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Riemannian manifolds with many geodesic loops

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Introduction.

This paper is the continuation of the previous one [8], in which we have investigated the (co)homology structure of an n ($\geqq 2$)-dimensional complete and connected Riemannian manifold M of class C^{∞} satisfying the conditions:

(d) there exists a point p such that all geodesics starting from p are geodesic loops,

(e) these geodesic loops are all of the same length $2l$.

The point p in the condition (d) is called the *basic point* and the constant $2l$ in the condition (e) is called the *loop length*. We may normalize suitably the Riemannian metric tensor in such a way that the maximum of the sectional curvature of M is equal to 1, since M is necessarily compact in our case. Then the loop length $2l$ is greater than or equal to π. The purpose of the present paper is to investigate the isometric structure of M under the most standard restrictions, that is, to prove the following

THEOREM. *Let M be an $n(\geqq 2)$-dimensional complete and connected Riemannian manifold satisfying the conditions* (d) *and* (e), *and suppose that the maximum of the sectional curvature is equal to* 1.

(1) *If $l=\pi/2$, then M is isometric to an n-dimensional real projective space $PR^n(1)$ with constant curvature* 1.

(2) *If $\pi/2<l<\pi$, then M has the same homology group as that of PR^n and the universal covering manifold of M is homeomorphic to a sphere.*

(3) *If $l=\pi$ in an odd dimensional simply connected M, then M is isometric to an n-dimensional sphere $S^n(1)$ with constant curvature* 1.

In § 1, we recall the fundamental theorem obtained in the previous paper [8] and prepare some results and notations for the later use. In § 2, we shall obtain a sufficient condition under which M is isometric to a sphere, and in the last section we shall prove the main theorem stated above.

§ 1. Preliminaries.

Throughout the paper, we assume that an $n(\geqq 2)$-dimensional complete and connected Riemannian manifold M satisfies the conditions (d) and (e) mentioned in the introduction. We refer to [8] for properties of M summarized in this section. Let γ be a geodesic loop at p parametrized by arc length in such a way that $\gamma(0)=p$. We denote by $\rho(X, Y)$ the plane section spanned by two vectors X and Y belonging to the tangent space $T(M)_x$ and G_M the union of all sets G_x, x being an arbitrary point of M. For a geodesic loop γ, we denote by $G_{\gamma(s)}$ the set of all plane sections $\rho(\gamma'(s), Y(s))$ at $\gamma(s)$, $Y(s)$ being an arbitrary vector field along γ, where $\gamma'(s)$ is the tangent vector of γ at $\gamma(s)$. We denote by $K(\rho)=K(X, Y)$ the sectional curvature corresponding to a plane section $\rho=\rho(X, Y)$, which is given by $K(X, Y)=-g(R(X, Y)X, Y)/\{g(X, X)g(Y, Y)-g(X, Y)^2\}$, where g and R denote the Riemannian metric tensor and the Riemannian curvature tensor on M, respectively. Let $K(M)$ be the set of all sectional curvatures $K(\rho)$, ρ being an arbitrary plane section. In the sequel, we normalize the Riemannian metric tensor in such a way that the relation $K(\rho)\leqq 1$ for $\rho\in G_M$ holds. Then we see easily that the conjugate distance from a point to its first conjugate point is greater than or equal to π. We denote by $Q(p)$ the locus of the first conjugate points of p and by $C(p)$ the cut locus, i. e., the locus of the minimal points of p along all geodesic loops emanating from p, respectively.

It has been proved in [8] that M is simply connected or the fundamental group $\pi_1(M)$ is of order 2. Concerning the relation between the fundamental group and the index of geodesic loops at p, we proved in [8]

THEOREM A. *Let M be an $n(\geqq 2)$-dimensional complete and connected Riemannian manifold satisfying the conditions* (d) *and* (e).

(1) *All geodesic loops at p are of the same index, which is equal to or less than $n-1$.*

(2) *There exists a geodesic loop at p of index zero if and only if the fundamental group of M is of order* 2.

(3) *There exists a geodesic loop at p of positive index if and only if M is simply connected.*

Concerning the (co)homology structure of M determined in the previous paper [8], we can state the following

THEOREM B. *Let M be an $n(\geqq 2)$-dimensional complete and connected Riemannian manifold satisfying the conditions* (d) *and* (e).

