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Embedding and Existence theorems of infinite Lie algebra

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In [3] and [5], V. W. Guillemin, I. M. Singer and S. Sternberg gave Existence and Uniqueness theorem and Realization theorem concerning the abstract transitive Lie algebra. In this paper we shall give some extensions of these theorems, i. e. Embedding theorem (Section 5) and Existence theorem (Section 7). The former involves as its applications Realization theorem and also theorems concerning the graded Lie algebra, the latter may be said to be a relative existence theorem with the higher order structure constant. We begin by giving an abstract definition of an infinite Lie algebra and a truncated Lie algebra of any order. Roughly speaking, we shall construct Lie algebra as a projective limit of truncated Lie algebras. By this method, we can simplify the proofs, especially of Existence theorem, and also state the properties of the higher order structure constant. (It is shown in Section 3 that our infinite Lie algebra is equivalent to the complete filtered Lie algebra [4], and hence also to the abstract transitive Lie algebra.)

1. Throughout this paper, all vector spaces and Lie algebras are assumed to be defined over a commutative field of characteristic 0. Suppose that a collection of

a sequence of finite dimensional vector spaces $V_0, V_1, \cdots, V_n, \cdots$,

a sequence of maps $0 \xleftarrow{\pi_0} V_0 \xleftarrow{\pi_1} V_1 \longleftarrow \cdots \xleftarrow{\pi_n} V_n \longleftarrow \cdots$, and

a sequence of bracket products $[\ ,\ ]'_n : V_n \times V_n \ni (x, y) \mapsto [x, y]'_n \in V_{n-1}$, $n = 0, 1, 2, \cdots, (V_{-1} = 0)$ is given, and that the following conditions (a)—(f) are satisfied for all $n \geqq 1$.

(a) π_n is linear and surjective;

(b) $\pi_{n-1}[x, y]'_n = [\pi_n x, \pi_n y]'_{n-1}$, for all $x, y \in V_n$;

(c) $[\ ,\ ]'_n$ is bilinear and anti-symmetric;

(d) $J'_n(x, y, z) = 0$ for all $x, y, z \in V_n$, where J'_n is a trilinear anti-symmetric map of $V_n \times V_n \times V_n$ into V_{n-2} defined by

$$J'_n(x, y, z) = [[x, y]'_n, \pi_n z]'_{n-1} + [[y, z]'_n, \pi_n x]'_{n-1} + [[z, x]'_n, \pi_n y]'_{n-1};$$

(e) Denote by g_n the kernel of the map $\pi_n: V_n \to V_{n-1}$. Then $[g_n, g_n]'_n = 0$;
(f) Let $a \in g_n$. If $[a, x]'_n = 0$ for all $x \in V_n$, then $a = 0$.

Set $L = \text{pr lim}\, V_n$. (L is the vector space consisting of all sequences $(x_0, x_1, \cdots)$ such that $x_n \in V_n$ and $\pi_n x_n = x_{n-1}$.) For any elements $X = (x_0, x_1, \cdots)$ and $Y = (y_0, y_1, \cdots)$ of L, we define bracket product $[,]$ by

$$[X, Y] = ([x_1, y_1]'_1, [x_2, y_2]'_2, \cdots). \tag{1.1}$$

It follows from (b) and (d) that $[,]$ satisfies the usual Jacobi identity, and consequently this product makes L into a Lie algebra. (Note that $\dim L < \infty$, if and only if $g_n = 0$ for all sufficiently large n).

DEFINITION. An infinite Lie algebra is the Lie algebra $L = \text{pr lim}\, V_n$, determined by a collection $\{\{V_n\}, \{\pi_n\}, \{[,]'_n\}\}$, satisfying the conditions (a)—(f), for all $n \geqq 1$.

Subsequently we shall denote the infinite Lie algebra simply by $\text{pr lim}\, V_n$, omitting π_n and $[,]'_n$, since no confusion will occur. V_n is called the n-th truncation of L, π_n the projection.

DEFINITION. A truncated Lie algebra of order p is a collection of a finite sequence of finite dimensional vector spaces $V_0, V_1, \cdots, V_p$, a sequence of maps $0 \overset{\pi_0}{\leftarrow} V_0 \leftarrow \cdots \overset{\pi_p}{\leftarrow} V_p$ and a sequence of bracket products $[,]'_n: V_n \times V_n \to V_{n-1}$, $n = 0, \cdots, p$, satisfying the conditions (a)—(f) for all $n \geqq 1$ and $n \leqq p$. ($p \geqq 0$).

We shall denote this truncated Lie algebra simply by $\{V_0, \cdots, V_p\}$, or more simply by V_p. A truncated Lie algebra is not a Lie algebra in the usual sense of the word. In this paper we shall be concerned with the question when a truncated Lie algebra $\{V_0, \cdots, V_p\}$ can be prolonged to an infinite Lie algebra $\text{pr lim}\, V_n$. This question was considered originally, we may say, by E. Cartan [1] in the third fundamental theorem of the infinite Lie group, and recently by V. Guillemin, I. M. Singer and S. Sternberg [3], [5], although their formulations were seemingly different from ours, and they usually reduced the problem to case $p = 1$.

Let V_p be a truncated Lie algebra of order p. If a subspace W_p of the vector space V_p satisfies $[W_p, W_p]'_p \subset \pi_p W_p$, then by setting $W_{n-1} = \pi_n W_n$, $p \geqq n \geqq 1$, we obtain a truncated Lie algebra $\{W_0, \cdots, W_p\}$, which we shall call a subalgebra W_p of V_p. A subalgebra W_p of V_p is said to be *transitive* if $W_0 = V_0$.

An infinite Lie algebra $M = \text{pr lim}\, W_n$ is said to be a subalgebra of L, if each W_n is a subalgebra of V_n. If $W_0 = V_0$ moreover, M is said to be a *transitive subalgebra* of L. We say that W_{n+1} is a prolongation of W_n in L, and M an infinite prolongation of W_n in L.

Let $L = \text{pr lim}\, V_n$ and $M = \text{pr lim}\, W_n$ be two infinite Lie algebras with the same symbols π_n and $[,]'_n$.

We say that an infinite sequence of maps $\{f_0, f_1, \cdots\}$ is an embedding of M into L, if it satisfies,

(1.2) $\qquad f_n$ is a linear injective map of W_n into V_n;

(1.3) $$\pi_{n-1}f_n = f_{n-1}\pi_n\,;$$

(1.4) $$[f_n x, f_n y]'_n = f_{n-1}[x, y]'_n \quad \text{for any } x, y \in W_n\,;$$

for all n.

In this case we say that $\{f_n\}$ is a lift of each f_n, and $f_p(p > n)$ a lift of f_n. If each f_n is surjective, $\{f_n\}$ is an isomorphism (an automorphism, if $L = M$) of M onto L.

