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On the uniqueness of solutions of the global Cauchy problem for a Kowalevskaja system

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§1. Introduction.

Consider a Kowalevskaja type system of partial differential equations

$$\frac{\partial u(t, x)}{\partial t}=L_t u \equiv \sum_{k=1}^{n} A_k(t, x) \frac{\partial u}{\partial x_k}+B(t, x)u\,, \tag{1.1}$$

where $u(t, x)$ is an m dimensional unknown vector function of $t \in \boldsymbol{R}^1$ and $x=(x_1, \cdots, x_n) \in \boldsymbol{R}^n$ and $A_k(t, x)$ $(k=1, \cdots, n)$ and $B(t, x)$ are $m \times m$ matrices. We assume that the components of A_k and B are functions of t and x which are analytic in x.

In order to state the hypotheses to be imposed on A_k and B more exactly, however, we need to introduce some notations. For an n dimensional vector $x=(x_1, \cdots, x_n)$ we put $\|x\|=\max_{1 \leqq i \leqq n}|x_i|$ and for a positive number β we denote by Ω_β *the strip domain* in the complex n dimensional space $\boldsymbol{C}^n$ defined by

$$\Omega_\beta=\{x+iy \mid x,\ y \in \boldsymbol{R}^n \text{ and } \|y\|<\beta\}\,.$$

We assume that the components of the matrices A_k and B are complex valued functions of $(t, z) \in [0, T) \times \Omega_\beta$ for some $T>0$ and $\beta>0$ which are regular analytic in z for any fixed t.

The purpose of this paper is to prove the following

THEOREM. *Let T and β be any positive numbers. Suppose that the components of A_k are bounded in $[0, T) \times \Omega_\beta$ and uniformly continuous in (t, z) there and that the components $b_{ij}(t, z)$ of B satisfy in $[0, T) \times \Omega_\beta$ the inequalities*

$$|b_{ij}(t, x+iy)| \leqq B_0 e^{a\|x\|} \tag{1.2}$$

for some positive constants B_0, a and $e^{-a\|x\|} b_{ij}(t, z)$ are uniformly continuous in (t, z) there. Let $v_i(t, x)$ $(i=1, \cdots, n)$ be functions of $(t, x) \in (0, T) \times \boldsymbol{R}^n$ which are measurable in x for any t and satisfy

$$|v_i(t, x)| \leqq C \exp(c e^{a\|x\|}) \qquad (i=1, \cdots, n) \tag{1.3}$$

almost everywhere for some positive constants C, c and a[1]. *Then, if*

$$u(t, x)=D_x^q v(t, x)\,, \tag{1.4}$$

where $q=(q_1, \cdots, q_n)$ *is any multi-index,* $D_x^q=\partial^{q_1+\cdots+q_n}/\partial x_1^{q_1}\cdots\partial x_n^{q_n}$ *and* $v(t, x)=(v_1(t, x), \cdots, v_n(t, x))$ *is a distribution solution*[2] *for* $0<t<T$ *of the Cauchy problem for the equation* (1.1) *with the homogeneous initial condition*

$$u(0, x)=0\,, \tag{1.5}$$

then $u(t, x)=0$ *as a distribution on the x-space for any* $t\in(0, T)$.

The above result is an improvement on that of M. Yamamoto's [2], [3] in a slightly imcomplete sense. Our result improves the theorem of Yamamoto in that we relax the boundedness condition on B in [2], [3] and that we establish the uniqueness of solutions in a wider class of distributions than in [2], [3] (only solutions of form (1.3)—(1.4) with $q=0$ were considered in [2], [3]). The above mentioned improvement is, however, incomplete in that the uniformity of the continuity in (t, z) of A_k and B were not assumed in [2], [3].

The above theorem is regarded also as an improvement on a theorem in an earlier paper [5] (Theorem 5.1) of the author's, where on A_k and B more restrictive and less natural conditions than in the above theorem were imposed. Note however, that the problem of the speed of propagation is not discussed in the present paper unlike in [5].

The remaining part of this paper is divided into four sections. From § 2 to § 4 are the preparations for § 5, where the proof of the theorem is completed.

In closing the introduction the author wishes to express his warmest thanks to Professor M. Nagumo, Dr. Y. Kōmura and Dr. S. Yosida for their valuable suggestions and constant encouragements.

§ 2. Function spaces.

In this section we shall introduce various function spaces which will play important roles in this paper and discuss the interrelations among them.

1. A notation. Suppose that E, F are topological spaces (or *pseudo* topological spaces in the sense that only the concepts of the convergence of the sequences of elements are introduced in them) such that $E\subset F$, that E is dense in F and that F induces a (pseudo) topology on E which is the same as or weaker than the original topology of E. In such a case we shall write

$$E\prec F\,.$$

1) The a's in (1.2) and (1.3) may be different, but there is no loss of generality in assuming that they coincide with each other.

2) We shall define in § 3 the exact meaning of *a distribution solution* of the Cauchy problem.

If E is a linear (pseudo) topological space, then we shall denote by E' the dual space of E, i. e. the totality of all continuous linear functionals on E. If two linear (pseudo) topological spaces E and F satisfy the relation $E \prec F$, we can regard F' as a subset of E' by means of an obvious identification mapping.