(1) *If M is simply connected, then the integral cohomology ring $H^*(M, Z)$ is a truncated polynomial ring generated by a unique element.*

(2) *If M is not simply connected, then M has the same (co)homology group as that of PR^n and the universal covering manifold of M is homeomorphic to a sphere.*

According to the cohomology theory (Adams [1] and Adem [2]), if the integral cohomology ring is a truncated polynomial ring generated by a unique element of dimension $\lambda+1$, then λ is necessarily equal to 1, 3, 7 or $n-1$, where n should be equal to 16 in the case $\lambda=7$. In particular, when n is odd, λ must be equal to $n-1$.

§ 2. The isometric structure.

In this section we shall consider a complete Riemannian manifold M, in which there exists a point p such that the cut locus $C(p)$ consists of a single point q. It is easily verified that M is compact and $C(q)$ consists also of a single point p. Thus, for the manifold M, the conditions (d) and (e) are satisfied and the index of each geodesic loop at p is equal to $n-1$. Therefore M is homeomorphic to a sphere. To determine the isometric structure of the manifold M, we shall prove the following

LEMMA. *If, in an $n(\geqq 2)$-dimensional complete and connected Riemannian manifold M, there exists a point p such that $C(p)$ consists of only one point q and $d(p, q)$, the distance between p and q, is equal to π, then we get $K(\rho)=1$ for any geodesic loop γ at p and any plane section ρ in G_γ.*

PROOF. We denote by $\bar{M}$ an n-dimensional sphere with constant curvature 1. Let $\bar{p}$ be an arbitrary but fixed point in $\bar{M}$ and ι_p an isometric isomorphism of $T(M)_p$ onto $T(\bar{M})_{\bar{p}}$. We define a mapping f of M onto $\bar{M}$ as follows: along a geodesic loop γ at p, the mapping f assigns to any point $\gamma(s)$ $(0\leqq s<\pi)$ in $M-\{q\}$ a point $\exp_{\bar{p}}(s\iota_p(\gamma'(0)))$ in $\bar{M}-\{\bar{q}\}$ and to the point q the fixed point $\bar{q}$, which is an antipodal point of $\bar{p}$. It is obvious that f is bijective, and f restricted to the domain $M-\{q\}$ is diffeomorphic. Hereafter any quantity in $\bar{M}$ corresponding to a quantity β in M under the mapping f is expressed by the corresponding symbol $\bar{\beta}$ with " $-$ ". We assume now that there exists a point $y=\gamma(s_1)$ $(0\leqq s_1<\pi)$ on a geodesic loop γ at p and a plane section ρ in $G_{\gamma(s_1)}$ such that $K(\rho)<1$. Then we define a mapping ϕ_s of $T(M)_{\gamma(s)}$ into $T(\bar{M})_{\bar{\gamma}(s)}$ as follows: for the geodesic $\bar{\gamma}(s)=f(\gamma(s))$ $(0\leqq s<\pi)$, ϕ_s assigns to any tangent vector X in $T(M)_{\gamma(s)}$ a tangent vector $\bar{\tau}_s^0(\iota_p(\tau_0^s X))$ in $T(\bar{M})_{\bar{\gamma}(s)}$, where τ_0^s denotes the parallel translation along γ from $\gamma(s)$ to $\gamma(0)$. Thus, the mapping ϕ_s is given by $\phi_s=\bar{\tau}_s^0\circ\iota_p\circ\tau_0^s$, which is necessarily an isometric isomorphism of $T(M)_{\gamma(s)}$ onto $T(\bar{M})_{\bar{\gamma}(s)}$. A mapping ϕ of $\mathfrak{X}(M)_\gamma$ into $\mathfrak{X}(\bar{M})_{\bar{\gamma}}$ is defined by $\phi X(s)=\phi_s(X(s))$ for each vector field $X(s)$ along γ, where $\mathfrak{X}(M)_\gamma$ is the vector space consisting of all vector fields along γ over the real number field. Calculating the index form $I(X(s), X(s))$ for a vector field $X(s)$ orthogonal to $\gamma'(s)$, we get easily $I(X(s), X(s))>I(\phi X(s), \phi X(s))$ along $\gamma([0, \pi])$. Since there are no conjugate points of p on $\gamma((0, \pi))$ and the multiplicity of p and q as conjugate points is equal to $n-1$, there is a non-zero Jacobi field $Y(s)$ along γ such that $Y(0)=Y(\pi)=0$ and $Y(s_1)\in\rho$. Then, taking account of the fact that the Jacobi field $Y(s)$ along γ is orthogonal to $\gamma'(s)$, we get $I(Y(s), Y(s))=0$ along $\gamma([0, \pi])$, and combining together two results obtained above, we get $I(\phi Y(s), \phi Y(s))<0$. By means of the isometric property of ϕ_s, we have $\phi Y(0)=\phi Y(\pi)=0$. Consequently, for a 1-parameter variation $\alpha(t, s)=\exp_{\bar{\gamma}(s)}(t\phi Y(s))$ $(-\varepsilon<t<\varepsilon,\ 0\leqq s\leqq\pi)$ of $\bar{\gamma}$ with variation vector field $\phi Y(s)$, there is a variation curve whose length is less than that of $\bar{\gamma}([0, \pi])$. This contradicts the choice of the segment γ. Consequently we get $K(\rho)=1$ for each ρ in $G_{\gamma(s_1)}$ $(0\leqq s_1<\pi)$. By means of the continuity of the Riemannian curvature, Lemma is proved completely.