Similarly, let $\{V_0, \cdots, V_p\}$ and $\{W_0, \cdots, W_p\}$ be two truncated Lie algebras. A map $f_p: W_p \to V_p$ is said to be an embedding of truncated Lie algebra W_p into V_p, if there exists a sequence of maps $\{f_0, \cdots, f_p\}$ such that (1.2), (1.3) and (1.4) are satisfied for all $n \leqq p$. Every map f_n $(n < p)$ is said to be a reduced map of f_p.

2. In this section, we shall prove some elementary properties of an infinite Lie algebra $L = \operatorname{pr\,lim} V_n$. Hereafter, we shall omit the subscript n on π_n, $[,]'_n$ and J'_n. We denote by π^n the n-th iterate of π. Thus π^n is the map $V_{n+k} \to V_k$ for any k. ($\pi^0 =$ identity map). We denote by G_n the kernel of the map $\pi^n: V_n \to V_0$. Clearly $G_0 = g_0 = V_0$, $G_1 = g_1$ and $G_n \supset g_n$. We shall keep these notations G_n and g_n for L throughout this section.

(2.1) *Let $a \in G_n$ $(n \geqq 1)$. If $[a, x]' = 0$ for all $x \in V_n$, then $a = 0$.*

Proof. Suppose $a \neq 0$, then for some m $(1 \leqq m \leqq n)$, $\pi^m a = 0$, $\pi^{m-1}a \neq 0$. Therefore $\pi^{m-1}a \in g_{n-m+1}$. Since $[\pi^{m-1}a, \pi^{m-1}x]' = \pi^{m-1}[a, x]' = 0$ for all $x \in V_n$, and $\pi^{m-1}: V_n \to V_{n-m+1}$ is surjective, we have $\pi^{m-1}a = 0$ by condition (f). This contradiction proves our assertion. QED.

(2.2) *If $a, b \in V_n$, $\pi^l a = 0$, $\pi^m b = 0$, $l \geqq 0$, $m \geqq 0$ and $l+m \leqq n+1$, then $[a, b]' = 0$.*

Proof. We prove by induction on n.

If $n = 0$ or 1, our assertion is trivial. We shall prove (2.2) under the assumption that this is true for V_{n-1} $(n \geqq 2)$. If $l = 0$ or $m = 0$, then $[a, b]' = 0$ is trivial. If $l \geqq 1$ and $m \geqq 1$, then $\pi a, \pi b \in V_{n-1}$ and $\pi^{l-1}(\pi a) = 0$, $\pi^{m-1}(\pi b) = 0$, $(l-1)+(m-1) \leqq n$. Therefore by the induction assumption, we have $[\pi a, \pi b]' = 0$. Since $\pi[a, b]' = [\pi a, \pi b]' = 0$, we have $[a, b]' \in g_{n-1}$. Let u be an arbitrary element of V_{n-1} and $\bar{u}$ an element of V_n such that $\pi\bar{u} = u$. By condition (d), $[[a, b]', u]' = [[a, \bar{u}]', \pi b]' - [[b, \bar{u}]', \pi a]'$. On the other hand, $\pi^l[a, \bar{u}]' = [\pi^l a, \pi^l \bar{u}]' = 0$, $\pi^{m-1}(\pi b) = 0$ and $l+(m-1) \leqq n$. Therefore, again by the induction assumption we have $[[a, \bar{u}]', \pi b]' = 0$. Similarly $[[b, \bar{u}]', \pi a]' = 0$. Hence

$[[a, b]', u]' = 0$ for any $u \in V_{n-1}$. Then $[a, b]' = 0$ follows from (f). QED.

As a particular case of (2.2), we have

(2.3) *If* $a \in G_n$ *and* $b \in g_n$, *then* $[a, b]' = 0$.

We say that $\bar{u} \in V_n$ is over $u \in V_k$, if $n > k$ and $\pi^{n-k}\bar{u} = u$. Let $a \in G_n$ ($n \geqq 1$), $u \in V_{n-1}$ and $\bar{u}$ be an arbitrary element of V_n over u. Then it follows from (2.3) that $[a, \bar{u}]'$ is determined by a and u, and is independent of the choice of $\bar{u}$. Thus the action of G_n on V_{n-1} is well defined, which we denote by $au = [a, \bar{u}]'$. It follows from (2.1) that the induced map $G_n \to \mathrm{Hom}(V_{n-1}, V_{n-1})$ is injective.

Let $a, b \in G_n$ ($n \geqq 1$) and $\tilde{a}, \tilde{b}$ be arbitrary elements of G_{n+1} over a, b respectively. Then we can easily see by making use of (2.2) that $[\tilde{a}, \tilde{b}]'$ is an element of G_n determined only by a and b. Thus we can define a product $[,]$ on G_n by $[a, b] = [\tilde{a}, \tilde{b}]'$. Then evidently

(2.4) $$\pi[a, b] = [a, b]', \qquad a, b \in G_n.$$

Moreover

(2.5) G_n *is a Lie algebra acting on* V_{n-1}. ($n \geqq 1$).

That is, $[a, b]u = a(bu) - b(au)$, *and consequently*

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0, \qquad a, b, c \in G_n, \ u \in V_{n-1}.$$

PROOF. Let $\bar{u} \in V_n$ be an element over u, and $\tilde{a}, \tilde{b}, \tilde{u} \in V_{n+1}$ be elements over $a, b, \bar{u}$ respectively. Then our assertion follows from $J'(\tilde{a}, \tilde{b}, \tilde{u}) = 0$. We can also prove in a similar device that

(2.6) g_n *is an ideal of* G_n *and acts trivially on* $G_{n-1} \subset V_{n-1}$. *Furthermore, if* $n \geqq 2$, g_n *is abelian.*

From now on we shall call g_n the n-th isotropy algebra of L.

3. In this section, we prove two propositions, by which we are assured that an infinite Lie algebra is algebraically the same with a complete filtered Lie algebra [**4**], and hence also with an abstract transitive Lie algebra [**3**], [**5**].

Definition of a complete filtered Lie algebra is as follows. A filtered Lie algebra is a Lie algebra L with a decreasing sequence of subalgebras $L = L_{-1} \supset L_0 \supset L_1 \supset L_2 \supset \cdots$, such that

(f1) $\bigcap_i L_i = 0$;

(f2) $[L_i, L_j] \subset L_{i+j}$ ($i+j \geqq -1$);

(f3) $\dim L_i/L_{i+1} < \infty$;

(f4) for every $A \in L_i$, $i \geqq 0$, such that $A \notin L_{i+1}$, there is an element $X \in L$, such that $[A, X] \notin L_i$.