2. The spaces $\mathcal{D}$ and $\mathcal{B}$. We denote by $\mathcal{D}$ the totality of all infinitely differentiable complex valued functions on $\boldsymbol{R}^n$ with compact supports and by $\mathcal{B}$ the totality of all infinitely differentiable complex valued functions on $\boldsymbol{R}^n$ all whose partial derivatives are bounded on $\boldsymbol{R}^n$. These two function spaces are those which play fundamental roles in L. Schwartz' distribution theory and we introduce the same (pseudo) topologies in these spaces as in the distribution theory. In particular, we shall say that a sequence of elements $\varphi_j(x)$ $(j=1, 2, \cdots)$ of $\mathcal{B}$ converges to 0 in $\mathcal{B}$ as $j \to \infty$, if and only if for any multi-index q the functions $D^q\varphi_j(x)$ $(j=1, 2, \cdots)$ are uniformly bounded on $\boldsymbol{R}^n$ and the sequence $D^q\varphi_j(x)$ converges to 0 as $j \to \infty$ uniformly on any compact subset of $\boldsymbol{R}^n$. According to L. Schwartz' theory we shall call an element of the space $\mathcal{D}'$ a distribution (on $\boldsymbol{R}^n$).

The following lemma is what was proved in Yamanaka [**5**] and is necessary in this paper, too.

LEMMA 2.1. *Let F be a locally convex linear topological space consisting of infinitely differentiable functions on $\boldsymbol{R}^n$ such that $\mathcal{D} \prec F$ and which satisfies the following two conditions:*

[2.1] If $\varphi(x) \in F$ and $\psi(x) \in \mathcal{B}$, then $\varphi(x)\psi(x) \in F$;

[2.2] If $\varphi(x) \in F$ and if $\psi_j(x) \in \mathcal{B}$ $(j=1, 2, \cdots)$ converges to 0 in $\mathcal{B}$ as $j \to \infty$, then $\varphi(x)\psi_j(x)$ converges to 0 as $j \to \infty$ in F.

Next let E be a linear subspace of F satisfying the following three conditions:

[2.3] E is non-trivial, i. e. E contains at least one element $\varphi(x) \not\equiv 0$;

[2.4] If $\varphi(x) \in E$ and $h \in \boldsymbol{R}^n$, then $\varphi(x-h) \in E$;

[2.5] If $\varphi(x) \in E$ and $\sigma \in \boldsymbol{R}^n$, then $\varphi(x)e^{i(x,\sigma)} \in E$, where $(x, \sigma) = x_1\sigma_1 + \cdots + x_n\sigma_n$.

Under the above mentioned conditions the linear subspace E is dense in F.

For the proof see [**5**], § 2, 3 (pp. 70-73).

3. The space $\mathcal{B}_{\alpha,a}$. Let a be a positive constant and let α be a real constant. We denote by $\mathcal{B}_{\alpha,a}$ or simply by $\mathcal{B}_\alpha$ (The letter a may be omitted, since its value is fixed arbitrarily. On the contrary α is a parameter whose value always moves.) the totality of all infinitely differentiable functions on $\boldsymbol{R}^n$ such that for any multi-index q and for any α' with $\alpha' < \alpha$

$$\sup_{x \in R^n} |D^q\varphi(x)| \cdot \exp(e^{a(\|x\|+\alpha')}) < \infty .$$

We can regard the space $\mathcal{B}_\alpha$ as a $K\{M_p\}$ type space in the sense of Gel'fand

and Šilov [**6**]. In fact, if we put

$$M_p(x)=\exp\left(e^{a(\|x\|+\alpha-1/p)}\right) \tag{2.1}$$

for $p=1, 2, \cdots$, then the two spaces $\mathscr{B}_{\alpha,a}$ and $K\{M_p\}$ obviously coincide with each other as simple sets. Therefore we introduce into the space $\mathscr{B}_\alpha$ the topology of the space $K\{M_p\}$ defined by (2.1). In other words, the topology of the space $\mathscr{B}_\alpha$ is defined by the countable system of norms

$$\|\varphi\|_{(p)}=\sup_{\substack{x\in \boldsymbol{R}^n \\ |q|_1\leqq p}} |D^q\varphi(x)|\cdot\exp\left(e^{a(\|x\|+\alpha-1/p)}\right) \qquad (p=1, 2, \cdots),$$

where $q=(q_1, \cdots, q_n)$ and $|q|_1=q_1+\cdots+q_n$. Moreover, note that the functions $M_p(x)$ defined by (2.1) clearly satisfy the condition for the perfectness

$$\lim_{|x|\to\infty}\frac{M_p(x)}{M_{p+1}(x)}=0\,. \tag{2.1$'$}$$

Hence (see [**6**], Vol. 2, p. 123), the space $\mathscr{B}_\alpha$ contains the space $\mathscr{D}$ as a dense subset. It is clear that $\mathscr{B}_\alpha$ induces on $\mathscr{D}$ a topology which is weaker than the original one of $\mathscr{D}$. Hence we have the relations $\mathscr{D}\prec\mathscr{B}_\alpha$ and $(\mathscr{B}_\alpha)'\subset\mathscr{D}'$. Further, if $u(x)$ is a measurable function on $\boldsymbol{R}^n$ such that for some $\alpha'<\alpha$

$$|u(x)|\leqq C\exp\left(e^{a(\|x\|+\alpha')}\right), \tag{2.2}$$

then the functional on $\mathscr{D}$ defined by

$$\int_{\boldsymbol{R}^n} u(x)D^q\varphi(x)dx \qquad (\varphi(x)\in\mathscr{D})$$

is clearly continuous with respect to the topology of $\mathscr{B}_\alpha$. Hence, a distribution of the form $D^qu(x)$, where $u(x)$ is a measurable function satisfying an inequality of form (2.2), is an element of the space $(\mathscr{B}_\alpha)'$. It is also clear that, if $u(t, x)$ is a measurable function of $x\in\boldsymbol{R}^n$ with a parameter t satisfying an inequality of the form

$$|u(t, x)|\leqq C\exp\left(e^{a(\|x\|+\alpha')}\right)$$

independently of t, then the distribution $D_x^q u(t, x)$ on $\boldsymbol{R}^n$ runs through a bounded set of the space $(\mathscr{B}_\alpha)'$ as the parameter t moves.

