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Remarks on pseudo-differential operators*

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§ 0. Introduction.

In a recent paper [2] Hörmander defined pseudo-differential operators through a function class $S^m_{\rho,\delta}(\Omega)$, $0\leqq\delta$, $0<\rho$, for an open set Ω in R^n. We say $p(x;\xi)\in S^m_{\rho,\delta}(\Omega)$, when $p(x;\xi)$ belongs to C^∞ $(R^n\times R^n)$ and, for every compact set $K\subset\Omega$ and all α, β, there exist constants $C_{\alpha,\beta,K}$ such that

$$|\partial_x^\alpha\partial_\xi^\beta p(x;\xi)|\leqq C_{\alpha,\beta,K}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}, \qquad x\in K,\quad \xi\in R^n,$$

where $\alpha=(\alpha_1,\cdots,\alpha_n)$ and $\beta=(\beta_1,\cdots,\beta_n)$ are multi-indices whose elements are non-negative integers and

$$\partial_{x_j}=\frac{\partial}{\partial x_j},\qquad \partial_{\xi_j}=\frac{\partial}{\partial\xi_j},\qquad \partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n},\qquad \partial_\xi^\beta=\partial_{\xi_1}^{\beta_1}\cdots\partial_{\xi_n}^{\beta_n},$$

$$|\alpha|=\alpha_1+\cdots+\alpha_n,\qquad |\beta|=\beta_1+\cdots+\beta_n.$$

In the present paper we shall study the H_s theory of pseudo-differential operators for the special case: $0\leqq\delta<\rho\leqq1$, $\Omega=R^n$ and $C_{\alpha,\beta,K}=C_{\alpha,\beta}$ (independent of K). In this case Hörmander [2] proved an inequality of the form

$$\|p(X;D_x)u\|_0\leqq C_p\|u\|_0,$$

when $m=0$, and Lax-Nirenberg [7] proved a sharp form of Gårding's inequality:

$$\mathcal{R}e\,(\boldsymbol{p}(X;D_x)\boldsymbol{u},\boldsymbol{u})\geqq -K\|\boldsymbol{u}\|_0^2,$$

when $m=1$, $\rho=1$ and $\delta=0$. But we must remark here that it is complicated to derive the corresponding inequalities when m is an arbitrary real number and the $\|\cdot\|_0$ norm is replaced by the $\|\cdot\|_s$ norm for real s. In the present note the space $\mathcal{B}$, i.e., the set of C^∞ functions in R^n (or $R^n\times R^n$) whose derivatives are all bounded, plays an important role.

In Section 1 we define the operator class $\mathcal{S}^m_{\rho,\delta}$, $0\leqq\delta<\rho\leqq1$, and, through it, the class $\mathcal{L}^m_{\rho,\delta}$ of pseudo-differential operators. The main theorems, which

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are found in Friedrichs [1], Kohn-Nirenberg [6], and Lax-Nirenberg [7], will be stated here (and proved in Section 3).

In Section 2 we prove the basic asymptotic expansion theorems concerning adjoints and products of operators of class $S^m_{\rho,\delta}$. Here we shall often make use of operators $p(X; D_x|X_1)$ of multiple symbol which are found in [1] and [8]. The method of Kuranishi (to appear) will be applied in the asymptotic expansion theorem for the behavior of operators of class $S^m_{\rho,\delta}$ under coordinate transformation, when $1-\rho\leqq\delta<\rho\leqq 1$.

Section 3 is devoted to the proofs of the theorems of Section 1, making use of the results of Section 2.

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§ 1. Definitions and Main Theorems.

Let $\mathcal{B}$ be the space of complex valued C^∞ functions, defined in R^n, whose derivatives are all bounded, and let $\mathcal{S}$ be the subspace of $\mathcal{B}$ consisting of functions, together with all their derivatives, which die down faster than any power of $|x|$ at infinity. $\mathcal{S}'$ denotes the dual space of $\mathcal{S}$.

For $u\in\mathcal{S}$ we define the Fourier transform of u by

$$\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx\,, \qquad x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n\,, \tag{1.1}$$

and for any real number s we define the norm $\|u\|_s$ by

$$\|u\|_s^2=\int\langle\xi\rangle^{2s}|\hat{u}(\xi)|^2 d\!\!{}^-\xi\,. \tag{1.2}$$

Here we used the Friedrichs notation in [1]:

$$\langle\xi\rangle=(1+|\xi|^2)^{1/2}\,, \qquad d\!\!{}^-\xi=(2\pi)^{-n}d\xi\,.$$

By H_s we denote the Hilbert space obtained as the completion of $\mathcal{S}$ in the norm $\|\cdot\|_s$, and set

$$H_{-\infty}=\bigcup_s H_s\,, \qquad H_\infty=\bigcap_s H_s\,.$$

For $u\in H_s$ and $v\in H_{-s}$, the inner product (u, v) is defined by

$$(u, v)=\int\hat{u}(\xi)\overline{\hat{v}(\xi)}d\!\!{}^-\xi\,.$$

DEFINITION 1.1. For any real number r we define an operator $\Lambda^r: H_{s+r}\to H_s$ by

$$\widehat{\Lambda^r u}(\xi)=\langle\xi\rangle^r\hat{u}(\xi)\,.$$

We have easily

$$\|u\|_s = \|\Lambda^s u\|_0, \qquad \|u\|_{s_1} \leqq \|u\|_{s_2} \qquad \text{for} \quad s_1 \leqq s_2 .$$

DEFINITION 1.2. i) For any real number m, we denote by $S^m_{\rho,\delta}$, $0 \leqq \delta < \rho \leqq 1$, the set of functions $p(x\,;\,\xi)$ which belong to C^∞ $(R^n \times R^n)$ and satisfy with constants $C_{\alpha,\beta}$

$$(1.3) \qquad |\partial_x^\alpha \partial_\xi^\beta p(x\,;\,\xi)| \leqq C_{\alpha,\beta} \langle \xi \rangle^{m+\delta|\alpha|-\rho|\beta|} \quad \text{in} \quad R^n \times R^n$$

for all α, β, and set

$$S^{-\infty} = \bigcap_m S^m \quad \text{where} \quad S^m = S^m_{1,0}.$$

ii) For $p(x\,;\,\xi) \in S^m_{\rho,\delta}$ we define an operator $p(X\,;\,D_x)$, which is called to be of class $\mathcal{S}^m_{\rho,\delta}$, by

$$(1.4) \qquad p(X\,;\,D_x)u(x) = \int e^{ix\cdot\xi} p(x\,;\,\xi)\hat{u}(\xi)d\xi\,, \qquad u \in \mathcal{S}\,,$$

and set

$$\mathcal{S}^{-\infty} = \bigcap_m \mathcal{S}^m \quad \text{where} \quad \mathcal{S}^m = \mathcal{S}^m_{1,0}.$$

For $p(x\,;\,\xi) \in S^m_{\rho,\delta}$ we shall often use a notation $|p|_{l_1,l_2} = |p|_{m,l_1,l_2}$ defined by

$$(1.5) \qquad |p|_{l_1,l_2} = \underset{|\alpha| \leqq l_1,\, |\beta| \leqq l_2}{\text{Max}} \ \sup_{R^n \times R^n} (|\partial_x^\alpha \partial_\xi^\beta p(x\,;\,\xi)| \langle \xi \rangle^{-(m+\delta|\alpha|-\rho|\beta|)}) < \infty\,.$$

REMARK. i) Let $p(x\,;\,-i\partial_x) = \sum_{|\alpha| \leqq m} a_\alpha(x)(-i\partial_x)^\alpha$ be a differential operator of order m with coefficients $a_\alpha(x)$ of class $\mathcal{B}$. Then $p(x\,;\,\xi) \in S^m$ and $p(x\,;\,-i\partial_x) = p(X\,;\,D_x) \in \mathcal{S}^m$.

ii) We can regard Λ^r as $\Lambda^r = \langle D_x \rangle^r \in \mathcal{S}^r$, and especially $\Lambda^r = \langle D_x \rangle^r$ coincides with a differential operator $(1-\Delta_x)^{r/2}$ when r is a non-negative even integer where $\Delta_x = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$. In what follows we often use this fact as in [4].

iii) $S^{m_1}_{\rho_1,\delta_1} \subset S^{m_2}_{\rho_2,\delta_2}$ when $m_1 \leqq m_2$, $\rho_1 \geqq \rho_2$, $\delta_1 \leqq \delta_2$.

iv) Using the fact

$$x^\alpha \partial_x^\beta p(X\,;\,D_x)u(x) = \sum_{\beta' \leqq \beta} C_{\beta,\beta'} \int e^{ix\cdot\xi} (i\partial_\xi)^\alpha \{\xi^{\beta'} \partial_x^{\beta-\beta'} p(x\,;\,\xi)\hat{u}(\xi)\} d\xi$$

it is easy to see that operator $p(X\,;\,D_x)$ is a continuous map $\mathcal{S}$ into $\mathcal{S}$.

THEOREM 1.1. *$p(X\,;\,D_x) \in \mathcal{S}^m_{\rho,\delta}$ is extended uniquely to a bounded operator: $H_{s+m} \to H_s$ for any s and we have with a constant $C_{p,s}$*

$$(1.6) \qquad \|p(X\,;\,D_x)u\|_s \leqq C_{p,s} \|u\|_{s+m}\,.$$

REMARK. This theorem, together with the corollary of Theorem 1.7, can be proved by means of interpolation theorems, if we only prove it for the integer $s = k$ (cf. [3]). We shall give here direct proofs without using interpolation theorems.

DEFINITION 1.3. We denote by $\mathcal{L}^{-\infty}$ the set of linear operators $G: H_{-\infty} \to H_\infty$ such that for all s_1, s_2 we have with constants C_{G,s_1,s_2}

(1.7) $$\|Gu\|_{s_1} \leqq C_{G,s_1,s_2}\|u\|_{s_2}\,.$$

We call G an infinitely smoothing operator.

DEFINITION 1.4. We denote by $\mathcal{L}^m_{\rho,\delta}$, $0 \leqq \delta < \rho \leqq 1$, the set of linear operators $G\colon H_{-\infty} \to H_{-\infty}$ such that there exist $p(x;\xi) \in S^m_{\rho,\delta}$ and

$$G - p(X;D_x) \in \mathcal{L}^{-\infty}\,,$$

and we call G a pseudo-differential operator of class $\mathcal{L}^m_{\rho,\delta}$ with the symbol $p(x;\xi) \in S^m_{\rho,\delta}$.

From the definition it is easy to see

$$\mathcal{L}^{m_1}_{\rho_1,\delta_1} \subset \mathcal{L}^{m_2}_{\rho_2,\delta_2} \quad \text{when} \quad m_1 \leqq m_2\,,\quad \rho_1 \geqq \rho_2\,,\quad \delta_1 \leqq \delta_2\,,$$
$$S^m_{\rho,\delta} \subset \mathcal{L}^m_{\rho,\delta}\,,\qquad S^{-\infty} \subset \mathcal{L}^{-\infty}\,.$$

Now, let $G \in \bigcap_m \mathcal{L}^m_{\rho,\delta}$. Then, for any s_1, s_2, we can select $p(X;D_x) \in S^m_{\rho,\delta}$ for $m = s_2 - s_1$ such that $G - p(X;D_x) \in \mathcal{L}^{-\infty}$. By means of Theorem 1.1 and the definition of $\mathcal{L}^{-\infty}$ we have

$$\|Gu\|_{s_1} \leqq \|(G - p(X;D_x))u\|_{s_1} + \|p(X;D_x)u\|_{s_1} \leqq C_{s_1,s_2}\|u\|_{s_2}\,.$$

This means $G \in \mathcal{L}^{-\infty}$, so that $\bigcap_m \mathcal{L}^m_{\rho,\delta} \subset \mathcal{L}^{-\infty}$. Since $\mathcal{L}^{-\infty} \subset \bigcap_m \mathcal{L}^m_{\rho,\delta}$ is clear, we have

(1.8) $$\mathcal{L}^{-\infty} = \bigcap_m \mathcal{L}^m_{\rho,\delta}\,.$$

Let $\phi(\xi)$ be a bounded and non-continuous function which vanishes outside a compact set, and define an operator Ψ by $\widehat{\Psi u}(\xi) = \phi(\xi)\hat{u}(\xi)$. Then, it is easy to see $\Psi \in \mathcal{L}^{-\infty}$. But in view of Remark iv) $\Psi \notin S^{-\infty}$ since $\phi(\xi)\hat{u}(\xi) \notin \mathcal{S}$ for some $u \in \mathcal{S}$. This means

$$S^{-\infty} \subsetneqq \mathcal{L}^{-\infty}\,.$$

THEOREM 1.2. i) *Let $G \in \mathcal{L}^m_{\rho,\delta}$. Then, for any s we have with a constant* $C_{G,s}$

