in H_+^0} \mathcal{H}_\nu^h \qquad \text{(direct sum)}.$$

$\mathcal{H}_\nu^h$ is an invariant subspace of $U_\nu | K$. We denote by $U_\nu(h)|K$ the representation: $k \to U(k)$ of K on $\mathcal{H}_\nu^h$.

LEMMA 3-4. *$U_\nu(h)|K$ is equivalent to* $\operatorname{Ind}_{K^h \uparrow K} \nu^h$ $(h \in H_+^0)$.

PROOF. This is a special case of Mackey's "Subgroup Theorem" (see Curtis-Reiner [1] p. 324).

For every $\mu \in \hat{K}$, and $h \in H_+$, we denote by $\mu | K^h$ the representation of K^h on V_μ obtained by restricting μ to K^h. Let $i(\mu | K^h, \nu^h)$ be the dimension of the vector space formed by linear mappings S of V_ν to V_μ satisfying the following condition:

$$\mu(k)S = S\nu^h(k) \qquad \text{for every } k \in K^h \ (h \in H_+).$$

Using Lemma 3-4 and the Frobenius' reciprocity theorem, we get the following.

LEMMA 3-5. *For every $\mu \in \hat{K}$ and $h \in H_+^0$, we denote by $i(U_\nu(h)|K, \mu)$ the multiplicity of μ in $U_\nu(h)|K$. Then we have $i(U_\nu(h)|K, \mu) = i(\mu | K^h, \nu^h)$.*

COROLLARY. *We have*

$$i(U_\nu | K, \mu) = \sum_{h \in H_+^0} i(\mu | K^h, \nu^h).$$

Put

$$J_\pi = \begin{pmatrix} 0 & 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 1 & 0 \cdots\cdots & 0 \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ \pi & & & & 0 \end{pmatrix} \in GL(n, P).$$

Put $K^{(j)} = J_\pi^{-j} K J_\pi^j$ $(1 \leqq j \leqq n-1)$ and define the representation $\nu^{(j)}$ of $K^{(j)}$ on V_ν and the representation $U_\nu^{(j)}$ of G on $\mathcal{H}_\nu$ as follows:

$$\nu^{(j)}(k) = \nu(J_\pi^j k J_\pi^{-j}) \qquad (k \in K^{(j)}).$$

$$U_\nu^{(j)}(g) = U_\nu(J_\pi^j g J_\pi^{-j}) \qquad (g \in G).$$

LEMMA 3-6[7].

$$G = \bigcup_{h \in H_+^j} K^{(j)} J_\pi^{-j} h K \qquad (\textit{disjoint union}).$$

PROOF. It is well-known that

$$GL(n, P) = \tilde{G} = \bigcup_{h \in H_+} KhK \qquad \text{(disjoint union)}.$$

Hence

7) This lemma was communicated to the author by Professor N. Iwahori to whom the author expresses his hearty thanks.

$$\tilde{G} = \bigcup_{h \in H_+} J_\pi^{-j} K h K = \bigcup_{h \in H_+} K^{(j)} J_\pi^{-j} h K \qquad \text{(disjoint union)}.$$

Since K, $K^{(j)} \subset G$, we have

$$G = \bigcup_{\substack{h \in H_+ \\ J_\pi^{-j} h \in G}} K^{(j)} J_\pi^{-j} h K = \bigcup_{h \in H_+^j} K^{(j)} J_\pi^{-j} h K \qquad \text{(disjoint union)}.$$

q. e. d.

We denote by $\mathcal{H}_{\nu(j)}$ the Hilbert space formed by V_ν-valued functions on G satisfying following conditions:

(i) $$f(kg) = \nu^{(j)}(k) f(g) \qquad (\forall k \in K^{(j)},\ \forall g \in G).$$

(ii) $$\int_G (f(g), f(g)) dg < \infty .$$

We define the representation $\tilde{U}_\nu^{(j)}$ of G on $\mathcal{H}_{\nu(j)}$ as follows:

$$(\tilde{U}_\nu^{(j)}(g) f)(g_0) = f(g_0 g) \qquad (g, g_0 \in G).$$

LEMMA 3-7. *$U_\nu^{(j)}$ and $\tilde{U}_\nu^{(j)}$ are mutually equivalent.*

PROOF. We define the isometric linear mapping T of $\mathcal{H}_\nu$ onto $\mathcal{H}_{\nu(j)}$ as follows:

$$(Tf)(g) = f(J_\pi^j g J_\pi^{-j}) \qquad (f \in \mathcal{H}_\nu).$$

We have for every $g_0 \in G$,

$$T(U_\nu^{(j)}(g_0) f)(g) = f(J_\pi^j (g g_0) J_\pi^{-j}) = \{\tilde{U}_\nu(g_0)(Tf)\}(g).$$

q. e. d.

We denote by $i(\tilde{U}_\nu^{(j)} | K, \mu)$ (resp. $i(U_\nu^{(j)} | K, \mu)$) the multiplicity of μ in the restriction of $\tilde{U}_\nu^{(j)}$ (resp. $U_\nu^{(j)}$) to K.

LEMMA 3-8. *We have*

$$i(\tilde{U}_\nu^{(j)} | K, \mu) = \sum_{h \in H_+^j} i(\mu | K^h, \nu^h).$$

PROOF. Using Lemma 3-6, this lemma can be proved in the same manner as corollary to Lemma 3-5. q. e. d.

COROLLARY. *We have*

$$i(U_\nu^{(j)} | K, \mu) = \sum_{h \in H_+^j} i(\mu | K^h, \nu^h).$$

PROOF. This corollary follows from Lemma 3-7 and Lemma 3-8. q. e. d.

3-3. For every integer j such that $1 \leq j \leq n-1$, we define the subgroup N_j of G as follows:

$$N_j = \left\{ \begin{pmatrix} 1_j & 0 \\ X & 1_{n-j} \end{pmatrix} ;\ X \in M(n-j, j\,;\, P) \right\}.$$

For every $\mu \in \hat{K}$, we denote by $r(\mu)$ the smallest non-negative integer such that μ is trivial on $K_{r(\mu)}=\{1+\pi^{r(\mu)} M(n, \mathfrak{O})\}$. Take $h=h(m_1, m_2, \cdots, m_n)$

$$=\begin{pmatrix} \pi^{m_1} & & & \\ & \pi^{m_2} & & \\ & & \ddots & \\ & & & \pi^{m_n} \end{pmatrix} \in H_+ .$$

LEMMA 3-9. *We assume that* $i(\mu|K^h, \nu^h)>0$ $(\mu \in \hat{K})$ *and that* $m_j-m_{j+1} \geqq r(\mu)-l$ *for some natural numbers* l *and* j $(1\leqq j\leqq n-1)$. *Then the restriction of* ν *to* $N_j \cap K_l$ *contains the identity representation of* $N_j \cap K_l$.

PROOF. By the assumption, there exists a non-zero linear mapping S of V_ν into V_μ such that $\mu(k)S=S\nu^h(k)=S\nu(hkh^{-1})$ for every $k\in K^h=K\cap h^{-1}Kh$. When $k\in N_j\cap K_l$, we have $h^{-1}kh\in K^h\cap K_{l+m_j-m_{j+1}}$. Since $r(\mu)\leqq l+m_j-m_{j+1}$, we have $S\nu(k)=S$ for every $k\in N_j\cap K_l$. Since $S\neq 0$, the restriction of ν to $N_j\cap K_l$ contains the identity representation of $N_j\cap K_l$. q. e. d.

In the following we assume that $r=r(\nu)\geqq 2$ and put $s=\left[\frac{r}{2}\right]$. We recall that χ_x^r $(x\in M(n, \mathfrak{R}_s))$ is the one-dimensional representation of K_{r-s} defined as follows:

$$\chi_x^r(k)=\chi(\pi^{-r} \operatorname{trace} x(k-1)) \qquad (k\in K_{r-s}).$$

We also recall that $(V_\nu)_x$ $(x\in M(n, \mathfrak{R}_s))$ is the subspace of V_ν defined as follows:

$$(V_\nu)_x=\{v\in V_\nu\,;\ \nu(k)v=\chi_x^r(k)v \qquad \text{for any } k\in K_{r-s}\}.$$

By Theorem 1, there exists a subset O_ν of $M(n, \mathfrak{R}_s)$, on which the adjoint action of $GL(n, \mathfrak{R}_s)$ acts transitively and such that $V_\nu=\sum_{x\in O_\nu}(V_\nu)_x$ (direct sum). Let t be an indeterminate. We put

$$\begin{aligned} C_\nu(t) &= \det(t\cdot 1-x) \qquad (x\in O_\nu) \\ &= t^n+\sum_{i=1}^n C_i t^{n-i} \qquad (C_i\in \mathfrak{R}_s) \end{aligned}$$

(Note that $C_\nu(t)$ does not depend on the choice of x.) For every natural number m $(m\leqq s)$, we define the polynomial $\varphi_m\cdot C_\nu$ over $\mathfrak{R}_m$ as follows:

$$(\varphi_m\cdot C)(t)=t^n+\sum_{i=1}^n \varphi_m^s(C_i)t^{n-i}.$$

We say that a polynomial over $\mathfrak{R}_m$ of degree j is monic when the coefficient of t^j is 1. We say that a monic polynomial over $\mathfrak{R}_m$ is irreducible if it cannot be expressed as product of two monic polynomials of lower degrees.

LEMMA 3-10. *We assume that the restriction of* ν *to* $N_j\cap K_{r-m}$ *contains the identity representation. Then* $\varphi_m\cdot C_\nu$ *decomposes into product of two monic polynomials over* $\mathfrak{R}_m$ *whose degrees are* j *and* $n-j$ $(1\leqq j\leqq n-1,\ 1\leqq m\leqq s)$.

PROOF. From the assumption, there exists a non-zero vector v of V_ν such that $\nu(k)v=v$ for every $k\in N_j\cap K_{r-m}$. Put $v=\sum_{x\in O_\nu} v_x$ $(v_x\in (V_\nu)_x)$. From definitions we have

$$\sum_{x\in O_\nu} v_x = v = \nu(k)v = \sum_{x\in O_\nu} \chi_x^r(k)v_x\,,$$

for every $k\in N_j\cap K_{r-m}$. Take an $x\in O_\nu$ such that $v_x\neq 0$. Then we have $\chi_x^r(k)=1$ for all $k\in N_j\cap K_{r-m}$. We write $x=\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4\end{pmatrix}$, where $x_1\in M(j, \mathfrak{R}_s)$, $x_2\in M(j, n-j\,; \mathfrak{R}_s)$, $x_3\in M(n-j, j\,; \mathfrak{R}_s)$ and $x_4\in M(n-j, \mathfrak{R}_s)$. Then $\chi(\pi^{-m}\text{ trace } x_2X)=1$ for all $X\in M(n-j, j\,; \mathfrak{O})$. Hence $\varphi_m^s(x_2)=0$. Thus we have

$$(\varphi_m\cdot C_\nu)(t)=\det(t\cdot 1-\varphi_m^s(x_1))\det(t\cdot 1-\varphi_m^s(x_4))\,.$$

q. e. d.

COROLLARY. *We assume that $\varphi_m(C_\nu(t))$ is irreducible. Then $i(\mu\,|\,K^h, \nu^h)=0$, when $\operatorname*{Max}_{1\leqq i\leqq n-1}(m_i-m_{i+1})\geqq r(\mu)-r(\nu)+m$, where we put $h=h(m_1, m_2, \cdots, m_n)\in H_+$.*

PROOF. This follows from Lemma 3-9 and Lemma 3-10. q. e. d.

LEMMA 3-11. *We assume that $C_\nu(t)$ is irreducible. Then we have $i(U_\nu\,|\,K, \mu)<\infty$ for any $\mu\in\hat{K}$.*

PROOF. From Corollary to Lemma 3-5, we have

$$i(U_\nu\,|\,K, \mu)=\sum_{h\in H_+^0} i(\mu\,|\,K^h, \nu^h)\,.$$

Take an element $h=h(m_1, m_2, \cdots, m_n)\in H_+^0$. We put $d(h)=\operatorname*{Max}_{1\leqq i\leqq n-1}(m_i-m_{i+1})$. When $d(h)\geqq r(\mu)-r(\nu)+s$, we have $i(\mu\,|\,K^h, \nu^h)=0$ by Corollary to Lemma 3-10. Hence $i(\mu\,|\,K^h, \nu^h)=0$ except a finite number of $h\in H_+^0$. Hence $i(U_\nu\,|\,K, \mu)<\infty$.

q. e. d.

LEMMA 3-12. *Let ν, μ be two elements of $\hat{K}$ such that $r=r(\nu)=r(\mu)\geqq 2$. Take an element $h=h(m_1, m_2, \cdots, m_n)$ in H_+. We assume that $m_1-m_n<s$ and $i(\mu\,|\,K^h, \nu^h)>0$. Then there exist $x\in O_\nu$ and $y\in O_\mu$ such that*

$$\varphi_{s-m_1+m_n}(h^{-1}Xh)=\varphi_{s-m_1+m_n}^s(y)$$

for every $X\in M(n, \mathfrak{O})$ satisfying $\varphi_s(X)=x$.

PROOF. From the assumption, there exists a non-zero linear mapping S of V_ν into V_μ such that $\mu(k)S=S\nu(hkh^{-1})$ for every $k\in K^h$. We have $K_{r-s+m_1-m_n}\subset K_{r-s}$ and $hK_{r-s+m_1-m_n}h^{-1}\subset K_{r-s}$.

For every $z\in O_\nu$, take a non-zero $v_z\in(V_\nu)_z$. Then we have $\mu(k)Sv_z=\chi_z^r(hkh^{-1})Sv_z$ for every $k\in K_{r-s+m_1-m_n}$. Since S is a non-zero linear mapping, we can take $x\in O_\nu$ and $v_x\in V_x$ such that $Sv_x\neq 0$. Then there exists $y\in O_\mu$ such that $\chi_y^r(k)=\chi_x^r(hkh^{-1})$ for every $k\in K_{r-s+m_1-m_n}$. We take $X, Y\in M(n, \mathfrak{O})$ such that $\varphi_s(X)=x$ and $\varphi_s(Y)=y$. Put $X=(x_{ij})$ $(x_{ij}\in\mathfrak{O})$, $Y=(y_{ij})$ $(y_{ij}\in\mathfrak{O})$ and $k=1+\pi^{r-s+m_1-m_n}(k_{ij})$ $(k_{ij}\in\mathfrak{O})$. We have

$$\chi(\pi^{m_1-m_n-s}\sum_{i,j=1}^n y_{ij}k_{ji})=\chi(\sum x_{ij}\pi^{m_1-m_n+m_j-m_i-s}k_{ji})$$

for every $k_{ij}\in\mathfrak{O}$.

This means that $y_{ij}-x_{ij}\pi^{m_j-m_i}\in\pi^{s-m_1+m_n}\mathfrak{O}$ $(1\leqq i, j\leqq n)$. Since $s>m_1-m_n$, we have $h^{-1}Xh\in M(n, \mathfrak{O})$ and $\varphi_{s-(m_1-m_n)}(h^{-1}Xh)=\varphi_{s-(m_1-m_n)}(Y)=\varphi^s_{s-(m_1-m_n)}(y)$.

q. e. d.

3-4. Let $Q(t)=t^n+\sum_{i=1}^{n}C_it^{n-i}$ $(C_i\in\mathfrak{O})$ be a monic polynomial over $\mathfrak{O}$ of degree n. For every natural number m, we put $(\varphi_m\cdot Q)(t)=t^n+\sum_{i=1}^{n}\varphi_m(C_i)t^{n-i}$. $\varphi_m\cdot Q$ is a monic polynomial of degree n over $\mathfrak{R}_m$.