A complete filtered Lie algebra is a filtered Lie algebra such that

(f5) if we define a uniform topology in L, by taking $\{L_i\}$ as a basis for the neighborhood system of the origin, then L is complete with respect to this uniformity.

PROPOSITION 1. *Let $L = \mathrm{pr}\lim V_n$ be an infinite Lie algebra. Denote by L_n the kernel of the natural projection $L \to V_n$, $n \geqq 0$. Then $L \supset L_0 \supset L_1 \supset \cdots$ is a complete filtered Lie algebra.*

PROOF. Let $X = (0, \cdots, 0, x_{i+1}, x_{i+2}, \cdots) \in L_i$ and $Y = (0, \cdots, 0, y_{j+1}, y_{j+2}, \cdots) \in L_j$. Then, by applying (2.2) to V_{i+j+1}, we have $[x_{i+j+1}, y_{i+j+1}]' = 0$. Hence by (1.1), $[X, Y] \in L_{i+j}$. Thus we have proved (f2). Since $L_{i+j} \subset L_i$ $(j \geqq 0)$, we see that L_i is a subalgebra of Lie algebra L. Completeness of L with respect to the uniformity defined by $\{L_i\}$ is clear from the definition of the projective limit. Other conditions for L to be a filtered Lie algebra are easily checked and our assertion is verified. QED.

PROPOSITION 2. *Let $L^0 \supset L_0^0 \supset L_1^0 \supset \cdots$ be a complete filtered Lie algebra. Then we can construct an infinite Lie algebra $L = \mathrm{pr}\lim V_n$, such that $\{L_i^0\}$ is isomorphic to the complete filtered Lie algebra $\{L_i\}$ determined by L just as stated in Proposition* 1.

PROOF. Let $\{L_i^0\}$ be a filtered Lie algebra which is not necessarily complete for a moment. We define an infinite Lie algebra as follows. Denote by V_n the quotient vector space L^0/L_n^0, and by p_n the natural projection $L^0 \to V_n$, $n = 0, 1, \cdots$. Then by (f3), $\dim V_n < \infty$. Since L_n^0 is a subspace of L_{n-1}^0, projection $\pi: V_n \to V_{n-1}$ is naturally defined. Next, for any $x, y \in V_n$, we take $X, Y \in L^0$ such that $p_n X = x$, $p_n Y = y$, and define $[,]'$ by $[x, y]' = p_{n-1}[X, Y]$. (Since $[L^0, L_n^0] \subset L_{n-1}^0$, $p_{n-1}[X, Y]$ depends only on x and y). We can check easily that these vector spaces $\{V_n\}$, projections $\{\pi\}$ and bracket products $\{[,]'\}$ satisfy all the conditions (a)—(f). Thus we have obtained an infinite Lie algebra $L = \mathrm{pr}\lim V_n$. Let $\{L_i\}$ be the complete filtered Lie algebra determined by L just in the way stated in Proposition 1. We define a map $f: L^0 \to L$ by $fX = (p_0 X, p_1 X, \cdots) \in L$, $X \in L^0$. Then we can easily verify that (1) f is injective, (2) f is a homomorphism of Lie algebra L^0 into Lie algebra L, (3) $fL_n^0 = L_n \cap fL^0$ and (4) fL^0 is dense in L. Now, by these properties (1)—(4) of f, we can conclude that L^0 is complete if and only if $fL^0 = L$. Therefore, under the assumption that $\{L_i^0\}$ is a complete filtered Lie algebra, the map f is a Lie algebra isomorphism of L^0 onto L, which sends L_n^0 onto L_n. QED.

4. In this section we shall briefly describe the associated graded infinite

Lie algebra of an infinite Lie algebra and the related homology groups. From now on, in this paper we always denote by L the infinite Lie algebra determined by $\{V_n\}$, and by g_n the n-th isotropy algebra of L.

We define a bracket product $[,]^0$ in the direct sum $\sum_{n=0}^{\infty} g_n$ as follows. Let $a \in g_i$ and $b \in g_j$. Set $[a, b]^0 = [\bar{a}, \bar{b}]'$, where $\bar{a}$ and $\bar{b}$ are arbitrary elements of V_{i+j} over a and b respectively. Then we can easily check, using (2.2), that $[a, b]^0$ is a well defined element of g_{i+j-1}, and also that the product $[,]^0$ satisfies the usual Jacobi identity. Thus we obtain the so-called graded Lie algebra $\sum_{n=0}^{\infty} g_n$ with bracket product $[,]^0$.

Setting $L_i = \sum_{n=i+1}^{\infty} g_n$ $(i \geqq -1)$, we have a filtered Lie algebra $\{L_i\}$. Then as in the proof of Proposition 2, we can construct the corresponding infinite Lie algebra, which we shall denote by $Gr(L) = \overline{\sum_{n \geqq 0}} g_n$. An infinite Lie algebra L is said to be graded, if it is isomorphic to $Gr(L)$.

If $a \in g_n$, $v \in g_0 = V_0$, we shall write $a \cdot v$ instead of $[a, v]^0$, thus we have a map $g_n \times V_0 \ni (a, v) \mapsto a \cdot v \in g_{n-1}$.

Let U and V be any vector spaces. We identify the vector space of all l-linear anti-symmetric maps of $U \times \cdots \times U$ (l times) into V with the space $V \otimes \wedge^l(U^*)$, where U^* is the dual space of U. We define the boundary operator $\partial: g_{i+1} \otimes \wedge^{j-1}(V_0^*) \to g_i \otimes \wedge^j(V_0^*)$ by $(\partial f)(v_1, v_2, \cdots, v_j) = \sum_{k=1}^{j} (-)^{j-k} f(v_1, \cdots, \hat{v}_k, \cdots, v_j) \cdot v_k$, $v_1, \cdots, v_j \in V_0$, $f \in g_{i+1} \otimes \wedge^{j-1}(V_0^*)$, where the symbol $\hat{v}$ indicates that the argument v is omitted. We can easily see that $\partial^2 = 0$. We denote by $H^{ij}(L)$ the homology group at $g_i \otimes \wedge^j(V_0^*)$, $i, j \geqq 0$, with respect to the boundary operator ∂. Suppose more generally that U is a vector space and a projection $\pi: U \to V_0$ is given. Then $\partial: g_{i+1} \otimes \wedge^{j-1}(U^*) \to g_i \otimes \wedge^j(U^*)$ is defined by $(\partial f)(u_1, \cdots, u_j) = \sum_{k=1}^{j} (-1)^{j-k} f(u_1, \cdots, \hat{u}_k, \cdots, u_j) \cdot (\pi u_k)$, $u_1, \cdots, u_j \in U$. In this case also we have $\partial^2 = 0$.