(1.9) $$\|Gu\|_s \leqq C_{G,s}\|u\|_{s+m}\,.$$

ii) *Let $G \in \mathcal{L}^m_{\rho,\delta}$ with the symbol $p(x;\xi) \in S^m_{\rho,\delta}$. Then, G^* in the sense*

(1.10) $$(Gu, v) = (u, G^*v)\,,\qquad u \in \mathcal{S}\,,\quad v \in \mathcal{S}\,,$$

exists as an element of $\mathcal{L}^m_{\rho,\delta}$ and has the symbol $p^*(x;\xi) \in S^m_{\rho,\delta}$ such that

(1.11) $$p^*(x;\xi) - \sum_{j=0}^{N-1} p^*_j(x;\xi) \in S^{m-(\rho-\delta)N}_{\rho,\delta} \qquad \textit{for any} \quad N$$

where $p^*_j(x;\xi) \in S^{m-(\rho-\delta)j}_{\rho,\delta}$, $j = 0, 1, \cdots$, and are defined by

(1.12) $$p^*_j(x;\xi) = \sum_{|\alpha|=j} \frac{1}{\alpha!}\partial^\alpha_x(-i\partial_\xi)^\alpha\overline{p(x;\xi)}\,.$$

iii) *Let* $G_1\in\mathcal{L}^{m_1}_{\rho,\delta}$, $G_2\in\mathcal{L}^{m_2}_{\rho,\delta}$ *with the symbols* $p_1(x;\xi)\in S^{m_1}_{\rho,\delta}$, $p_2(x;\xi)\in S^{m_2}_{\rho,\delta}$, *respectively. Then we have* $G_1G_2\in\mathcal{L}^{m_1+m_2}_{\rho,\delta}$ *with the symbol* $r(x;\xi)\in S^{m_1+m_2}_{\rho,\delta}$ *such that*

$$r(x;\xi)-\sum_{j=0}^{N-1}r_j(x;\xi)\in S^{m_1+m_2-(\rho-\delta)N}_{\rho,\delta}\qquad \textit{for any}\quad N, \tag{1.13}$$

where $r_j(x;\xi)\in S^{m_1+m_2-(\rho-\delta)j}_{\rho,\delta}$, $j=0,1,\cdots$, *and are defined by*

$$r_j(x;\xi)=\sum_{|\alpha|=j}\frac{1}{\alpha!}(-i\partial_\xi)^\alpha p_1(x;\xi)\partial_x^\alpha p_2(x;\xi). \tag{1.14}$$

COROLLARY. *Let* $G_1\in\mathcal{L}^{m_1}_{\rho,\delta}$, $G_2\in\mathcal{L}^{m_2}_{\rho,\delta}$. *Then, the commutator*

$$[G_1,G_2]=G_1G_2-G_2G_1\in\mathcal{L}^{m_1+m_2-(\rho-\delta)}_{\rho,\delta}. \tag{1.15}$$

THEOREM 1.3. *Let* $G\in\mathcal{L}^{-\infty}$. *Then, there exists the kernel* $K(x;y)\in\mathcal{B}(R^n\times R^n)$ *of* G *such that*

$$\|\partial_x^\alpha K(x;\cdot)\|_{s,y}\leqq C_{\alpha,s}\qquad \textit{for any}\quad \alpha,\ s, \tag{1.16}$$

and we have

$$Gu(x)=\int K(x;y)u(y)dy. \tag{1.17}$$

THEOREM 1.4. $p(X;D_x)\in S^m_{\rho,\delta}\cap\mathcal{L}^{-\infty}$ *if and only if* $p(X;D_x)\in S^{-\infty}$.

COROLLARY. *Let* $G\in\mathcal{L}^m_{\rho,\delta}$. *Then, the symbol* $p(x;\xi)\in S^m_{\rho,\delta}$ *is uniquely determined* (mod $S^{-\infty}$).

Now let $\boldsymbol{G}=(G_{ij})$ be an $l\times l$ matrix of $G_{ij}\in\mathcal{L}^m_{\rho,\delta}$ and let $\boldsymbol{p}(x;\xi)=(p_{ij}(x;\xi))$ be an $l\times l$ matrix of $p_{ij}(x;\xi)\in S^m_{\rho,\delta}$ which are the symbols of G_{ij}. Then, we write

$$\boldsymbol{G}\in\boldsymbol{\mathcal{L}}^m_{\rho,\delta},\qquad \boldsymbol{p}(x;\xi)\in\boldsymbol{S}^m_{\rho,\delta},\qquad \boldsymbol{p}(X;D_x)\in\boldsymbol{S}^m_{\rho,\delta}$$

and call $\boldsymbol{p}(x;\xi)$ the symbol of $\boldsymbol{G}$. We denote $\boldsymbol{u}=(u_1,\cdots,u_l)\in\boldsymbol{\mathcal{S}}$ ($\in\boldsymbol{H}_s$) when each $u_j\in\mathcal{S}$ ($\in H_s$), $j=1,\cdots,l$.

Then, we have

THEOREM 1.5 (Lax-Nirenberg). *Let* $\boldsymbol{G}\in\boldsymbol{\mathcal{L}}^m_{\rho,\delta}$. *Suppose there exists a hermitian symmetric and non-negative matrix* $\boldsymbol{p}_0(x;\xi)\in\boldsymbol{S}^m_{\rho,\delta}$ *such that*

$$\boldsymbol{G}-\boldsymbol{p}_0(X;D_x)\in\boldsymbol{\mathcal{L}}^{m-(\rho-\delta)}_{\rho,\delta}. \tag{1.18}$$

(*We call* $\boldsymbol{p}_0(x;\xi)$ *the principal symbol of* $\boldsymbol{G}$.) *Then we have with a constant* K_0

$$\mathcal{Re}\,(\boldsymbol{Gu},\boldsymbol{u})\geqq -K_0\|\boldsymbol{u}\|^2_{(m-(\rho-\delta))/2}. \tag{1.19}$$

THEOREM 1.6. *Let* $\boldsymbol{G}\in\boldsymbol{\mathcal{L}}^0_{\rho,\delta}$ *with the symbol* $\boldsymbol{p}(x;\xi)\in\boldsymbol{S}^0_{\rho,\delta}$ *and set*

$$|\boldsymbol{p}(x;\xi)|^\infty_{\sup}=\overline{\lim_{|\xi|\to\infty}}\ \sup_x|\boldsymbol{p}(x;\xi)|, \tag{1.20}$$

where $|\boldsymbol{p}(x;\xi)|$ *is defined by*

$$|\boldsymbol{p}(x;\xi)|=\operatorname*{Max}_{|u|=1}\{|\boldsymbol{p}(x;\xi)\boldsymbol{u}|\} \tag{1.21}$$

with constant vectors $\boldsymbol{u}=(u_1,\cdots,u_l)$. *Then, we have*

$$\inf_{T\in\mathcal{L}^{-(\rho-\delta)}_{\rho,\delta}}\|\boldsymbol{G}-\boldsymbol{T}\|\leqq|\boldsymbol{p}|^{\infty}_{\sup}\leqq\|\boldsymbol{p}(X;D_x)\| \tag{1.22}$$

where $\|\boldsymbol{G}\|=\sup_{\|u\|_0=1}\|\boldsymbol{Gu}\|_0$.

Next, we consider a C^∞ coordinate transformation $x(y)=(x_1(y),\cdots,x_n(y))$ such that we have with a constant $C>0$

$$\partial_{y_j}x_i(y)\in\mathcal{B}_y\,,\quad i,j=1,\cdots,n\,,\quad C^{-1}\leqq|\partial_y x(y)|\leqq C \tag{1.23}$$

where $\partial_y x(y)=(\partial_{y_j}x_i(y))$ is the Jacobian matrix and $|\partial_y x(y)|$ denotes its determinant. Then, we have

THEOREM 1.7. *Let* $G\in\mathcal{L}^m_{\rho,\delta,x}$ *with the symbol* $p(x;\xi)\in S^m_{\rho,\delta,x}$. *Suppose* $1-\rho\leqq\delta<\rho$. *Then,* $Q=Q_G$ *defined by*

$$Qw(y)=(Gu)(x(y))\qquad\text{for}\quad w(y)=u(x(y)) \tag{1.24}$$

belongs to $\mathcal{L}^m_{\rho,\delta,y}$ *and has the symbol* $q(y;\eta)\in S^m_{\rho,\delta,y}$ *such that*

$$q(y;\eta)-\sum_{j=1}^{N-1}q_j(y;\eta)\in S^{m-(\rho-\delta)N}_{\rho,\delta,y}\qquad\text{for any}\quad N \tag{1.25}$$

where $q_j(y;\eta)\in S^{m-(\rho-\delta)j}_{\rho,\delta,y}$, $j=0,1,\cdots$, *and are defined by*

$$q_j(y;\eta)=\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial^\alpha_{y_1}\{(-i\partial_\eta)^\alpha p(x(y);\partial_y x(y,y_1)^{TI}\eta)\cdot|\partial_y x(y,y_1)|^{-1}|\partial_y x(y_1)|\}_{y_1=y} \tag{1.26}$$

with

$$\partial_y x(y,y_1)=\int_0^1\partial_y x(y_1+t(y-y_1))dt\,. \tag{1.27}$$

COROLLARY. *The space* H_s *is invariant under the coordinate transformation, which satisfies* (1.23), *in the sense* $H_{s,x}\ni u(x)$ *if and only if* $w(y)=u(x(y))\in H_{s,y}$ *for any* s.

§ 2. Properties of operators of class $S^m_{\rho,\delta}$.

First we give a fundamental

LEMMA 2.1 (Hörmander). *Let* $p(X;D_x)\in S^m_{\rho,\delta}$, $0\leqq\delta<\rho\leqq1$. *Then, for any non-negative integer* k, $p(X;D_x)$ *can be uniquely extended to a bounded operator:* $H_{k+m}\to H_k$ *and we have with a constant* $C_{m,k}$

$$\|p(X;D_x)u\|_k\leqq C_{m,k}\operatorname*{Max}_{l_1+l_2=N_0}\{|p|_{l_1+k,l_2}\}\|u\|_{k+m} \tag{2.1}$$

where $N_0=\operatorname{Max}\{2\delta(n+1)/(\rho-\delta)+1,\ [(n+1)/\rho]+1\}$.

PROOF. In the case $m=0$, $k=0$, we follow carefully Hörmander's proof in [2], p. 154, by setting $\varepsilon=(\rho+\delta)/2$. Then we get (2.1) for $m=0$, $k=0$. For general $p(X; D_x)\in S^m_{\rho,\delta}$, we note that

$$\|p(X; D_x)u\|_k \leqq \sum_{|\alpha|\leqq k} \|\partial_x^\alpha p(X; D_x)u\|_0$$

and

$$\partial_x^\alpha p(X; D_x)u(x)=\sum_{\alpha'\leqq\alpha} C_{\alpha,\alpha'}\int e^{ix\cdot\xi}\partial_x^{\alpha'}p(x; \xi)\xi^{\alpha-\alpha'}\langle\xi\rangle^{-(k+m)}\widehat{\Lambda^{k+m}u}(\xi)d\xi .$$

Then, since

$$\partial_x^{\alpha'}p(x; \xi)\xi^{\alpha-\alpha'}\langle\xi\rangle^{-(k+m)}\in S^0_{\rho,\delta} \qquad \text{for} \quad \alpha'\leqq\alpha , \quad |\alpha|\leqq k ,$$

we have, by means of (2.1) for $m=0$, $k=0$,

$$\|p(X; D_x)u\|_k \leqq C_{m,k} \operatorname*{Max}_{l_1+l_2=N_0} \{|p|_{l_1+k,l_2}\}\|\Lambda^{k+m}\|_0 .$$

Noting that $\|\Lambda^{k+m}u\|_0=\|u\|_{k+m}$, we get (2.1). Q. E. D.