LEMMA 3-13. *We assume that $Q(t)$ is irreducible. Then there exists a natural number m_0 such that $\varphi_m\cdot Q$ is irreducible as a monic polynomial over $\mathfrak{R}_m$ whenever $m\geqq m_0$.*

PROOF. For every natural number m, we denote by n_m the minimum of degrees of monic polynomials over $\mathfrak{R}_m$ which divide $\varphi_m\cdot Q$. It is obvious that $n_1\leqq n_2\leqq\cdots\leqq n_m\leqq\cdots\leqq n$. Hence there exists a natural number m_0 such that $n_{m_0}=n_{m_0+1}=\cdots$. Assume that $n'=n_{m_0}<n$. For every natural number m $(m\geqq m_0)$, there exist $d_{1,m}, d_{2,m},\cdots, d_{n',m}; d'_{1,m}, d'_{2,m},\cdots. d'_{n-n',m}\in\mathfrak{O}$ such that

$$\varphi_m\cdot Q(t)=\varphi_m((t^{n'}+\sum_{i=1}^{n'}d_{i,m}t^{n'-i})(t^{n-n'}+\sum_{j=1}^{n-n'}d'_{j,m}t^{n-n'-j}))\,.$$

There exists a subsequence $\{m_s; s=1, 2,\cdots\}$ of $\{m; m\geqq m_0\}$ such that $\lim_{s\to\infty}d_{i,m_s}$ and $\lim_{s\to\infty}d'_{j,m_s}$ exist for every i and j $(1\leqq i\leqq n', 1\leqq j\leqq n-n')$. Put $d_i=\lim_{s\to\infty}d_{i,m_s}$ and $d'_j=\lim_{s\to\infty}d'_{j,m_s}$. We have

$$Q(t)=(t^{n'}+\sum_{i=1}^{n'}d_it^{n'-i})(t^{n-n'}+\sum_{j=1}^{n-n'}d'_jt^{n-n'-j})\,.$$

This contradicts the assumption that $Q(t)$ is irreducible. Hence we must have $n'=n$. $\varphi_m\cdot Q$ is irreducible whenever $m\geqq m_0$. q. e. d.

LEMMA 3-14. *Let X be an element of $M(n, \mathfrak{O})$ such that the centralizer of X in G is contained in K and that the characteristic polynomial of X is irreducible. Then there exists an open neighbourhood $\mathfrak{U}$ of X in $M(n, \mathfrak{O})$ such that*

$$\mathfrak{U}\cap\{gXg^{-1}; g\in G\}=\mathfrak{U}\cap\{kXk^{-1}; k\in K\}\,.$$

PROOF. For every $h\in H^0_+$, we put

$$S^h_X=\{gXg^{-1}; g\in KhK\}\,.$$

Since the centralizer of X in G is contained in K, we have

$$\{gXg^{-1}; g\in G\}=\bigcup_{h\in H^0_+}S^h_X \qquad \text{(disjoint union)}.$$

Put $C_X(t)=\det(t\cdot 1-X)$. Since $C_X(t)$ is an irreducible monic polynomial over $\mathfrak{O}$ by the assumption, there exists a natural number m such that $\varphi_m\cdot C_X$ is irreducible, by Lemma 3-13.

Put $d(h)=\underset{1\leqq i\leqq n-1}{\mathrm{Max}}(m_i-m_{i+1})$ for every $h=h(m_1, m_2, \cdots, m_n)\in H_+^0$. Let us show that $S_X^h\cap M(n, \mathfrak{O})=\phi$ for every $h\in H_+^0$ satisfying $d(h)\geqq m$. Suppose there exists an h such that $d(h)\geqq m$ and that $S_X^h\cap M(n, \mathfrak{O})\neq\phi$. There exist $k_1, k_2\in K$ such that $k_1hk_2Xk_2^{-1}h^{-1}k_1^{-1}=Y\in M(n, \mathfrak{O})$. Take i such that $m_i-m_{i+1}=d(h)$ and write $k_1^{-1}Yk_1=hk_2Xk_2^{-1}h^{-1}=\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4\end{pmatrix}$, $(z_1\in M(i, \mathfrak{O}), z_2\in M(i, n-i; \mathfrak{O})$, $z_3\in M(n-i, i; \mathfrak{O})$ and $z_4\in M(n-i, \mathfrak{O}))$. Then we have $\varphi_m(z_2)=0$. Hence we have $\det(t\cdot 1-\varphi_m(hk_2Xk_2^{-1}h^{-1}))=\det(t\cdot 1-\varphi_m(z_1))\det(t\cdot 1-\varphi_m(z_4))$. On the other hand, we have

$$\begin{aligned}\det(t\cdot 1-\varphi_m(hk_2Xk_2^{-1}h^{-1}) &=\varphi_m(\det(t\cdot 1-hk_2Xk_2^{-1}h^{-1}))\\ &=\varphi_m(\det(t\cdot 1-X))=\varphi_m\cdot C_X(t).\end{aligned}$$

This contradicts the fact that $\varphi_m\cdot C_X$ is irreducible. Hence we have $S_X^h\cap M(n, \mathfrak{O})=\phi$, when $d(h)\geqq m$. Since $\{S_X^h; h\in H_+^0, d(h)<m\}$ is a finite family of mutually disjoint compact subsets of $M(n, P)$ and $S_X^1\subset M(n, \mathfrak{O})$, there exists an open subset $\mathfrak{U}$ of S_X^1 in $M(n, \mathfrak{O})$ such that $\mathfrak{U}\cap S_X^h=\phi$ whenever $h\neq 1$ and $d(h)<m$. When $d(h)\geqq m$, we have $\mathfrak{U}\cap S_X^h=\phi$. Hence $\mathfrak{U}\cap S_X^h=\phi$ whenever $h\neq 1$. We have

$$\mathfrak{U}\cap\{gXg^{-1}; g\in G\}=\mathfrak{U}\cap\{\bigcup_{h\in H_+^0} S_X^h\}=\mathfrak{U}\cap S_X^1=\mathfrak{U}\cap\{kXk^{-1}; k\in K\}.$$

q. e. d.

For every element $X\in M(n, \mathfrak{O})$ such that $\varphi_1(x)\neq 0$ and every natural number l (we assume $l\geqq 2$), we put $I_X^l=\{k\in K; \varphi_{[\frac{l}{2}]}(kXk^{-1})=\varphi_{[\frac{l}{2}]}(X)\}$. We define the one-dimensional representation χ_X^l of $K_{l-[\frac{l}{2}]}$ as follows:

$$\chi_X^l(k)=\chi(\pi^{-l}\operatorname{trace} X(k-1)) \qquad (k\in K_{l-[\frac{l}{2}]}).$$

We denote by π_X^l the set of equivalence classes of irreducible unitary representations λ of I_X^l satisfying $\lambda(k)=\chi_X^l(k)\cdot 1$ for every $k\in K_{l-[\frac{l}{2}]}$. Put $\nu_\lambda=\underset{I_X^l\uparrow K}{\mathrm{Ind}}\,\lambda$. From Theorem 1, ν_λ is an irreducible unitary representation of K. We have $r(\nu_\lambda)=l$ and $O_{\nu_\lambda}=\{\varphi_{[\frac{l}{2}]}(kXk^{-1}); k\in K\}$. We also have $C_{\nu_\lambda}(t)=\det(t\cdot 1-\varphi_{[\frac{l}{2}]}(X))$.

THEOREM 3. *Take $X\in M(n, \mathfrak{O})$ such that $\varphi_1(X)\neq 0$. We assume that the centralizer of X in G is contained in K and that the characteristic polynomial of X is irreducible*[8].

There exists a natural number l_0 such that for every $l\geqq l_0$ and for every λ in π_X^l, $U_{\nu_\lambda}=\underset{K\uparrow G}{\mathrm{Ind}}\,\nu_\lambda$ is an irreducible untary representation of G.

PROOF. From Lemma 3-13, there exists a natural number j_1 such that

8) An example of such X will be given in Lemma 4-5.

$\det(t\cdot 1-\varphi_{j_1}(X))$ is an irreducible monic polynomial over $\mathfrak{R}_{j_1}$. From Lemma 3-14, there exists a natural number j_2 such that

$$\{X+z\,;\,z\in\pi^{j_2}M(n,\mathfrak{O})\}\cap\{gXg^{-1}\,;\,g\in G\}\subset\{kXk^{-1}\,;\,k\in K\}\,.$$

Put $l_0=2\{(n-1)j_1+j_2\}$. We assume that $l\geqq l_0$ and $\lambda\in\pi_X^l$. From corollary to Lemma 3-5, we have $i(U_{\nu_\lambda}|K,\nu_\lambda)=\sum_{h\in H_+^0}i(\nu_\lambda|K^h,\nu_\lambda^h)$. Take $h=h(m_1,m_2,\cdots,m_n)\in H_+^0$. Since $\varphi_{j_1}^{[\frac{l}{2}]}(C_{\nu_\lambda}(t))=\det(t\cdot 1-\varphi^{j_1}(X))$ is irreducible $\left(\left[\frac{l}{2}\right]\geqq\left[\frac{l_0}{2}\right]=(n-1)j_1+j_2>j_1\right)$, we have $i(\nu_\lambda|K^h,\nu_\lambda^h)=0$ when $\operatorname{Max}_{1\leqq i\leqq n-1}(m_i-m_{i+1})\geqq j_1$, by corollary to Lemma 3-10. Now we assume that $\operatorname{Max}_{1\leqq i\leqq n-1}(m_i-m_{i+1})<j_1$ and $i(\nu_\lambda|K^h,\nu_\lambda^h)>0$. We have $m_1-m_n=\sum_{i=1}^{n-1}(m_i-m_{i+1})<(n-1)j_1<\left[\frac{l_0}{2}\right]\leqq\left[\frac{l}{2}\right]$. From Lemma 3-12, there exist $Y_1,Y_2\in M(n,\mathfrak{O})$ such that $\varphi_{[\frac{l}{2}]}(Y_1),\varphi_{[\frac{l}{2}]}(Y_2)\in O_{\nu_\lambda}$ and that $\varphi_{[\frac{l}{2}]-m_1+m_n}(h^{-1}Y_1h)=\varphi_{[\frac{l}{2}]-m_1+m_n}(Y_2)$. Since $O_{\nu_\lambda}=\{\varphi_{[\frac{l}{2}]}(kXk^{-1})\,;\,k\in K\}$, there exist $k_1,k_2\in K$ such that

$$\varphi_{[\frac{l}{2}]}(k_iXk_i^{-1})=\varphi_{[\frac{l}{2}]}(Y_i)\qquad(i=1,2)\,.$$

Therefore we have

$$h^{-1}k_1Xk_1^{-1}h\in M(n,\mathfrak{O})$$

and

$$\varphi_{[\frac{l}{2}]-m_1+m_n}(h^{-1}k_1Xk_1^{-1}h)=\varphi_{[\frac{l}{2}]-m_1+m_n}(k_2Xk_2^{-1})\,.$$

Since $\left[\frac{l}{2}\right]-m_1+m_n\geqq\left[\frac{l_0}{2}\right]-m_1+m_n>j_2$, we have

$$k_2^{-1}h^{-1}k_1Xk_1^{-1}hk_2\in\{gXg^{-1}\,;\,g\in G\}$$
$$\cap\{X+z\,;\,z\in\pi^{j_2}M(n,\mathfrak{O})\}\subset\{kXk^{-1}\,;\,k\in K\}\,.$$

Hence there exists $k_3\in K$ such that $k_2^{-1}h^{-1}k_1Xk_1^{-1}hk_2=k_3Xk_3^{-1}$. Since the centralizer of X in G is contained in K, we have

$$k_3^{-1}k_2^{-1}h^{-1}k_1\in K\quad\text{and}\quad h\in H_+^0\cap K=1\,.$$

Thus we have proved that $i(\nu_\lambda|K^h,\nu_\lambda^h)=0$ unless $h=1$. Hence we have $i(U_{\nu_\lambda}|K,\nu_\lambda)=i(\nu_\lambda|K^1,\nu_\lambda^1)=1$. Hence U_{ν_λ} is irreducible by corollary to Lemma 3-1. q. e. d.

COROLLARY 1. *U_{ν_λ} is square-integrable and the formal degree of U_{ν_λ} is equal to the dimension of the representation space of ν_λ.*

PROOF. This follows from Lemma 3-2. q. e. d.

COROLLARY 2. *For every Schwartz-Bruhat function f on G we put $U_{\nu_\lambda}(f)=\int_G f(g)U_{\nu_\lambda}(g)dg$. Then U_{ν_λ} is of trace-class and we have*

$$\text{Trace } U_{\nu_\lambda}(f)=\int_G dg\int_K f(gkg^{-1}) \text{ trace } \nu_\lambda(k)dk\,.$$

PROOF. This follows from Lemma 3-11 and Lemma 3-3. q. e. d.

§ 4. Irreducible unitary representations of G parametrized by certain characters of compact Cartan subgroups of G.

4-1. We call a maximal abelian subgroup A of G a Cartan subgroup when every element of A is semi-simple. For every Cartan subgroup A, we denote by $\mathcal{A}$ the subalgebra of $M(n, P)$ generated by elements of A.

LEMMA 4-1. *When A is a Cartan subgroup of G, $\mathcal{A}$ is a maximal abelian semi-simple subalgebra of $M(n, P)$ and we have $A=\mathcal{A}\cap G$.*

PROOF. From the definition of $\mathcal{A}$, we have

$$\mathcal{A}=\{\sum_{i=1}^{l} x_i a_i\,;\, x_1, \cdots, x_l\in P\,;\, a_1, \cdots, a_l\in A\}\,.$$

Since A is commutative and every element of A is semi-simple, $\mathcal{A}$ is a commutative semi-simple subalgebra of $M(n, P)$. Take an element x of $M(n, P)$ which commutes with $\mathcal{A}$. Take a natural number l such that $\pi^l x\in M(n, \mathfrak{O})$. Then it is obvious that $1+\pi^{l+1}x\in G$ and that $1+\pi^{l+1}x$ commute with A. Since A is maximal abelian in G, we have $1+\pi^{l+1}x\in A$ and $x\in\mathcal{A}$. Hence $\mathcal{A}$ is a maximal abelian subalgebra of $M(n, P)$. It is obvious that $A\subset G\cap\mathcal{A}$. Since A is maximal abelian in G and every element of $\mathcal{A}\cap G$ commutes with A, we have $\mathcal{A}\cap G\subset A$. Hence $A=\mathcal{A}\cap G$. q. e. d.

LEMMA 4-2. *When A is a compact Cartan subgroup of G, $\mathcal{A}$ is an extension field of P of degree n and A is the unit group of $\mathcal{A}$.*

PROOF. From Lemma 4-1, $\mathcal{A}$ is a commutative semi-simple subalgebra of $M(n, P)$. From Dedekind's theorem, $\mathcal{A}$ is a direct sum of extensions of $P:\mathcal{A}=\bigoplus_{i=1}^{r}\mathcal{A}_i$, where $\mathcal{A}_i$ is an extension of P. Let $1=\sum_{i=1}^{r}1_i$ $(1_i\in\mathcal{A}_i)$ be the decomposition of the unit element of $\mathcal{A}$. We put $(P)^n=M(n, 1\,; P)$, $(P)^n_i=1_i\cdot(P)^n$ and $n_i=\dim(P)^n_i$ $(1\leqq i\leqq r)$. We assume that $r\geqq 2$. Then we have $n_1, n_2>0$.

Put $\alpha_l=\pi^{-ln_2}\cdot 1_1+\pi^{ln_1}\cdot 1_2+\sum_{i=3}^{r}1_i$. Obviously we have $\alpha_l\in\mathcal{A}\cap G=A$ for every integer l. Since A is compact, there exists a sequence $l_1<l_2<l_3<\cdots$ of natural numbers such that $\lim_{j\to\infty}\alpha_{l_j}$ exists. Take $0\neq v\in(P)^n_1$, then we have $\alpha_{l_j}v=\pi^{-l_jn_2}v$. But $\lim_{j\to\infty}\pi^{-l_jn_2}v$ does not exist unless $n_2=0$. Thus we get a contradiction.

We have $r=1$ and $\mathcal{A}=\mathcal{A}_1$ is an extension of P. Since $\mathcal{A}$ is maximal abelian in $M(n, P)$, the degree of $\mathcal{A}$ over P is n. For every $\alpha\in\mathcal{A}$, we have $\det\alpha=\text{Norm}\,\mathcal{A}/P(\alpha)$. Hence

$$A=\{\alpha\in\mathcal{A};\ \det\alpha\in\mathfrak{O}^*\}$$
$$=\{\alpha\in\mathcal{A};\ \mathrm{Norm}_{\mathcal{A}/P}\alpha\in\mathfrak{O}^*\},$$

so A is the unit group of $\mathcal{A}$.