Let $\{V_0 = h_0, h_1, \cdots\}$ be a sequence of subspaces h_n of g_n, such that $h_n \cdot V_0 \subset h_{n-1}$ for all n. Then the homology group at $h_i \otimes \wedge^j(V_0^*)$ is similarly defined, which we denote by $H^{ij}(\{h\})$.

LEMMA 1. *Let U be a vector space, and π a projection of U onto V_0. Let $f \in h_i \otimes \wedge^m(U^*)$ $(m \geqq 1)$ and assume $\partial f = 0$. If $H^{ij}(\{h\}) = 0$, for $j = 1, \cdots, m$, then there exists an element $\sigma \in h_{i+1} \otimes \wedge^{m-1}(U^*)$, such that $\partial \sigma = f$.*

PROOF. Let G be the kernel of the projection π. Let $\varphi: V_0 \to U$ be a linear map such that $\pi\varphi =$ identity map of V_0. Then we have $U = G \oplus \varphi V_0$, where the symbol $\oplus$ means the direct sum. Let $a_1, \cdots, a_{m-j} \in G$. We define $f_j(a_1, \cdots, a_{m-j}) \in h_i \otimes \wedge^j(V_0^*)$ by $f_j(a_1, \cdots, a_{m-j})(v_1, \cdots, v_j) = f(a_1, \cdots, a_{m-j}, \varphi v_1, \cdots, \varphi v_j)$, $v_1, \cdots, v_j \in V_0$, $j = 0, \cdots, m$. From $(\partial f)(a_1, \cdots, a_{m-j}, \varphi v_1, \cdots, \varphi v_{j+1}) = 0$, we obtain $\partial(f_j(a_1, \cdots, a_{m-j}))=0$. (In particular, we have $f(a_1, \cdots, a_m)=0$, for $j=0$.) Therefore, by the assumption $H^{ij}(\{h\})=0$, there exists an element $\sigma_j(a_1, \cdots, a_{m-j})$ of $h_{i+1}\otimes\wedge^{j-1}(V_0^*)$ such that $\partial(\sigma_j(a_1, \cdots, a_{m-j}))=f_j(a_1, \cdots, a_{m-j})$, for each $j=1, \cdots, m$. Moreover we can assume that $\sigma_j: G\times\cdots\times G\ni(a_1, \cdots, a_{m-j})\mapsto\sigma_j(a_1, \cdots, a_{m-j})\in h_{i+1}\otimes\wedge^{j-1}(V_0^*)$ is $(m-j)$-linear and anti-symmetric. Then the desired σ is given by $\sigma(u_1, \cdots, u_{m-1})=\sum_{j=1}^{m}\sum_{\tau}\operatorname{sgn}\tau\ \sigma_j(a_{\tau(1)}, \cdots, a_{\tau(m-j)})(v_{\tau(m-j+1)}, \cdots, v_{\tau(m-1)})$, where $u_k=a_k+\varphi v_k\in G\oplus\varphi V_0$, $k=1, \cdots, m-1$ and the sum $\sum_{\tau}$ is taken over all permutations τ of $\{1, \cdots, m-1\}$ such that $\tau(1)<\cdots<\tau(m-j)$ and $\tau(m-j+1)<\cdots<\tau(m-1)$. QED.

5. Throughout this section, we assume that $\{W_0, W_1, \cdots, W_p, W_{p+1}\}$ is a truncated Lie algebra of order $p+1$ such that $\dim W_0=\dim V_0$. Let $\varphi: W_p\to W_{p+1}$ be an arbitrary linear section. (By a linear section φ we mean a linear map such that $\pi\varphi=$ identity map of W_p.) Then we have a transitive subalgebra $\varphi W_p=\{W_0, \cdots, W_p, \varphi W_p\}$ of W_{p+1}.

LEMMA 2. *Let $\varphi: W_p\to W_{p+1}$ be a linear section. Assume that there exists an embedding $f': \varphi W_p\to V_{p+1}$, and that $H^{p1}(L)=0$. Then there exists a unique embedding $f: W_{p+1}\to V_{p+1}$, which coincides with f' on φW_p.*

PROOF. We denote by f_k the embedding $W_k\to V_k$ reduced from f', $(k\leqq p)$. Let h be the kernel of the projection $W_{p+1}\to W_p$. Then we have $W_{p+1}=h\oplus\varphi W_p$. In order to extend f' on φW_p to f on W_{p+1}, satisfying (1.3) and (1.4) with n replaced by $p+1$, we have only to define a map $f: h\to g_{p+1}$ satisfying

(5.1) $$[fa, f'x]'=f_p[a, x]' \quad \text{for any } a\in h \text{ and } x\in\varphi W_p\,.$$

Putting $f_pW_p=U$ and $\pi f'x=u$, we have from (5.1)

$$(fa)u=f_p[a, \varphi f_p^{-1}u]'\,,\qquad a\in h,\ u\in U\,.$$

Since $\pi f_p[a, \varphi f_p^{-1}u]'=f_{p-1}[\pi a, f_p^{-1}u]'=0$, we have $f_p[a, \varphi f_p^{-1}u]\in g_p$. Define $T_a\in g_p\otimes U^*$ by $T_a(u)=f_p[a, \varphi f_p^{-1}u]'$, $u\in U$. Then $\partial T_a=0$. In fact, $\partial T_a(u, v)=[f_p[a, \varphi f_p^{-1}u]', v]'-[f_p[a, \varphi f_p^{-1}v]', u]'=f_{p-1}[\pi a, [\varphi f_p^{-1}u, \varphi f_p^{-1}v]']'=0$. $u, v\in U$. Besides, we can see that since $\dim W_0=\dim V_0$, $\pi^p: U\to V_0$ is a surjection. Therefore by Lemma 1, there exists an element, say fa, of g_{p+1} such that $\partial(fa)=T_a$ i.e. $(fa)u=T_a(u)$ for all $u\in U$. Moreover, since $\partial: g_{p+1}\to g_p\otimes U^*$ is an injection, fa is uniquely determined. Finally, injectivity of the map $h\ni a\mapsto fa\in g_{p+1}$ is obvious. Therefore the resulting map f defined on W_{p+1} also satisfies (1.2) with n replaced by $p+1$. QED.