Also, as a further consequence of the relation (2.1′), we have the following fact (see [**6**], Vol. 2, Chap. 2, § 2, 3):

A sequence $\varphi_j(x)$ $(j=1, 2, \cdots)$ of elements of the space $\mathscr{B}_\alpha$ converges to 0 in $\mathscr{B}_\alpha$ as $j\to\infty$, if and only if the set $\{\varphi_j(x)\}$ of elements of $\mathscr{B}_\alpha$ is bounded with respect to the topology of $\mathscr{B}_\alpha$ and the sequence of functions $D^q\varphi_j(x)$ $(j=1, 2, \cdots)$ converges to 0 as $j\to\infty$ uniformly on any compact subset of $\boldsymbol{R}^n$ for any multi-index q.

From this fact we further conclude that

if $\psi_j(x)$ $(j=1, 2, \cdots)$ is a sequence of elements of the space $\mathscr{B}$ and converges *to* 0 *in* $\mathscr{B}$ *as* $j\to\infty$, *then for any* $\varphi(x)\in\mathscr{B}_\alpha$ *the sequence* $\varphi(x)\psi_j(x)$ $(j=1, 2, \cdots)$ *converges to* 0 *in* $\mathscr{B}_\alpha$ *as* $j\to\infty$.

Thus the space $\mathscr{B}_\alpha$ satisfies the conditions [2.1] and [2.2] of Lemma 2.1.

4. Continuous families of Banach spaces. Let Λ be a (finite or infinite) real interval whose left end is open and let there correspond to each $\lambda\in\Lambda$ a Banach space $\tilde{\Phi}_\lambda$ with a norm $\|\varphi\|_\lambda$ in such a way that, if $\lambda, \mu\in\Lambda$ and $\lambda\leqq\mu$, then $\tilde{\Phi}_\mu\subset\tilde{\Phi}_\lambda$ and $\|\varphi\|_\lambda\leqq\|\varphi\|_\mu$ for any $\varphi\in\tilde{\Phi}_\mu$. Then for each $\lambda\in\Lambda$ we put

$$\Phi_\lambda=\bigcap_{\substack{\lambda'<\lambda\\ \lambda'\in\Lambda}}\tilde{\Phi}_{\lambda'} \tag{2.3}$$

and introduce into the linear space Φ_λ the topology that is the weakest among those for which all the inclusion mappings $\Phi_\lambda\to\tilde{\Phi}_{\lambda'}$ $(\lambda'<\lambda,\ \lambda'\in\Lambda)$ are continuous. In other words, Φ_λ is a locally convex linear topological space for which the family of the countable subsets

$$U_n=\left\{\varphi\in\Phi_\lambda \,\middle|\, \|\varphi\|_{\lambda-1/n}<\frac{1}{n}\right\} \qquad (n=1, 2, \cdots)$$

form a fundamental system of neighbourhoods of the origin. The completeness of the space Φ_λ follows directly from the completeness of the spaces $\tilde{\Phi}_{\lambda'}$ and the relation (2.3). Hence Φ_λ is what Bourbaki calls a Fréchet space or what Gel'fand and Šilov call a countably normed space (except that the condition of the concordance of norms is not necessarily satisfied).

5. The space $\tilde{\mathscr{A}}^\beta_{\alpha,a}$ and $\mathscr{A}^\beta_{\alpha,a}$. Let α be a real constant and let β, a be two positive constants. For a regular function $\varphi(z)$ on the complex m dimensional strip domain Ω_β we define a norm $\|\varphi\|^\beta_{\alpha,a}$ (or simply $\|\varphi\|^\beta_\alpha$) by

$$\|\varphi\|^\beta_{\alpha,a}=\sup_{z\in\Omega_\beta}|\varphi(z)|\exp\left(e^{a(\|x\|+\alpha)}\right)$$

and we denote by $\tilde{\mathscr{A}}^\beta_{\alpha,a}$ (or simply by $\tilde{\mathscr{A}}^\beta_\alpha$) the totality of all regular functions $\varphi(z)$ on Ω_β for which the norm $\|\varphi\|^\alpha_{\beta,a}$ take finite values. It can be shown that the space $\tilde{\mathscr{A}}^\beta_{\alpha,a}$ is non-trivial, if and only if $\alpha\beta<\pi/2$. For the proof of this fact see Mandelbrojt [1], chap. II, 2, i.

It is clear that $\tilde{\mathscr{A}}^\beta_{\alpha,a}$ becomes a Banach space with respect to the norm $\|\varphi\|^\beta_{\alpha,a}$ and also that, if $\alpha'\leqq\alpha$ and $0<\beta'\leqq\beta$, then

$$\tilde{\mathscr{A}}^\beta_{\alpha,a}\subset\tilde{\mathscr{A}}^{\beta'}_{\alpha',a} \quad\text{and}\quad \|\varphi\|^{\beta'}_{\alpha',a}\leqq\|\varphi\|^\beta_{\alpha,a} \qquad (\varphi(z)\in\tilde{\mathscr{A}}^\beta_{\alpha,a})\,.$$

Hence we can constract a family of countably normed spaces $\{\mathscr{A}^\beta_{\alpha,a}\}$ from the family of Banach spaces $\{\tilde{\mathscr{A}}^\beta_{\alpha,a}\}$, just like the construction of Φ_λ from $\tilde{\Phi}_\lambda$ in the preceding subsection, by putting

$$\mathscr{A}^\beta_{\alpha,a}\ (\text{or } \mathscr{A}^\beta_\alpha)=\bigcap_{\substack{\alpha'<\alpha\\ 0<\beta'<\beta}}\tilde{\mathscr{A}}^{\beta'}_{\alpha',a}\,.$$

The topology of the space $\mathscr{A}^\beta_\alpha$ is defined by the countable system of norms

$\|\varphi\|^{\beta-1/p}_{\alpha-1/p}$ ($p=p_0,\ p_0+1,\ \cdots$; p_0 being a natural number such that $\beta-1/p_0>0$).