LEMMA 4-3. *Let $\mathcal{A}$ be a subalgebra of $M(n, P)$ which is an extension of P of degree n. Then $A=\mathcal{A}\cap G$ is a compact Cartan subgroup of G and $\mathcal{A}$ is generated by elements of A.*

PROOF. From the assumption, it is easily proved that $\mathcal{A}$ is a maximal abelian semi-simple subalgebra of $M(n, P)$ and that $\mathcal{A}\cap G$ is the unit group of $\mathcal{A}$. Therefore it is obvious that A is compact maximal abelian subgroup of G whose elements are semi-simple. We can take an element a of the unit group of $\mathcal{A}$ such that $\mathcal{A}=P[a]$. Then $a\in A$ and a generates $\mathcal{A}$. q. e. d.

We put $\tilde{G}=GL(n, P)$. Two subgroups L_1 and L_2 of G are said to be $\tilde{G}$-conjugate if there exists an element $g\in\tilde{G}$ such that $gL_1g^{-1}=L_2$. Two extensions P_1 and P_2 of P are said to be P-conjugate if there exists an isomorphism which maps P_1 onto P_2 and fixes every element of P.

LEMMA 4-4. *Let A_1 and A_2 be two compact Cartan subgroups of G. A_1 and A_2 are $\tilde{G}$-conjugate if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ are P-conjugate.*

PROOF. Assume that A_1 and A_2 are $\tilde{G}$-conjugate, then there exists an element g of $\tilde{G}$ such that $A_2=gA_1g^{-1}$. We have $\mathcal{A}_2=g\mathcal{A}_1g^{-1}$ and the isomorphism: $\alpha\to g\alpha g^{-1}$ $(\alpha\in\mathcal{A}_1)$ maps $\mathcal{A}_1$ onto $\mathcal{A}_2$ and fixes every element of P. Hence $\mathcal{A}_1$ and $\mathcal{A}_2$ are P-conjugate. Conversely, we assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are P-conjugate. There exists an isomorphism σ which maps $\mathcal{A}_1$ onto $\mathcal{A}_2$ and

$$\sigma\begin{pmatrix} x & & & \\ & x & & \\ & & \ddots & \\ & & & x\end{pmatrix}=\begin{pmatrix} x & & & \\ & x & & \\ & & \ddots & \\ & & & x\end{pmatrix}\quad \text{for every } x\in P.$$

Take an element $\alpha_1\in\mathcal{A}_1$ such that $\mathcal{A}_1=P[\alpha_1]$. We denote by C_{α_1} the minimal polynomial of α_1 over P. Then C_{α_1} is of degree n and irreducible. C_{α_1} is the minimal polynomial of $\sigma(\alpha_1)$ and characteristic polynomials of α_1 and α_2 are identical with C_{α_1}. Hence there exists $g\in G$ such that $\sigma(\alpha_1)=g\alpha_1g^{-1}$. Since $\mathcal{A}_1=P[\alpha_1]$ and $\mathcal{A}_2=\sigma\mathcal{A}_1=P[\sigma(\alpha_1)]$, it is obvious that $\sigma\alpha=g\alpha g^{-1}$ for every $\alpha\in\mathcal{A}_1$. Hence A_1 and A_2 are $\tilde{G}$-conjugate. q. e. d.

Since every extension of P of degree n can be isomorphically imbedded in $M(n, P)$, we have the following proposition from Lemma 4-1$\sim$Lemma 4-4.

PROPOSITION 4-1[9]. *There exists one to one correspondence between the set of $\tilde{G}$-conjugate classes of compact Cartan subgroups of G and the set of P-*

9) This proposition is communicated to the author by Dr. H. Hijikata and Dr. Y. Ihara to whom the author expresses his hearty gratitude.

conjugate classes of extensions of P of degree n. Every compact Cartan subgroup of G is isomorphic to the unit group of the corresponding extension of P.

4-2. We denote by $\mathcal{E}(P, n)$ a complete set of representatives of P-conjugate classes of extensions of P of degree n. It is known that $\mathcal{E}(P, n)$ is a finite set.

For every element $\tilde{P}_i$ of $\mathcal{E}(P, n)$ we take an injective homomorphism τ_i of $\tilde{P}_i$ into $M(n, P)$ as algebras over P. Put $A_i = G \cap \tau_i(\tilde{P}_i)$. Then A_i is a compact Cartan subgroup of G and $\{A_i\,;\ \tilde{P}_i \in \mathcal{E}(P, n)\}$ is a complete set of representatives of $\tilde{G}$-conjugate classes of compact Cartan subgroups of G.

LEMMA 4-5. *Assume that $\tau_i(\tilde{\mathfrak{O}}_i) \subset M(n, \mathfrak{O})$, where $\tilde{\mathfrak{O}}_i$ is the ring of integral elements of $\tilde{P}_i$. Let x be any integral regular element of $\tilde{P}_i$*[10] *and write $X = \tau_i(x) \in M(n, \mathfrak{O})$. Then the centralizer of X in G is contained in K and the characteristic polynomial of X is irreducible.*

PROOF. Since $x \in \tilde{P}_i$ is regular, the minimal polynomial of x over P is of degree n and coincides with the characteristic polynomial of $X = \tau_i(x)$. Therefore the characteristic polynomial of X is irreducible. As $X = \tau_i(x)$ generates $\tau_i(\tilde{P}_i)$, every element of $M(n, P)$ which commutes with X lies in $\tau_i(\tilde{P}_i)$. Hence the centralizer of X in G is $\tau_i(\tilde{P}_i) \cap G = \tau_i(\tilde{\mathfrak{O}}_i^*) \subset K$. q. e. d.

In the following, we assume that p (the characteristic of the residue class field $\mathfrak{O}/\mathfrak{P}$) is prime to n. We construct for every $\tilde{P}_i \in \mathcal{E}(P, n)$ an injective homomorphism τ_i which is convenient for our applications. We omit the index i.

Denote by $\tilde{\mathfrak{O}}$ and $\tilde{\mathfrak{P}}$ the ring of integral elements of $\tilde{P}$ and the maximal ideal of $\tilde{\mathfrak{O}}$ respectively. Denote by e and f the ramification index and the modular degree of $\tilde{P}$ over P respectively. We denote by $P^{(f)}$ the unramified extension of P of degree f. $P^{(f)}$ is a subfield of $\tilde{P}$. We put $\mathfrak{O}^{(f)} = \tilde{\mathfrak{O}} \cap P^{(f)}$ and $\mathfrak{P}^{(f)} = \tilde{\mathfrak{P}} \cap P^{(f)}$. Since p is prime to n, it is known that there exists a prime element $\tilde{\pi}$ of $\tilde{\mathfrak{P}}$ such that $\tilde{\pi}^e$ lies in $\mathfrak{O}^{(f)}$ and generates $\mathfrak{P}^{(f)}$ in $\mathfrak{O}^{(f)}$. (See Lang's book [4] p. 38). Take such a $\tilde{\pi}$. Then we have

$$\tilde{P} = P^{(f)}[\tilde{\pi}] \quad \text{and} \quad \tilde{\mathfrak{O}} = \mathfrak{O}^{(f)}[\tilde{\pi}]\,.$$

$\{1, \tilde{\pi}, \cdots, \tilde{\pi}^{e-1}\}$ is a $P^{(f)}$-base of $\tilde{P}$. The regular representation of $\tilde{P}$ with respect to this base defines an injective homomorphism ι' of $\tilde{P}$ into $M(e, \mathfrak{P}^{(f)})$. We have

$$\begin{pmatrix} x \\ x\tilde{\pi} \\ \vdots \\ x\tilde{\pi}^{e-1} \end{pmatrix} = \iota'(x) \begin{pmatrix} 1 \\ \tilde{\pi} \\ \vdots \\ \tilde{\pi}^{e-1} \end{pmatrix} \qquad \text{for every } x \in \tilde{P}\,.$$

We denote by $\mathrm{Gal}(P^{(f)}/P)$ the Galois group of $P^{(f)}$ with respect to P. $\mathrm{Gal}(P^{(f)}/P)$ is a cyclic group of order f. Let σ be a generator of this group.

10) We call an element x of $\tilde{P}_i$ regular if $\tilde{P} = \tilde{P}[x]$.

For every natural number m, put $\mathfrak{R}_m^{(f)}=\mathfrak{O}^{(f)}/(\mathfrak{P}^{(f)})^m$. $\mathfrak{R}_m^{(f)}$ is a finite ring and $\mathrm{Gal}\,(P^{(f)}/P)$ operates naturally on this ring. The homomorphism φ_m of $\mathfrak{R}$ onto $\mathfrak{R}_m$ extends naturally to the homomorphism of $\mathfrak{O}^{(f)}$ onto $\mathfrak{R}_m^{(f)}$. We denote this extension by the same symbol. We also denote by φ_m the naturally defined mapping of matrices over $\mathfrak{O}^{(f)}$ onto matrices over $\mathfrak{R}_m^{(f)}$. Put

$$J=\begin{pmatrix} 0 & 1_e & 0 & \cdots\cdots & 0 \\ 0 & 0 & 1_e & 0\cdots\cdots & 0 \\ \vdots & & & 1_e & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1_e \\ 1_e & & & & 0 \end{pmatrix},$$

where 1_e is the e by e identity matrix. We define the P-submodule $\mathfrak{M}$ of $M(n, P^{(f)})$ as follows:

$$\mathfrak{M}=\{x\in M(n, P^{(f)})\,;\ \sigma x=JxJ^{-1}\}\,.$$

Put $\mathfrak{M}_0=\mathfrak{M}\cap M(n, \mathfrak{O}^{(f)})$ and for every natural number m, put $\mathfrak{K}_m=\mathfrak{M}\cap\{1+\pi^m\mathfrak{M}_0\}$. Take an element $g_0\in GL(n, \mathfrak{O}^{(f)})$ such that $\sigma g_0=Jg_0$.

Take an element $x\in\mathfrak{O}^{(f)}$ such that $\varphi_1(\prod_{k=1}^{f-1}(x-\sigma^k x))\neq 0$. Put

$$h_0=\begin{pmatrix} 1\ x\ x^2 \cdots x^{f-1} & & & \\ & 1\ x\ x^2\cdots x^{f-1} & & \\ & & \ddots & \\ & & & 1\ x\ x^2\cdots x^{f-1} \end{pmatrix}\in M(e, n\,;\mathfrak{O}^{(f)})\,.$$

We can put

$$g_0=\begin{pmatrix} h_0 \\ \sigma h_0 \\ \vdots \\ \sigma^{f-1}h_0 \end{pmatrix}.$$

We denote by κ the automorphism of $M(n, P^{(f)})$ defined as follows:

$$\kappa(X)=g_0^{-1}Xg_0 \qquad (X\in M(n, P^{(f)}))\,.$$

We have $\kappa(\mathfrak{M})=M(n, P)$, $\kappa(\mathfrak{M}_0)=M(n, \mathfrak{O})$ and $\kappa(\mathfrak{K}_m)=K_m$ $(m=1, 2, \cdots)$. We put $\mathfrak{K}=\kappa^{-1}(K)$. Now we define the injective homomorphism ι of $\tilde{P}$ into $\mathfrak{M}$ (as algebras over P) as follows:

$$\iota(x)=\begin{pmatrix} \iota'(x) & & & \\ & \sigma\iota'(x) & & \\ & & \ddots & \\ & & & \sigma^{f-1}\iota'(x) \end{pmatrix} \qquad (x\in\tilde{P})\,.$$

Put $\tau=\kappa\cdot\iota$. Then τ is an injective homomorphism of $\tilde{P}$ into $M(n, P)$. When $x\in\tilde{\mathfrak{O}}$, we have $\iota'(x)\in M(e, \mathfrak{O}^{(f)})$, $\iota(x)\in\mathfrak{M}_0$ and $\tau(x)\in M(n, \mathfrak{O})$. Hence we have $\tau(\tilde{\mathfrak{O}})\subset M(n, \mathfrak{O})$. Put $A=\tau(\tilde{P})\cap G=\tau(\tilde{\mathfrak{O}}^*)$ and $\mathfrak{A}=\iota(\tilde{\mathfrak{O}}^*)$. A is a compact Cartan subgroup of G contained in K.

Since $\{1, \tilde{\pi}, \tilde{\pi}^2\ \cdots, \tilde{\pi}^{e-1}\}$ is an $\mathfrak{O}^{(f)}$-base of $\tilde{\mathfrak{O}}$, for every $x\in\tilde{\mathfrak{O}}$, there exist

$\mu_1, \mu_2, \cdots, \mu_e \in \mathfrak{O}^{(f)}$ such that $x = \sum_{j=1}^{e} \mu_j \tilde{\pi}^{j-1}$. Put $\Pi = \tilde{\pi}^e$. From the assumption, we have $\Pi \in \mathfrak{P}^{(f)}$.

LEMMA 4-6. *We have*

$$\iota'(x) = \sum_{j=1}^{e} \mu_j \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 \ddots & \\ & & & \ddots & 1 \\ \Pi & & & & 0 \end{pmatrix}^{j-1}.$$

Putting $\bar{\mu}_j = \varphi_1(\mu_j)$ $(1 \leq j \leq e)$, *we have*

$$\varphi_1(\iota'(x)) = \begin{pmatrix} \bar{\mu}_1 & \bar{\mu}_2 & \cdots\cdots & \bar{\mu}_e \\ & \bar{\mu}_1 & \bar{\mu}_2 & \vdots \\ & & \ddots & \bar{\mu}_2 \\ 0 & & & \bar{\mu}_1 \end{pmatrix}.$$

PROOF. This lemma follows from the definition of ι'.

LEMMA 4-7. *Put* $x = \sum_{j=1}^{e} \mu_j \tilde{\pi}^{j-1} \in \tilde{\mathfrak{O}}$ $(\mu \in \mathfrak{O}^{(f)})$. *Then* $\tilde{\mathfrak{O}} = \mathfrak{O}[x]$ *if and only if the following two conditions are satisfied:*

(i) $\varphi_1(\mu_1) \neq \varphi_1(\sigma^k \mu_1)$ $(k = 1, 2, \cdots, f-1)$,

(ii) $\mu_2 \in \mathfrak{O}^{(f)*}$ (*when* $e > 1$).

PROOF. Assume that $\tilde{\mathfrak{O}} = \mathfrak{O}[x]$.

Put $\tilde{\mathfrak{R}} = \tilde{\mathfrak{O}}/\tilde{\mathfrak{P}}$ and denote by Φ the natural projection of $\tilde{\mathfrak{O}}$ onto $\tilde{\mathfrak{R}}$. We have

$$\mathfrak{R}_1^{(f)} = \tilde{\mathfrak{R}} = \mathfrak{R}_1[\Phi_1(x)] = \mathfrak{R}_1[\varphi_1(\mu_1)].$$

Hence $\varphi_1(\sigma^k(\mu_1)) \neq \varphi_1(\mu_1)$ $(k = 1, 2, \cdots, f-1)$. We further assume that $e > 1$. There exist $c_0, c_1, \cdots, c_k \in \mathfrak{O}$ such that $\tilde{\pi} = \sum_{j=0}^{k} c_j x^j$. We have $\sum_{j=0}^{k} c_j \mu_1^j \in \tilde{\mathfrak{P}} \cap \mathfrak{O}^{(f)} = \mathfrak{P}^{(f)} \subset \tilde{\mathfrak{P}}^e$. On the other hand, we have $\tilde{\pi} - \sum_{j=0}^{k} c_j(\mu_1^j + j\mu_1^{j-1}\mu_2\tilde{\pi}) \in \tilde{\mathfrak{P}}^2$. Therefore we have

$$\tilde{\pi}\{1 - \mu_2 \sum_{j=0}^{k} j c_j \mu_1^{j-1}\} \in \tilde{\mathfrak{P}}^2.$$

Hence $\mu_2 \in \mathfrak{O}^{(f)*}$.