Taking account of the lemma mentioned above, we shall prove the following

THEOREM. *If, in an $n(\geqq 2)$-dimensional complete and connected Riemannian manifold M, there exists a point such that $C(p)$ consists of only one point q and $d(p, q)=\pi$, then M is isometric to an n-dimensional sphere $S^n(1)$ with constant curvature* 1.

PROOF. In order to prove the theorem, it is sufficient to show that f is distance-preserving, that is, it satisfies $d(x, y)=\bar{d}(f(x), f(y))$ for arbitrary two points x and y in M [7]. For any point x in $M-\{q\}$, let γ be a geodesic loop at p passing through x such that $x=\gamma(s)$. Let $U(p, \pi)$ be an open ball in $T(M)_p$ with center at the origin and with radius π. Since the exponential mapping $\exp_p$ has the maximal rank in $U(p, \pi)$, there is a vector A in $T(T(M)_p)_{s\gamma'(0)}$ such that $A=(d\exp_p)^{-1}X$ for each vector X in $T(M)_x$. From $f\circ\exp_p=\exp_{\bar{p}}\circ\iota_p$, we get $\bar{g}(dfX, dfX)=\bar{g}(d\exp_{\bar{p}}(d\iota_pA), d\exp_{\bar{p}}(d\iota_pA))$. Taking account of the fact that $\exp_{\bar{p}}$ has the maximal rank in $\iota_pU(p, \pi)$, and by virtue of the lemma stated above, we can apply the Rauch's comparison theorem [**10**] for M and $S^n(1)$, and hence we get $\bar{g}(dfX, dfX)=g(X, X)$. Thus f restricted to $M-\{q\}$ is an isometry. By means of the continuity of the distance function this shows that f is distance-preserving everywhere. This completes the proof of the theorem.

REMARK 2.1. We see that if the first conjugate locus $Q(p)$ consists of a single point q which is different from p, then conjugate distances at p are constant [**9**]. This implies that $C(p)$ consists also of only one point q. Thus the condition stated in the theorem is equivalent, for a simply connected M, to the property that there is a point p such that $Q(p)$ consists of a single point.

REMARK 2.2. For a 2-dimensional ovaloid, we know the following Klingenberg's theorem [**6**]: if, in a 2-dimensional compact and simply connected Riemannian manifold N with $0<K(N)\leqq 1$, there exists a closed geodesic of length 2π, then N is isometric to a 2-dimensional sphere $S^2(1)$ with constant curvature 1. According to his proof, the theorem stated in this section is proved also

in the case $n=2$.

REMARK 2.3. We note here that M is not necessarily assumed to be of positive curvature. We know [3] that adding the suitable pinching condition, a generalization of the Klingenberg's theorem for the ovaloid is proved. Then there is no need for the assumption of the structure concerning the cut locus for the manifold.

REMARK 2.4. When in a 2-dimensional M each point has its cut locus consisting of a single point, M is simply connected Wiedersehensfläche [4] and hence is isometric to $S^2(1)$.