Now let us compare two spaces $\mathcal{B}_{\alpha,a}$ and $\mathcal{A}^{\beta}_{\alpha',a}$ with $\alpha\leqq\alpha'$. Of course we can regard $\mathcal{A}^{\beta}_{\alpha',a}$ as a subset of $\mathcal{B}_{\alpha,a}$. It is easily verified that the topology of $\mathcal{A}^{\beta}_{\alpha',a}$ is stronger than the one which is induced by $\mathcal{B}_{\alpha,a}$ (see Lemma 5.3). By that lemma we see that for any natural number p there exists a natural number p' and a positive constant $C_{pp'}$ such that $\|\varphi\|_{(p)}\leqq C_{pp'}\|\varphi\|^{\beta-1/p'}_{\alpha'-1/p'}$ for any $\varphi\in\mathcal{A}^{\beta}_{\alpha',a}$). As already mentioned, for any $a>0$ we can take a positive number β for which $\mathcal{A}^{\beta}_{\alpha',a}$ is non-trivial. For such a pair of values of a and β, therefore, the condition [2.3] of Lemma 2.1 is satisfied. It is almost self-evident that $\mathcal{A}^{\beta}_{\alpha'}$ satisfies the conditions [2.4] and [2.5]. Also we already know that the space $\mathcal{B}_{\alpha,a}$ satisfies the conditions [2.1] and [2.2]. Hence all the conditions necessary in order to apply Lemma 2.1 are now satisfied and we conclude that

a non-trivial space $\mathcal{A}^{\beta}_{\alpha',a}$ is dense in the space $\mathcal{B}_{\alpha,a}$ with $\alpha\leqq\alpha'$ and accordingly the relations

$$\mathcal{A}^{\beta}_{\alpha',a}\prec\mathcal{B}_{\alpha,a}\quad\text{and}\quad(\mathcal{B}_{\alpha,a})'\subset(\mathcal{A}^{\beta}_{\alpha',a})'$$

hold.

§3. Classical solution, weak solution and distribution solution of the Cauchy problem.

1. Conventional notation and phraseology. Let Φ be a set. If $\varphi=(\varphi_j)$ is a vector of length, say, m whose components φ_j belong to Φ, then we say that φ is a vector (of length m) in Φ and write simply $\varphi\in\Phi$. If Φ is a function space, then we call a vector in Φ a function vector in Φ. By analogy we shall also use such expressions as a functional vector, a distribution vector etc. In general, if all components φ_j of $\varphi=(\varphi_j)$ have a common property P, then we say that the vector φ has the property P. Let Φ be a linear topological space and let Φ' be the dual space of Φ. We denote the scalar product of $u\in\Phi'$ and $\varphi\in\Phi$ by $\langle u,\varphi\rangle$. For a vector $u=(u_j)\in\Phi'$ and a vector $\varphi=(\varphi_j)\in\Phi$ we put $\langle u,\varphi\rangle=\sum_j\langle u_j,\varphi_j\rangle$.

2. Classical solution of the Cauchy problem. We say that $u(t,x)$ is a classical solution for $0\leqq t<T$ of the Cauchy problem (1.1)—(1.5), if and only if $u(t,x)$ is a function vector of length m which is continuous with respect to (t,x) on $[0,T)\times\boldsymbol{R}^n$, is continuously differentiable with respect to t and x on $(0,T)\times\boldsymbol{R}^n$ and actually satisfies (1.1) (for $(t,x)\in(0,T)\times\boldsymbol{R}^n$) and (1.5) (for $x\in\boldsymbol{R}^n$).

3. Weak solution and distribution solution of the Cauchy problem. For a function vector $\varphi(x)$ of length m which is defined and differentiable on $\boldsymbol{R}^n$ we put

$$L_t^*\varphi(x)=-\sum_{k=1}^{n}\frac{\partial({}^tA_k(t,\,x)\varphi(x))}{\partial x_k}+{}^tB(t,\,x)\varphi(x)\,,$$

where ${}^tA_k(t, x)$ and ${}^tB(t, x)$ are the transposed matrices of $A_k(t, x)$ and $B(t, x)$ respectively.

Now let Φ be a linear topological space consisting of complex valued functions on $\boldsymbol{R}^n$ such that, if $\varphi(x)$ is a function vector in Φ of length m, then $L_t^*\varphi(x)\in\Phi$. We say that

$u(t, x)$ is a weak solution for $0<t<T$ of the Cauchy problem (1.1)-(1.5) with respect to the space Φ, *if and only if $u(t, x)$ is a functional vector $\in\Phi'$ of length m for each t, $0<t<T$, and for any function vector $\varphi(x)\in\Phi$ of length m*

$$\frac{d}{dt}\langle u(t,\,x),\,\varphi(x)\rangle=\langle u(t,\,x),\,L_t^*\varphi(x)\rangle \qquad (0<t<T) \tag{3.1}$$

and

$$\lim_{t\to+0}\langle u(t,\,x),\,\varphi(x)\rangle=0\,. \tag{3.2}$$

In particular, since the matrices $A_k(t, x)$ and $B(t, x)$ are in the C^∞ class with respect to x, a weak solution of the problem (1.1)-(1.5) with respect to the space $\mathcal{D}$ is defined. We call a weak solution with respect to $\mathcal{D}$ a distribution solution. It is clear that any classical solution is a distribution solution.

4. Relation between the concepts of weak solutions with respect to different spaces. Suppose that the concepts of weak solutions with respect to two spaces Φ and Φ_1 such that $\Phi\prec\Phi_1$ are defined. The following proposition gives a sufficient condition in order that a weak solution with respect to Φ is at the same time a weak solution with respect to Φ_1.