Conversely we assume that conditions (i) and (ii) are satisfied. It follows from (i) that $\tilde{\mathfrak{R}} = \mathfrak{R}_1^{(f)} = \mathfrak{R}_1[\Phi_1(x)]$.

Put $Q(t) = \prod_{k=0}^{f-1} (t - \sigma^k \mu_1)$ (t is an indeterminate). $Q(t)$ is a polynomial over $\mathfrak{O}$. From assumptions, we have $Q(x) \in \tilde{\mathfrak{P}}$ and $Q(x) \notin \tilde{\mathfrak{P}}^2$. Now it is obvious that $\tilde{\mathfrak{O}} = \mathfrak{O}[x]$. q. e. d.

COROLLARY 1. *We assume that* $\tilde{\mathfrak{O}} = \mathfrak{O}[x]$ $(x \in \tilde{\mathfrak{O}})$. *Then the minimal polynomial of* $\varphi_1(\tau(x))$ *over* $\mathfrak{R}_1$ *is the characteristic polynomial of* $\varphi_1(\tau(x))$ *which is the e-th power of an irreducible polynomial over* $\mathfrak{R}_1$ *of degree f.*

PROOF. Put $x=\sum_{j=1}^{e}\mu_j\tilde{\pi}^{j-1}$ ($\mu_j\in\mathfrak{O}^{(f)}$) and put $\bar{\mu}_j=\varphi_1(\mu_j)$. From Lemma 4-6, we have

$$\varphi_1(\iota(x))=\begin{pmatrix} Y & & & \\ & \sigma Y & & \\ & & \ddots & \\ & & & \sigma^{f-1}Y \end{pmatrix},\ \text{where } Y=\begin{pmatrix} \bar{\mu}_1 & \bar{\mu}_2 & \cdots & \bar{\mu}_e \\ & \bar{\mu}_1 & \ddots & \vdots \\ & & \ddots & \bar{\mu}_2 \\ & & & \bar{\mu}_1 \end{pmatrix}.$$

Therefore $\det(t\cdot 1-\varphi_1(\tau(x)))=\det(t\cdot 1-\varphi_1(\iota(x)))=\{\prod_{k=0}^{f-1}(t-\sigma^k\bar{\mu}_1)\}^e$. From Lemma 4-7, $\psi(t)=\prod_{k=0}^{f-1}(t-\sigma^k\bar{\mu}_1)$ is an irreducible polynomial over $\mathfrak{R}_1$ of degree f. Therefore, the minimal polynomial over $\mathfrak{R}_1$ of $\varphi_1(\tau(x))$ is equal to $\psi(t)^{e'}$ for some natural number e' not exceeding e. If $\psi(\varphi_1(\tau(x)))^{e'}=0$ we must have $\psi(\varphi_1(\iota(x)))^{e'}=0$. Since $\bar{\mu}_2\neq 0$ (Lemma 4-7), we have $e'\geqq e$. Hence $e=e'$. q. e. d.

COROLLARY 2. *We assume that* $\tilde{\mathfrak{O}}=\mathfrak{O}[x]$ ($x\in\tilde{\mathfrak{O}}$). *Put*

$$c_x(t)=\det(t\cdot 1-\tau(x))$$
$$=t^n+\sum_{i=1}^{n}c_it^{n-i}\qquad(c_i\in\mathfrak{O}).$$

When $e=1$, $\varphi_1(c_x(t))=t^n+\sum_{i=1}^{n}\varphi_1(c_i)t^{n-i}$ *is an irreducible polynomial over* $\mathfrak{R}_1$. *When* $e>1$, $\varphi_2(c_x(t))=t^n+\sum_{i=1}^{n}\varphi_2(c_i)t^{n-i}$ *is an irreducible monic polynomial over* $\mathfrak{R}_2$.

PROOF. When $e=1$, this corollary is a consequence of Corollary 1. Now we assume that $e>1$ and that $\varphi_2(C_x(t))$ is reducible. Then there exist $d_1, d_2, \cdots, d_j; d'_j, \cdots, d'_{n-j}\in\mathfrak{O}$ ($1\leqq j<n$) such that

$$\varphi_2(C_x(t))=\varphi_2(\{t^j+\sum_{i=1}^{j}d_jt^{j-i}\}\{t^{n-j}+\sum_{i=1}^{n-j}d'_it^{n-j-i}\}).$$

Since $C_x(x)=0$, we have

$$(x^j+\sum_{i=1}^{j}d_ix^{j-i})(x^{n-j}+\sum_{i=1}^{n-j}d'_ix^{n-j-i})\in\mathfrak{P}^2\tilde{\mathfrak{O}}=\tilde{\mathfrak{P}}^{2e},$$

Therefore, we may assume that

$$x^j+\sum_{i=1}^{j}d_ix^{j-i}\in\tilde{\mathfrak{P}}^e.$$

Hence $\varphi_1(\tau(x))^j+\sum_{i=1}^{j}\varphi_1(d_i)\varphi_1(\tau(x))^{j-i}=0$. Therefore the minimal polynomial of $\varphi_1(\tau(x))$ over $\mathfrak{R}_1$ is of degree less than n. This contradicts Corollary 1. $\varphi_2(C_x(t))$ is irreducible. q. e. d.

LEMMA 4-8. *We assume that* $\tilde{\mathfrak{O}}=\mathfrak{O}[x]$ ($x\in\tilde{\mathfrak{O}}$). *Then the centralizer of* $\tau(x)$ *in* G *is* A *and is contained in* K.

PROOF. It is obvious that $\tilde{P}=P[x]$ and x is a regular integral element.

The lemma is an immediate consequence of Lemma 4-5. q. e. d.

4-3. We assume that p is prime to n and use notations in 4-2 without further references.

For every natural number m, we define the subgroup $\mathfrak{U}_m$ of $\tilde{\mathfrak{O}}^*$ as follows:

$$\mathfrak{U}_m = \{x \in \tilde{\mathfrak{O}}^* ; x-1 \in \tilde{\mathfrak{P}}^m\} .$$

Let ξ be a continuous character of $\tilde{\mathfrak{O}}^*$. There exists a natural number m such that ξ is trivial on $\mathfrak{U}_m$ and is not trivial on $\mathfrak{U}_{m-1}$. We assume that $m \geqq 2$. Take a character χ of the additive group of P which is trivial on $\mathfrak{O}$ and is not trivial on $\pi^{-1}\mathfrak{O}$. Since p is prime to n, $\tilde{P}$ is tamely ramified over P. Therefore the mapping: $x \rightarrow \chi(\mathrm{trace}_{\tilde{P}/P}\tilde{\pi}^{1-e}x)$ gives a character of the additive group $\tilde{P}$ which is trivial on $\tilde{\mathfrak{O}}$ and is not trivial on $\tilde{\pi}^{-1}\tilde{\mathfrak{O}}$. Since the mapping: $a \rightarrow \tilde{\pi}^{[\frac{m}{2}]-m}(a-1)$ gives an isomorphism of $\mathfrak{U}_{m-[\frac{m}{2}]}/\mathfrak{U}_m$ onto $\tilde{\mathfrak{O}}/\tilde{\mathfrak{P}}^{[\frac{m}{2}]}$, there exists an element x'_ξ of $\tilde{\mathfrak{O}}^*$ such that we have

$$\xi(a) = \chi(\mathrm{trace}_{\tilde{P}/P}\{\tilde{\pi}^{1-e-m}x'_\xi(a-1)\}) \qquad \text{for every } a \in \mathfrak{U}_{m-[\frac{m}{2}]} .$$

Put $m-1 = m'e-\delta$ (m' and δ are integers such that $0 \leqq \delta < e$). We have $\tilde{\pi}^{1-e-m} = \tilde{\pi}^{\delta-(m'+1)e}$, and $\tilde{\pi}^{+e(m'+1)} = \Pi^{+(m'+1)}$ is an element of $(\tilde{\mathfrak{P}}^{(f)})^{+(m'+1)}$. Hence there exists an element x_ξ of $\tilde{\mathfrak{P}}^\delta$ such that $\xi(a) = \chi(\pi^{-(m'+1)}\,\mathrm{trace}_{\tilde{P}/P}x_\xi(a-1))$ for every $a \in \mathfrak{U}_{m-[\frac{m}{2}]}$.

We call ξ a *strongly regular character* of $\tilde{\mathfrak{O}}^*$, when we can take x_ξ so that we have $\tilde{\mathfrak{O}} = \mathfrak{O}[x_\xi]$. When ξ is strongly regular, we have $x_\xi \in \tilde{\mathfrak{O}}^*$ by Lemma 4-7. Since $x_\xi \in \tilde{\mathfrak{P}}^\delta$, we have $\delta = 0$ and $m = m'e+1$. Since $m \geqq 2$ from the assumption, we have $m' \geqq 1$.

4-4. We assume p is prime to n and use notations in 4-3. Let ξ be a strongly regular character of $\tilde{\mathfrak{O}}^*$. ξ is trivial on $\mathfrak{U}_m$ and is not trivial on $\mathfrak{U}_{m-1}$. We can write $m = m'e+1$, where m' is a natural number. There exists $x_\xi \in \tilde{\mathfrak{O}}$ such that $\tilde{\mathfrak{O}} = \mathfrak{O}[x_\xi]$ and that

$$\xi(a) = \chi(\pi^{-(m'+1)}\,\mathrm{trace}_{\tilde{P}/P}\, x_\xi(a-1)), \qquad \text{for every } a \in \mathfrak{U}_{m-[\frac{m}{2}]} .$$

We write $l = \left[\frac{m'+1}{2}\right]$ and $X = \tau(x_\xi)$. We define the function $\chi_X^{m'+1}$ on K as follows:

$$\chi_X^{m'+1}(k) = \chi(\pi^{-(m'+1)}\,\mathrm{trace}\, X(k-1)) \qquad (k \in K,\ X = \tau(x_\xi)) .$$

From Lemma 2-2, $\chi_X^{m'+1}$ defines a one-dimensional representation of $K_{m'+1-l}$ which is trivial on $K_{m'+1}$ and coincides with ξ on $A \cap K_{m'+1}$[11] ($A = \tau(\tilde{\mathfrak{O}}^*)$).

LEMMA 4-9. *The centralizer of the one-dimensional representation* $\chi_X^{m'+1}$ *of*

11) We define the character ξ of A by

$$\xi(\tau(\alpha)) = \xi(\alpha) \qquad (\alpha \in \tilde{\mathfrak{O}}^*) .$$

$K_{m'+1-l}$ *in* K *is* AK_l.

PROOF. The centralizer of $X=\tau(x_\xi)$ in K is A by Lemma 4-8. Since $\tau(x_\xi)$ is a quasi-regular element in $M(n, \mathfrak{O})$ by Corollary 1 to Lemma 4-7, the lemma follows from Lemma 2-2 and Lemma 2-3. q. e. d.

Now we assume that m' is odd. We have $l=\frac{m'+1}{2}$ and $K_{m'+1-l}=K_l$. We define the function μ_ξ on AK_l as follows:

$$\mu_\xi(ak)=\xi(a)\chi_X^{m'+1}(k) \qquad (k\in K_l,\ a\in A).$$

Then μ_ξ is a one-dimensional representation of $A\cdot K_l$ which coincides with $\chi_X^{m'+1}$ on K_l. (Although the residue class of x_ξ mod.$\mathfrak{P}^{\left[\frac{m}{2}\right]}$ is uniquely determined by ξ, x_ξ is not uniquely determined by ξ. Hence, μ_ξ is not uniquely determined by ξ.)

PROPOSITION 4-2. *We assume that* m' *is odd. Put* $\nu_\xi=\operatorname{Ind}_{A\cdot K_l\uparrow K}\mu_\xi$. $\left(l=\frac{m'+1}{2}\right)$. *Then* ν_ξ *is a continuous irreducible unitary representation of* K.

PROOF. This proposition follows from Lemma 4-9 and Theorem 2. q. e. d.

COROLLARY 1. *The dimension of the representation space of* ν_ξ *is* $q^{\frac{n(n-1)}{2}m'}(q^n-q^{n-f})^{-1}\prod_{k=1}^{n}(q^k-1)$.

PROOF. From Proposition 4-2, the dimension of the representation space of ν_ξ is

$$\begin{aligned}\sharp^{12)}(K/AK_l)&=\sharp\{(K/K_l)/(A/A\cap K_l)\}\\&=\sharp(GL(n,\mathfrak{R}_l))/\sharp(\tilde{\mathfrak{O}}^*/\mathfrak{U}_{el})\\&=q^{(l-1)n^2}q^{\frac{n(n-1)}{2}}\prod_{k=1}^{n}(q^k-1)/q^{f(el-1)}(q^f-1)\\&=q^{(2l-1)\frac{n(n-1)}{2}}\prod_{k=1}^{n}(q^k-1)/q^{n-f}(q^f-1)\\&=q^{m'\frac{n(n-1)}{2}}(q^n-q^{n-f})^{-1}\prod_{k=1}^{n}(q^k-1)\,.\end{aligned}$$

q. e. d.

COROLLARY 2. *Using notations in* § 2, *we have*

$$r(\nu_\xi)=m'+1\,,\qquad s=l$$

and

$$O_{\nu_\xi}=\{\varphi_l(k\tau(x_\xi)k^{-1})\,;\ k\in K\}\,.$$

4-5. We assume that p is prime to n. We use notations in 4-4 and assume that m' is even. Then $l=\left[\frac{m'+1}{2}\right]=\frac{m'}{2}$. We shall construct a certain irre-

12) In the following we denote by $\sharp(S)$ the number of elements of a finite set S.

ducible unitary representation μ_ξ of $A\cdot K_l$ which coincides with $\chi_X^{m'+1}\cdot 1$ $(X=\tau(x_\xi))$ on $K_{l+1}=K_{m'+1-l}$.

We recall certain notations in 4-2. We put

$$\mathfrak{M}_0=\{x\in M(n, \mathfrak{O}^{(f)})\,;\ \sigma x=JxJ^{-1}\}$$

and

$$\mathfrak{K}_s=\{x\in\mathfrak{M}_0\,;\ x-1\in\pi^s\mathfrak{M}_0\}\qquad (s=1, 2, \cdots).$$

The mapping κ (conjugation by some element $g_0^{-1}\in GL(n, \mathfrak{O}^{(f)})$) gives an isomorphism of $\mathfrak{K}_s$ onto K_s $(s=1, 2, \cdots)$. We put $\mathfrak{K}=\kappa^{-1}(K)=GL(n, \mathfrak{O}^{(f)})\cap\mathfrak{M}_0$ and put $\mathfrak{A}=\kappa^{-1}(A)=\iota(\tilde{\mathfrak{H}}^*)$. We identify $\tilde{\mathfrak{H}}^*$ with $\mathfrak{A}$ by means of mapping ι(ξ is then a character of $\mathfrak{A}$).

We put $\mathfrak{X}=\iota(x_\xi)$. We define the function $\chi_{\mathfrak{X}}^{m'+1}$ on $\mathfrak{K}$ as follows:

$$\chi_{\mathfrak{X}}^{m'+1}(k)=\chi(\pi^{-(m'+1)}\operatorname{trace}\mathfrak{X}(k-1))\qquad (k\in\mathfrak{K}).$$

We have $\chi_{\mathfrak{X}}^{m'+1}=\chi_X^{m'+1}\cdot\kappa$ and $\chi_{\mathfrak{X}}^{m'+1}$ is one dimensional representation of $\mathfrak{K}_{m'+1-l}=\mathfrak{K}_{l+1}$ which coincides with ξ on $\mathfrak{A}\cap\mathfrak{K}_{l+1}$. Put

$$\bar{\mathfrak{M}}^1=\{x\in M(n, \mathfrak{R}_1^{(f)})\,;\ \sigma x=JxJ^{-1}\}$$

$\bar{\mathfrak{M}}^1$ is an $\mathfrak{R}_1$-module. For every $\mathfrak{R}_1$-submodule T of $\bar{\mathfrak{M}}^1$, put

$$\mathfrak{K}_l(T)=\{g\in\mathfrak{K}_l\,;\ \varphi_1(\pi^{-l}(g-1))\in T\}.$$

$\mathfrak{K}_l(T)$ is a normal subgroup of $\mathfrak{K}_l$. Put

$$\alpha_\xi(Y_1, Y_2)=\operatorname{trace}\varphi_1(\mathfrak{X})(Y_1Y_2-Y_2Y_1)\qquad (Y_1, Y_2\in\bar{\mathfrak{M}}^1, \mathfrak{X}=\iota(x_\xi)).$$

α_ξ is an $\mathfrak{R}_1$-bilinear skew-symmetric form on $\bar{\mathfrak{M}}^1$. For every $\mathfrak{R}_1$-submodule T of $\bar{\mathfrak{M}}^1$, put $T^\perp=\{x\in\bar{\mathfrak{M}}^1\,;\ \alpha_\xi(x, y)=0$ for every $y\in T\}$.