PROPOSITION 3. *Assume that $H^{p1}(L)=H^{p2}(L)=0$ and that there exists an embedding $f_p: W_p\to V_p$. Then we can lift f_p to an embedding $f: W_{p+1}\to V_{p+1}$.*

PROOF. Set $f_pW_p=U$. Let $\theta: U\to V_{p+1}$ and $\varphi: W_p\to W_{p+1}$ be arbitrary linear sections. Define $F\in V_p\otimes\wedge^2(U^*)$ by

(5.2) $$F(u, v) = f_p[\varphi f_p^{-1}u, \varphi f_p^{-1}v]' - [\theta u, \theta v]' \qquad \text{for } u, v \in U.$$

Then it is easily checked using $J'=0$ that F is a cycle $\in g_p \otimes \wedge^2(U^*)$. Therefore there is an element $\sigma \in g_{p+1} \otimes U^*$ and $\partial\sigma = F$. Define $f': \varphi W_p \to V_{p+1}$ by $f'(\varphi x) = \theta'(f_p x)$, where $\theta' = \theta + \sigma$, $x \in W_p$. Then for any $x, y \in W_p$, $[f'\varphi x, f'\varphi y]' = [\theta f_p x, \theta f_p y]' + (\partial\sigma)(f_p x, f_p y) = f_p[\varphi x, \varphi y]'$. Thus f' is an embedding of φW_p into V_{p+1}. Now we can apply Lemma 2 and our assertion is verified. QED.

We generalize Lemma 2 as follows.

PROPOSITION 4. *Let* $\{W'_0, \cdots, W'_{p+1}\}$ *be a transitive subalgebra of* $\{W_0, \cdots, W_{p+1}\}$. *Assume that there exist embeddings* $f_p: W_p \to V_p$ *and* $f': W'_{p+1} \to V_{p+1}$, *such that* f' *is a lift of the restriction of* f_p *to* W'_p. *If* $H^{p1}(L)=0$, *then we can lift* f_p *to an embedding* f_{p+1} *of* W_{p+1} *into* V_{p+1}, *which coincides with* f' *on* W'_{p+1}. *Furthermore such* f_{p+1} *is uniquely determined.*

PROOF. Let $\varphi: W_p \to W_{p+1}$ be a linear section such that $\varphi W'_p \subset W'_{p+1}$. Then we can take a linear section $\theta: U = f_p W_p \to V_{p+1}$ such that

$$f'\varphi x = \theta f_p x \qquad \text{for all } x \in W'_p.$$

Define $F \in g_p \otimes \wedge^2(U^*)$ by (5.2) with new φ and θ. Then $F(u, v)=0$, if $u, v \in f_p W'_p$.

Now we wish to solve the equation $\partial\sigma = F$, with unknown $\sigma \in g_{p+1} \otimes U^*$, under the condition $\sigma(u)=0$ for $u \in f_p W'_p$. Since $\dim W'_0 = \dim V_0$, we have $\pi^p f_p W'_p = V_0$. Therefore $U = f_p W'_p \oplus G$, where G is a subspace of the kernel of the projection $\pi^p: U \to V_0$. Since $\partial F = 0$, we have $F(a, b)=0$ and $\partial(F(a))=0$ for $a, b \in G$, where $F(a)$ is an element of $g_p \otimes U^*$ defined by $F(a)(u) = F(a, u)$, in the same way as in the proof of Lemma 1. Owing to these properties of F, our equation reduces to $\partial(\sigma(a)) = F(a)$, $a \in G$. Then, using Lemma 1, we have a unique solution $\sigma(a) \in g_{p+1}$.

Define $f_{p+1}: \varphi W_p \to V_{p+1}$, by $f_{p+1}(\varphi x) = \theta f_p x + \sigma(a)$, where x is an arbitrary element of W_p, and a is the G component of $f_p x$ with respect to the direct sum $U = f_p W'_p \oplus G$. Then we can see easily that $[f_{p+1}\varphi x, f_{p+1}\varphi y]' = f_p[\varphi x, \varphi y]'$, for any $x, y \in W_p$. Thus $f_{p+1}: \varphi W_p \to V_{p+1}$ is an embedding. Applying Lemma 2, we can extend f_{p+1} uniquely to the embedding $W_{p+1} \to V_{p+1}$. QED.

By repeated applications of Proposition 4, we have the following theorem.

THEOREM 1. *Let* $L = \operatorname{pr\,lim} V_n$ *and* $M = \operatorname{pr\,lim} W_n$ *be infinite Lie algebras such that* $\dim V_0 = \dim W_0$, *and let* $M' = \operatorname{pr\,lim} W'_n$ *be a transitive subalgebra of* M. *Assume that there exist embeddings* $f_p: W_p \to V_p$ (*for some* p) *and* $\{f'_n\}: M' \to L$, *such that* f_p *coincides with* f'_p *on* W'_p. *Assume further that* $H^{i1}(L)=0$ *for all* $i \geqq p$.

Then there exists a unique embedding $\{f_n\}: M \to L$ *which is a lift of* f_p *and coincides with* $\{f'_n\}$ *on* M'.

THEOREM 2 (EMBEDDING THEOREM). *Let* $L = \operatorname{pr\,lim} V_n$ *and* $M = \operatorname{pr\,lim} W_n$

be infinite Lie algebras such that $\dim V_0 = \dim W_0$. *Assume that there exists an embedding* $f_p: W_p \to V_p$ *for some* p *and* $H^{i1}(L) = H^{i2}(L) = 0$ *for all* $i \geqq p$.

Then f_p *can be lifted to an embedding* $\{f_n\}: M \to L$. *Furthermore, if* $\{\tilde{f}_n\}$ *is another such lift of* f_p, *then there exists an automorphism* $\{h_n\}$ *of* L, *such that* h_n *sends* $f_n W_n$ *onto* $\tilde{f}_n W_n$ *for each* n.

PROOF. The first assertion follows from repeated applications of Proposition 3. The existence of $\{h_n\}$ is proved as follows. Set $L' = \operatorname{pr}\lim f_n W_n$. Applying Theorem 1 to L and L', we can extend the embedding $\{\tilde{f}_n f_n^{-1}\}: L' \to L$ to the embedding $\{h_n\}: L \to L$. Since $\dim V_n < \infty$, each h_n is a bijection, hence $\{h_n\}$ is an automorphism of L. QED.

APPLICATIONS. Set $D(V_0) = \overline{\sum_{n \geqq 0}} V_0 \otimes S^n(V_0^*)$. This can be regarded as the graded infinite Lie algebra of all formal power series vector fields on V_0. It is known that $H^{ij}(D(V_0)) = 0$ for all $i, j \geqq 0$, $(i, j) \neq (0, 0)$, and indeed $D(V_0)$ is characterized by this property.

THEOREM 3 (REALIZATION THEOREM). *Every infinite Lie algebra* L *is isomorphic to a transitive subalgebra of* $D(V_0)$. *Furthermore such subalgebra is determined up to an automorphism of* $D(V_0)$.

PROOF. Since $H^{ij}(D(V_0)) = 0$ for all $i \geqq 0$, $j \geqq 1$, we can apply Theorem 2 to $D(V_0)$ and L, with $f_0 =$ identity map of V_0. QED.