PROPOSITION 3.1. *Suppose that Φ_1 is a countably normed space and suppose that $L_t^*\varphi(x)$ depends continuously both on $\varphi(x)$ and t, $0\leqq t<T$, with respect to the topology of Φ_1. Then, if $u(t, x)$ is a weak solution for $0<t<T$ of the Cauchy problem* (1.1)-(1.5) *with respect to Φ and if $u(t, x)$ stays in a bounded set of Φ' as t moves in $(0, T)$, then $u(t, x)$ is a weak solution for $0<t<T$ of the problem* (1.1)-(1.5) *with respect to Φ_1, too.*

PROOF. Take a function vector $\varphi(x)\in\Phi_1$ of length m arbitrarily. We can take a sequence $\varphi_j(x)$ $(j=1, 2, \cdots)$ of function vectors $\in\Phi$ which converges to $\varphi(x)$ as $j\to\infty$ in Φ_1. Since $L_t^*\varphi(x)$ depends continuously on $\varphi(x)$ in Φ_1, $L_t^*\varphi_j(x)$ converges to $L_t^*\varphi(x)$ as $j\to\infty$ in Φ. And since the weak solution $u(t, x)$ belongs to Φ_1', we have

$$\lim_{j\to\infty}\langle u(t,\,x),\,\varphi_j(x)\rangle=\langle u(t,\,x),\,\varphi(x)\rangle \tag{3.3}$$

and

$$\begin{aligned}\lim_{j\to\infty}\frac{d}{dt}\langle u(t,\,x),\,\varphi_j(x)\rangle&=\lim_{j\to\infty}\langle u(t,\,x),\,L_t^*\varphi_j(x)\rangle\\&=\langle u(t,\,x),\,L_t^*\varphi(x)\rangle\,.\end{aligned} \tag{3.4}$$

Further, since $u(t, x)$ is in a bounded set of Φ_1' for $0<t<T$ and since the initial and strong topologies coincide with each other in the countably normed space Φ_1, the convergence in (3.4) is uniform with respect to t, $0<t<T$. It is also easily seen that

$$(d/dt)\langle u(t, x), \varphi_j(x)\rangle = \langle u(t, x), L_t^*\varphi_j(x)\rangle$$

is continuous with respect to t as a consequence of the continuous dependence on t of $L_t^*\varphi_j(x)$ in Φ_1. Hence we can apply *the termwise differentiation theorem* in the calculus and conclude that the limit function $\langle u(t, x), \varphi(x)\rangle$ of the sequence $\langle u(t, x), \varphi_j(x)\rangle$ is differentiable with respect to t, $0<t<T$ and

$$\frac{d}{dt}\langle u(t, x), \varphi(x)\rangle = \langle u(t, x), L_t^*\varphi(x)\rangle . \tag{3.5}$$

Next let us examine the limit relation (3.2). Let $\varphi(x)$ and $\varphi_j(x)$ $(j=1, 2, \cdots)$ be the same as above. Since $u(t, x)$ is a weak solution with respect to Φ, we have for $j=1, 2, \cdots$

$$\lim_{t\to+0} \langle u(t, x), \varphi_j(x)\rangle = 0 . \tag{3.6}$$

On the other hand $u(t, x)$ is in a bounded set of Φ_1' and a bounded set of the dual space Φ_1' of a countably normed space Φ_1 is equi-continuous. Hence for any given positive number ε we can find a neighbourhood U of the origin of the space Φ_1 such that $\psi(x)\in U$ implies $|\langle u(t, x), \psi(x)\rangle| < \varepsilon$ for $0<t<T$. For the given element $\varphi(x)$ of Φ_1 and for the above chosen neighbourhood U take an index j such that $\varphi(x)-\varphi_j(x)\in U$ and fix it. For this j we can take a positive number τ such that, if $0<t<\tau$, then $|\langle u(t, x), \varphi_j(x)\rangle| < \varepsilon$, because of (3.6). Therefore $0<t<\tau$ implies

$$|\langle u(t, x), \varphi(x)\rangle| \leqq |\langle u(t, x), \varphi(x)-\varphi_j(x)\rangle| + |\langle u(t, x), \varphi_j(x)\rangle| < 2\varepsilon .$$

This shows that (3.2) holds for any $\varphi(x)\in\Phi_1$. q. e. d.

REMARK. We can relax the hypotheses in the above proposition that Φ_1 is a countably normed space. It will suffice for our purpose to assume that Φ_1 is a tonnelé space in Bourbaki's sense. In that case we only have to replace in the proof the word *sequence* by the word *filter*.

5. Weak solution with respect to the space $\mathscr{B}_\alpha$ and distribution solution. Under the conditions on $A_k(t, x)$ and $B(t, x)$ in Theorem, it is easy to see that $L_t^*\varphi(x)$ is defined for $\varphi(x)\in\mathscr{B}_{\alpha,a}$ and $0\leqq t<T$ and depends continuously on $\varphi(x)$ and t in $\mathscr{B}_{\alpha,a}$. Further we know that $\mathscr{D}\prec\mathscr{B}_{\alpha,a}$. Hence, by virtue of Prop. 3.1 we obtain the following

PROPOSITION 3.2. *Suppose that $A_k(t, x)$ and $B(t, x)$ satisfy the conditions in Theorem. Then, if $v(t, x)=(v_j(t, x))$ is a function vector of length m which is measurable in x for $0<t<T$ and satisfies an inequality of the form*

$$\| v(t, x)\| \leqq C\exp(e^{a(\|x\|+\alpha')}) \qquad (\alpha'<\alpha)$$

almost everywhere with respect to x for $0<t<T$ *and if* $u(t, x)=D_x^q v(t, x)$ *where q is any multi-index is a distribution solution for* $0<t<T$ *of the Cauchy problem* (1.1)-(1.5), *then* $u(t, x)$ *is a weak solution for* $0<t<T$ *of the problem* (1.1)-(1.5) *with respect to* $\mathcal{B}_{\alpha,a}$, *too.*

§ 4. Strong solutions of the Cauchy problem, Holmgren's principle.