LEMMA 4-10. *Let* Z_1 *and* Z_2 *be two elements of* $\mathfrak{K}_l$. *Putting* $Z_i=1+\pi^l z_i$ $(z_i\in\mathfrak{M}_0,\ i=1, 2)$, *we have*

$$\varphi_{2l+1}(Z_1Z_2Z_1^{-1}Z_2^{-1})=1+\varphi_{2l+1}(\pi^{2l}\{z_1z_2-z_2z_1\}).$$

PROOF. We have

$$\varphi_{2l+1}(Z_i^{-1})=1-\varphi_{2l+1}(\pi^l z_i)+\varphi_{2l+1}(\pi^{2l}z_i^2)\qquad (i=1, 2).$$

Therefore we have

$$\begin{aligned}&\varphi_{2l+1}(Z_1Z_2Z_1^{-1}Z_2^{-1})\\&=\varphi_{2l+1}\{(1+\pi^l z_1)(1+\pi^l z_2)(1-\pi^l z_1+\pi^{2l}z_1^2)(1-\pi^l z_2+\pi^{2l}z_2^2)\}\\&=1+\varphi_{2l+1}(\pi^{2l}(z_1z_2-z_2z_1)).\end{aligned}$$

q. e. d.

LEMMA 4-11. *Let* T *be an* $\mathfrak{R}_1$*-submodule of* $\bar{\mathfrak{M}}^1$ *and let* η *be any one-dimensional representation of* $\mathfrak{K}_l(T)$ *which coincides with* $\chi_{\mathfrak{X}}^{m'+1}$ *on* $\mathfrak{K}_{l+1}$. *Then the centralizer of* η *in* $\mathfrak{K}_l$ *is* $\mathfrak{K}_l(T^\perp)$.

PROOF. Take $g\in\mathfrak{K}_l$ and $h\in\mathfrak{K}_l(T)$. Write $g=1+\pi^l x$ and $h=1+\pi^l y$ $(x, y\in\mathfrak{M}_0)$. We have $(g\cdot\eta)(h)=\eta(g^{-1}hg)=\eta(h)\eta(g^{-1}hgh^{-1})$. Therefore g centralizes η if and only if $\eta(g^{-1}hgh^{-1})=1$ for every $h\in\mathfrak{K}_l(T)$. It follows from Lemma 4-10 that

$$\begin{aligned}\eta(g^{-1}hgh^{-1})&=\chi_{\mathfrak{X}}^{m'+1}(1-\pi^{2l}(xy-yx))\\&=\chi(-\pi^{-1}\alpha_\xi(\varphi_1(x), \varphi_1(y)))\,.\end{aligned}$$

Hence g centralizes η if and only if $\varphi_1(x)\in T^\perp$. q. e. d.

Let $z_1, z_2, \cdots, z_f$ be elements of $M(e, \mathfrak{O}^{(f)})$ (resp. $M(e, \mathfrak{R}_1^{(f)})$), we define $\mathcal{M}(z_1, z_2, \cdots, z_f)\in\mathfrak{M}_0$ (resp. $\bar{\mathfrak{M}}^1$) as follows:

$$\mathcal{M}(z_1, z_2, \cdots, z_f)=\begin{pmatrix} z_1 & z_2 & \cdots & z_f\\ \sigma z_f & \sigma z_1 & \cdots & \sigma z_{f-1}\\ & & & \\ \sigma^{f-1}z_2 & & \cdots & \sigma^{f-1}z_1\end{pmatrix}.$$

We define the $\mathfrak{R}_1$-submodule W of $\bar{\mathfrak{M}}^1$ as follows: when f is odd,

$$W=\{\mathcal{M}(z_1, z_2, \cdots, z_{\frac{f+1}{2}}, \overbrace{0, 0, \cdots, 0}^{\frac{f-1}{2}})\,;\ z_1, z_2, \cdots, z_{\frac{f+1}{2}}\in M(e, \mathfrak{R}_1^{(f)})$$
$$\text{and } z_1 \text{ is an upper triangular matrix}\}\,,$$

when f is even,

$$W=\{\mathcal{M}(z_1, z_2, \cdots, z_{\frac{f}{2}+1}, \overbrace{0, 0, \cdots, 0}^{\frac{f}{2}-1})\,;\ z_1, z_2, \cdots, z_{\frac{f}{2}+1}\in M(e, \mathfrak{R}_1^{(f)}),$$
$$z_1 \text{ is an upper triangular matrix and } z_{\frac{f}{2}+1} \text{ is an upper triangular matrix with diagonal elements zero}\}\,.$$

When f is even, we define the $\mathfrak{R}_1$-submodule E of $\bar{\mathfrak{M}}^1$ as follows:

$$E=\{\mathcal{M}(\overbrace{0, 0, \cdots, 0}^{\frac{f}{2}}, z, \overbrace{0, \cdots, 0}^{\frac{f}{2}-1})\,;$$
$$z \text{ is a diagonal matrix in } M(e, \mathfrak{R}_1^{(f)})\}\,.$$

It follows from Lemma 4-6 that $\mathfrak{K}_l(W)$ and $\mathfrak{K}_l(W+E)$ are normal subgroups of $\mathfrak{A}\cdot\mathfrak{K}_l$.

LEMMA 4-12. *We have*

$$W^\perp=\begin{cases} W\,, & \textit{when } f \textit{ is odd,}\\ W+E\,, & \textit{when } f \textit{ is even.}\end{cases}$$

PROOF. We assume $\omega=\mathcal{M}(w_1, w_2, \cdots, w_f)\in W^\perp$ $(w_1, w_2, \cdots, w_f\in M(e, \mathfrak{R}_1^{(f)}))$. Then $\alpha_\xi(\omega, y)=0$ for every $y\in W$. Write $y=\mathcal{M}(z_1, z_2, \cdots, z_{[\frac{f}{2}]+1}, 0, \cdots, 0)\in W$ $(z_1, z_2, \cdots, z_{[\frac{f}{2}]+1}\in M(e, \mathfrak{R}_1^{(f)}))$. We have

$$\sum_{k=0}^{f-1}\sigma^k[\text{trace }\bar{\mathfrak{X}}'\{z_1w_1-w_1z_1+Y\}]=0, \quad \text{where } \bar{\mathfrak{X}}'=\varphi_1(\iota'(x_\xi))$$

and

$$Y=\sum_{j=2}^{[\frac{f}{2}]+1} z_j\sigma^{j-1}w_{f+2-j}-\sum_{j=f+1-[\frac{f}{2}]}^{f} w_j\sigma^{j-1}z_{f+2-j}(\natural).$$

Putting $Z_j=\delta_{ij}Z$ $\left(2\leqq i\leqq\left[\frac{f+1}{2}\right]\right)$ in ($\natural$), we have

$$\sum_{k=0}^{f-1}\sigma^k \text{ trace } \bar{\mathfrak{X}}'(z\sigma^{i-1}w_{f+2-i}-w_{f+2-i}\sigma^{f+1-i}z)$$

$$=\sum_{k=0}^{f-1}\sigma^k \text{ trace } z(\sigma^{i-1}w_{f+2-i}\cdot\bar{\mathfrak{X}}'-\sigma^{i-1}\bar{\mathfrak{X}}'\cdot\sigma^{i-1}w_{f+2-i})=0$$

for every $z\in M(e,\mathfrak{R}_1^{(f)})$. We have $\sigma^{i-1}w_{f+2-i}\cdot\bar{\mathfrak{X}}'=\sigma^{i-1}\bar{\mathfrak{X}}'\cdot\sigma^{i-1}w_{f+2-i}$. It follows from Lemma 4-6 that

$$\bar{\mathfrak{X}}'=\varphi_1(\iota'(x_\xi))=\begin{pmatrix}\bar{\mu}_1 & \bar{\mu}_2 & \cdots\cdots & \bar{\mu}_e\\ & \bar{\mu}_1 & \ddots & \vdots\\ & & \ddots & \bar{\mu}_2\\ 0 & & & \bar{\mu}_1\end{pmatrix},$$

where we put $x_\xi=\sum_{i=1}^{e}\mu_i\tilde{\pi}^{i-1}$ and $\bar{\mu}_i=\varphi_1(\mu_i)$. By Lemma 4-7, we have $\bar{\mu}_2\neq 0$ and $\bar{\mu}_1\neq\sigma^k\bar{\mu}_1$ $(k=1,\cdots,f-1)$. Hence we have $w_{f+2-i}=0$ $\left(2\leqq i\leqq\left[\frac{f+1}{2}\right]\right)$. Putting $z_j=\delta_{j1}z$ in ($\natural$), we have

$$\sum_{k=0}^{f-1}\sigma^k \text{ trace } \bar{\mathfrak{X}}'(zw_1-w_1z)=0$$

for every upper triangular matrix $z\in M(e,\mathfrak{R}_1^{(f)})$. Therefore w_1 is an upper triangular matrix. Finally we assume that f is even and put $z_1=\cdots=z_{\frac{f}{2}}=0$ in ($\natural$). We have

$$\sum_{k=0}^{f-1}\sigma^k \text{ trace } Z_{\frac{f}{2}+1}(\sigma^{\frac{f}{2}}w_{\frac{f}{2}+1}\cdot\bar{\mathfrak{X}}'-\sigma^{\frac{f}{2}}\bar{\mathfrak{X}}'\cdot\sigma^{\frac{f}{2}}w_{\frac{f}{2}+1})=0$$

for every upper triangular matrix $Z_{\frac{f}{2}+1}$ whose diagonal elements are zero. Therefore $w_{\frac{f}{2}+1}$ is an upper triangular matrix. It is proved that

$$\omega\in W \quad \text{when } f \text{ is odd,}$$

and that

$$\omega\in W+E \quad \text{when } f \text{ is even.}$$

Therefore $W^\perp\subset W$ when f is odd and $W^\perp\subset W+E$ when f is even. Since the inverse inclusion relation is obvious, lemma is proved. q. e. d.

Now, we construct a one-dimensional representation η_ξ of $\mathfrak{K}_l(W)$ which satisfies following conditions:

(i) η_ξ coincides with $\chi_{\mathfrak{X}}^{m'+1}$ on $\mathfrak{K}_{l+1}$,

(ii) η_ξ coincides with ξ on $\mathfrak{K}_l(W)\cap\mathfrak{A}$,

(iii) Every element of $\mathfrak{A}$ centralizes η_ξ.

We define an $\mathfrak{R}_1$-submodule W_0 of W as follows:

$$W_0=\{\mathcal{M}(z_1, z_2, \cdots, z_f)\in W;\ z_1 \text{ is an upper triangular matrix with diagonal elements zero}\}.$$

It follows from Lemma 4-6 that $\mathfrak{K}_l(W_0)$ is a normal subgroup of $\mathfrak{A}\cdot\mathfrak{K}_l$.

LEMMA 4-13. *$\chi_{\mathfrak{X}}^{2l+1}$ is a one-dimensional representation of $\mathfrak{K}_l(W_0)$ which coincides with ξ on $\mathfrak{A}\cap\mathfrak{K}_l(W_0)$. Every element of $\mathfrak{A}$ centralizes $\chi_{\mathfrak{X}}^{2l+1}$.*

PROOF. Let $k_1=1+\pi^l x_1$ and $k_2=1+\pi^l x_2$ be elements of $\mathfrak{K}_l(W_0)$ ($x_1, x_2\in\mathfrak{M}_0$). Since $k_1k_2=1+\pi^l(x_1+x_2)+\pi^{2l}x_1x_2$, we have

$$\chi_{\mathfrak{X}}^{2l+1}(k_1k_2)=\chi_{\mathfrak{X}}^{2l+1}(k_1)\chi_{\mathfrak{X}}^{2l+1}(k_2)\chi(\pi^{-1}\operatorname{trace}\mathfrak{X}x_1x_2).$$

Put

$$x_1=\mathcal{M}(z_1, z_2, \cdots, z_f)$$

and

$$x_2=\mathcal{M}(z_1', z_2', \cdots, z_f') \qquad (z_i, z_i'\in M(e, \mathfrak{O}^{(f)})).$$

We have

$$\chi(\pi^{-1}\operatorname{trace}\mathfrak{X}x_1x_2)$$

$$=\chi[\pi^{-1}\sum_{k=0}^{f-1}\sigma^k \operatorname{trace}\iota'(x_\xi)\{z_1z_1'+Z\}], \qquad \text{where } Z=\sum_{i=2}^{f} z_i\sigma^{i-1}z'_{f+2-i}.$$

Note that $\varphi_1(x_1)$ and $\varphi_1(x_2)$ are in W_0 and that $\varphi_1(\iota'(x_\xi))$ is an upper triangular matrix. Hence we have $\operatorname{trace}\iota'(x_\xi)\{z_1z_1'+Z\}\in\mathfrak{P}^{(f)}$. Since χ is trivial on $\mathfrak{O}$, we have

$$\chi(\pi^{-1}\operatorname{trace}\mathfrak{X}x_1x_2)=1.$$

Therefore $\chi_{\mathfrak{X}}^{2l+1}(k_1k_2)=\chi_{\mathfrak{X}}^{2l+1}(k_1)\chi_{\mathfrak{X}}^{2l+1}(k_2)$ for every $k_1, k_2\in\mathfrak{K}_l(W_0)$. Hence $\chi_{\mathfrak{X}}^{2l+1}$ is a one-dimensional representation of $\mathfrak{K}_l(W_0)$. We have $\mathfrak{K}_l(W_0)\cap\mathfrak{A}=\iota(\mathfrak{U}_{le+1})$.

Remember that $m=2le+1$ and $m-\left[\frac{m}{2}\right]=le+1$. When $a\in\mathfrak{U}_{le+1}$, we have from the definition of x_ξ,

$$\xi(a)=\chi(\pi^{-(2l+1)}\operatorname{trace}_{\tilde{P}/P}(x_\xi)(a-1))$$

$$=\chi(\pi^{-(2l+1)}\operatorname{trace}\mathfrak{X}(\iota(a)-1))=\chi_{\mathfrak{X}}^{2l+1}(\iota(a)).$$

Therefore $\chi_{\mathfrak{X}}^{2l+1}$ coincides with ξ on $\mathfrak{A}\cap\mathfrak{K}_l(W_0)$. It is obvious that every element of $\mathfrak{A}$ centralizes $\chi_{\mathfrak{X}}^{2l+1}$. q. e. d.