THEOREM 4. *Let* L *be an infinite Lie algebra such that* $H^{i1}(L) = H^{i2}(L) = 0$ *for all* $i \geqq 1$. *Then* L *is graded, if and only if the first structure constant of* L *is* 0. (As for the structure constant, see Section 7).

PROOF. Let L be graded. Since the first structure constant of $Gr(L)$ is 0, that of L must also be 0.

Conversely, assume that the first structure constant of L is 0. Then we can see easily that there is an isomorphism of the first truncation V_1 of L onto that of $Gr(L)$. Since the homology groups of $Gr(L)$ and L are the same, we can apply Theorem 2 to obtain an embedding $L \to Gr(L)$. Since the dimension of the truncation of any order of L is equal to that of the same order of $Gr(L)$, the embedding is an isomorphism. QED.

We say that an infinite Lie algebra $L = \operatorname{pr}\lim V_n$ is abelian if $[V_n, V_n]' = 0$ for all n. An infinite Lie algebra is said to be *flat*, if it contains an abelian transitive subalgebra.

The first structure constant of a flat infinite Lie algebra is 0. Every graded infinite Lie algebra is evidently flat. Therefore the following theorem follows from Theorem 1.

THEOREM 5. *Let* L *be an infinite Lie algebra such that* $H^{i1}(L) = 0$ *for all* $i \geqq 1$. *If* L *is flat, then it is also graded.*

6. Let h_n be any subspace of g_n. We define $h_n^{(1)} \subset g_{n+1}$ by $h_n^{(1)} = \{a \in g_{n+1};$

$a \cdot v \in h_n$ for all $v \in V_0\}$. Let W_p be a transitive subalgebra of V_p. We say that W_p is prolongable in L, if there exists a linear section $\theta: W_p \to V_{p+1}$ such that $[\theta W_p, \theta W_p]' \subset W_p$. Let $h_p = g_p \cap W_p$ and h_{p+1} be an arbitrary subspace of $h_p^{(1)}$. If W_p is prolongable by a linear section θ, then $h_{p+1} + \theta W_p$, is clearly a subalgebra of V_{p+1}. Conversely, if W_{p+1} is a subalgebra of V_{p+1} such that $\pi W_{p+1} = W_p$, then the kernel of the projection $W_{p+1} \to W_p$ is contained in $h_p^{(1)}$. If it is identical with $h_p^{(1)}$, W_{p+1} is called the normal prolongation of W_p in L. We denote by $\{H^{ij}(W_p; L)\}$ the homology groups determined by the sequence $\{h_i\}$, where $h_i = g_i \cap W_i$, if $i \leqq p$, and h_i $(i > p)$ is the subspace of g_i defined inductively by $h_n = h_{n-1}^{(1)}$.

PROPOSITION 5. *Let W_p $(p \geqq 1)$ be a transitive subalgebra of V_p. If $H^{p-1,j}(W_p; L) = 0$, $j = 1, 2, 3$ and $H^{pj}(L) = 0$, $j = 1, 2$, then W_p is prolongable in L.*

PROOF. Let $\psi: W_{p-1} \to W_p$ and $\theta: W_p \to V_{p+1}$ be arbitrary linear sections. Set $F(u, v) = [\theta u, \theta v]' - \psi[u, v]'$, $u, v \in W_p$. Then $\pi F(u, v) = 0$, therefore F is an element of $g_p \otimes \wedge^2(W_p^*)$. By definition of ∂ and by $J' = 0$, we have $(\partial F)(u, v, w) = -\sum [\psi[u, v]', w]'$, for any $u, v, w \in W_p$, where $\sum$ is the sum over all cyclic permutations of u, v, w. The right hand side of this identity shows that ∂F is a cycle belonging to $h_{p-1} \otimes \wedge^3(W_p^*)$. It follows from the assumption $H^{p-1,j}(W_p; L) = 0$, $j = 1, 2, 3$, that there exists an element f of $h_p \otimes \wedge^2(W_p^*)$ such that $\partial f = \partial F$. Then $f - F$ is a cycle belonging to $g_p \otimes \wedge^2(W_p^*)$. Again, by the assumption $H^{pj}(L) = 0$, $j = 1, 2$, there is an element σ of $g_{p+1} \otimes W_p^*$ such that $\partial \sigma = f - F$. Define a linear section $\theta': W_p \to V_{p+1}$ by setting $\theta' = \theta + \sigma$. Then, for any $u, v \in W_p$, we have $[\theta' u, \theta' v]' = [\theta u, \theta v]' + (\partial \sigma)(u, v) = f(u, v) + \psi[u, v]'$. Since $f(u, v)$ and $\psi[u, v]' \in W_p$, we have $[\theta' u, \theta' v] \in W_p$. Thus W_p is prolongable by θ'. QED.

THEOREM 6. *Let $L = \mathrm{pr}\lim V_n$ be an infinite Lie algebra and W_p $(p \geqq 1)$ a transitive subalgebra of V_p. Assume that $H^{ij}(W_p; L) = 0$ for all $i \geqq p-1$ and $j = 1, 2, 3$ and also $H^{ij}(L) = 0$ for all $i \geqq p$ and $j = 1, 2$.*

Then there exists a normal infinite prolongation M of W_p in L, such that for any infinite prolongation M' of W_p in L, there is an automorphism of L which embeds M' into M.

PROOF. By Proposition 5, W_p is prolongable in L. Let W_{p+1} be its normal prolongation in L. In this case we have $H^{ij}(W_p; L) = H^{ij}(W_{p+1}; L)$. Therefore W_{p+1} is again prolongable in L. Continuing in this fashion, we can obtain a normal infinite prolongation of W_p in L, which we denote by M. Since we have $H^{ij}(M) = H^{ij}(W_p, L) = 0$, $i \geqq p$, $j = 1, 2$, there is an embedding of any infinite prolongation M' of W_p into M, owing to Theorem 2. Then, it follows from Theorem 1, that we can extend the embedding $M' \to M$ to the embedding of L into L, which is clearly an automorphism of L. QED.

7. In this section, we shall make some observation on the properties of the structure constants of the transitive subalgebra of L, and state existence theorem.