1. Let Φ be a linear topological space consisting of complex valued functions on $\boldsymbol{R}^n$, let I be a real interval and let t_0 be a real number $\in I$. We say that a function vector $\varphi(t, x)=(\varphi_1(t, x), \cdots, \varphi_m(t, x))$ whose components are complex valued functions of $(t, x) \in I \times \boldsymbol{R}^n$ is a strong solution on I of the Cauchy problem

$$\begin{cases} \dfrac{\partial \varphi}{\partial t}=-L_t^* \varphi, & (4.1) \\ \varphi(t_0, x)=\varphi_0(x) & (4.1') \end{cases}$$

with respect to the space Φ, if and only if $\varphi(t, x)$ is a function vector $\in \Phi$ for any t, $t \in I$, which depends continuously on t in Φ, satisfies (4.1′) and

$$\frac{\partial \varphi(t, x)}{\partial t}=\lim_{\substack{h \to 0 \\ t+h \in I}} \frac{\varphi(t+h, x)-\varphi(t, x)}{h}=-L_t^* \varphi,$$

where the limit is taken in the sense of the topology of Φ.

Between the existence of strong solutions of the Cauchy problem (4.1)-(4.1′) with respect to Φ and the uniqueness of the weak solution of the Cauchy problem (1.1)-(1.5) with respect to Φ there is the following remarkable relation called Holmgren's principle.

PROPOSITION 4.1. *Suppose that* Φ *and* $\Phi_{(1)}$ *are linear topological spaces consisting of complex valued functions on* $\boldsymbol{R}^n$ *satisfying the relation* $\Phi \prec \Phi_{(1)}$. *Let* $\Phi_{(0)}$ *be a subset of* Φ *which is dense in* $\Phi_{(1)}$ *(but not necessarily in* Φ*). Then, if there exists a positive number* τ *such that for any real interval* $[t_1, t_0]$ *with* $|t_0-t_1|<\tau$ *and* $[t_1, t_0] \subset [0, T)$ *and for any function vector* $\varphi_0(x) \in \Phi_{(0)}$ *the Cauchy problem* (4.1)-(4.1′) *has a strong solution on* $[t_1, t_0]$ *with respect to* Φ, *then the only weak solution for* $0<t<T$ *of the Cauchy problem* (1.1)-(1.5) *with respect to* $\Phi_{(1)}$ *is* $u(t, x) \equiv 0$.

The above proposition being only a slight modification of a theorem of Gel'fand and Šilov's ([**6**], Vol. 3, chap. 2, § 3, 2), the reader is requested to consult with [**6**], Vol. 3 for its proof.

2. A sufficient condition for the existence of strong solutions. Let $\{\tilde{\Phi}_\lambda\}_{\lambda \in \Lambda}$ be a continuous family of Banach spaces such as discussed in § 2, 4. We suppose that $\tilde{\Phi}_\lambda$ consists of complex valued functions on $\boldsymbol{R}^n$. From $\{\tilde{\Phi}_\lambda\}$ we construct a continuous family of countably normed spaces $\{\Phi_\lambda\}$ by means of

(2.3). For a function vector $\varphi(x)=(\varphi_1(x), \cdots, \varphi_m(x))\in\Phi_\lambda$ we put

$$\|\varphi\|_\lambda=\max_{1\leqq i\leqq m}\|\varphi_i\|_\lambda,$$

$\|\varphi_i\|_\lambda$ being the norm of $\varphi_i(x)$ in the space Φ_λ.

Let L_t^* be a linear operator matrix from $\tilde{\Phi}=\bigcup_\lambda\tilde{\Phi}_\lambda$ into itself for any t, $0<t<T$, such that if $\lambda<\mu$ and $\lambda, \mu\in\Lambda$, then L_t^* is a bounded linear operator matrix from $\tilde{\Phi}_\mu$ into $\tilde{\Phi}_\lambda$. (Note, accordingly, that L_t^* is a continuous linear operator matrix from Φ_λ into itself for any $\lambda\in\Lambda$.) We suppose further that, for any $\varphi\in\tilde{\Phi}_\mu$, $L_t^*\varphi$ depends continuously on t in $\tilde{\Phi}_\lambda$ with $\lambda<\mu$ and

$$\|L_t^*\varphi\|_\lambda\leqq\frac{C}{\mu-\lambda}\|\varphi\|_\mu \tag{4.3}$$

for some constant C which is independent of λ, μ and φ.

PROPOSITION 4.2. *Let λ, μ be any number such that $\lambda, \mu\in\Lambda$ and $\lambda<\mu$. Put $\Phi=\Phi_\lambda$ and $\Phi_{(0)}=\Phi_\mu$. Under the above mentioned conditions on the operator matrix L_t^* the Cauchy problem (4.1)-(4.1′) has a strong solution with respect to Φ for any $\varphi_0\in\Phi_{(0)}$, and any $t_0\in(0, T)$ on any t-interval I such that $t_0\in I\subset[0, T)$ and $|I|<(\mu-\lambda)/Ce$, $|I|$ being the length of I.*

For the proof see [**4**].

§ 5. Proof of the Theorem.

Now let $u(t, x)$ be a distribution solution in the sense explained in § 3 for $0<t<T$ of the Cauchy problem (1.1)-(1.5) which is of the form (1.3)-(1.4). For the constant c in (1.3) take a constant α such that $e^{a\alpha}>c$. Then it follows from Prop. 3.2 that $u(t, x)$ is also a weak solution of the Cauchy problem with respect to the space $\mathscr{B}_{\alpha,a}$. In order to prove our theorem, therefore, it is enough to establish the uniqueness of the weak solution of the Cauchy problem with respect to $\mathscr{B}_{\alpha,a}$.