LEMMA 2.2. *Let* $p(y; \xi)\in S^m_{\rho,\delta}$ *and set*

$$F(\xi)=\int e^{-iy\cdot\xi}p(y; \xi)u(y)dy , \qquad u\in\mathcal{S} .$$

Then, we have for any N

$$|F(\xi)| \leqq C_{u,N}|p|_{N,0}\langle\xi\rangle^{m-(1-\delta)N} . \tag{2.2}$$

The proof is clear, since

$$\xi^\alpha F(\xi)=\int e^{-iy\cdot\xi}(-i\partial_y)^\alpha(p(y; \xi)u(y))dy$$

and

$$|(-i\partial_y)^\alpha(p(y; \xi)u(y))| \leqq C_\alpha|p|_{|\alpha|,0}\langle\xi\rangle^{m+\delta|\alpha|}\sum_{\alpha'\leqq\alpha}|\partial_y^{\alpha'}u(y)| .$$

Now, let $p(x; \xi|x_1)$ be a C^∞ function in $R^n\times R^n\times R^n$ which satisfies

$$|\partial_x^\alpha\partial_\xi^\beta\partial_{x_1}^\gamma p(x; \xi|x_1)| \leqq C_{\alpha,\beta,\gamma}\langle\xi\rangle^{m+\delta|\alpha+\gamma|-\rho|\beta|} , \tag{2.3}$$

and define $|p|_{l_1,l_2,l_3}=|p|_{m,l_1,l_2,l_3}$ by

$$|p|_{l_1,l_2,l_3}=\operatorname*{Max}_{|\alpha|\leqq l_1,\,|\beta|\leqq l_2,\,|\gamma|\leqq l_3}\ \sup_{R^n\times R^n\times R^n}(|\partial_x^\alpha\partial_\xi^\beta\partial_{x_1}^\gamma p(x; \xi|x_1)| \cdot\langle\xi\rangle^{-(m+\delta|\alpha+\gamma|-\rho|\beta|)}) .$$

Then, we define an operator $p(x; D_x|X_1)$ by

$$p(X; D_x|X_1)u(x)=\int e^{ix\cdot\xi}\int e^{-ix_1\cdot\xi}p(x; \xi|x_1)u(x_1)dx_1d\xi \tag{2.4}$$

and call this an operator of multiple symbol.

We have

LEMMA 2.3. *Let* m *be a negative number such that* $m<-(n+k_1+2k_2)$.

Then, for the operator $p(X; D_x|X_1)$ *of multiple symbol in* (2.4), *we have*

$$|\partial_x^\alpha p(X; D_x|X_1)u(x)| \leqq A\int \langle x-x_1\rangle^{-(n+1)} |\langle i\partial_{x_1}\rangle^{-2k_2}u(x_1)|dx_1, \quad u\in\mathcal{S}, \quad |\alpha|\leqq k_1, \tag{2.5}$$

and consequently we have

$$\|p(X; D_x|X_1)u\|_{k_1} \leqq A'\|u\|_{-2k_2}, \qquad u\in\mathcal{S}, \tag{2.6}$$

where A, A' *are constants of the form*

$$C_{m,n,k_1,k_2}|p|_{k_1,n+1,2k_2}. \tag{2.7}$$

PROOF. By the assumption we can write

$$p(X; D_x|X_1)u(x) = \int K(x, x_1)u(x_1)dx_1$$

where

$$K(x, x_1) = \int e^{i(x-x_1)\cdot\xi}p(x; \xi|x_1)d\xi.$$

Then we have for $|\nu|\leqq n+1$, $|\beta|\leqq k_2$, $|\gamma|\leqq 2k_2$,

$$\begin{aligned}|(x-x_1)^\nu\partial_x^\beta\partial_{x_1}^\gamma K(x, x_1)| &= |\sum_{\beta'\leqq\beta,\gamma'\leqq\gamma} C_{\beta,\beta',\gamma,\gamma'}\int e^{i(x-x_1)\cdot\xi}(i\partial_\xi)^\nu \\ &\quad\cdot\{\xi^{\beta'+\gamma'}\partial_x^{\beta-\beta'}\partial_{x_1}^{\gamma-\gamma'}p(x; \xi|x_1)\}d\xi| \\ &\leqq C_{n,k_1,k_2}|p|_{k_1,n+1,2k_2},\end{aligned}$$

since

$$|(i\partial_\xi)^\nu\{\xi^{\beta'+\gamma'}\partial_x^{\beta-\beta'}\partial_{x_1}^{\gamma-\gamma'}p(x; \xi|x_1)\}| \leqq C_{n,k_1,k_2}\langle\xi\rangle^{m+k_1+2k_2} \in L^1_{(\xi)}.$$

This means

$$\partial_x^\beta\partial_y^\gamma K(x, x_1) \in L^1_{(x-x_1)} \qquad \text{for} \quad |\beta|\leqq k_1, \ |\gamma|\leqq 2k_2.$$

We write as in [4]

$$\partial_x^\beta p(X; D_x|X_1)u(x) = \int \partial_x^\beta K(x, x_1)\langle i\partial_{x_1}\rangle^{2k_2}\langle i\partial_{x_1}\rangle^{-2k_2}u(x_1)dx_1.$$

Then, integrating by parts

$$\begin{aligned}|\partial_x^\beta p(X; D_x|X_1)u(x)| &= \left|\int \partial_x^\beta\langle i\partial_{x_1}\rangle^{2k_2}K(x, x_1)\cdot\langle i\partial_{x_1}\rangle^{-2k_2}u(x_1)dx_1\right| \\ &\leqq A_1\int\langle x-x_1\rangle^{-(n+1)}|\langle i\partial_{x_1}\rangle^{-2k_2}u(x_1)|dx_1, \\ &\qquad |\beta|\leqq k_1.\end{aligned}$$

Hence, we get (2.5). By Schwarz's inequality we have

$$|\partial_x^\beta p(X; D_x|X_1)u(x)|^2 \leqq A_2\int\langle x-x_1\rangle^{-(n+1)}|\langle i\partial_{x_1}\rangle^{-2k_2}u(x_1)|^2dx_1,$$

and, integrating both sides with respect to x, we obtain (2.6). Q. E. D.

THEOREM 2.1. *Let $p(x;\xi|x_1)$ be a C^∞ function which satisfies* (2.3) *and let $p(X;D_x|X_1)$ be the corresponding operator of multiple symbol. We define $p_j(x;\xi)\in S^{m-(\rho-\delta)j}_{\rho,\delta}$, $j=0,1,\cdots$, by*

$$p_j(x;\xi)=\sum_{|\alpha|=j}\frac{1}{\alpha!}(-i\partial_\xi)^\alpha\partial^\alpha_{x_1}p(x;\xi|x_1)_{x_1=x}. \tag{2.8}$$

Then, for any integer $k_1, k_2\geqq 0$ we have

$$|\partial^\alpha_x R_N u(x)|\leqq A\int\langle x-x_1\rangle^{-(n+1)}|\langle i\partial_{x_1}\rangle^{-2k_2}u(x_1)|dx_1, \qquad |\alpha|\leqq k_1, \tag{2.9}$$

and consequently we have

$$\|R_N u\|_{k_1}\leqq A'\|u\|_{-2k_2}, \tag{2.10}$$

where $R_N=R_N(X;D_x|X_1)$ is defined by

$$R_N u(x)=\Big(p(X;D_x|X_1)-\sum_{j=0}^{N-1}p_j(X;D_x)\Big)u(x), \tag{2.11}$$

N is an arbitrary positive integer which is bigger than $(m+n+k_1+2k_2)/(\rho-\delta)$, and A, A' are constants of the form

$$C_{n,m,k_1,k_2,N}|p|_{k_1,N+n+1,N+k_1+2k_2}. \tag{2.12}$$

PROOF. Since

$$\begin{aligned}
&p_j(X;D_x)u(x)\\
&=\sum_{|\alpha|=j}\frac{1}{\alpha!}\int(-i\partial_\xi)^\alpha\partial^\alpha_{x_1}p(x;\xi|x_1)_{x_1=x}\int e^{i(x-x_1)\cdot\xi}u(x_1)dx_1d\xi\\
&=\iint e^{i(x-x_1)\cdot\xi}\sum_{|\alpha|=j}\frac{1}{\alpha!}(x_1-x)^\alpha\partial^\alpha_{x_1}p(x;\xi|x_1)_{x_1=x}u(x_1)dx_1d\xi,
\end{aligned}$$

we have

$$R_N u(x)=\int\Big\{\int e^{i(x-x_1)\cdot\xi}\sum_{|\alpha|=N}\frac{N}{\alpha!}(x_1-x)^\alpha p_\alpha(x;\xi|x,x_1)u(x_1)dx_1\Big\}d\xi \tag{2.13}$$

where

$$p_\alpha(x;\xi|x,x_1)=\int_0^1(1-t)^{N-1}\partial^\alpha_{x_1}p(x;\xi|x+t(x_1-x))dt. \tag{2.14}$$

Now, let $\phi(\xi)$ be a C_0^∞ function such that

$$\phi(\xi)=1 \quad\text{on}\quad \{\xi;\ |\xi|\leqq 1\}$$

and set $\phi_\varepsilon(\xi)=\phi(\varepsilon\xi)$, $\varepsilon>0$. Then, $\phi_\varepsilon(\xi)$ has the properties:

$$\begin{aligned}
&\text{i)}\quad \phi_\varepsilon(\xi)\in C_0^\infty, \quad \phi_\varepsilon(\xi)\to 1 \text{ as } \varepsilon\to 0 \quad \text{for any } \xi,\\
&\text{ii)}\quad |\partial^\alpha_x\phi_\varepsilon(\xi)|\leqq C_\alpha\varepsilon^\tau\langle\xi\rangle^{-(|\alpha|-\tau)} \quad \text{for any } 0\leqq\tau\leqq|\alpha|,
\end{aligned} \tag{2.15}$$

with a constant C_α independent of $\varepsilon>0$. By means of Lemma 2.2, the function in brackets in (2.13) belongs to $L^1_{(\xi)}$ for any fixed x, so that we have by means of Lebesgue's theorem

$$R_N u(x) = \sum_{|\alpha|=N} \frac{N}{\alpha!} \lim_{\varepsilon\to 0} \iint (i\partial_\xi)^\alpha e^{i(x-x_1)\cdot\xi} \phi_\varepsilon(\xi) p_\alpha(x; \xi | x, x_1) u(x_1) d\xi dx_1$$

$$= \sum_{|\alpha|=N} \frac{N}{\alpha!} \lim_{\varepsilon\to 0} \iint e^{i(x-x_1)\cdot\xi} \Big\{ \phi_\varepsilon(\xi)(-i\partial_\xi)^\alpha p_\alpha(x; \xi | x, x_1)$$

$$+ \sum_{\alpha'<\alpha} C_{\alpha,\alpha'} \partial_\xi^{\alpha-\alpha'} \phi_\varepsilon(\xi) \partial_\xi^{\alpha'} p_\alpha(x; \xi | x, x_1) \Big\} u(x_1) d\xi dx_1 .$$

Then, making use of (2.15) and noting that $N > (m+n)/(\rho-\delta)$, we have for small fixed $0 < \tau_1 < \tau_2$,

$$|\phi_\varepsilon(\xi)(-i\partial_\xi)^\alpha p_\alpha(x; \xi | x, x_1)| \leqq C_N \langle \xi \rangle^{-(n+\tau_2)} \in L^1_{(\xi)} ,$$

$$|\partial_\xi^{\alpha-\alpha'} \phi_\varepsilon(\xi) \partial_\xi^{\alpha'} p_\alpha(x; \xi | x, x_1)| \leqq C_N \varepsilon^{\tau_1} \langle \xi \rangle^{-(n+\tau_2-\tau_1)} \in L^1_{(\xi)} .$$

Hence, again by means of Lebesgue's theorem, we have

$$R_N u(x) = \sum_{|\alpha|=N} \frac{N}{\alpha!} \iint e^{i(x-x_1)\cdot\xi} (-i\partial_\xi)^\alpha p_\alpha(x; \xi | x, x_1) u(x_1) d\xi dx_1$$

$$= \sum_{|\alpha|=N} \frac{N}{\alpha!} p_\alpha(X; D_x | X_1) u(x) ,$$

where $p_\alpha(X; D_x | X_1)$ are operators, of multiple symbol, defined by

$$p_\alpha(x; \xi | x_1) = (-i\partial_\xi)^\alpha p_\alpha(x; \xi | x, x_1) .$$

Then, by the definition (2.14) of $p_\alpha(x; \xi | x, x_1)$, replacing m by $m-(\rho-\delta)N$, $p_\alpha(x; \xi | x_1)$ satisfy the condition (2.3). Applying Lemma 2.3 to $p_\alpha(X; D_x | X_1)$ we get (2.9), (2.10) from (2.5), (2.6), respectively. Noting that

$$|p_\alpha|_{k_1, n+1, 2k_2} \leqq C_{n,k_1,k_2,N} |p|_{k_1, N+n+1, N+k_1+2k_2} ,$$

we can see that constants A, A' have the form (2.12). Q.E.D.