We define the isomorphic imbedding δ of $(\mathfrak{O}^{(f)*})^e=\overbrace{\mathfrak{O}^{(f)*}\times\cdots\times\mathfrak{O}^{(f)*}}^{e}$ into $\mathfrak{K}$ as follows:

$$\delta(x_1, x_2, \cdots, x_e) = \mathcal{M}\left(\begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_e \end{pmatrix}, \overbrace{0, \cdots, 0}^{e-1}\right).$$

We denote by $\mathfrak{D}$ the image of $(\mathfrak{O}^{(f)*})^e$ by δ. We put $\mathfrak{D}_s = \mathfrak{D} \cap \mathfrak{K}_s$ $(s = 1, 2, \cdots)$. It is obvious that $\mathfrak{K}_l(W) = \mathfrak{D}_l \cdot \mathfrak{K}_l(W_0)$. and that $\mathfrak{D}_l \cap \mathfrak{K}_l(W_0) = \mathfrak{D}_{l+1}$. We put $\mathfrak{U}_s^{(f)} = \mathfrak{U}_{es} \cap \mathfrak{O}^{(f)}$ $(s = 1, 2, \cdots)$. Then $\mathfrak{D}_s$ is the image of $(\mathfrak{U}_s^{(f)})^e$ by δ. Since p is prime to n, the mapping: $\alpha \to \alpha^e$, of $\mathfrak{U}_s^{(f)}$ into itself defines an automorphism of $\mathfrak{U}_s^{(f)}$ $(s = 1, 2, \cdots; n = ef)$. Therefore, for every element $\alpha \in \mathfrak{U}_s^{(f)}$ there exists unique $\alpha^{\frac{1}{e}} \in \mathfrak{U}_s^{(f)}$ such that $(\alpha^{\frac{1}{e}})^e = \alpha$. We define a character ρ of $\mathfrak{D}_l$ as follows:

$$\rho(\delta(x_1, x_2, \cdots, x_e)) = \xi((x_1 \cdot \cdots \cdot x_e)^{\frac{1}{e}}) \qquad (x_1, \cdots, x_e \in \mathfrak{U}_s^{(f)}).$$

Then ρ coincides with $\chi_{\mathfrak{x}}^{2l+1}$ on $\mathfrak{D}_{l+1}$. (Note that diagonal elements of $\iota'(x_\xi)$ are all equal by Lemma 4-6.) It is obvious that ρ coincides with ξ on $\mathfrak{D}_l \cap \mathfrak{A}$. Now we define the function η_ξ on $\mathfrak{K}_l(W)$ as follows:

$$\eta_\xi(dk) = \rho(d)\chi_{\mathfrak{x}}^{2l+1}(k) \qquad (d \in \mathfrak{D}_l, \ k \in \mathfrak{K}_l(W_0)).$$

Well-definedness of η_ξ is obvious. It follows from Lemma 4-11 and 4-12 that every element of $\mathfrak{D}_l$ centralizes the one-dimensional representation $\chi_{\mathfrak{x}}^{2l+1}$ of $\mathfrak{K}_l(W_0)$. Hence η_ξ is a one-dimensional representation of $\mathfrak{K}_l(W)$ which coincides with $\chi_{\mathfrak{x}}^{2l+1}$ on $\mathfrak{K}_{l+1}$ and coincides with ξ on $\mathfrak{K}_l(W) \cap \mathfrak{A}$.

LEMMA 4-14. *Every element of $\mathfrak{A}$ centralizes η_ξ.*

PROOF. Let a be any element of $\mathfrak{A}$ and k be any element of $\mathfrak{K}_l(W)$. Write $k = dh$, where $d \in \mathfrak{D}_l$ and $h \in \mathfrak{K}_l(W_0)$. We have

$$a^{-1}ka = a^{-1}dad^{-1} \cdot d \cdot a^{-1}ha \quad \text{and} \quad a^{-1}dad^{-1}, \ a^{-1}ha \in \mathfrak{K}_l(W_0).$$

Therefore

$$\eta(a^{-1}ka) = \eta(d)\chi_{\mathfrak{x}}^{2l+1}(a^{-1}ha)\chi_{\mathfrak{x}}^{2l+1}(a^{-1}dad^{-1}) = \eta(k)\chi_{\mathfrak{x}}^{2l+1}(a^{-1}dad^{-1}).$$

Put

$$d = \delta(x_1, x_2, \cdots, x_e), \ d' = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_e \end{pmatrix} \qquad (x_i \in \mathfrak{U}_l^{(f)}; \ i = 1, 2, \cdots, e).$$

We have

$$\chi_{\mathfrak{x}}^{2l+1}(a^{-1}dad^{-1}) = \chi[\pi^{-(2l+1)}\sum_{k=0}^{f-1}\sigma^k \operatorname{trace} \iota'(x_\xi)\{\iota'(\alpha)^{-1}d'\iota'(\alpha)d'^{-1}\}],$$

where $a = \iota'(\alpha)$. Write $x_i = 1 + \pi^l y_i$ $(y_i \in \mathfrak{O}^{(f)}; i = 1, 2, \cdots, e)$ and $y' = \begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_e \end{pmatrix}$, then we have

$$\iota'(\alpha)^{-1}d'\iota'(\alpha)d'^{-1}=\{1+\pi^l\iota'(\alpha)^{-1}y'\iota'(\alpha)\}\{1+\sum_{k=1}^{\infty}(-1)^k\pi^{lk}y'^k\}$$
$$=1+\pi^l\{\iota'(\alpha)^{-1}y'\iota'(\alpha)-y'\}+\pi^{2l}(y'^2-\iota'(\alpha)^{-1}y'\iota'(\alpha)y')+\pi^{3l}Z$$
$$(Z\in M(e,\mathfrak{O}^{(f)}))\,.$$

Since $\iota'(x_\xi)$ commutes with $\iota'(\alpha)$ and $\varphi_1(\iota'(x_\xi))$ is an upper triangular matrix we have

$$\pi^{-(2l+1)}\,\mathrm{trace}\,\iota'(x_\xi)\{\iota'(\alpha)^{-1}d'\iota'(\alpha)d'^{-1}-1\}\in\mathfrak{O}^{(f)}\,.$$

Hence $\chi_{\mathfrak{X}}^{2l+1}(a^{-1}dad^{-1})=1$ and a centralizes η_ξ. q. e. d.

COROLLARY. *The centralizer of* η_ξ *in* $\mathfrak{A}\cdot\mathfrak{K}_l$ *is* $\mathfrak{A}\cdot\mathfrak{K}_l(W^\perp)$.

PROOF. Corollary follows from Lemma 4-11 and Lemma 4-14. q. e. d.

The restriction of $\chi_{\mathfrak{X}}^{2l+1}$ to $\mathfrak{K}_l(W_0)$ depends only on the residue class of x_ξ mod. $\mathfrak{P}^{[\frac{m}{2}]}=\mathfrak{P}^{le}$ and therefore uniquely determined by ξ. Therefore η_ξ is uniquely determined by ξ. Now we assume that f is odd. We define the function ψ_ξ on $\mathfrak{A}\cdot\mathfrak{K}_l(W)$ as follows: $\psi_\xi(ak)=\xi(a)\eta_\xi(k)$ $(a\in\mathfrak{A},\ k\in\mathfrak{K}_l(W))$. Then ψ_ξ is a one-dimensional representation of $\mathfrak{A}\cdot\mathfrak{K}_l(W)$ which coincides with η_ξ on $\mathfrak{K}_l(W)$.

LEMMA 4-15. *We assume that f is odd and put*

$$\tilde{\mu}_\xi=\operatorname*{Ind}_{\mathfrak{A}\cdot\mathfrak{K}_l(W)\uparrow\mathfrak{A}\cdot\mathfrak{K}_l}\psi_\xi\,.$$

Then $\tilde{\mu}_\xi$ *is an irreducible unitary representation of* $\mathfrak{A}\cdot\mathfrak{K}_l$ *which coincides with* $\chi_{\mathfrak{X}}^{2l+1}\cdot 1$ *on* $\mathfrak{K}_{l+1}$.

PROOF. It follows from Corollary to Lemma 4-14 and Lemma 4-12 that the centralizer of η_ξ in $\mathfrak{A}\cdot\mathfrak{K}_l$ is $\mathfrak{A}\cdot\mathfrak{K}_l(W)$. Since ψ_ξ is a one-dimensional representation of $\mathfrak{A}\cdot\mathfrak{K}_l(W)$ which coincides with η_ξ on $\mathfrak{K}_l(W)$, $\tilde{\mu}_\xi$ is irreducible by Lemma 1-2. The remaining part is obvious. q. e. d.

COROLLARY 1. *The dimension of the representation space of* $\tilde{\mu}_\xi$ *is* $q^{\frac{1}{2}n(n-1)}$.

COROLLARY 2. *We put* $\mu_\xi=\tilde{\mu}_\xi\cdot\kappa^{-1}$. *Then* μ_ξ *is an irreducible unitary representation of* $A\cdot K_l$ *which coincides with* $\chi_X^{2l+1}\cdot 1$ *on* K_{l+1}.

Now we assume that f is even. We denote by M^* the group of root of 1 in $\tilde{\mathfrak{O}}^*$ of orders prime to p. Then, we have $\tilde{\mathfrak{O}}^*=M^*\mathfrak{U}_1$ and $M^*\cap\mathfrak{U}_1=\{1\}$. M^* can be naturally identified with $\mathfrak{R}_1^{(f)*}$. We put $\mathfrak{A}_0=\iota(M^*)$ and $\mathfrak{A}_1=\iota(\mathfrak{U}_1)$. It is obvious that $\mathfrak{A}_1\mathfrak{K}_l(W)$ is a normal subgroup of $\mathfrak{A}\mathfrak{K}_l(W^\perp)$. We define the function Λ_ξ on $\mathfrak{A}_1\cdot\mathfrak{K}_l(W)$ as follows:

$$\Lambda_\xi(ak)=\xi(a)\eta_\xi(k)\qquad(a\in\mathfrak{A}_1,\ k\in\mathfrak{K}_l(W))\,.$$

Then Λ_ξ is a one-dimensional representation of $\mathfrak{A}_1\cdot\mathfrak{K}_l(W)$ which coincides with η_ξ on $\mathfrak{K}_l(W)$.

LEMMA 4-16. *Every element of* $\mathfrak{A}\cdot\mathfrak{K}_l(W^\perp)$ *centralizes* Λ_ξ.

PROOF. It is obvious that every element of $\mathfrak{A}$ centralizes Λ_{ξ}. Take $ah \in \mathfrak{A}_1 \cdot \mathfrak{K}_l(W)$ ($a \in \mathfrak{A}_1$, $h \in \mathfrak{K}_l(W)$ and $g \in \mathfrak{K}_l(W^{\perp})$. We have by the definition of Λ_{ξ} that

$$\begin{aligned}\Lambda_{\xi}(g^{-1}ahg) &= \Lambda_{\xi}(ah)\Lambda_{\xi}(g^{-1}aga^{-1}) \\ &= \Lambda_{\xi}(ah)\eta_{\xi}(g^{-1}aga^{-1}) \qquad (g^{-1}aga^{-1} \in \mathfrak{K}_l(W_0)) \\ &= \Lambda_{\xi}(ah)\chi_{x}^{2l+1}(g^{-1}aga^{-1}).\end{aligned}$$

Put $g = 1+\pi^l x$ and $a = \iota(\alpha)$ ($\alpha \in \mathfrak{U}_1$). We have

$$\begin{aligned}\chi_{x}^{2l+1}(g^{-1}aga^{-1}) &= \chi_{x}^{2l+1}[(1-\pi^l x+\pi^{2l}x^2)(1+\pi^l axa^{-1})] \\ &= \chi[\pi^{-(2l+1)} \operatorname{trace} \iota(x_{\xi})\pi^l\{axa^{-1}-x\}] \\ &\quad \times \chi[\pi^{-(2l+1)} \operatorname{trace} \iota(x_{\xi})\pi^{2l}(x^2-xaxa^{-1})] \\ &= \chi[\pi^{-1} \operatorname{trace} \iota(x_{\xi})(x^2-xaxa^{-1})].\end{aligned}$$

Note that $\varphi_1(x) \in W^{\perp}$. We further remember that $\varphi_1(\iota'(x_{\xi}))$ is an upper triangular matrix and that $\varphi_1(\iota'(\alpha))$ is an upper triangular matrix with diagonal elements 1. Using these, we can easily prove that $\operatorname{trace} \iota(x_{\xi})(x^2-xaxa^{-1}) \in \mathfrak{P}$. Hence

$$\chi[\pi^{-1} \operatorname{trace} (x_{\xi})(x^2-xaxa^{-1})] = 1.$$

Therefore every element of $\mathfrak{K}_l(W^{\perp})$ centralizes Λ_{ξ}. q. e. d.

For every z, $w \in M(e, \mathfrak{O}^{(f)})$, we put

$$\Delta(z, w) = \mathcal{M}(z, \overbrace{0, \cdots, 0}^{\frac{f}{2}-1}, w, \overbrace{0, \cdots, 0}^{\frac{f}{2}-1}).$$

We define the subgroup $\mathfrak{E}$ of $\mathfrak{K}_l(E)$ as follows:

$\mathfrak{E} = \{\Delta(1+\pi^{2l}z, \pi^l w);\ z$ and w are diagonal matrices of $M(e, \mathfrak{O}^{(f)}\}$.

It is obvious that $\mathfrak{A}_0$ normalizes $\mathfrak{E}$ and $\mathfrak{A}_0 \cap \mathfrak{E} = \{1\}$. We have

$$\mathfrak{A} \cdot \mathfrak{K}_l(W^{\perp}) = (\mathfrak{A}_0 \cdot \mathfrak{E})(\mathfrak{A}_1 \cdot \mathfrak{K}(W))$$

and

$$\mathfrak{A}_0 \cdot \mathfrak{E} \cap \mathfrak{A}_1 \cdot \mathfrak{K}_l(W) = \mathfrak{E} \cap \mathfrak{K}_{l+1}.$$

It the following, we construct an irreducible unitary representation Θ_{ξ} of $\mathfrak{A}_0 \cdot \mathfrak{E}$ which coincides with $\chi_{x}^{2l+1} \cdot 1$ on $\mathfrak{E} \cap \mathfrak{K}_{l+1}$. We denote by D the set of diagonal matrices of $M(e, \mathfrak{R}_1^{(f)})$. We put $D_{\pm} = \{\zeta \in D,\ \sigma^{\frac{f}{2}}\zeta = \pm\zeta\}$. We denote by $L^2(D_+)$ the Hilbert space of complex valued functions on D_+ endowed with the following inner-product:

$$(f, g) = \sum_{\zeta \in D_+} f(\zeta)\overline{g(\zeta)} \qquad (f, g \in L^2(D_+)).$$

The dimension of $L^2(D_+)$ is $q^{\frac{n}{2}}$. We define the character λ_x of the additive group D as follows:

$$\lambda_x(\zeta)=\chi(\pi^{-1}\sum_{k=0}^{f-1}\{\sigma^k \text{ trace } \iota'(x_\xi)\zeta\}).$$

Take $\varepsilon\in\mathfrak{R}_1^{(f)}$ such that $\sigma^{\frac{f}{2}}\varepsilon=-\varepsilon$. We denote by $\mathfrak{R}_1^{(f)}(+)$ the subfield of $\mathfrak{R}_1^{(f)}$ formed by $\sigma^{\frac{f}{2}}$-fixed elements. We denote by sgn the Legendre symbol of the multiplicative group of $\mathfrak{R}_1^{(*)}(+)$. For every element g of $\mathfrak{A}_0\cdot\mathfrak{E}$, we define the linear operator $\Theta_\xi(g)$ on $L^2(D_+)$ as follows.

(i) When $g=\Delta(1+\pi^{2l}z, \pi^l w)\in\mathfrak{E}$,

$$\{\Theta_\xi(g)f(\zeta)=f(\zeta+\bar{w}_+)\lambda_x(-2\zeta\bar{w}_-)\lambda_x(\bar{z}-\frac{1}{2}\bar{w}\bar{w}'-\bar{w}_+\bar{w}_-)\},$$

where we put $\bar{z}=\varphi_1(z)$, $\bar{w}=\varphi_1(w)$, $\bar{w}'=\sigma^{\frac{f}{2}}\bar{w}$, $2\bar{w}_+=\bar{w}+\bar{w}'$ and $2\bar{w}_-=\bar{w}-\bar{w}'$ ($f\in L^2(D_+)$, $\zeta\in D_+$).