On the other hand, if we put $\Phi_{(1)}=\mathscr{B}_{\alpha,a}$, $\Phi=\mathscr{A}_{\alpha,a}^\beta$ and $\Phi_{(0)}=\mathscr{A}_{\alpha',a}^{\beta'}$, where $\alpha'-\alpha=\beta'-\beta>0$ and $\mathscr{A}_{\alpha'}^{\beta'}$ is nontrivial, then it follows from the discussions in § 2, 5 that these three spaces $\Phi_{(0)}$, Φ and $\Phi_{(1)}$ satisfy the hypotheses in Prop. 4.1. In order to establish the uniqueness of the weak solution of the Cauchy problem (1.1)-(1.5) with respect to $\mathscr{B}_{\alpha,a}$, therefore, it is enough to show the existence of a strong solution of the Cauchy problem (4.1)-(4.1′) with respect to the space $\mathscr{A}_{\alpha,a}^\beta$. Let us show the existence of a strong solution by means of Prop. 4.2. For this purpose it is necessary to prepare the following three lemmas.

LEMMA 5.1. *If $A_k(t, x)$ $(k=1, \cdots, n)$ satisfy the hypotheses in Theorem, then for any $\varphi(x)=(\varphi_j(x))\in\tilde{\mathscr{A}}_{\alpha,a}^\beta$ the products ${}^tA_k(t, x)\varphi(x)$ are again in $\tilde{\mathscr{A}}_\alpha^\beta$, depend continuously on t in $\tilde{\mathscr{A}}_\alpha^\beta$ and there exists a positive constant C_A not* *depending on* $\varphi(x)$ *such that*

$$\|{}^tA_k(t, x)\varphi(x)\|_\alpha^\beta \leqq C_A\|\varphi\|_\alpha^\beta \qquad (k=1, \cdots, n). \tag{5.1}$$

This is almost self-evident.

LEMMA 5.2. *If* b_{ij} (t, x)'*s satisfy the hypotheses in Theorem and if* $\alpha<\alpha'$, *then for any* $\varphi(x)\in \tilde{\mathcal{A}}_{\alpha',a}^\beta$ ${}^tB(t, x)\varphi$ $(t\in[0, T))$ *is in* $\tilde{\mathcal{A}}_\alpha^\beta$, *depends continuously on* t *in* $\tilde{\mathcal{A}}_\alpha^\beta$ *and*

$$\|{}^tB(t, x)\varphi\|_\alpha^\beta \leqq \frac{mB_0a^{-1}e^{-a\alpha}}{\alpha'-\alpha}\|\varphi\|_{\alpha'}^\beta. \tag{5.2}$$

PROOF. Suppose $\varphi(z)\in\tilde{\mathcal{A}}_{\alpha',a}^\beta$. For $z\in\Omega_\beta$ we have

$$\|\varphi(z)\| \leqq \|\varphi\|_\alpha^\beta \exp(-e^{a(\|x\|+\alpha')}).$$

From this inequality and (1.2)

$$\begin{aligned}\|{}^tB(t, z)\varphi(z)\| &\leqq mB_0\|\varphi\|_{\alpha'}^\beta \exp(-e^{a(\|x\|+\alpha')}+a\|x\|)\\ &\leqq mB_0\|\varphi\|_{\alpha'}^\beta \exp(-e^{a(\|x\|+\alpha)}-(e^{a\alpha'}-e^{a\alpha})e^{a\|x\|}+a\|x\|).\end{aligned} \tag{5.3}$$

But, if we put $p=e^{a\alpha'}-e^{a\alpha}$ and $s=a\|x\|$ in the easily provable inequality

$$-pe^s+s<-\log p \qquad (p>0,\ s\text{—real}),$$

we obtain from (5.3)

$$\begin{aligned}\|{}^tB(t, z)\varphi(z)\| &\leqq mB_0\|\varphi\|_{\alpha'}^\beta \exp(-e^{a(\|x\|+\alpha)}-\log(e^{a\alpha'}-e^{a\alpha}))\\ &=\frac{mB_0\|\varphi\|_{\alpha'}^\beta}{e^{a\alpha'}-e^{a\alpha}}\exp(-e^{a(\|x\|+\alpha)})\\ &\leqq \frac{mB_0\|\varphi\|_{\alpha'}^\beta}{(\alpha'-\alpha)ae^{a\alpha}}\exp(-e^{a(\|x\|+\alpha)}),\end{aligned}$$

from which (5.2) is obtained directly. The fact that ${}^tB(t, x)\varphi$ depends continuously on t in $\tilde{\mathcal{A}}_\alpha^\beta$ is also easily proved by using (1.2) and the uniform continuity of $e^{-a\|x\|}b_{ij}(t, z)$. q. e. d.

LEMMA 5.3. *If* $\alpha'-\alpha=\beta'-\beta>0$, *then for any* $\varphi(x)\in\tilde{\mathcal{A}}_{\alpha'}^{\beta'}$ $\dfrac{\partial\varphi(x)}{\partial x_i}$ $(i=1, 2, \cdots, n)$ *is in* $\tilde{\mathcal{A}}_\alpha^\beta$ *and*

$$\left\|\frac{\partial\varphi}{\partial x_i}\right\|_\alpha^\beta \leqq \frac{1}{\alpha'-\alpha}\|\varphi\|_{\alpha'}^{\beta'} \qquad (i=1, \cdots, n). \tag{5.4}$$

PROOF. We shall prove the lemma for the case $n=1$ only, since the proof for the general case is quite similarly performed except for some notational complications.