Now, according to Friedrichs [1], we define the reversed operator $p^R(X; D_x)$ of $p(X; D_x) \in S^m_{\rho,\delta}$ by

$$(2.16) \qquad p^R(X; D_x) u(x) = \int e^{ix\cdot\xi} \int e^{-ix_1\cdot\xi} p(x_1; \xi) u(x_1) dx_1 d\xi , \qquad u \in \mathcal{S} .$$

We have

THEOREM 2.2. *Let $p(X; D_x) \in S^m_{\rho,\delta}$ and $p^R(X; D_x)$ be the reversed operator of $p(X; D_x)$. We define $p_j(x; \xi) \in S^{m-(\rho-\delta)j}_{\rho,\delta}$, $j = 0, 1, \cdots$, by*

$$(2.17) \qquad p_j(x; \xi) = \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_x^\alpha (-i\partial_\xi)^\alpha p(x; \xi) .$$

Then, we have

$$p^R(X; D_x) \sim \sum_{j=0}^{\infty} p_j(X; D_x) , \qquad p(X; D_x) \sim \sum_{j=0}^{\infty} (-1)^j p_j^R(X; D_x)$$

in the sense: for any integer $k_1, k_2 \geqq 0$ *we have*

$$\left\| \left(p^R(X; D_x) - \sum_{j=0}^{N-1} p_j(X; D_x) \right) u \right\|_{k_1} \leqq A \|u\|_{-2k_2}, \qquad u \in \mathcal{S}, \tag{2.18}$$

$$\left\| \left(p(X; D_x) - \sum_{j=0}^{N-1} (-1)^j p_j^R(X; D_x) \right) u \right\|_{k_1} \leqq A' \|u\|_{-2k_2}, \qquad u \in \mathcal{S}, \tag{2.18$'$}$$

where N is an arbitrary positive integer which is bigger than $(m+n+k_1+2k_2)/(\rho-\delta)$, *and* A, A' *are constants of the form*

$$C_{n,m,k_1,k_2,N} |p|_{N+k_1+2k_2, N+n+1}\,. \tag{2.19}$$

Proof. We can consider $p^R(X; D_x)$ as an operator of multiple symbol: $p^R(X; D_x) = p(X; D_x | X_1)$ where $p(X; D_x | X_1)$ is defined by $p(x; \xi | x_1) = p(x_1; \xi)$. Then, applying Theorem 2.1 and noting that $p_j(x; \xi) = \sum_{|\alpha|=j} \frac{1}{\alpha!} (-i\partial_\xi)^\alpha \partial_{x_1}^\alpha p(x_1; \xi)_{x_1=x}$, we easily get (2.18). It is easy to see that a constant A has the form (2.19). We adopt a function $\phi_\varepsilon(\xi)$ which has properties (2.15). Then, as in the proof of Theorem 1.1, we have

$$\begin{aligned} p_j^R(X; D_x)u(x) &= \int e^{ix\cdot\xi} \int e^{-ix_1\cdot\xi} \sum_{|\alpha|=j} (-i\partial_\xi)^\alpha \partial_{x_1}^\alpha p(x_1; \xi) u(x_1) dx_1 d\xi \\ &= (-1)^j \iint e^{i(x-x_1)\cdot\xi} \sum_{|\alpha|=j} \frac{1}{\alpha!} (x-x_1)^\alpha \partial_{x_1}^\alpha p(x_1; \xi) u(x_1) dx_1 d\xi\,. \end{aligned}$$

Hence, writing

$$p(X; D_x)u(x) = \iint e^{i(x-x_1)\cdot\xi} p(x; \xi) u(x_1) dx_1 d\xi\,,$$

we have

$$\begin{aligned} R_N u(x) &\equiv \left(p(X; D_x) - \sum_{j=0}^{N-1} (-1)^j p_j^R(X; D_x) \right) u(x) \\ &= \sum_{|\alpha|=N} \frac{N}{\alpha!} \iint e^{i(x-x_1)\cdot\xi} (x-x_1)^\alpha p_\alpha(x, x_1; \xi) u(x_1) dx_1 d\xi \end{aligned}$$

where

$$p_\alpha(x, x_1; \xi) = \int_0^1 (1-t)^{N-1} p(x_1 + t(x-x_1); \xi) dt\,.$$

Then, again, making use of $\phi_\varepsilon(\xi)$, we get

$$\begin{aligned} R_N u(x) &= \sum_{|\alpha|=N} \frac{N}{\alpha!} \iint e^{i(x-x_1)\cdot\xi} (i\partial_\xi)^\alpha p_\alpha(x, x_1; \xi) u(x_1) dx_1 d\xi \\ &= \sum_{|\alpha|=N} \frac{N}{\alpha!} p_\alpha(X; D_x | X_1) u(x)\,, \end{aligned}$$

where $p_\alpha(X; D_x | X_1)$ are defined by $p_\alpha(x; \xi | x_1) = (i\partial_\xi)^\alpha p_\alpha(x, x_1; \xi)$. Applying Lemma 2.3 to $p_\alpha(X; D_x | X_1)$ we get (2.18)′. Q. E. D.

Theorem 2.3. *Let* $p_1(X; D_x) \in S_{\rho,\delta}^{m_1}$, $p_2(X; D_x) \in S_{\rho,\delta}^{m_2}$, *respectively. We define*

$r_j(x;\xi)\in S^{m_1+m_2-(\rho-\delta)j}_{\rho,\delta}$, $j=0, 1, \cdots$, *by*

$$r_j(x;\xi)=\sum_{|\alpha|=j}\frac{1}{\alpha!}(-i\partial_\xi)^\alpha p_1(x;\xi)\partial_x^\alpha p_2(x;\xi). \tag{2.20}$$

Then, we have

$$p_1(X;D_x)p_2(X;D_x)\sim\sum_{j=0}^{\infty}r_j(X;D_x)$$

in the sense: for any integer $k_1, k_2\geqq 0$ *we have*

$$\Big\|\Big(p_1(X;D_x)p_2(X;D_x)-\sum_{j=0}^{N-1}r_j(X;D_x)\Big)u\Big\|_{k_1}\leqq \text{const.}\,\|u\|_{-2k_2} \tag{2.21}$$

where N is an arbitrary positive integer which is bigger than $(n+m_2+k_1'+2k_2)/(\rho-\delta)$ *with* $k_1'=\text{Max}\{k_1+m_1, 0\}$.

PROOF. Set

$$p_{2,j}=\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_x^\alpha(-i\partial_\xi)^\alpha p_2(x;\xi)$$

and write

$$\begin{aligned} p_1(X;D_x)&p_2(X;D_x) \\ &=p_1(X;D_x)\sum_{j=0}^{N-1}(-1)^j p_{2,j}^R(X;D_x)+p_1(X;D_x)(p_2(X;D_x) \\ &\quad -\sum_{j=0}^{N-1}(-1)^j p_{2,j}^R(X;D_x)). \end{aligned} \tag{2.22}$$

Then, by means of Lemma 2.1 and (2.18)′ in Theorem 2.2, we have

$$\begin{aligned} &\Big\|p_1(X;D_x)\Big(p_2(X;D_x)-\sum_{j=0}^{N-1}(-1)^j p_{2,j}^R(X;D_x)\Big)u\Big\|_{k_1} \\ &\leqq \text{const.}\Big\|\Big(p_2(X;D_x)-\sum_{j=0}^{N-1}(-1)^j p_{2,j}^R(X;D_x)\Big)u\Big\|_{k_1'} \\ &\leqq \text{const.}\|u\|_{-2k_2}. \end{aligned} \tag{2.23}$$

Set

$$p(x;\xi\,|\,x_1)=p_1(x;\xi)\sum_{j=0}^{N-1}(-1)^j p_{2,j}(x_1;\xi).$$

Then, by definition, we have

$$p_1(X;D_x)\sum_{j=0}^{N-1}(-1)^j p_{2,j}^R(X;D_x)=p(X;D_x\,|\,X_1). \tag{2.24}$$

Hence, setting

$$r_j'(x;\xi)=\sum_{|\beta|=j}\frac{1}{\beta!}(-i\partial_\xi)^\beta\partial_{x_1}^\beta p(x;\xi\,|\,x_1)_{x_1=x}, \qquad j=0, 1, \cdots, \tag{2.25}$$

we have by means of Theorem 2.1

$$\Big\|\Big(p(X;D_x\,|\,X_1)-\sum_{j=0}^{N-1}r_j'(X;D_x)\Big)u\Big\|_{k_1}\leqq \text{const.}\,\|u\|_{-2k_2}. \tag{2.26}$$

By definition

$$\begin{aligned}
&\sum_{j=0}^{N-1} r_j'(x;\xi)\\
&=\sum_{|\beta|<N}\frac{1}{\beta!}(-i\partial_\xi)^\beta\Big\{p_1(x;\xi)\sum_{|\alpha|<N}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_x^{\alpha+\beta}(-i\partial_\xi)^\alpha p_2(x;\xi)\Big\}\\
&=\sum_{|\alpha|<N,|\beta|<N}\frac{(-1)^{|\alpha|}}{\alpha!}\sum_{\beta'\leqq\beta}\frac{1}{\beta'!(\beta-\beta')!}(-i\partial_\xi)^{\beta'}p_1(x;\xi)\partial_x^{\alpha+\beta}(-i\partial_\xi)^{\alpha+\beta-\beta'}p_2(x;\xi)\\
&=\sum_{|\alpha+\beta|<N}\frac{(-1)^{|\alpha|}}{\alpha!}\{''\}+\sum_{\substack{|\alpha+\beta|\geqq N\\|\alpha|<N,|\beta|<N}}\frac{(-1)^{|\alpha|}}{\alpha!}\{''\}\\
&\equiv I_N^{(1)}(x;\xi)+I_N^{(2)}(x;\xi).
\end{aligned}\tag{2.27}$$

Then, $I_N^{(2)}(x;\xi)\in S_{\rho,\delta}^{m_1+m_2-(\rho-\delta)N}\subset S_{\rho,\delta}^{-(k_1+2k_2)}$, so that by means of Lemma 2.1, we have

$$\|I_N^{(2)}(X;D_x)u\|_{k_1}\leqq\text{const.}\ \|u\|_{-2k_2}.\tag{2.28}$$

On the other hand

$$\begin{aligned}
I_N^{(1)}(x;\xi)=&\sum_{|\gamma|<N,\beta'\leqq\gamma}\frac{(-1)^{|\gamma|}}{\beta'!}(-i\partial_\xi)^{\beta'}p_1(x;\xi)\\
&\cdot\sum_{\beta'\leqq\beta\leqq\gamma}\frac{(-1)^{|\beta|}}{(\beta-\beta')!(\gamma-\beta)!}\partial_x^\gamma(-i\partial_\xi)^{\gamma-\beta'}p_2(x;\xi).
\end{aligned}$$

Then, since

$$\sum_{\beta'\leqq\beta\leqq\gamma}\frac{(-1)^{|\beta|}}{(\beta-\beta')!(\gamma-\beta)!}=\begin{cases}1 & \text{when}\quad\beta'=\gamma\\ 0 & \text{when}\quad\beta'<\gamma\end{cases},$$

we have

$$\begin{aligned}
I_N^{(1)}(x;\xi)&=\sum_{|\gamma|<N}\frac{1}{\gamma!}(-i\partial_\xi)^\gamma p_1(x;\xi)\partial_x^\gamma p_2(x;\xi)\\
&=\sum_{j=0}^{N-1}r_j(x;\xi).
\end{aligned}\tag{2.29}$$

From (2.22)-(2.29), we obtain (2.21). Q. E. D.

THEOREM 2.4. *Let an* $l\times l$ *matrix* $\boldsymbol{p}(x;\xi)=(p_{ij}(x;\xi))$ *belong to* $\boldsymbol{S}_{\rho,\delta}^{\rho-\delta}$. *Suppose* $\boldsymbol{p}(x;\xi)$ *is hermitian symmetric and non-negative. Then, there exists a constant K such that*

$$\mathcal{R}e\,(\boldsymbol{p}(X;D_x)\boldsymbol{u},\boldsymbol{u})\geqq-K\|\boldsymbol{u}\|_0,\qquad u\in\mathcal{S}.\tag{2.30}$$

PROOF. We follow the method of Friedrichs in [1].

I) First we assume that every $p_{ij}(x;\xi)$ has compact support with respect to x. Let $q(z)$ be a non-negative valued and even function of class C_0^∞, such that

$$\int q^2(z)dz=1,\qquad \operatorname{supp} q\subset\{z;\ |z|\leqq1\},\tag{2.31}$$

and define an operator $r(X; D_x) \in S^{\rho-\delta}_{\rho,\delta}$ by

$$r(x;\xi)=\int p(x;\xi+\langle\xi\rangle^{\varepsilon}z)q^2(z)dz \tag{2.32}$$

$$=\int p(x;\zeta)F^2(\zeta;\xi)d\zeta \qquad \text{for} \quad \varepsilon=(\rho+\delta)/2\,,$$

where

$$F(\zeta;\xi)=q((\zeta-\xi)\langle\xi\rangle^{-\varepsilon})\langle\xi\rangle^{-\varepsilon n/2}\,. \tag{2.33}$$

Then, setting

$$q_l(z)=\operatorname*{Max}_{|\alpha|\leqq l}|\partial_z^{\alpha}q(z)|$$

and

$$F_l(\zeta;\xi)=q_l((\zeta-\xi)\langle\xi\rangle^{-\varepsilon})\langle\xi\rangle^{-\varepsilon n/2}$$

we have easily

$$|\partial_\xi^{\alpha}F(\zeta;\xi)|\leqq C_{|\alpha|}\langle\xi\rangle^{-\varepsilon|\alpha|}F_{|\alpha|}(\zeta;\xi)\,. \tag{2.34}$$

We define another operator R_0 by

$$\widehat{R_0u}(\xi)=\int\left\{\int F(\zeta;\xi)\hat{p}(\xi-\eta;\zeta)F(\zeta;\eta)d\zeta\right\}\hat{u}(\eta)d\eta \tag{2.35}$$

where $\hat{p}(\chi;\xi)$ is the Fourier transform of $p(x;\xi)$ with respect to x.