(ii) When $g=\iota(\alpha)$ ($\alpha\in M^*$) and $\alpha\neq\pm\sigma^{\frac{f}{2}}\alpha$ (we note that M^* is naturally identified with $\mathfrak{R}_1^{(f)*}$),

$$\{\Theta_\xi(g)f\}(\zeta)=\xi(\alpha)\operatorname{sgn}(\varepsilon\gamma)^e c^{-1}\sum_{\zeta'\in D_+}\lambda_x[\gamma^{-1}\{-\beta\zeta^2+2\zeta\zeta'-\beta\zeta'^2\}]f(\zeta'),$$

where we put

$$2\beta=\frac{\alpha'}{\alpha}+\frac{\alpha}{\alpha'}\ (\alpha'=\sigma^{\frac{f}{2}}\alpha),\qquad 2\gamma=\frac{\alpha'}{\alpha}-\frac{\alpha}{\alpha'}\quad\text{and}\quad c=\sum_{\zeta\in D_+}\lambda_x(-\varepsilon\zeta^2).$$

(iii) When $g=\iota(\alpha)$ ($\alpha\in M^*$) and $\alpha=\pm\sigma^{\frac{f}{2}}\alpha$,

$$(\Theta_\xi(g)f)(\zeta)=\left(\operatorname{sgn}\frac{\alpha}{\alpha'}\right)^e f\left(\frac{\alpha}{\alpha'}\zeta\right)\xi(\alpha),\qquad\text{where } \alpha'=\sigma^{\frac{f}{2}}\alpha.$$

(iv) When $g=ah$ ($a\in\mathfrak{A}_0$, $h\in\mathfrak{E}$),

$$\Theta_\xi(g)=\Theta_\xi(a)\Theta_\xi(h).$$

LEMMA 4-17. *Θ_ξ is an irreducible unitary representation of $\mathfrak{A}_0\cdot\mathfrak{E}$ on $L^2(D_+)$ which coincides with $\chi_x^{2l+1}\cdot 1$ on $\mathfrak{E}\cap\mathfrak{K}_{l+1}$.*

PROOF. We denote by $A(D)$ the group whose underlying space is $D_+\times D_-\times T$[13] and whose composition rule is given as follows:

$$(u, v, t)(u', v', t')=(u+u', v+v', tt'\langle u, v'\rangle)$$

($u, u'\in D_+$, $v, v'\in D_-$ and $t, t'\in T$), where we put $\langle u, v'\rangle=\lambda_x(-2uv)$. We define a unitary representation U of $A(D)$ on $L^2(D_+)$ as follows:

$$\{U((u, v, t))f\}(\zeta)=f(\zeta+u)\langle\zeta, v\rangle t.$$

We denote by Ξ the mapping of $\mathfrak{E}$ into $A(D)$ given as follows:

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13) We denote by T the multiplicative group of complex numbers of modulus 1.

$$\Xi(\Delta(1+\pi^{2l}z, \pi^l w)) = \left(\bar{w}_+, \bar{w}_-, \lambda_x\left(\bar{z}-\frac{1}{2}\bar{w}\bar{w}'-\bar{w}_+\bar{w}_-\right)\right),$$

where we put $\bar{z}=\varphi_1(z)$, $\bar{w}=\varphi_1(w)$, $\bar{w}'=\sigma^{\frac{f}{2}}\bar{w}$, $\bar{w}_+=\frac{1}{2}(\bar{w}+\bar{w}')$ and $\bar{w}_-=\frac{1}{2}(\bar{w}-\bar{w}')$. By direct computation, it is proved that Ξ is a homomorphism of $\mathfrak{G}$ into $A(D)$ and that

$$\Theta_\xi(g) = U\cdot\Xi(g) \qquad \text{for every} \quad g\in\mathfrak{G}.$$

Hence Θ_ξ is a unitary representation of $\mathfrak{G}$ on $L^2(D_+)$. It is easily proved that this is an irreducible representation of $\mathfrak{G}$. For any $g_\alpha=\iota(\alpha)\in\mathfrak{A}_0$ $(\alpha\in M^*)$, we define the automorphism S_α of $A(D)$ as follows:

$$S_\alpha((u, v, t)) = \left(u', v', \left\langle\frac{1}{2}u', v'\right\rangle\left\langle\frac{1}{2}u, v\right\rangle^{-1}t\right),$$

where we put

$$u'=\beta u+\gamma v \quad \text{and} \quad v'=\gamma u+\beta v\,,$$

$$\left(\beta=\frac{1}{2}\left(\frac{\alpha'}{\alpha}+\frac{\alpha}{\alpha'}\right) \quad \text{and} \quad \gamma=\frac{1}{2}\left(\frac{\alpha'}{\alpha}-\frac{\alpha}{\alpha'}\right)\right).$$

Then we have

$$\Xi(g_\alpha^{-1}gg_\alpha) = S_\alpha(\Xi(g)) \qquad \text{for every} \quad g\in\mathfrak{G}.$$

Put

$$\{R(g_\alpha)f\}(\zeta) = q^{-\frac{n}{4}}\sum_{\zeta'\in D_+}\left\langle\frac{1}{2}\zeta, \zeta\beta\gamma^{-1}\right\rangle\langle\zeta, -\zeta^1\gamma^{-1}\rangle\left\langle\frac{1}{2}\zeta', \zeta'\beta\gamma^{-1}\right\rangle f(\zeta'),$$

if $\alpha'\neq\pm\alpha$ and

$$\{R(g_\alpha)f\}(\zeta) = f\left(\frac{\alpha}{\alpha'}\zeta\right) \qquad \text{if} \quad \alpha=\pm\alpha'.$$

Then, Weil's result (see Weil [8] Chap. I; see also Tanaka [7] Section 2) shows that R is a projective unitary representation of $\mathfrak{A}_0$ which satisfies the following relation for every $g\in\mathfrak{G}$:

$$U\cdot\Xi(g_\alpha^{-1}gg_\alpha) = R(g_\alpha)^{-1}\cdot U\cdot\Xi(g)R(g_\alpha).$$

Put

$$\tilde{R}(g_\alpha) = \operatorname{sgn}(\varepsilon\gamma)^e\left(q^{-\frac{n}{4}}\sum_{\zeta\in D_+}\left\langle\frac{1}{2}\zeta, \varepsilon\zeta\right\rangle\right)^{-1}R(g_\alpha) \qquad \text{if} \quad \alpha\neq\pm\alpha'$$

and

$$\tilde{R}(g_\alpha) = \operatorname{sgn}\left(\frac{\alpha}{\alpha'}\right)^e R(g_\alpha) \qquad \text{if} \quad \alpha=\pm\alpha'.$$

Then, easy computation shows that R is a unitary representation of $\mathfrak{A}_0$ and we have

$$\Theta_\xi(g_\alpha) = \xi(\alpha)\tilde{R}(g_\alpha).$$

Therefore Θ_ξ is a unitary representation of $\mathfrak{A}_0$ which satisfies

$$\Theta_\xi(g_\alpha^{-1}gg_\alpha)=\Theta_\xi(g_\alpha)^{-1}\Theta_\xi(g)\Theta_\xi(g_\alpha) \quad \text{for every} \quad g\in\mathfrak{E}.$$

Now it is obvious that Θ_ξ is an irreducible unitary representation of $\mathfrak{A}_0\cdot\mathfrak{E}$ which coincides with $\chi_x^{2l+1}\cdot 1$ on $\mathfrak{E}\cap\mathfrak{K}_{l+1}$. q. e. d.

COROLLARY. *Let* $g=\iota(\alpha)\in\mathfrak{A}_0$, *we have*

$$\text{trace } \Theta_\xi(g)=\begin{cases}(-1)^e\xi(\alpha)\operatorname{sgn}(\alpha\alpha')^e, & \text{when } \alpha\neq\pm\alpha' \\ 0 & \text{when } \alpha=-\alpha' \\ \xi(\alpha)q^{\overline{\frac{n}{2}}} & \text{when } \alpha=\alpha'.\end{cases}$$

Now we define the operator-valued function Ψ_ξ on $\mathfrak{A}\cdot\mathfrak{K}_l(W^\perp)$ as follows:

$$\Psi_\xi(g_1g_2)=\Theta_\xi(g_1)\Lambda_\xi(g_2),$$

where

$$g_1\in\mathfrak{A}_0\cdot\mathfrak{E} \quad \text{and} \quad g_2\in\mathfrak{A}_1\cdot\mathfrak{K}_l(W).$$

Then it follows from Lemma 4-16 and Lemma 4-17 that Ψ_ξ is an irreducible unitary representation of $\mathfrak{A}\cdot\mathfrak{K}_l(W^\perp)$ on $L^2(D_+)$ which coincides with $\eta_\xi\cdot 1$ on $\mathfrak{K}_l(W)$.

LEMMA 4-18. *We assume that f is even. Put*

$$\tilde{\mu}_\xi=\operatorname*{Ind}_{\mathfrak{A}\cdot\mathfrak{K}_l(W^\perp)\uparrow\mathfrak{A}\cdot\mathfrak{K}_l}\Psi_\xi.$$

Then $\tilde{\mu}_\xi$ *is an irreducible unitary representation of* $\mathfrak{A}\cdot\mathfrak{K}_l$ *which coincides with* $\chi_x^{2l+1}\cdot 1$ *on* $\mathfrak{K}_{l+1}$.

PROOF. From Corollary to Lemma 4-14, it follows that $\mathfrak{A}\cdot\mathfrak{K}_l(W^\perp)$ is the centralizer of η_ξ in $\mathfrak{A}\cdot\mathfrak{K}_l$. Since Ψ_ξ is an irreducible unitary representation of $\mathfrak{A}\cdot\mathfrak{K}_l(W^\perp)$ which coincides with $\eta_\xi\cdot 1$ on $\mathfrak{K}_l(W)$, it follows from Lemma 1-2 that $\tilde{\mu}_\xi$ is an irreducible representation of $\mathfrak{A}\cdot\mathfrak{K}_l$. It is obvious that $\tilde{\mu}_\xi$ coincides with $\chi_x^{2l+1}\cdot 1$ on $\mathfrak{K}_{l+1}$. q. e. d.

COROLLARY 1. *The dimension of the representation space of* $\tilde{\mu}_\xi$ *is* $q^{\frac{1}{2}n(n-1)}$.

COROLLARY 2. *We put* $\mu_\xi=\tilde{\mu}_\xi\circ\kappa^{-1}$. *Then* μ_ξ *is an irreducible unitary representation of* $A\cdot K_l$ *which coincides with* $\chi_X^{2l+1}\cdot 1$ *on* K_{l+1}.

PROPOSITION 4-3. *Let* μ_ξ *be the irreducible unitary representation of* $A\cdot K_l$ *which is defined in Lemma* 4-15 *when f is odd and is defined in Lemma* 4-18 *when f is even. Put* $\nu_\xi=\operatorname*{Ind}_{A\cdot K_l\uparrow K}\mu_\xi$. *Then* ν_ξ *is a continuous irreducible unitary representation of K.*

PROOF. This proposition follows from Theorem 1, Lemma 4-9, Lemma 4-15 and Lemma 4-18. q. e. d.

COROLLARY 1. *The dimension of the representation space of* ν_ξ *is equal to* $q^{\frac{(n-1)}{2}m'}(q^n-q^{n-f})^{-1}\prod_{k=1}^{n}(q^k-1)$.

PROOF. Since the dimension of the representation space of μ_ξ is $q^{\frac{1}{2}n(n-1)}$, the dimension of the representation space of ν_ξ is equal to $q^{\frac{1}{2}n(n-1)}\#(K/A\cdot K_l)$. We have

$$q^{\frac{1}{2}n(n-1)}\#(K/AK_l)=q^{2l\cdot\frac{n(n-1)}{2}}(q^n-q^{n-f})^{-1}\prod_{k=1}^{n}(q^k-1)$$

$$=q^{\frac{n(n-1)}{2}m'}(q^n-q^{n-f})^{-1}\prod_{k=1}^{n}(q^k-1)\,. \qquad \text{q. e. d.}$$

COROLLARY 2. *Using notations in* § 2, *we have*

$$r(\nu_\xi)=m'+1\,, \qquad s=l$$

and

$$O_{\nu_\xi}=\{\varphi_l(k\tau(x_\xi)k^{-1})\,;\ k\in K\}\,.$$

4-6. We assume that p is prime to n. We denote by ν_ξ an irreducible unitary representation of K constructed in 4-4 and 4-5, corresponding to a strongly regular character ξ of $\tilde{\mathfrak{O}}^*$. ξ is trivial on $\mathfrak{U}_m$ and is not trivial on $\mathfrak{U}_{m-1}$. We put $m=m'e+1$. There exists $x_\xi\in\tilde{\mathfrak{O}}$ such that $\tilde{\mathfrak{O}}=\mathfrak{O}[x_\xi]$ and that, for every $a\in\mathfrak{U}_{m-[\frac{m}{2}]}$, $\xi(a)=\chi(\pi^{-(m'+1)}\operatorname{trace} x_\xi(a-1))$. We use notations in 4-4 and 4-5. We also use notations in § 3.

THEOREM 4. *We assume that* $l=\left[\frac{m'+1}{2}\right]>e-1$ *and put* $U_\xi=\operatorname{Ind}_{K\uparrow G}\nu_\xi$. *Then* U_ξ *is a continuous irreducible unitary representation of* G.

PROOF. From Corollary to Lemma 3-5, we have

$$i(U_\xi|K,\nu_\xi)=\sum_{h\in H_+^0}i(\nu_\xi|K^h,\nu_\xi^h)\,.$$

Put $h=h(m_1,\cdots,m_n)\in H_+^0$. From Corollaries to Proposition 4-2 and 4-3, we have

$$O_{\nu_\xi}=\{\varphi_l(k\tau(x_\xi)k^{-1})\,;\ k\in K\}\,.$$

When $e=1$, $\det(t\cdot 1-\varphi_1(\tau(x_\xi)))$ is an irreducible polynomial over $\mathfrak{R}_1$ by Corollary 1 to Lemma 4-7. Hence, by Corollary to Lemma 3-10, $i(\nu_\xi|K^h,\nu_\xi^h)=0$ if $\operatorname{Max}_{1\leqq i\leqq n-1}(m_i-m_{i+1})\geqq 1$. Therefore $i(U_\xi|K,\nu_\xi)=i(\nu_\xi|K^1,\nu_\xi^1)=1$. When $e>2$, $\det(t\cdot 1-\varphi_2(\tau(x_\xi)))$ is an irreducible monic polynomial over $\mathfrak{R}_2$, by Corollary 2 to Lemma 4-7. Hence, by Corollary to Lemma 3-10, $i(\nu_\xi|K^h,\nu_\xi^h)=0$ when $\operatorname{Max}_{1\leqq i\leqq n-1}(m_i-m_{i+1})\geqq 2$. Now we assume that $\operatorname{Max}_{1\leqq i\leqq n-1}(m_i-m_{i+1})\leqq 1$ and that $i(\nu_\xi|K^h,\nu_\xi^h)>0$.

We assume that $m_j-m_{j+1}=1$ for some j $(1\leqq j\leqq n-1)$. From Lemma 3-9 and Lemma 3-10, $\det(t\cdot 1-\varphi_1(\tau(x_\xi)))$ decomposes into two polynomials over $\mathfrak{R}_1$ of degrees j and $n-j$. From Corollary 1 to Lemma 4-7, $\det(t\cdot 1-\varphi_1(\tau(x_\xi)))$ is an e-th power of an irreducible polynomial of degree f over $\mathfrak{R}_1$. Hence $j\equiv 0 \bmod f$. So we have $m_1-m_n\leqq e-1<l$. Since $i(\nu_\xi|K^h,\nu_\xi^h)>0$, we have, from Lemma 3-12, that there exist $k_1, k_2\in K$ such that

$$\varphi_{l-m_1+m_n}(h^{-1}k_1\tau(x_\xi)k_1^{-1}h)=\varphi_{l-m_1+m_n}(k_2\tau(x_\xi)k_2^{-1}).$$

Since $k_2\tau(x_\xi)k_2^{-1}$ is a quasi-regular element in $M(n, \mathfrak{O})$ by Corollary 1 to Lemma 4-7, $h^{-1}k\tau(x_\xi)k_1^{-1}h$ is also a quasi-regular element in $M(n, \mathfrak{O})$. From Corollary 4 to Lemma 2-1, there exists $k_3 \in K$ such that

$$k_3\tau(x_\xi)k_3^{-1}=h^{-1}k_1\tau(x_\xi)k_1^{-1}h.$$

From Lemma 4-8, the centralizer of $\tau(x_\xi)$ in G is contained in K. Therefore $k_1^{-1}hk_3 \in K$. Hence $h\in K\cap H_+^0=1$. We have

$$i(U_\xi|K, \nu_\xi)=i(\nu_\xi|K^1, \nu_\xi^1)=1.$$

From Corollary to Lemma 3-1, U_ξ is irreducible. q. e. d.