Suppose $\varphi(x)\in\tilde{\mathcal{A}}_{\alpha'}^{\beta'}$. For $z\in\Omega_{\beta'}$ we have

$$|\varphi(z)| \leqq \|\varphi\|_{\alpha'}^{\beta'} \exp(-e^{a(|x|+\alpha')}). \tag{5.5}$$

Using (5.5) we shall estimate the absolute value of the derivative $\varphi'(z)=\varphi'(x+iy)$ on the segment $AB=\{x+iy \mid x=x^0,\ |y|\leqq\beta\}$, where x^0 is an arbitrary real

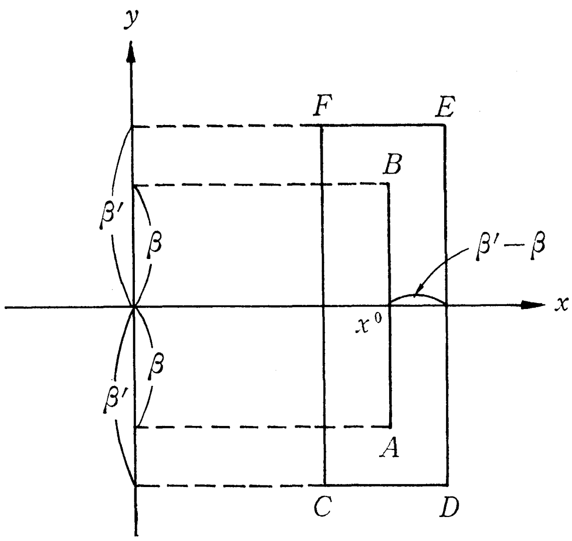

Fig. 5.1.

number, in the complex plane (see Fig. 5.1). Consider a rectangle $CDEF$ as shown in Fig. 5.1 drawn around the segment AB. The distance from each point of AB to the boundary of the rectangle $CDEF$ is equal to $\beta'-\beta=\alpha'-\alpha$. Therefore, if we put

$$M=\sup\{|\varphi(z)| \mid z\in \text{the interior of the rectangle}\},$$

then we have, by Cauchy's integral formula,

$$|\varphi'(z)|\leqq\frac{M}{\alpha'-\alpha}\qquad(z\in AB). \tag{5.6}$$

But, if $z=x+iy$ is in the interior of the rectangle, then $|x|>|x^0|-(\alpha'-\alpha)$. Hence we obtain from (5.5)

$$\begin{aligned} M&\leqq \|\varphi\|_{\alpha'}^{\beta'}\exp(-e^{\alpha(|x^0|-(\alpha'-\alpha)+\alpha')})\\ &=\|\varphi\|_{\alpha'}^{\beta'}\exp(-e^{\alpha(|x^0|+\alpha)}). \end{aligned} \tag{5.7}$$

Combining (5.7) with (5.6) and considering the arbitrariness of x^0 we have for any $z=x+iy\in\Omega_\beta$

$$|\varphi'(z)|\leqq\frac{1}{\alpha'-\alpha}\|\varphi\|_{\alpha'}^{\beta'}\exp(-e^{\alpha(|x|+\alpha)}),$$

from which (5.4) is obtained directly. q. e. d.

From the above three lemmas we conclude that, *if* $\alpha'-\alpha=\beta'-\beta>0$ *and* $\varphi(x)\in\mathcal{A}_{\alpha',a}^{\beta'}$, *then* $L_t^*\varphi$ *is in* $\mathcal{A}_\alpha^\beta$ *for* $0\leqq t<T$, *depends continuously on* t *in* $\mathcal{A}_\alpha^\beta$ *and*

$$\|L_t^*\varphi\|_\alpha^\beta\leqq\frac{nC_A+mB_0a^{-1}e^{-a\alpha}}{\alpha'-\alpha}\|\varphi\|_{\alpha'}^{\beta'}.$$

Now for the given α, α', β and β' such that $\alpha'-\alpha=\beta'-\beta>0$ set

$$\tilde{\Phi}_\lambda = \tilde{\mathcal{A}}^{\beta'+\lambda}_{\alpha'+\lambda} \qquad (0 \leqq \lambda \leqq \alpha' - \alpha)$$

and $C = nC_A + mB_0 a^{-1} e^{-a\alpha}$. Then for any λ, μ such that $0 \leqq \lambda < \mu \leqq \alpha' - \alpha$ we have the inequality (4.3), where $\|\varphi\|_\lambda = \|\varphi\|^{\beta'+\lambda}_{\alpha'+\lambda}$. Therefore the operator matrix L_t^* completely satisfies the hypotheses in Prop. 4.2. Hence the Cauchy problem (4.1)-(4.1′) has a strong solution with respect to $\Phi = \mathcal{A}^{\beta}_{\alpha,a}$ for any $\varphi_0 \in \Phi_{(0)} = \tilde{\mathcal{A}}^{\beta'}_{\alpha',a}$ and any $t_0 \in (0, T)$ on any t-interval I such that $t_0 \in I \subset [0, T)$ and $|I| < (\alpha' - \alpha)/(Ce)$.

By the discussions at the beginning of this section the proof of Theorem is now completed.

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References

[1] S. Mandelbrojt, Series adherentes, regularisation des suites, applications, Gauthier-Villars, Paris, 1952.

[2] M. Yamamoto, On Cauchy's problem for a linear system of partial differential equations of first order, Proc. Japan Acad., 17 (1966), 555-559.

[3] M. Yamamoto, On the Cauchy problem for Kowalewski systems, Osaka J. Math., 4 (1967), 5-13.

[4] T. Yamanaka, Note on Kowalevskaja's system of partial differential equations, Comment. Math. Univ. St. Paul., 9 (1961), 7-10.

[5] T. Yamanaka, On the Cauchy problem for Kowalevskaja systems of partial differential equations, Comment. Math. Univ. St. Paul., 15 (1967), 67-89.

[6] И. М. Гельфанд и Г. Е. Шилов, Обобщенные Функции, Вып. 2 и 3, Москва, 1958.