Then, noting that $p(x;\zeta)$ is hermitian symmetric and non-negative, we have

$$(R_0u, u)=\int\left\{\int\overline{v(\zeta;x)}\cdot p(x;\zeta)v(\zeta;x)dx\right\}d\zeta\geqq 0\,, \tag{2.36}$$

where $v(\zeta;x)$ is defined by $\widehat{v(\zeta;\cdot)}(\xi)=F(\zeta;\xi)\hat{u}(\xi)$. Now, we fix an integer N such that

$$N\geqq 2\{\delta(n+1)/(\rho-\delta)+1\}\,. \tag{2.37}$$

1. Since $q(z)$ is an even function, noting (2.31), we can write down

$$r(x;\xi)=p(x;\xi)+\sum_{1<|\alpha|<N}\frac{1}{\alpha\,!}r_{\alpha}(x;\xi)+R_N(x;\xi) \tag{2.38}$$

where

$$r_{\alpha}(x;\xi)=\partial_\xi^{\alpha}p(x;\xi)\langle\xi\rangle^{\varepsilon|\alpha|}\int z^{\alpha}q^2(z)dz$$

and

$$R_N(x;\xi)=\sum_{|\alpha|=N}\frac{N}{\alpha\,!}\int\left\{\int_0^1(1-t)^{N-1}\partial_\xi^{\alpha}p(x;\xi+\langle\xi\rangle^{\varepsilon}tz)dt\right\}\langle\xi\rangle^{\varepsilon N}z^{\alpha}q^2(z)dz\,.$$

Then, it is easy to see $r_{\alpha}(x;\xi)\in S^{(\rho-\delta)-(\rho-\delta)|\alpha|/2}_{\rho,\delta}\subset S^0_{\rho,\delta}$ for $|\alpha|\geqq 2$, so that by means of Lemma 2.1 we have

$$\|r_{\alpha}(X;D_x)u\|_0\leqq \text{const.}\,\|u\|_0\,. \tag{2.39}$$

Noting (2.37), we have

$$(1+|\chi|)^{n+1}|\hat{R}_N(\chi\,;\,\xi)| \leqq \sum_{|\beta|\leqq n+1}\int |\partial_x^\beta R_N(x\,;\,\xi)|\,dx \leqq C_{n,N,p}$$

where $\hat{R}_N(\chi\,;\,\xi)$ is the Fourier transform of $R_N(x\,;\,\xi)$ with respect to x. Here we used the assumption (2.37).

Then we have

$$\|R_N(X\,;\,D_x)u\|_0=\left\|\int \hat{R}_N(\xi-\eta\,;\,\eta)\hat{u}(\eta)d\eta\right\|_{L^2_{(\xi)}} \tag{2.40}$$

$$\leqq \left(\int \sup_\eta |\hat{R}_N(\chi\,;\,\eta)|\,d\chi\right)\|u\|_0 = \text{const.}\ \|u\|_0\,.$$

From (2.38)-(2.40) we obtain

$$\|(r(X\,;\,D_x)-p(X\,;\,D_x))u\|_0 \leqq \text{const.}\ \|u\|_0\,. \tag{2.41}$$

2. Next we estimate $\|(R_0-r(X\,;\,D_x))u\|_0$. Set

$$F_N(\zeta\,;\,\xi,\eta)\equiv F(\zeta\,;\,\xi)-\sum_{|\alpha|<N}\frac{(\xi-\eta)^\alpha}{\alpha\,!}\partial_\eta^\alpha F(\zeta\,;\,\eta)\,.$$

Then we can write

$$\widehat{R_0u}(\xi)=\widehat{r(X\,;\,D_x)u}(\xi)+\sum_{0<|\alpha|<N}\frac{1}{\alpha\,!}\widehat{r'_\alpha(X\,;\,D_x)u}(\xi)+\widehat{R'_Nu}(\xi) \tag{2.42}$$

where $r'_\alpha(X\,;\,D_x)$ and R'_N are defined by

$$r'_\alpha(x\,;\,\xi)=\int(-i\partial_x)^\alpha p(x\,;\,\zeta)\partial_\xi^\alpha F(\zeta\,;\,\xi)F(\zeta\,;\,\xi)d\zeta$$

and

$$\widehat{R'_Nu}(\xi)=\int\left\{\int F_N(\zeta\,;\,\xi,\eta)\hat{p}(\xi-\eta,\zeta)F(\zeta\,;\,\eta)d\zeta\right\}\hat{u}(\eta)d\eta\,,$$

respectively. Noting (2.34) and $\langle\zeta\rangle\leqq 2\langle\eta\rangle$ on $\operatorname{supp}F(\zeta\,;\,\eta)$, it is easy to see

$$r'_\alpha(x\,;\,\xi)\in S_{\varepsilon,\delta}^{(\rho-\delta)-(\varepsilon-\delta)|\alpha|}\subset S_{\varepsilon,\delta}^0 \qquad \text{for}\quad |\alpha|\geqq 2\,. \tag{2.43}$$

Since

$$\partial_{\xi_j}F(\zeta\,;\,\xi)=-\Big\{\sum_{k=1}^n \partial_{\sigma_k}q((\zeta-\xi)\langle\xi\rangle^{-\varepsilon})(\langle\xi\rangle^{-\varepsilon}\delta_{jk}+\varepsilon(\zeta-\xi)\langle\xi\rangle^{-\varepsilon-1}\partial_{\xi_j}\langle\xi\rangle)$$

$$+\frac{\varepsilon n}{2}q((\zeta-\xi)\langle\xi\rangle^{-\varepsilon})\langle\xi\rangle^{-1}\partial_{\xi_j}\langle\xi\rangle\Big\}\langle\xi\rangle^{-\varepsilon n/2}\,,$$

we have for $\alpha_j=(0,\cdots,\overset{j}{\overbrace{1}},\cdots,0)$, $j=1,\cdots,n$,

$$r'_{\alpha_j}(x\,;\,\xi)=\int i\partial_{x_j}p(x\,;\,\xi+\sigma\langle\xi\rangle^\varepsilon)\langle\xi\rangle^{-\varepsilon}\partial_{\sigma_j}q(\sigma)q(\sigma)d\sigma \tag{2.44}$$

$$+\int i\partial_{x_j}p(x\,;\,\zeta)F'_j(\zeta\,;\,\xi)F(\zeta\,;\,\xi)d\zeta$$

$$\equiv r^{(1)\prime}_{\alpha_j}(x\,;\,\xi)+r^{(2)\prime}_{\alpha_j}(x\,;\,\xi)$$

where $F_j'(\zeta;\xi)$ are functions satisfying

$$|\partial_\xi^\alpha F_j'(\zeta;\xi)| \leqq C_\alpha \langle\xi\rangle^{-1-\varepsilon|\alpha|} F_{|\alpha|+1}(\zeta;\xi).$$

Then we get

$$r_{\alpha_j}^{(2)\prime}(x;\xi) \in S_{\varepsilon,\delta}^{\rho-1} \subset S_{\varepsilon,\delta}^{0}. \tag{2.45}$$

Noting that $\partial_{\sigma_j} q(\sigma) q(\sigma)$ are odd functions, we can write

$$r_{\alpha_j}^{(1)\prime}(x;\xi) = \int i\partial_{x_j}(p(x;\xi+\sigma\langle\xi\rangle^\varepsilon) - p(x;\xi))\langle\xi\rangle^{-\varepsilon}\partial_{\sigma_j}q(\sigma)q(\sigma)d\sigma$$

$$= \sum_{k=1}^n \int\left\{\int_0^1 i\partial_{x_j}\partial_{\xi_k} p(x;\xi+\theta\sigma\langle\xi\rangle^\varepsilon)d\theta\right\}\sigma_k\partial_{\sigma_j}q(\sigma)q(\sigma)d\sigma.$$

This means

$$r_{\alpha_j}^{(1)\prime}(x;\xi) \in S_{\rho,\delta}^{0} \subset S_{\varepsilon,\delta}^{0}. \tag{2.46}$$

Hence from (2.43)-(2.46) we obtain by means of Lemma 2.1

$$\left\| \sum_{0<|\alpha|<N} \frac{1}{\alpha!} r_\alpha'(X;D_x)u \right\|_0 \leqq \text{const.}\ \|u\|_0. \tag{2.47}$$

Now, in order to estimate $F_N(\zeta;\xi,\eta)$ we shall use an elementary formula (see [5], p. 82):

$$\begin{aligned}
&\Big(f(1) - \sum_{j=0}^{N-1}\frac{1}{j!}f^{(j)}(0)\Big)g(0) \\
&= \sum_{j=0}^{N-1}(-1)^{j+1}\int_0^1 \phi_{N,j}(\theta)f^{(N-j)}(\theta)g^{(j)}(\theta)d\theta \\
&\quad + (-1)^N\int_0^1 \frac{\theta^{N-1}}{(N-1)!}(f(1)-f(\theta))g^{(N)}(\theta)d\theta \qquad \text{for } f(\theta), g(\theta) \in C_{[0,1]}^N,
\end{aligned} \tag{2.48}$$

where

$$\phi_{N,0}(\theta) = -\frac{(1-\theta)^{N-1}}{(N-1)!},$$

$$\phi_{N,j}(\theta) = \frac{(1-\theta)^{N-j}}{(N-j)!}\frac{\theta^{j-1}}{(j-1)!} - \frac{(1-\theta)^{N-j-1}}{(N-j-1)!}\frac{\theta^j}{j!}, \qquad j=1,\cdots,N-1.$$

Setting $f(\theta) = F(\zeta;\eta+\theta(\xi-\eta))$ and $g(\theta) = \langle\eta+\theta(\xi-\eta)\rangle^{\varepsilon N}$ in (2.48) and using (2.34), we have

$$|F_N(\zeta;\xi,\eta)\langle\eta\rangle^{\varepsilon N}| = \left|\Big(f(1) - \sum_{j=0}^{N-1}\frac{1}{j!}f^{(j)}(0)\Big)g(0)\right|$$

$$\leqq \text{const.}\ \langle\xi-\eta\rangle^N \int_0^1 (F(\zeta;\xi) + F_N(\zeta;\eta+\theta(\xi-\eta)))d\theta.$$

Noting

$$|\langle\xi-\eta\rangle^{n+1+N}\hat{p}(\xi-\eta;\zeta)| \leqq \text{const.}\ \langle\zeta\rangle^{(n+1+N)\delta+(\rho-\delta)}$$

$$\leqq \text{const.}\ \langle\eta\rangle^{(n+1+N)\delta+(\rho-\delta)} \quad \text{on} \quad \operatorname{supp} F(\zeta;\eta).$$

We obtain by means of Schwarz's inequality

$$|\widehat{R'_N u}(\xi)| \leqq \text{const.} \int \left\{ \int\int_0^1 (F(\zeta\,;\,\xi)^2 + F_N(\zeta\,;\,\eta+\theta(\xi-\eta))^2) d\theta d\zeta \right\}^{1/2}$$

$$\times \left\{ \int F(\zeta\,;\,\xi)^2 d\zeta \right\}^{1/2} \langle \xi-\eta \rangle^{-(n+1)} |\hat{u}(\eta)| d\eta$$

$$\leqq \text{const.} \int \langle \xi-\eta \rangle^{-(n+1)} |\hat{u}(\eta)| d\eta\,.$$

Here we used $(n+1+N)\delta+(\rho-\delta) \leqq \varepsilon N$ by the assumption (2.37). Consequently we have

$$\|R'_N u\|_0^2 = \int |\widehat{R'_N u}(\xi)|^2 d\xi \leqq \text{const.}\ \|u\|_0^2\,. \tag{2.49}$$

From (2.42), (2.47), (2.49) we obtain

$$\|(R_0 - r(X\,;\,D_x))u\|_0 \leqq \text{const.}\ \|u\|_0 \tag{2.50}$$

and from (2.41) and (2.50) we get

$$\|(R_0 - p(X\,;\,D_x))u\|_0 \leqq \text{const.}\ \|u\|_0\,. \tag{2.51}$$

Then, writing

$$\mathcal{R}e\,(p(X\,;\,D_x)u,\,u) = \mathcal{R}e\,((p(X\,;\,D_x)-R_0)u,\,u) + \mathcal{R}e\,(R_0 u,\,u)\,,$$

and using (2.36) and (2.51), we obtain (2.30).