Let W_p $(p \geqq 1)$ be a transitive subalgebra of V_p, and $h_p = W_p \cap g_p$. We define an element $c \in W_{p-1} \otimes \wedge^2(W_{p-1}^*)$ by $c(u, v) = [\varphi u, \varphi v]'$, $u, v \in W_{p-1}$, where $W_{p-1} = \pi W_p$ and φ is an arbitrary linear section $W_{p-1} \to W_p$. Let φ' be another such linear section and $c'(u, v) = [\varphi' u, \varphi' v]'$. Then there is an element S of $h_p \otimes W_{p-1}^*$ such that $\varphi' = \varphi + S$, and we have $c' = c + \partial S$. That is, $c' = c$ (mod. $\partial(h_p \otimes W_{p-1}^*)$). Therefore, a class $\boldsymbol{c} = \{c\} \in W_{p-1} \otimes \wedge^2(W_{p-1}^*)/\partial(h_p \otimes W_{p-1}^*)$ is determined independently of the choice of φ. We call $\boldsymbol{c}$ the structure constant of W_p. If $W_p = V_p$, $\boldsymbol{c}$ is called the p-th order structure constant of L. Clearly we have $\pi(c(u, v)) = [u, v]'$ and $(\partial c)(u, v, w) = J'(\varphi u, \varphi v, \varphi w) = 0$, for $u, v, w \in W_{p-1}$, where ∂c is an element of $W_{p-2} \otimes \wedge^3(W_{p-1}^*)$, defined by $(\partial c)(u, v, w) = [c(u, v), w]' + [c(v, w), u]' + [c(w, u), v]'$. Since these properties of c are independent of the choice of the representative c of the class $\boldsymbol{c}$, we can write them as follows,

$$\pi \boldsymbol{c} = [\,,\,]' \quad \text{and} \quad \partial \boldsymbol{c} = 0\,, \tag{7.1}$$

where π and ∂ are understood as maps of $W_{p-1} \otimes \wedge^2(W_{p-1}^*)/\partial(h_p \otimes W_{p-1}^*)$ into $W_{p-2} \otimes \wedge^2(W_{p-1}^*)$ and $W_{p-2} \otimes \wedge^3(W_{p-1}^*)$ respectively.

Now we assume that W_p is prolongable in L. Then by the same argument as in Section 2, we have

(7.2) *h_p is a Lie algebra acting on W_{p-1}.*

Next, take a linear section $\theta: W_p \to V_{p+1}$, such that $[\theta W_p, \theta W_p]' \subset W_p$, and set

$$Y(u, v) = [\theta\varphi u, \theta\varphi v]' - \varphi c(u, v) \quad \text{and} \quad Z(a)u = [\theta a, \theta\varphi u]' - \varphi(au)\,,$$

where $u, v \in W_{p-1}$ and $a \in h_p$.

Then we can see that $Y \in h_p \otimes \wedge^2(W_{p-1}^*)$ and $Z(a) \in h_p \otimes W_{p-1}^*$. From $J'(\theta\varphi u, \theta\varphi v, \theta a) = 0$, we obtain

$$ac - \partial(Z(a)) = 0 \qquad \text{for any } a \in h_p\,, \tag{7.3}$$

where ac is an element of $W_{p-1} \otimes \wedge^2(W_{p-1}^*)$ defined by $(ac)(u, v) = a(c(u, v)) - c(au, v) - c(u, av)$.

Similarly from $J'(\theta\varphi u, \theta\varphi v, \theta\varphi w) = 0$, we obtain

$$c^2 + \partial Y = 0\,, \tag{7.4}$$

where c^2 is an element of $W_{p-1} \otimes \wedge^3(W_{p-1}^*)$, defined by $c^2(u, v, w) = c(c(u, v), w)$ $c(c(v, w), u) + c(c(w, u), v)$, $u, v, w \in W_{p-1}$. E. Cartan's original statements corresponding to (7.2), (7.3) and (7.4) are respectively 2°, 3° and 4° in Chapter II, Section 23, [**1**]. Formulations like (7.3) and (7.4) are due to [**3**] and [**5**]. We shall show that they can be written in terms of $\boldsymbol{c}$. It is easily seen that for any $a\in h_p$, $a\boldsymbol{c}$ is a well defined element of $W_{p-1}\otimes\wedge^2(W^*_{p-1})/\partial(h_p\otimes W^*_{p-1})$, and (7.3) becomes

$$a\boldsymbol{c}=0 \quad \text{for any } a\in h_p\,. \tag{7.5}$$

As for c^2, we wish to prove that if we set $c'=c+\partial S$ for any element $S\in h_p\otimes W^*_{p-1}$, then $(c')^2=c^2+\partial T$ for some element $T\in h_p\otimes\wedge^2(W^*_{p-1})$. This does not hold in general, but under the assumptions (7.2) and (7.3), we can verify by a straightforward computation that such T is given by $T(u,v)=S(c(u,v))+S(S(u)v-S(v)u)-[S(u),S(v)]-Z(S(u))v+Z(S(v))u$, where $Z(\cdot)$ is an element of $h_p\otimes W^*_{p-1}$ which satisfies (7.3). Thus c^2 determines a well defined element $\boldsymbol{c}^2$ of $W_{p-1}\otimes\wedge^3(W^*_{p-1})/\partial(h_p\otimes\wedge^2(W^*_{p-1}))$ and (7.4) can be written as

$$\boldsymbol{c}^2=0\,. \tag{7.6}$$

PROPOSITION 6. *Let W_{p-1} be a transitive subalgebra of V_{p-1}, and h_p a subalgebra of the Lie algebra g_p satisfying $h_pW_{p-1}\subset W_{p-1}$. Assume that W_p and W'_p are prolongations of W_{p-1} in L and $h_p=g_p\cap W_p=g_p\cap W'_p$. Assume further that $H^{p1}(L)=0$. Then there exists an isomorphism of W_p onto W'_p which induces the identity map of W_{p-1}, if and only if the structure constants of W_p and W'_p are identical.*

PROOF. We shall only prove the existence of an isomorphism f_p of W_p onto W'_p, under the assumption that they have the same structure constant, because the converse is nearly evident.

Let $\varphi: W_{p-1}\to W_p$ and $\varphi': W_{p-1}\to W'_p$ be arbitrary linear sections. Then $\varphi'=\varphi+S$ for some $S\in g_p\otimes W^*_{p-1}$. It follows from the assumption on the structure constant that $\partial S=\partial\sigma$ for some $\sigma\in h_p\otimes W^*_{p-1}$. Thus $S-\sigma$ is a cycle $\in g_p\otimes W^*_{p-1}$. Hence $S-\sigma=\partial T$ for some element $T\in g_{p+1}$, and we have $\varphi'=\varphi+\sigma+\partial T$. Define a map $f_p: W_p=\varphi W_{p-1}\oplus h_p\to W'_p=\varphi' W_{p-1}\oplus h_p$ by $f_p(\varphi u+a)=\varphi' u+(a-\sigma(u))$, where $u\in W_{p-1}$ and $a\in h_p$. Then by a simple calculation we see that f_p is an isomorphism which we want. QED.