COROLLARY 1. *U_ξ is square integrable and the formal degree of U_ξ is*

$$q^{\frac{1}{2}n(n-1)m'}(q^n-q^{n-f})^{-1}\prod_{k=1}^{n}(q^k-1).$$

PROOF. This corollary follows from Lemma 3-2, Proposition 4-2 and Proposition 4-3.

COROLLARY 2. *Put*

$$U_\xi(f)=\int_G f(g)U_\xi(g)dg,$$

where f is a Schwartz-Bruhat function on G. Then $U_\xi(f)$ is of trace class and we have

$$\begin{aligned}\text{Trace } U_\xi(f)&=\int_G dg\int_K f(gkg^{-1})\,\text{trace } \nu_\xi(k)dk\\&=q^{\frac{n(n-1)}{2}(2l-1)}(q^n-q^{n-f})^{-1}\prod_{k=1}^{n}(q^k-1)\\&\quad\int_G dg\int_{A\cdot K_l} f(gkg^{-1})\,\text{trace } \mu_\xi(k)dk.\end{aligned}$$

PROOF. This follows from Lemma 3-11 and Lemma 3-3.

4-7. We assume that p is prime to n. We use notations in 4-2. Let $\tilde{P}_1, \tilde{P}_2 \in \mathcal{E}(P, n)$ ($\mathcal{E}(P, n)$ is a complete set of representatives of P-conjugate classess of extensions of P of degree n). Let ξ_i be a character of $\tilde{\mathfrak{O}}_i^*$ $(i=1, 2)$. ξ_i is trivial on $\mathfrak{U}_{m_i}^{(i)}$ and is not trivial on $\mathfrak{U}_{m_i-1}^{(i)}$ $(i=1, 2)$. In the following, we assume that ξ_1 and ξ_2 are both strongly regular. Then we can put $m_i=m_i'e_i+1$ $(m_i'\geqq 1)$ $(i=1, 2)$. There exists an element x_{ξ_i} of $\tilde{\mathfrak{O}}_i$ such that $\tilde{\mathfrak{O}}_i=\mathfrak{O}[x_{\xi_i}]$ and that

$$\xi_i(a)=\chi(\pi^{-(m_i'+1)}\,\text{trace}_{\tilde{P}_i/P}\, x_{\xi_i}(a-1)) \qquad \text{for every} \quad a\in \mathfrak{U}_{m_i-[\frac{m_i}{2}]}^{(i)} \qquad (i=1, 2).$$

We assume that

$$l_i=\left[\frac{m_i'+1}{2}\right]>e_i-1 \qquad (i=1, 2).$$

Let ν_{ξ_i} be the irreducible unitary representation of K corresponding to ξ_i constructed in 4-4 and 4-5 ($i=1, 2$). Put $U_{\xi_i}=\operatorname{Ind}_{K\uparrow G}\nu_{\xi_i}$ ($i=1, 2$). Then U_{ξ_i} is an irreducible unitary representation of G from Theorem 4 ($i=1, 2$).

PROSOSITION 4-4. *U_{ξ_1} and U_{ξ_2} are mutually equivalent if and only if ν_{ξ_1} and ν_{ξ_2} are mutually equivalent.*

PROOF. It is sufficient to prove that ν_{ξ_1} and ν_{ξ_2} are equivalent under the assumption that U_{ξ_1} and U_{ξ_2} are equivalent. The formal degree of U_{ξ_i} is equal to

$$q^{\frac{1}{2}n(n-1)m_i'}(q^n-q^{n-f_i})^{-1}\prod_{k=1}^{n}(q^k-1)$$

by Corollary 1 to Theorem 4. Hence we have $m_1'=m_2'$ and $f_1=f_2$. Let us compute $i(U_{\xi_1}|K, \nu_{\xi_2})$ (we use notations in § 3). From Corollary to Lemma 3-5, we have

$$i(U_{\xi_1}|K, \nu_{\xi_2})=\sum_{h\in H_+^0} i(\nu_{\xi_2}|K^h, \nu_{\xi_1}^h).$$

Repeating the argument in the proof of Theorem 4, we can prove

$$i(\nu_{\xi_2}|K^h, \nu_{\xi_1}^h)=0 \qquad \text{if} \quad h\neq 1.$$

Therefore we have $i(U_{\xi_1}|K, \nu_{\xi_2})=i(\nu_{\xi_2}|K^1, \nu_{\xi_1}^1)$. Since U_{ξ_1} and U_{ξ_2} are mutually equivalent, we have $i(U_{\xi_1}|K, \nu_{\xi_2})=i(U_{\xi_2}|K, \nu_{\xi_2})=1$. Hence we have $i(\nu_{\xi_2}|K^1, \nu_{\xi_1}^1)=1$. Therefore ν_{ξ_1} and ν_{ξ_2} are equivalent. q. e. d.

Now we study influences of outer automorphisms of G on U_ξ. Put

$$J_\pi=\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & \ddots & 1 \\ \pi & & & & 0 \end{pmatrix}.$$

We define the representation $U_{\xi_1}^{(j)}$ of G as follows ($1\leqq j\leqq n-1$):

$$U_{\xi_1}^{(j)}(g)=U_{\xi_1}(J_\pi^j g J_\pi^{-j}).$$

Then $U_{\xi_1}^{(j)}$ is obviously irreducible.

PROPOSITION 4-5. *If j is not divisible by f_1, $U_{\xi_1}^{(j)}$ and U_{ξ_2} are mutually inequivalent.*

PROOF. It is obvious that $U_{\xi_1}^{(j)}$ is square integrable and that the formal degree of $U_{\xi_1}^{(j)}$ is equal to that of U_{ξ_1}. Hence, $U_{\xi_1}^{(j)}$ and U_{ξ_2} is not mutually equivalent unless $m_1'=m_2'$ and $f_1=f_2$.

Now we assume that $m_1'=m_2'$ and $f_1=f_2$. Let us compute $i(U_{\xi_1}^{(j)}|K, \nu_{\xi_2})$. From Lemma 3-8, we have

$$i(U_{\xi_1}^{(j)}|K, \nu_{\xi_2})=\sum_{h\in H_+^j} i(\nu_{\xi_2}|K^h, \nu_{\xi_1}^h).$$

Put $h=h(s_1, s_2, \cdots, s_n)\in H_+^j$. Suppose $i(\nu_{\xi_2}|K^h, \nu_{\xi_1}^h)>0$. Repeating the argument in the proof of Theorem 4, we can prove that $s_k=s_{k+1}$ for every k not divisible by $f_1=f_2$. Therefore we have

$$j=\sum_{k=1}^{n} s_k=f_1(s_1+s_{f_1+1}+s_{2f_1+1}+\cdots+s_{(e_1-1)f_1+1}).$$

Hence j is divisible by f_1. This is impossible by the assumption. We have $i(\nu_{\xi_2}|K^h, \nu_{\xi_1}^h)=0$ for every $h\in H_+^j$. Therefore we have proved that $i(U_{\xi_1}^{(j)}|K^h, \nu_{\xi_2})=0$.

If we assume that $U_{\xi_1}^{(j)}$ and U_{ξ_2} are equivalent, we must have

$$i(U_{\xi_1}^{(j)}|K, \nu_{\xi_2})=i(U_{\xi_2}|K, \nu_{\xi_2})=1.$$

This is impossible. q. e. d.

4-8. We assume that p is prime to n. We return to the notation of 4-6. Put

$$SL(n, \mathfrak{O})=K'\subset K$$

and

$$SL(n, P)=G'\subset G.$$

We take $\tilde{P}\in\mathcal{E}(P, n)$ and put $A=\tau(\tilde{\mathfrak{O}}^*)$.

LEMMA 4-19. *The double coset space* $K'\backslash K/AK_r$ $(r\geqq 1)$ *is a finite set and we have*

$$\#(K'\backslash K/AK_r)=G.C.D.(q-1, e).$$

PROOF. We define the isomorphism d of $K'\backslash K$ onto $\mathfrak{O}^*$ by $d(k)=\det k$ $(k\in K)$. Then $K'\backslash K/AK_r$ can be identified with $\mathfrak{O}^*/d(AK_r)$.

For any $x\in\mathfrak{O}^*$, we have $d(\tau(x))=x^n$. Therefore $d(AK_r)\supset\{x^n;\ x\in\mathfrak{O}^*\}$. Since we assume that p is prime to n, we have $\{x^n;\ x\in\mathfrak{O}^*\}\supset 1+\mathfrak{P}$. Hence we have

$$\mathfrak{O}^*/d(AK_r)=\mathfrak{O}^*/(1+\mathfrak{P})/d(A)/(1+\mathfrak{P}),$$
$$=\mathfrak{R}_1^*/(\mathfrak{R}_1^*)^e,$$

where $(\mathfrak{R}_1^*)^e=\{x^e;\ x\in\mathfrak{R}_1^*\}$. q. e. d.

In the following we write

$$(q-1, e)=G.C.D.(q-1, e).$$

Take a representative system $k_1=1, k_2, \cdots, k_{(q-1,e)}$ of the double coset space $A\cdot K_l\backslash K/K'$ $\left(l=\left[\frac{m'+1}{2}\right]\right)$. For every i $(1\leqq i\leqq(q-1, e))$ we put

$$(AK_l)_i'=k_i^{-1}(AK_l\cap K')k_i.$$

We define a representation $(\mu_\xi^{14)})_i'$ of $(AK_l)_i'$ as follows:

$$(\mu_\xi)_i'(k)=\mu_\xi(k_ikk_i^{-1}) \qquad (k\in(AK_l)_i').$$

14) μ_ξ is the representation of $A\cdot K_l$ defined in Lemmas 4-15 and 4-18.

We denote by $(\chi_X^{m'+1})'_i$ a one-dimensional representation of $K'_{m'+1-l}=K_{m'+1-l}\cap K'$ defined as follows:

$$(\chi_X^{m'+1})'_i(k)=\chi(\pi^{-(m'+1)}\text{ trace }X\{k_i k k_i^{-1}-1\})\qquad (X=\tau(x_\xi),\ k\in K'_{m'+1-l}).$$

LEMMA 4-20. *$(\mu_\xi)'_i$ is an irreducible unitary representation of $(AK_l)'_i$ which coincides with $(\chi_X^{m'+1})'_i\cdot 1$ on $K'_{m'+1-l}$ $(1\leqq i\leqq (e,q-1))$.*

PROOF. It follows easily from the definition of μ_ξ that the restriction of μ_ξ to K_l is still irreducible. Since K_l is the direct product of $K'_l=K_l\cap K'$ and the center of K_l, the restriction of μ_ξ to K'_l is still irreducible. Therefore $(\mu_\xi)'_1$ is an irreducible unitary representation of $(AK_l)'_i$. Since μ_ξ coincides with $\chi_X^{m'+1}\cdot 1$ on $K_{m'+1-l}$, $(\mu_\xi)'_1$ coincides with $(\chi_X^{m'+1})'_1\cdot 1$ on $K'_{m'+1-l}$. Now the Lemma is obvious. q. e. d.

LEMMA 4-21. *The centralizer in K' of the one-dimensional representation $(\chi_X^{m'+1})'_i$ of $K'_{m'+1-l}$ is $(AK_l)'_i$ $(i=1,\cdots,(e,q-1))$.*

PROOF. This lemma follows immediately from Lemma 4-9.

LEMMA 4-22. *Put*

$$(\nu_\xi)'_i=\operatorname*{Ind}_{(AK_l)'_i\uparrow K'}(\mu_\xi)'_i\qquad (1\leqq i\leqq (e,q-1)).$$

Then $(\nu_\xi)'_1,(\nu_\xi)'_2,\cdots,(\nu_\xi)'_{(e,n-1)}$ are mutually inequivalent irreducible unitary representations of K'.

PROOF. It follows from Lemma 4-19, Lemma 4-20, Lemma 4-21 and Lemma 1-2 that $(\nu_\xi)'_i$ is an irreducible unitary representation of K' $(1\leqq i\leqq (e,q-1))$. Since $(\chi_X^{m'+1})'_i$ and $(\chi_X^{m'+1})'_j$ $(i\neq j)$ lie on different K'-orbits in the set of all one-dimensional representations of $K'_{m'+1-l}$, $(\nu_\xi)'_i$ and $(\nu_\xi)'_j$ are mutually inequivalent. q. e. d.

PROPOSITION 4-6. *We denote by ν'_ξ the representation of K' obtained by restricting ν_ξ to K'. Then ν'_ξ is equivalent to $\bigoplus_{j=1}^{(q-1,e)}(\nu_\xi)'_j$.*

PROOF. This proposition follows easily from "Subgroup Theorem" (see Curtis-Reiner [1] p. 324) and Lemma 4-22. q. e. d.

COROLLARY. *The dimension of the representation space of $(\nu_\xi)'_j$ is*

$$(q-1,e)^{-1}q^{\frac{1}{2}n(n-1)m'}(q^n-q^{n-f})^{-1}\prod_{k=1}^{n}(q^k-1)\qquad (1\leqq j\leqq (e,q-1)).$$

PROPOSITION 4-7. *We put*

$$(U_\xi)'_i=\operatorname*{Ind}_{K'\uparrow G'}(\nu_\xi)'_i\qquad (1\leqq i\leqq (e,n-1)).$$

When $l=\left[\frac{m'+1}{2}\right]>e-1$, $(U_\xi)'_i$ is a square-integrable irreducible unitary representation of G'.

PROOF. Proof of this proposition is an easy modification of the proof of theorem 4. q. e. d.

Corollary 1. *We normalize Haar measure $d'g$ on G' so that the volume of K' is 1. Then the formal degree of $(U_\xi)'_i$ is*

$$(q-1, e)^{-1}q^{\frac{1}{2}n(n-1)m'}(q^n-q^{n-f})^{-1}\prod_{k=1}^{n}(q^k-1) \qquad (1\leqq i\leqq (q-1, e)).$$

Corollary 2. *For a Schwartz-Bruhat function f on G' we put*

$$(U_\xi)'_i(f)=\int_{G'} f(g)(U_\xi)'_i(g)d'g .$$

Then $(U_\xi)'_i(f)$ is of trace class and we have

$$\mathrm{Trace}\,(U_\xi)'_i(f)=\int_{G'} d'g\int_{K'} f(g'kg'^{-1})\,\mathrm{trace}\,(\nu_\xi)'_i(k)d'k .$$

Since $K'\backslash G'$ is identified naturally with $K\backslash G$, we get the following proposition.

Proposition 4-8. *We denote by $(U_\xi)'$ the representation of G' obtained by restricting U_ξ to G'. Then $(U_\xi)'$ is equivalent to $\bigoplus_{i=1}^{(e,q-1)}(U_\xi)'_i$.*

Proposition 4-9. *$(U_{\xi_1})'_{i_1}$ and $(U_{\xi_2})'_{i_2}$ are mutually equivalent if and only if $(\nu_{\xi_1})'_{i_1}$ and $(\nu_{\xi_2})'_{i_2}$ are mutually equivalent (we assume that $l_i>e_i-1$ $(i=1, 2)$).*

Proof. Proof of this proposition is similar to that of Proposition 4-4.

q. e. d.

We define the representation $(U_{\xi_1})'_{i,r}$ of G' as follows:

$$(U_{\xi_1})'_{i,r}(g)=(U_{\xi_1})'_i(J_\pi^r g J_\pi^{-r})^{15)} .$$

When $l_1>e_1-1$, it follows from Proposition 4-7 that $(U_{\xi_1})'_{i,r}$ is an irreducible unitary representation of G'.

Proposition 4-10. *When r is not divisible by f_1, $(U_{\xi_1})'_{i,r}$ is not equivalent to $(U_{\xi_2})'_j$.*

Proof. Proof of this proposition is similar to that of Proposition 4-5.

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