II) For general $p(X\,;\,D_x) \in \dot{S}^{\rho-\delta}_{\rho,\delta}$. Let $\psi(x)$, $\phi(x)$ be non-negative valued C_0^∞ functions such that

$$\begin{aligned} &\int \psi(x)dx = 1, \qquad \operatorname{supp}\psi \subset \{x\,;\,|x| \leqq \tau_0\}\,, \\ &\phi(x) = 1 \quad \text{on} \quad \{x\,;\,|x| \leqq 2\tau_0\}, \quad \operatorname{supp}\phi \subset \{x\,;\,|x| \leqq 3\tau_0\} \end{aligned} \tag{2.52}$$

for a fixed $\tau_0 > 0$.

We define $p_z(X\,;\,D_x) \in S^{\rho-\delta}_{\rho,\delta}$ by

$$p_z(x\,;\,\xi) = \psi(z+x)p(x\,;\,\xi)\,, \tag{2.53}$$

and set

$$u_z^{(1)}(x) = \phi(z+x)u(x), \qquad u_z^{(2)}(x) = (1-\phi(z+x))u(x)\,. \tag{2.54}$$

Then,

$$\begin{aligned} &\mathcal{R}e\,(p(X\,;\,D_x)u,\,u) \\ &= \int \mathcal{R}e\,(p_z(X\,;\,D_x)u_z^{(1)},\,u)dz + \int \mathcal{R}e\,(p_z(X\,;\,D_x)u_z^{(2)},\,u)dz\,. \end{aligned} \tag{2.55}$$

Noting that $\phi(z+x)=1$ on $\operatorname{supp}\psi(z+x)$, we have from the result of I)

$$
(2.56)\quad \begin{aligned} &\int \mathcal{R}e\,(\boldsymbol{p}_z(X\,;D_x)\boldsymbol{u}_z^{(1)},\,\boldsymbol{u})dz=\int \mathcal{R}e\,(\boldsymbol{p}_z(X\,;D_x)\boldsymbol{u}_z^{(1)},\,\boldsymbol{u}_z^{(1)})dz \\ &\geqq -K\iint \phi(z+x)^2|\boldsymbol{u}(x)|^2dx\,dz\geqq -K'\|\boldsymbol{u}\|_0^2. \end{aligned}
$$

Here, we must remark that from the proof of I) the constant K has the form $C_{N,M}|\boldsymbol{p}_z|_{l_1,l_2}$ with l_1, l_2 depending only on M, N and $|\boldsymbol{p}_z|_{l_1,l_2}\leqq C_{N,M,\psi}|\boldsymbol{p}|_{l_1,l_2}$.

Noting again $\phi(z+x)=1$ on $\operatorname{supp}\psi(z+x)$, we have

$$
(2.57)\quad \begin{aligned} &\left|\int \mathcal{R}e\,(\boldsymbol{p}_z(X\,;D_x)\boldsymbol{u}_z^{(2)},\,\boldsymbol{u})dz\right|=\left|\int \mathcal{R}e\,(\boldsymbol{p}_z(X\,;D_x)\boldsymbol{u}_z^{(2)},\boldsymbol{u}_z^{(1)})dz\right| \\ &\leqq\int(\|\boldsymbol{p}_z(X\,;D_x)\boldsymbol{u}_z^{(2)}\|_0^2+\|\boldsymbol{u}_z^{(1)}\|_0^2)dz \\ &\leqq\iint|\boldsymbol{p}_z(X\,;D_x)\boldsymbol{u}_z^{(2)}(x)|^2dx\,dz+\text{const.}\,\|\boldsymbol{u}\|_0^2. \end{aligned}
$$

Then, we have

$$
\begin{aligned} \boldsymbol{p}_z(X\,;D_x)\boldsymbol{u}_z^{(2)}(x)&=\int e^{ix\cdot\xi}\int e^{-ix_1\cdot\xi}\psi(z+x)\boldsymbol{p}(x\,;\xi)(1-\phi(z+x_1))\cdot\boldsymbol{u}(x_1)dx_1d\xi \\ &=\boldsymbol{p}_z(X\,;D_x|X_1)\boldsymbol{u}(x) \end{aligned}
$$

where $\boldsymbol{p}_z(X\,;D_x|X_1)$ is defined by

$$
\boldsymbol{p}_z(x\,;\xi|x_1)=\psi(z+x)\boldsymbol{p}(x\,;\xi)(1-\phi(z+x)).
$$

Noting that $(-i\partial_\xi)^\alpha\partial_{x_1}^\alpha\boldsymbol{p}_z(x\,;\xi|x_1)=0$ for any α, we have by means of Theorem 2.1

$$
(2.58)\quad |\boldsymbol{p}_z(X\,;D_x)\boldsymbol{u}_z^{(2)}(x)|^2\leqq\text{const.}\,\phi(z+x)\int\langle x-x_1\rangle^{-(n+1)}|\boldsymbol{u}(x_1)|^2dx_1,
$$

and get by (2.57)

$$
(2.59)\quad \left|\int \mathcal{R}e\,(\boldsymbol{p}_z(X\,;D_x)\boldsymbol{u}_z^{(2)},\,\boldsymbol{u})dz\right|\leqq\text{const.}\,\|\boldsymbol{u}\|_0^2.
$$

From (2.55), (2.56), (2.59), we get (2.30) for general $\boldsymbol{p}(X\,;D_x)\in\mathcal{S}_{\rho,\delta}^{\rho-\delta}$. This completes the proof. Q. E. D.

Now, let $x=x(y)$ be a coordinate transformation which satisfies the condition (1.23).

LEMMA 2.4. *For any integer k we have with a constant C_k*

$$
(2.60)\quad C_k^{-1}\|u\|_{k,x}\leqq\|w\|_{k,y}\leqq C_k\|u\|_{k,x},\qquad u\in\mathcal{S},\ w(y)=u(x(y)).
$$

PROOF. When k is a non-negative integer, making use of the equivalence norm $\sum_{|\alpha|\leqq k}\|\partial_x^\alpha u\|_0$ we can easily get (2.60). For negative k, using $\|u\|_k=\sup_{v\neq0}\dfrac{|(u,v)|}{\|v\|_{-k}}$, we also get (2.60). Q. E. D.

THEOREM 2.5. *Let $p(X\,;D_x)\in S_{\rho,\delta,x}^m$ and let Q_p be an operator defined by*

(2.61) $$Q_p w(y) = (p(X\,;D_x)u)(x(y))\,, \qquad u \in \mathcal{S},\ w(y) = u(x(y))\,.$$

Suppose $1-\rho \leqq \delta < \rho$. *Then, for any integer* $k_1,\ k_2 \geqq 0$ *we have*

(2.62) $$\Big\|\Big(Q_p - \sum_{j=0}^{N-1} q_j(Y\,;D_y)\Big)w\Big\|_{k_1,y} \leqq A\|w\|_{-2k_2,y}\,,$$

where $q_j(y\,;\eta) \in S^{m-(\rho-\delta)j}_{\rho,\delta,y}$, $j=0, 1, \cdots$, *and are defined by* (1.26), N *is an arbitrary positive integer which is bigger than* $(m+n+k_1+2k_2)/(\rho-\delta)$, *and* A *is a constant of the form*

(2.63) $$C_{n,m,k_1,k_2,N}\,|p|_{k_1,2N+n+1+k_1+2k_2}\,.$$

PROOF. Let $\psi(x)$, $\phi(x)$ be non-negative valued C_0^∞ functions which satisfy the conditions (2.52) where τ_0 is a small positive number such that for any z

(2.64) $$(2C)^{-1} \leqq |\partial_y x(y, y_1)| \leqq 2C \quad \text{on} \quad \operatorname*{supp}_{(y,y_1)} \psi(z+x(y))\phi(z+x(y_1))\,,$$

where $\partial_y x(y, y_1)$ is the matrix defined in (1.27). Now we write $Q_p w$ as

(2.65) $$Q_p w(y) = \int p_z(X\,;D_x)u_z^{(1)}(x(y))dz + \int p_z(X\,;D_x)u_z^{(2)}(x(y))dz$$

where $p_z(X;D_x)$, $u_z^{(1)}(x)$, $u_z^{(2)}(x)$ are defined as in (2.53), (2.54), respectively.

I) First we consider

$$\partial_x^\alpha \int p_z(X\,;D_x)u_z^{(2)}(x)dz = \int \partial_x^\alpha p_z(X\,;D_x)u_z^{(2)}(x)dz, \qquad |\alpha| \leqq k_1\,.$$

We have

$$p_z(X\,;D_x)u_z^{(2)}(x) = \iint e^{i(x-x_1)\cdot\xi} p_z(x\,;\xi)(1-\phi(z+x_1))u(x_1)dx_1 d\xi$$

so that, as in (2.58), we have by Theorem 2.1

(2.66) $$\begin{aligned} &|\partial_x^\alpha p_z(X\,;D_x)u_z^{(2)}(x)|^2 \\ &\leqq A_1 \psi(z+x)\int \langle x-x_1\rangle^{-(n+1)} |\langle i\partial x_1\rangle^{-2k_2} u(x_1)|^2 dx_1 \end{aligned}$$

with a constant A_1 of the form

(2.67) $$C_{n,m,k_1,k_2,N,\psi,\phi}\,|p|_{k_1,N+n+1}\,.$$

Then, we have with a constant A_2 of the form (2.67)

(2.68) $$\begin{aligned} \Big\|\int p_z(X\,;D_x)u_z^{(2)}dz\Big\|_{k_1,x} &\leqq \sum_{|\alpha|\leqq k_1} \Big\|\int \partial_x^\alpha p_z(X\,;D_x)u_z^{(2)}(x)\,dz\Big\|_0 \\ &\leqq A_2 \|u\|_{-2k_2}\,. \end{aligned}$$

II) We follow the method of Kuranishi. Using a function $\phi_\varepsilon(\xi)$ which has the properties (2.15), we can write

$$p_z(X;D_x)u_z^{(1)}(x(y))=\int e^{ix(y)\cdot\xi}p_z(x(y);\xi)\int e^{-ix_1\cdot\xi}u_z^{(1)}(x_1)dx_1d\xi$$
$$=\lim_{\varepsilon\to 0}\int\Big\{\int e^{i(x(y)-x(y_1))\cdot\xi}\phi_\varepsilon(\xi)p_z(x(y);\xi)d\xi\Big\}u_z^{(1)}(x(y_1))|\partial_y x(y_1)|dy_1\,.$$

Now, we take a change of variable $\partial_y x(y,y_1)^T\xi=\eta$. Then, we have

$$\begin{aligned}&p_z(X;D_z)u_z^{(1)}(x(y))\\ &=\lim_{\varepsilon\to 0}\int e^{iy\cdot\eta}\Big\{\int e^{-iy_1\cdot\eta}\phi_\varepsilon(\partial_y x(y,y_1)^{TI}\eta)q_z(y;\eta|y_1)u(x(y_1))dy_1\Big\}d\eta\end{aligned}\tag{2.69}$$

where

$$q_z(y;\eta|y_1)=\phi(z+x(y))p(x(y);\partial_y x(y,y_1)^{TI}\eta)|\partial_y x(y,y_1)|^{-1}|\partial_y x(y_1)|\phi(z+x(y_1))\,.$$

From the assumption: $1-\rho\leqq\delta<\rho$, it is easy to see that $q_z(y;\eta|y_1)$ satisfies the condition (2.3). Since $|\partial_{y_1}^\alpha\phi_\varepsilon(\partial_y(y,y_1)^{TI}\eta)|\leqq C_\alpha$ (with a constant C_α independent of $\varepsilon>0$) on $\operatorname{supp}\phi(z+x(y))\phi(z+x(y_1))$, by means of Lemma 2.2 the function in the brackets in (2.69) is estimated by an $L^1_{(\eta)}$ function independent of ε. Letting $\varepsilon\to 0$, we have

$$p_z(X;D_x)u_z^{(1)}(x(y))=q_z(Y;D_y|Y_1)w(y)\,.\tag{2.70}$$

Set

$$q_{z,j}(y;\eta)=\sum_{|\alpha|=j}\frac{1}{\alpha!}(-i\partial_\eta)^\alpha\partial_{y_1}^\alpha q_z(y;\eta|y_1)_{y_1=y,j=0,1,\cdots},\tag{2.71}$$

and

$$R_{z,N}w(y)=\Big(q_z(Y;D_y|y_1)-\sum_{j=0}^{N-1}q_{z,j}(Y;D_y)\Big)w(y)\,.\tag{2.72}$$

Then, by means of Theorem 2.1 we have for $|\beta|\leqq k_1$

$$|\partial_y^\beta R_{z,N}w(y)|^2\leqq A_1\phi(z+x(y))\int\langle y-y_1\rangle^{-(n+1)}|\langle i\partial_{y_1}\rangle^{-2k_2}w(y_1)|^2dy_1\,,$$

so that we have

$$\Big\|\int R_{z,N}w(y)dz\Big\|_{k_1,y}\leqq A_2\|w\|_{-2k_2,y}\,,\tag{2.73}$$

where A_1, A_2 are constants of the form (2.12) which are estimated by a constant of the form (2.63). Noting that $\phi(z+x(y_1))=1$ in a small neighborhood of $\operatorname{supp}_y\phi(z+x(y))$, it is easy to see that

$$q_j(y;\eta)=\int q_{z,j}(y;\eta)dz\,.$$

Then, we have

$$\int p_z(X;D_x)u_z^{(1)}(x(y))dz=\sum_{j=0}^{N-1}q_j(Y;D_y)w(y)+\int R_{z,N}w(y)dy\,.\tag{2.74}$$

From (2.65), (2.68) (2.73), (2.74), applying Lemma 2.4 we get (2.62). Q. E. D.