§ 3. The loop length.

In this section we shall prove the theorem stated in the introduction. First of all, we shall prove the following

THEOREM 3.1. *If, in an $n(\geqq 2)$-dimensional complete and connected Riemannian manifold M satisfying the conditions* (d) *and* (e), *the inequality $\pi/2\leqq l<\pi$ holds, then M has the same (co)homology group as that of PR^n and the universal covering manifold of M is homeomorphic to a sphere S^n.*

PROOF. In order to prove this theorem, it is sufficient by virtue of the second assertion of the fundamental theorem B to show that M is not simply connected. We assume now that there is a geodesic loop γ at p such that $\gamma(s')$ $(0<s'<2l)$ is a conjugate point of p along γ. Then, in the inverse geodesic loop γ^{-1} at p defined by $\gamma^{-1}(s)=\gamma(2l-s)$, the point $\gamma^{-1}(2l-s')=\gamma(s')$ is also conjugate to $p=\gamma^{-1}(0)$ along γ^{-1}, because of the condition (e). Accordingly we have $s'\geqq\pi$ and $2l-s'\geqq\pi$ and consequently $l\geqq\pi$. This contradicts the assumption of the theorem. Thus all geodesic loops at p are of index 0. By means of the second assertion of Theorem A, this means that M is not simply connected.

Taking account of the proof stated above and of the well known properties of the cut locus [5], we find easily

LEMMA 3.2. *If $l=\pi$, then the distance from p to each point in $C(p)$ is equal to π.*

It played an essential role in the proof of Lemma in the section 2 that $C(p)$ coincides with $Q(p)$ and the multiplicity is equal to $n-1$. Taking account of this fact and Lemma 3.2 and repeating the similar discussion to that developed in the proof of Lemma given in § 2, we have

LEMMA 3.3. *If $l=\pi$ and there exists a geodesic loop at p of index $n-1$, then we get $K(\rho)=1$ for any geodesic loop γ at p and any plane section ρ in G_γ.*

Making use of Lemmas 3.2 and 3.3, we shall prove

THEOREM 3.4. *If, an $n(\geqq 2)$-dimensional complete and connected Riemannian* *manifold M satisfying the conditions* (d) *and* (e), *there exists a geodesic loop at p of index $n-1$ and the equality $l=\pi$ holds, then M is isometric to an n-dimensional sphere $S^n(1)$ with constant curvature* 1.

PROOF. The theorem stated in § 2 shows that in order to prove Theorem 3.4 it is sufficient to show that $C(p)$ consists of only one point. Because of the condition (d), $C(p)$ contains at least one point, say q. We assume now that $C(p)$ does not coincide with $\{q\}$. Then, taking account of the properties of the cut locus, we see that for a sufficiently small number δ there is a spherical neighbourhood U with center at q and with radius δ such that the intersection of $U-\{q\}$ and $C(p)-\{q\}$ is not empty. For any point x in $U\cap C(p)-\{q\}$, let σ be a minimal geodesic segment joining q and x in such a way that $\sigma(0)=q$ and $\sigma(t_0)=x$, and let y be the point on σ which is the closest to p. Taking a geodesic loop γ at p passing through y, we put $y=\sigma(t_1)=\gamma(s_1)$ $(0\leqq t_1\leqq t_0, \pi-\delta<s_1\leqq\pi)$. Then, by means of the property of Lemma 3.2 and the Gauss' lemma, we see that γ is orthogonal to σ at y. We consider now a 1-parameter variation α of γ defined by $\alpha(t, s)=\exp_{\gamma(s)}(tX(s))$, where $X(s)=\sin(s/2)\tau_s^{s_1}\sigma'(t_1)$. If we calculate the first variation $L'(0)$ and the second variation $L''(0)$ with respect to the variation α, and take account of the fact that $X(s)$ is orthogonal to γ, we get $L'(0)=0$ and $L''(0)=(5\sin s_1-3s_1)/8$. δ being sufficiently small, we get the inequality $L''(0)<0$.

On the other hand, the locus of final points of the variation α lies in the segment σ. This implies that $\exp_p U(p, s_1)\cap\sigma=\{\phi\}$, which means that $L(t)\geqq L(0)$ for a sufficiently small t. This contradicts $L''(0)<0$. Thus $C(p)$ consists of a single point q. This completes the proof.