PROPOSITION 7. *Let W_p be a transitive subalgebra of V_p ($p\geqq 0$). Assume that $H^{p1}(L)=H^{p2}(L)=0$ and that there exists an element $c\in W_p\otimes\wedge^2(W^*_p)$ such that $\pi c=[\,,\,]'$ and $\partial c=0$. Then W_p is prolongable in L by a linear section $\phi: W_p\to V_{p+1}$ such that $c(u,v)=[\phi u,\phi v]'$ for $u,v\in W_p$.*

PROOF. Take an arbitrary linear section $\theta: W_p\to V_{p+1}$. Define $F\in V_p\otimes\wedge^2(W^*_p)$ by $F(u,v)=c(u,v)-[\theta u,\theta v]'$. Then we can see easily that F is a cycle $\in g_p\otimes\wedge^2(W^*_p)$. Hence there is an element $\sigma\in g_{p+1}\otimes W^*_p$ such that $\partial\sigma=F$. Then $\phi=\theta+\sigma$ is the linear section which we want. QED.

PROPOSITION 8. *Let W_p be a transitive subalgebra of V_p and $h_p=W_p\cap g_p$.* ($p\geqq 1$). *Assume that* (1) $H^{p1}(L)=H^{p2}(L)=0$, (2) *h_p is a subalgebra of the Lie algebra g_p and $h_pW_{p-1}\subset W_{p-1}$,* ($W_{p-1}=\pi W_p$), (3) *the structure constant $\boldsymbol{c}$ of*

W_p satisfies (7.5) *and* (7.6). *Then W_p is prolongable in L.*

PROOF. We take a representative c of $\boldsymbol{c}$ which is given by $\varphi: W_{p-1} \to W_p$. Then by the assumption (3), we have $c^2 + \partial Y = 0$ for some $Y \in h_p \otimes \wedge^2(W^*_{p-1})$ and $ac = \partial(Z(a))$ for some $Z(a) \in h_p \otimes W^*_{p-1}$, where we can assume that $h_p \ni a \mapsto Z(a)$ is a linear map.

Now we define an element $\bar{c} \in W_p \otimes \wedge^2(W^*_p)$, referring to the direct sum $W_p = \varphi W_{p-1} \oplus h_p$, by $\bar{c}(\varphi u, \varphi v) = \varphi c(u, v) + Y(u, v)$, $\bar{c}(a, \varphi u) = \varphi(au) + Z(a)u$, $\bar{c}(a, b) = [a, b]$, where $u, v \in W_{p-1}$ and $a, b \in h_p$. Then $\bar{c}$ satisfies $\pi\bar{c} = [\ ,\ ]'$ and $\partial\bar{c} = 0$. Indeed, $\pi\bar{c}(\varphi u + a, \varphi v + b) = c(u, v) + av - bu = [\varphi u + a, \varphi v + b]'$, and a simple computation shows $(\partial\bar{c})(\varphi u_1 + a_1, \varphi u_2 + a_2, \varphi u_3 + a_3) = c^2(u_1, u_2, u_3) + (\partial Y)(u_1, u_2, u_3) - \sum(a_1 c)(u_2, u_3) + \sum(Z(a_1))(u_2, u_3) + \sum([a_1, a_2]u_3 - a_1(a_2 u_3) + a_2(a_1 u_3)) + \sum[[a_1, a_2], a_3]$, where each $\sum$ indicates the sum taken over all cyclic permutations of subscripts 1, 2 and 3. Then the assumptions (2), (3) imply that $\partial\bar{c}$ vanishes. Finally by Proposition 7 our assertion is verified. QED.

Let h_p be a subspace of g_p, and let $\{h_p, h_{p+1}, \cdots\}$ be a sequence such that $h_{n+1} = h_n^{(1)}$ for all $n \geqq p$. We denote by $H^{ij}(h_p; L)$ the homology group at $h_i \otimes \wedge^j(V^*_0)$ which is defined only for $i \geqq p$.

THEOREM 7 (EXISTENCE THEOREM). *Let $L = \mathrm{pr}\lim V_n$ be an infinite Lie algebra. Let W_{p-1} be a transitive subalgebra of V_{p-1}, h_p a subspace of the p-th isotropy algebra g_p of L and $\boldsymbol{c}$ an element of $W_{p-1} \otimes \wedge^2(W^*_{p-1})/\partial(h_p \otimes W^*_{p-1})$. ($p \geqq 1$). Assume that*

(1) *$H^{ij}(h_p; L) = 0$ for all $i \geqq p$ and $j = 1, 2, 3$ and $H^{ij}(L) = 0$ for all $i \geqq p-1$ and $j = 1, 2$;*

(2) *h_p is a subalgebra of the Lie algebra g_p, satisfying $h_p W_{p-1} \subset W_{p-1}$;*

(3) *$\boldsymbol{c}$ satisfies* (7.1), (7.5) *and* (7.6).

Then there exists a subalgebra $M = \mathrm{pr}\lim W_n$ of L such that

(i) *M is an infinite prolongation of W_{p-1} in L;*

(ii) *h_p is the p-th isotropy algebra of M;*

(iii) *$\boldsymbol{c}$ is the p-th order structure constant of M.*

Furthermore, if M' is any subalgebra of L which satisfies (i), (ii), (iii) *with M replaced by M', then there is an automorphism of L which embeds M' into M.*

PROOF. Let c be a representative of $\boldsymbol{c}$. Then by the condition (7.1) and by Proposition 7 with p replaced by $p-1$, we have $c(u, v) = [\phi u, \phi v]'$, $u, v \in W_{p-1}$, for some $\phi: W_{p-1} \to V_p$.

Set $W_p = \phi W_{p-1} + h_p$, then the assumption (2) implies that W_p is a subalgebra of V_p. It follows from Proposition 8, that W_p is prolongable in L. Let W_{p+1} be a normal prolongation of W_p in L. Since $H^{ij}(W_{p+1}; L) = H^{ij}(h_p; L)$ for $i \geqq p$, we can apply Theorem 6 to W_{p+1} and we obtain the normal infinite prolongation M of W_{p+1} in L, which evidently satisfies (i), (ii) and (iii). Next, let $M' = \mathrm{pr}\lim W'_n$ be any subalgebra of L satisfying (i), (ii), (iii). Then by

Proposition 6, there is an isomorphism f_p of W'_p onto W_p. Since $H^{ij}(M)=H^{ij}(h_p; L)=0$ for all $i \geqq p$ and $j=1, 2$, we can lift f_p to an embedding $M' \to M$, by Theorem 2. Finally, by Theorem 1 we can extend this embedding to the embedding $L \to L$, which is clearly an automorphism of L. QED.

We remark that the Existence and Uniqueness theorem in [**3**] or [**5**] is implied in this theorem by taking $L=D(V_0)$ and $p=1$, condition (7.1) reducing null in case $p=1$.

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