§ 3. Proof of Theorems in Section 1.

PROOF OF THEOREM 1.1. In Theorem 2.3 we set $p_1(X; D_x) = \Lambda^s \in S^s \subset S^s_{\rho,\delta}$ and $p_2(X; D_x) = p(X; D_x) \in S^m_{\rho,\delta}$. Set $k_1 = 0$ and $k_2 = \text{Max}\{-[(s+m)/2]+1, 0\}$. Then, we have for a large N

$$\left\| \left(\Lambda^s p(X; D_x) - \sum_{j=0}^{N-1} r_j(X; D_x) \right) u \right\|_0 \leqq \text{const.} \|u\|_{-2k_2} \leqq \text{const.} \|u\|_{s+m}.$$

Since $r_j(X; D_x) \in S^{s+m}_{\rho,\delta}$, $j = 0, 1, \cdots$, we have by means of Lemma 2.1

$$\begin{aligned} \|p(X; D_x)u\|_s &= \|\Lambda^s p(X; D_x)u\|_0 \\ &\leqq \sum_{j=0}^{N-1} \|r_j(X; D_x)u\|_0 + \text{const.} \|u\|_{s+m} \leqq \text{const.} \|u\|_{s+m}. \end{aligned}$$

Q. E. D.

LEMMA 3.1. *Let $p_j(x; \xi)$, $j = 0, 1, \cdots$, be a sequence of functions of class $S^{m-(\rho-\delta)j}_{\rho,\delta}$. Then, there exists a $p(x; \xi) \in S^m_{\rho,\delta}$ such that*

$$p(x; \xi) - \sum_{j=0}^{N-1} p_j(x; \xi) \in S^{m-(\rho-\delta)N}_{\rho,\delta} \quad \textit{for any } N. \tag{3.1}$$

PROOF. Let $\phi(\xi)$ be a C^∞ function such that

$$\phi(\xi) = \begin{cases} 0 & \text{for } |\xi| \leqq 1, \\ 1 & \text{for } |\xi| \geqq 2, \end{cases}$$

and set

$$p(x; \xi) = \sum_{j=0}^{\infty} \phi(\xi/t_j) p_j(x; \xi),$$

where t_j, $j = 0, 1, \cdots$, are determined such that

$$|\partial_x^\alpha \partial_\xi^\beta p_j(x; \xi)| \leqq \frac{1}{2^j} \langle \xi \rangle^{m+\delta|\alpha|-\rho|\beta|+1} \quad \text{for } |\xi| \geqq t_j, \quad |\alpha|+|\beta| \leqq j,$$

and $t_j \to \infty$. Then, $p(x; \xi)$ is a desired one. Q. E. D.

PROOF OF THEOREM 1.2. Let $p(x; \xi) \in S^m_{\rho,\delta}$ be the symbol of G and write $G = (G - p(X; D_x)) + p(X; D_x)$. Then, by means of Theorem 1.1, we get (1.9).

ii) Since $|(Gu, v)| \leqq \|Gu\|_0 \|v\|_0 \leqq C_G \|u\|_m \|v\|_0$, there exists a unique element $w = G^* v \in H_{-m}$ such that

$$(Gu, v) = (u, G^* v) \quad \text{for } u \in H_m, \quad v \in H_0.$$

Now set $\bar{p}(x; \xi) = \overline{p(x; \xi)} \in S^m_{\rho,\delta}$. Then, by definition, it is easy to see

$$(p(X; D_x)u, v) = (u, \bar{p}^R(X; D_x)v).$$

By means of Lemma 3.1 we can construct $p^*(x; \xi) \in S^m_{\rho,\delta}$ which satisfies (1.11). We write down

$$G^*-p^*(X\,;D_x)=(G^*-\bar{p}^R(X\,;D_x))+\Big(\bar{p}^R(X\,;D_x)-\sum_{j=0}^{N-1}p_j^*(X\,;D_x)\Big)$$
$$+\Big(\sum_{j=0}^{N-1}p_j^*(X\,;D_x)-p^*(X\,;D_x)\Big)\,.$$

It is easy to see that $G-p(X\,;D_x)\in\mathcal{L}^{-\infty}$ derives $G^*-\bar{p}^R(X\,;D_x)\in\mathcal{L}^{-\infty}$. By means of Theorem 2.2, for any $k_1, k_2\geqq 0$, we have

$$\Big\|\Big(\bar{p}^R(X\,;D_x)-\sum_{j=0}^{N-1}p_j^*(X\,;D_x)\Big)u\Big\|_{k_1}\leqq \text{const.}\ \|u\|_{-2k_2}$$

for large N. Since $\sum_{j=0}^{N-1}p_j^*(X\,;D_x)-p^*(X\,;D_x)\in S_{\rho,\delta}^{m-(\rho-\delta)N}$, we have by Lemma 2.1

$$\Big\|\Big(\sum_{j=0}^{N-1}p_j^*(X\,;D_x)-p^*(X\,;D_x)\Big)u\Big\|_{k_1}\leqq \text{const.}\ \|u\|_{k_1+m-(\rho-\delta)N}\,.$$

Hence, for any s_1, s_2, taking $k_1\geqq s_1$, $-2k_2\leqq s_2$, and N such that $k_1+m-(\rho-\delta)N\leqq s_2$, we have

$$\|(G^*-p^*(X\,;D_x))u\|_{s_1}\leqq \text{const.}\ \|u\|_{s_2}\,.$$

This means $G^*-p^*(X\,;D_x)\in\mathcal{L}^{-\infty}$.

iii) By means of Lemma 3.1 we construct $r(x\,;\xi)\in S_{\rho,\delta}^{m_1+m_2}$ which satisfies (1.13) and we write

$$G_1G_2-r(X\,;D_x)=(G_1-p_1(X\,;D_x))G_2+p_1(X\,;D_x)(G_2-p_2(X\,;D_x))$$
$$+\Big(p_1(X\,;D_x)p_2(X\,;D_x)-\sum_{j=0}^{N-1}r_j(X\,;D_x)\Big)$$
$$+\Big(\sum_{j=0}^{N-1}r_j(X\,;D_x)-r(X\,;D_x)\Big)\,.$$

Then, by a way similar to the proof of ii), we get $G_1G_2-r(X\,;D_x)\in\mathcal{L}^{-\infty}$.

Q. E. D.

PROOF OF COROLLARY. Let $r(x\,;\xi)$, $r'(x\,;\xi)$ be the symbols of G_1G_2, G_2G_1, respectively. Then, $r_0(x\,;\xi)=r(x\,;\xi)-r'(x\,;\xi)$ is the symbol of $[G_1, G_2]$ and by definition $r_0(x\,;\xi)\in S_{\rho,\delta}^{m_1+m_2-(\rho-\delta)}$. Hence, $[G_1, G_2]\in\mathcal{L}_{\rho,\delta}^{m_1+m_2-(\rho-\delta)}$. Q. E. D.

PROOF OF THEOREM 1.3. I) We have by definition

$$|\partial_x^\alpha Gu(x)|\leqq\int\langle\xi\rangle^{-n}(\langle\xi\rangle^n|\xi^\alpha|\,|\widehat{Gu}(\xi)|)d\xi$$
$$\leqq C_n\|Gu\|_{n+|\alpha|}\leqq C_{n,|\alpha|,s}\|u\|_{-s}\,,\qquad \text{for any}\quad s\,,$$

so that there exists $K_\alpha(x\,;y)\in H_{\infty,y}$ such that

$$\|K_\alpha(x\,;\cdot)\|_{s,y}\leqq C_{n,|\alpha|,s} \tag{3.2}$$

and we can write

$$\partial_x^\alpha G(x) = \int K_\alpha(x\,;\,y)u(y)dy\,. \tag{3.3}$$

From this we get, for any fixed x, $K_\alpha(x\,;\,y) \in \mathcal{B}_y$ and

$$|\partial_y^\beta K_\alpha(x\,;\,y)| \leqq C_{n,|\beta|}\|K_\alpha(x\,;\,\cdot)\|_{n+|\beta|} \leqq C_{n,|\alpha|,|\beta|}\,. \tag{3.4}$$

We have

$$\begin{aligned} |K_\alpha(x+\varDelta x\,;\,y)-K_\alpha(x\,;\,y)| &\leqq C_n\|K_\alpha(x+\varDelta x\,;\,\cdot)-K_\alpha(x\,;\,\cdot)\|_{n,y} \\ &= C_n \sup_{u\neq 0} \frac{|\partial_\tau^\alpha(Gu(x+\varDelta x)-Gu(x))|}{\|u\|_{-n}} \\ &\leqq C_{n,|\alpha|}|\varDelta x|\,. \end{aligned} \tag{3.5}$$

From (3.4), (3.5) it follows that $K_\alpha(x\,;\,y)$ is bounded and uniformly continuous in $R^n \times R^n$.

II) Now set $K(x\,;\,y) = K_0(x\,;\,y)$ and

$$K_j(x\,;\,y) = \int_0^{x_j} K_{\alpha_j}(x_1, \cdots, \overset{j}{\overbrace{\tau}}, \cdots, x_n\,;\,y)d\tau\,, \qquad j = 1, \cdots, n\,,$$

where $\alpha_j = (0, \cdots, \overset{j}{\overbrace{1}}, \cdots, 0)$. We define $G_j u(x)$ by

$$G_j u(x) = \int K_j(x\,;\,y)u(y)dy\,.$$

Then, we have $\partial_{x_j}\{Gu(x) - G_j u(x)\} = 0$, so that $Gu(x) - G_j u(x)$ are independent of x_j for any $u \in \mathcal{S}$. Hence, we get

$$K(x\,;\,y) - \int_0^{x_j} K_{\alpha_j}(x_1, \cdots, \overset{j}{\overbrace{\tau}}, \cdots, x_n\,;\,y)d\tau$$

are independent of x_j and consequently we have $\partial_{x_j}K(x\,;\,y) = K_{\alpha_j}(x\,;\,y)$ in the classical sense. Since $K_{\alpha_j}(x\,;\,y)$ are continuous, we have $K(x\,;\,y) \in C^1(R^n \times R^n)$, and inductively we have $K(x\,;\,y) \in C^\infty(R^n \times R^n)$ and $\partial_x^\alpha \partial_y^\beta K(x\,;\,y) = \partial_y^\beta K_\alpha(x\,;\,y)$. In view of (3.2)-(3.4) this completes the proof. Q. E. D.

PROOF OF THEOREM 1.4. $S_{\rho,\delta}^m \cap \mathcal{L}^{-\infty} \supset S^{-\infty}$ is clear.

Assume $p(X\,;\,D_x) \in S_{\rho,\delta}^m \cap \mathcal{L}^{-\infty}$. Set

$$p_0(X\,;\,D_x) = p(X\,;\,D_x)\Lambda^{-(m+n+1)} \in S_{\rho,\delta}^{-(n+1)} \cap \mathcal{L}^{-\infty}\,. \tag{3.6}$$

Then, we can write

$$p_0(X\,;\,D_x)u(x) = \int F(x\,;\,x-y)u(y)dy \tag{3.7}$$

where

$$F(x\,;\,z) = \int e^{iz\cdot\xi}p_0(x\,;\,\xi)d\xi\,.$$

On the other hand, by means of Theorem 1.3, there exists $K(x\,;\,y) \in \mathcal{B}(R^n \times R^n)$

such that

$$\|\partial_x^\alpha K(x\,;\,\cdot)\|_{s,y}^2=\int\langle\xi\rangle^{2s}|\partial_x^\alpha\check{K}(x\,;\,\xi)|^2 d\xi\leqq C_{|\alpha|,s} \tag{3.8}$$

and we have

$$p_0(X\,;\,D_x)u(x)=\int K(x\,;\,y)u(y)dy \tag{3.9}$$

where

$$\check{K}(x\,;\,\xi)=\int e^{iy\cdot\xi}K(x\,;\,y)dy\,.$$

Since $F(x\,;\,x-y)$ and $K(x\,;\,y)$ are continuous, we have from (3.7) and (3.9) $F(x\,;\,x-y)=K(x\,;\,y)$, so that we have

$$e^{ix\cdot\xi}p_0(x\,;\,\xi)=\check{K}(x\,;\,\xi)\,. \tag{3.10}$$

Now assume that there exist α_0, $l_0>0$ and a sequence $\{x_\nu, \xi_\nu\}$ such that $|\xi_\nu|\to\infty$ $(\nu\to\infty)$,

$$|\partial_x^{\alpha_0}p_0(x_\nu\,;\,\xi_\nu)|\langle\xi_\nu\rangle^{l_0}\geqq C>0 \tag{3.11}$$

and

$$\sup_x(|\partial_x^\alpha p_0(x\,;\,\xi)|\langle\xi\rangle^l)\to 0 \quad \text{as} \quad |\xi|\to\infty\,, \quad \text{for every} \quad l,\ \alpha<\alpha_0\,. \tag{3.12}$$

Since $|\partial_{\xi_j}\partial_x^{\alpha_0}p_0(x\,;\,\xi)|\leqq C_{\alpha_0}\langle\xi\rangle^{-(n+1)-\rho+\delta|\alpha_0|}$, we have

$$|\partial_x^{\alpha_0}p_0(x_\nu\,;\,\xi)|\langle\xi\rangle^{l_0}\geqq C/2 \qquad \text{when} \quad |\xi-\xi_\nu|\leqq\langle\xi_\nu\rangle^{-N_0} \tag{3.13}$$

for a large fixed N_0.