As a direct consequence of Theorem 3.4, we have

COROLLARY 3.5. *If $l=\pi$ in a 2-dimensional simply connected M, then M is isometric to a 2-dimensional sphere $S^2(1)$ with constant curvature* 1.

REMARK 3.1. Corollary 3.5 is closely related to Klingenberg's theorem for a 2-dimensional ovaloid mentioned in Remark 2.2.

In a simply connected M, the first assertion of Theorem B shows that the integral cohomology ring $H^*(M, Z)$ is a truncated polynomial ring generated by a unique element of dimension $\lambda+1$. In the case n is odd, as remarked in the section 1, λ must be equal to $n-1$. This implies that all geodesic loops at p are of index $n-1$. Thus, as a direct consequence of Theorem 3.4, we have

COROLLARY 3.6. *If $l=\pi$ in an odd dimensional simply connected M, then M is isometric to an n-dimensional sphere $S^n(1)$ with constant curvature* 1.

REMARK 3.2. It is well known that a complex (quaternion or Cayley) projective space with canonical Riemannian metric such that the maximum of the sectional curvature is equal to 1 is an even dimensional simply connected and complete Riemannian manifold satisfying the property that all geodesics are closed and of the same length 2π.

Therefore we can not remove from Corollary 3.6 the assumption that M is of odd dimension. By means of the structure of these examples it seems to the author that if, in an even dimensional complete, simply connected and connected Riemannian manifold M satisfying the conditions (d) and (e), the equality $l=\pi$ holds, then M might be isometric to one of the simply connected symmetric spaces of rank 1 with canonical Riemannian metric such that $1/4 \leqq K(\rho) \leqq 1$ for any plane section ρ.

Combining Theorems 3.1 and 3.4 together, we can prove

COROLLARY 3.7. *If $l=\pi/2$, then M is isometric to an n-dimensional real projective space PR^n with constant curvature* 1.

PROOF. Since Theorem 3.1 shows that M is not simply connected, the fundamental group of M is of order 2. In order to prove the corollary, it is sufficient to show that the universal covering manifold $\tilde{M}$ of M is isometric to $S^n(1)$. We denote by $\{\tilde{p}_1, \tilde{p}_2\}$ the inverse image of p under the covering mapping. Making use of the properties of covering spaces and of the given assumptions, we see that $\tilde{M}$ satisfies the conditions (d) and (e) and that the points $\tilde{p}_1$ and $\tilde{p}_2$ are basic and the loop length of $\tilde{p}_i$ is equal to $4l=2\pi$. Summing up, we know that $\tilde{M}$ satisfies the assumptions of Theorem 3.4. Therefore $\tilde{M}$ is isometric to $S^n(1)$. Thus Corollary 3.7 is proved.

Taking account of Theorem 3.1 and Corollaries 3.6 and 3.7 we complete the proof of the theorem mentioned in the introduction.

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Bibliography

[1] J. F. Adams, On the non-existence of elements of Hopf invariant one, Ann. of Math., **72** (1960), 20-104.

[2] J. Adem, Relations on iterated reduced powers, Proc. Nat. Acad. Sci. U. S. A., **39** (1953), 636-638.

[3] M. Berger, Les variétés riemanniennes 1/4-pinchees, Ann. Scuola Nor. Sup. Pisa, **14** (1960), 161-170.

[4] L. W. Green, Auf Wiedersehensfläche, Ann. of Math., **78** (1963), 289-299.

[5] W. Klingenberg, Contributions to Riemannian geometry in the large, Ann. of Math., **69** (1959), 654-666.

[6] W. Klingenberg, Neue Ergebnisse über konvexe Flächen, Comm. Math. Helv., **34** (1960), 17-36.

[7] S. B. Myers and N. E. Steenrod, The group of isometries of a Riemannian manifold, Ann. of Math., **40** (1939), 400-416.

[8] H. Nakagawa, A note on theorems of Bott and Samelson, J. Math. Kyoto Univ., **7** (1967), 205-220.

[9] T. Ōtsuki, On focal elements and the spheres, Tôhoku Math. J., **17** (1965), 285-304.

[10] H. E. Rauch, A contribution to differential geometry in the large, Ann. of Math., **54** (1951), 38-55.