From (3.10) we can write

$$\partial_x^{\alpha_0}\check{K}(x\,;\,\xi)=e^{ix\cdot\xi}(\partial_x^{\alpha_0}p_0(x\,;\,\xi)+\sum_{\alpha'<\alpha}C_{\alpha,\alpha'}\xi^{\alpha-\alpha'}\partial_x^{\alpha'}p_0(x\,;\,\xi))\,.$$

Then, in view of (3.11), (3.12), we have

$$|\partial_x^{\alpha_0}\check{K}(x\,;\,\xi)|\langle\xi\rangle^{l_0}\geqq C/3 \qquad \text{when} \quad |\xi-\xi_\nu|\leqq\langle\xi_\nu\rangle^{-N_0}\,,$$

and by means of (3.8), for $M>l_0+nN_0/2$, we have

$$\begin{aligned} C_{|\alpha_0|,M} &\geqq \int_{|\xi-\xi_\nu|\leqq\langle\xi_\nu\rangle^{-N_0}}\langle\xi\rangle^{2M}|\partial_x^{\alpha_0}\check{K}(x\,;\,\xi)|^2 d\xi \\ &\geqq \frac{C}{3}\int_{|\xi-\xi_\nu|\leqq\langle\xi_\nu\rangle^{-N_0}}\langle\xi\rangle^{2(M-l_0)}d\xi=C_{M,N_0,l_0,n}\langle\xi_\nu\rangle^{2(M-l_0)-nN_0}\to\infty \\ &\qquad \text{as} \quad |\xi_\nu|\to\infty\,. \end{aligned}$$

This derives the contradiction.

Hence, we can conclude

$$\sup_x (|\partial_x^\alpha p_0(x;\xi)|\langle\xi\rangle^l) \to 0 \quad \text{as} \quad |\xi|\to\infty \tag{3.14}$$

for any α and l.

In general we have with constants C, C'

$$\begin{aligned}|\partial_{\xi_j}\partial_x^\alpha p_0(x;\xi)|^2 &\leqq C \sup_{|\xi'-\xi|\leqq 1} |\partial_x^\alpha p_0(x;\xi')| \{ \sup_{|\xi'-\xi|\leqq 1} |\partial_x^\alpha p_0(x;\xi')| + \operatorname{Max}_{|\beta|=2} \sup_{|\xi'-\xi|\leqq 1} |\partial_\xi^\beta \partial_x^\alpha p_0(x;\xi')| \} \\ &\leqq C' \langle\xi\rangle^{-(n+1)+\delta|\alpha|} \sup_{|\xi'-\xi|\leqq 1} |\partial_x^\alpha p_0(x;\xi')| .\end{aligned}$$

Then, by means of (3.14) we get

$$\sup_x (|\partial_\xi^\beta \partial_x^\alpha p_0(x;\xi)|\langle\xi\rangle^l) \to 0 \quad \text{as} \quad |\xi|\to\infty \tag{3.15}$$

for $|\beta|=1$ and any α, l, and inductively we get (3.15) for any α, β, l. This means that $p_0(x;\xi)$ and also $p(x;\xi)$ belong to $S^{-\infty}$.

The Corollary is clear. Q. E. D.

Proof of Theorem 1.5. Set $\boldsymbol{p}_1(X;D_x) = \boldsymbol{p}_0(X;D_x)\Lambda^{-m+(\rho-\delta)}$. Then, $\boldsymbol{p}_1(X;D_x) \in \boldsymbol{S}_{\rho,\delta}^{\rho-\delta}$, so that by means of Theorem 2.4 we have

$$\mathscr{R}e\,(\boldsymbol{p}_1(X;D_x)\boldsymbol{u}, \boldsymbol{u}) \geqq -K\|\boldsymbol{u}\|_0^2 . \tag{3.16}$$

Setting $\tau = (m-(\rho-\delta))/2$ and $\boldsymbol{v} = \Lambda^{-\tau}\boldsymbol{u}$, we then write

$$\begin{aligned}\mathscr{R}e\,(\boldsymbol{G}\boldsymbol{v}, \boldsymbol{v}) &= \mathscr{R}e\,(\Lambda^{-\tau}(\boldsymbol{G}-\boldsymbol{p}_0(X;D_x))\Lambda^{-\tau}\boldsymbol{u}, \boldsymbol{u}) \\ &\quad + \mathscr{R}e\,([\Lambda^{-\tau}, \boldsymbol{p}_0(X;D_x)]\Lambda^{-\tau}\boldsymbol{u}, \boldsymbol{u}) \\ &\quad + \mathscr{R}e\,(\boldsymbol{p}_1(X;D_x)\boldsymbol{u}, \boldsymbol{u}) .\end{aligned} \tag{3.17}$$

Then, by means of the assumption, Theorem 1.2 and the corollary of Theorem 1.2, we have

$$\Lambda^{-\tau}(\boldsymbol{G}-\boldsymbol{p}_0(X;D_x))\Lambda^{-\tau} \in \mathcal{L}_{\rho,\delta}^0, \qquad [\Lambda^{-\tau}, \boldsymbol{p}_0(X;D_x)]\Lambda^{-\tau} \in \mathcal{L}_{\rho,\delta}^0 . \tag{3.18}$$

Hence, from (3.16)-(3.18) we have

$$\mathscr{R}e\,(\boldsymbol{G}\boldsymbol{v}, \boldsymbol{v}) \geqq -K_0\|\boldsymbol{u}\|_0^2 = -K_0\|\boldsymbol{v}\|_{(m-(\rho-\delta))/2}^2 .$$

This completes the proof. Q. E. D.

Proof of Theorem 1.6. We may derive (1.22) for $\boldsymbol{p}(X;D_x)$. The first part is easily derived by making use of Theorem 1.5. For the second part we take a sequence $\{x_\nu;\xi_\nu\}$ such that $|\xi_\nu|\to\infty$ as $\nu\to\infty$ and $|\boldsymbol{p}(x_\nu;\xi_\nu)| \to |\boldsymbol{p}|_{\sup}^\infty$.

Let $\Theta(x)$, $\psi(\xi)$ be C_0^∞ functions such that

$$\Theta(x) = 1 \quad \text{for} \quad |x|\leqq 1, \qquad \psi(\xi) = 1 \quad \text{for} \quad |\xi|\leqq 1,$$

and set for $\tau > 0$

$$\Theta_{\nu,\tau}(x) = \Theta(\tau\langle\xi_\nu\rangle^\delta(x-x_\nu)), \qquad \psi_{\nu,\tau}(\xi) = \psi(\tau\langle\xi\rangle^{-\rho}(\xi-\xi_\nu)) .$$

Setting

(3.19) $$p^{(\nu)}(x\,;\,\xi)=p(x\,;\,\xi)-p(x_\nu\,;\,\xi_\nu)\,,$$

we write

$$\begin{aligned}p^{(\nu)}(x\,;\,\xi)=\Theta_{\nu,\tau}(x)p^{(\nu)}(x\,;\,\xi)\psi_{\nu,\tau}(\xi)&\\+(1-\Theta_{\nu,\tau}(x))p^{(\nu)}(x\,;\,\xi)\psi_{\nu,\tau}(\xi)&\\+p^{(\nu)}(x\,;\,\xi)(1-\psi_{\nu,\tau}(\xi))\,.&\end{aligned}$$

Then, we can verify that each term of the above right hand side belongs to $S^0_{\rho,\delta}$ and, for any integer l_1, $l_2\geqq 0$, the norm $|\cdot|_{l_1,l_2}$ is estimated with a constant independent of ν. Now, we take a C_0^∞ function $v(x)$ such that

$$\|v\|_0=1\,,\qquad \operatorname{supp} v\subset\{x\,;\,|x|\leqq 1\}\,,$$

and take constant vectors u_ν such that

(3.20) $$|u_\nu|=1\,,\qquad |p(x_\nu\,;\,\xi_\nu)u_\nu|=|p(x_\nu\,;\,\xi_\nu)|\,.$$

Then, if we set $u_{\nu,\tau}(x)=e^{ix\cdot\xi_\nu}v(\tau\langle\xi_\nu\rangle^\delta(x-x_\nu))\langle\xi_\nu\rangle^{\delta n/2}\tau^{n/2}u_\nu$, we get

$$\|u_{\nu,\tau}\|_0=1\,,\qquad \|p^{(\nu)}(X\,;\,D_x)u_{\nu,\tau}\|\leqq\varepsilon(\tau)+C_\tau\{\|u_{\nu,\tau}\|_{-(\rho-\delta)}+\|(1-\psi_{\nu,\tau}(D_x))u_{\nu,\tau}\|_0\}\,,$$

where $\varepsilon(\tau)\to 0$ as $\tau\to\infty$. Here we used $|\Theta_{\nu,\tau}p^{(\nu)}\psi_{\nu,\tau}|\to 0$ $(\tau\to\infty)$ and Theorem 1.5.

Since $\|u_{\nu,\tau}\|_{-(\rho-\delta)}\to 0$ and $\|(1-\psi_{\nu,\tau}(D_x))u_{\nu,\tau}\|_0\to 0$ as $\nu\to\infty$, we have

$$\|p^{(\nu)}(X\,;\,D_x)u_{\nu,\tau}\|\leqq 2\varepsilon(\tau)\qquad\text{for}\quad \nu\geqq\nu_0(\tau)\,,$$

so that for any $\varepsilon>0$ we have

$$\|p^{(\nu)}(x\,;\,D_x)u_{\nu,\tau}\|\leqq 2\varepsilon\qquad\text{when}\quad \tau\geqq\tau_0 \text{ and } \nu\geqq\nu_0(\tau_0)\,.$$

In view of (3.19), (3.20), this means

$$\|p(X\,;\,D_x)\|\geqq|p|^\infty_{\sup}\,.\qquad\text{(Cf. [1].)}$$ Q. E. D.

PROOF OF THEOREM 1.7. Since $G-p(X\,;\,D_x)\in\mathcal{L}_x^{-\infty}$, by means of Lemma 2.4 we get $Q_G-Q_p\in\mathcal{L}_y^{-\infty}$. By Lemma 3.1 we construct $q(y\,;\,\eta)\in S^m_{\rho,\delta}$ which satisfies (1.25). Then, from Theorem 2.5 we have $Q_p-q(X\,;\,D_x)\in\mathcal{L}_y^{-\infty}$, so that

$$Q_G-q(X\,;\,D_x)=(Q_G-Q_p)+(Q_p-q(X\,;\,D_x))\in\mathcal{L}_y^{-\infty}$$

and $Q_G\in\mathcal{L}^m_{\rho,\delta,y}$. Q. E. D.

PROOF OF COROLLARY. Let $w\in H_{s,y}$. Since $\Lambda^s\in\mathcal{L}^s_x$, by means of Theorem 1.7 Q_s defined by $Q_sw(y)=(\Lambda^su)(x(y))$ belongs to $\mathcal{L}^s_y$. Hence, we have

$$\begin{aligned}\|u\|^2_{s,x}=\|\Lambda^su\|^2_{0,x}=\int|(\Lambda^su)(x)|^2dx=\int|(\Lambda^su)(x(y))|^2\,|\partial_yx(y)|\,dy\\ \leqq\text{const.}\int|(Q_sw)(y)|^2dy\leqq\text{const.}\,\|w\|^2_{s,y}\,,\end{aligned}$$

and also we have

$$\|w\|_{s,y}^2 \leqq \text{const.}\ \|u\|_{s,x}^2. \qquad \text{Q.E.D.}$$

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