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The Plancherel formula for $SU(p, q)$

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§ 0. Introduction.

The Plancherel formula for connected real semisimple Lie groups has been discussed by many mathematicians, Harish-Chandra, R. Takahashi [8], L. Pukanszky [9(a) and (b)], K. Okamoto [5], B. D. Romm [6] and others. But the explicit form of this formula has not yet been published for general case. The purpose of this paper is to obtain the Plancherel formula for $SU(p, q)$. But the method employed here has certain generalities. Let us explain it.

First let us introduce some notations. Let G be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}_0, \mathfrak{h}_1, \cdots, \mathfrak{h}_q$ be a maximal set of Cartan subalgebras of $\mathfrak{g}$ which are not conjugate to each other under inner automorphisms of G, and let H_k be the Cartan subgroup of G corresponding to $\mathfrak{h}_k$, i.e., the centralizer of $\mathfrak{h}_k$ in G. Denote by $\mathfrak{g}^c$ and $\mathfrak{h}_k^c$ the complexifications of $\mathfrak{g}$ and $\mathfrak{h}_k$, respectively. Introduce a lexicographic order in the set of all roots of $(\mathfrak{g}^c, \mathfrak{h}_k^c)$, and denote by P_k, $S_{k,R}$ and $S_{k,I}$ the set of all positive, positive real and positive singular imaginary roots, respectively. Every element $\boldsymbol{h}_0$ of $\mathfrak{h}_k$ can be considered as a differential operator on H_k by

$$\boldsymbol{h}_0 f(h) = \frac{d}{dt} f(h \exp (t\boldsymbol{h}_0))|_{t=0} \qquad (h \in H_k),$$

and, therefore, so is every element of the symmetric algebra $S(\mathfrak{h}_k^c)$ of $\mathfrak{h}_k^c$. Let $L = L_k$ be the differential operator $\prod_{\alpha \in P_k} \boldsymbol{h}_\alpha$ on H_k, where $\boldsymbol{h}_\alpha$ is an element of $\mathfrak{h}_k^c$ such that $Sp(ad\, \boldsymbol{h}_\alpha\, ad\, \boldsymbol{h}) = \alpha(\boldsymbol{h})$ for all $\boldsymbol{h} \in \mathfrak{h}_k^c$.

Let G^c be a simply connected complex semisimple Lie group with Lie algebra $\mathfrak{g}^c$ and H_k^c be the Cartan subgroup of G^c corresponding to $\mathfrak{h}_k^c$. Put $\rho_k = 2^{-1} \sum_{\alpha \in P_k} \alpha$. Then there exists unique complex analytic homomorphism ξ_λ of H_k^c into $\boldsymbol{C}$ such that

$$\xi_\lambda(\exp \boldsymbol{h}) = e^{\lambda(\boldsymbol{h})} \qquad (\boldsymbol{h} \in \mathfrak{h}_k^c),$$

for $\lambda = \alpha$ $(\alpha \in P_k)$ and $\lambda = \rho_k$. Replacing G, if necessary, by a covering group of it which covers G only finitely many times, we can assume that the inclusion mapping of $\mathfrak{g}$ into $\mathfrak{g}^c$ can be extended to a homomorphism j of G into

G. Denote the homomorphism $\xi_\lambda \circ j$ also by ξ_λ and put for $h \in H_k$,[1]

$$\Delta^k(h) = \xi_{\rho_k}(h) \cdot \prod_{\alpha \in P_k} (1-\xi_\alpha(h)^{-1}), \qquad \Delta_I'^k(h) = \prod_{\alpha \in S_{k,I}} (1-\xi_\alpha(h)^{-1}),$$

$$\Delta_R'^k(h) = \prod_{\alpha \in S_{k,R}} (1-\xi_\alpha(h)^{-1}), \qquad \varepsilon_R^k(h) = \operatorname{sgn}(\Delta_R'^k(h)).$$

Let H_k', $H_k'(I)$ and $H_k'(R)$ be the subset of H_k defined by $\Delta^k(h) \neq 0$, $\Delta_I'^k(h) \neq 0$ and $\Delta_R'^k(h) \neq 0$, respectively.

Let dg and $d_k h$ be Haar measures on G and H_k, respectively, and $d_k \bar{g}$ an invariant measure on $\bar{G}^k = G/H_k$. It is known that there exist positive constants $\alpha_0, \alpha_1, \cdots, \alpha_q$ such that for any integrable function f on G

$$\int_G f(g)dg = \sum_{k=0}^{q} \alpha_k \int_{H_k} \left\{ \int_{\bar{G}^k} f(ghg^{-1}) d_k \bar{g} \right\} |\Delta^k(h)|^2 d_k h, \tag{1}$$

where $\bar{g} = gH_k \in \bar{G}^k$ in the k-th term. Define for $f \in C_0^\infty(G)$, a function K_f^k on H_k' by

$$K_f^k(h) = \varepsilon_R^k(h) \overline{\Delta^k(h)} \int_{\bar{G}^k} f(ghg^{-1}) d_k \bar{g} \qquad (h \in H_k'), \tag{2}$$

where $\overline{\Delta^k(h)}$ is the complex conjugate of $\Delta^k(h)$. Let π be an invariant eigendistribution on G. Then there exists an analytic function $\pi(g)$ on the set G' of all regular elements of G such that $(\pi, f) = \int_G f(g)\pi(g)dg$ ($f \in C_0^\infty(G)$). Therefore putting

$$\kappa^k(h) = \varepsilon_R^k(h) \Delta^k(h) \pi(h) \qquad (h \in H_k'), \tag{3}$$

we obtain a real analytic function $\kappa^k = \kappa_\pi^k$ on H_k' for every k. Then we see that

$$(\pi, f) = \int_G f(g)\pi(g)dg = \sum_{k=0}^{q} \alpha_k \int_{H_k} K_f^k(h) \kappa^k(h) d_k h \qquad (f \in C_0^\infty(G)). \tag{4}$$

Note that, for the character π of a representation $g \to T(g)$ ($g \in G$), the function $\pi(g)$ is independent of the choice of the Haar measure dg, because

$$(\pi, f) = \int_G f(g)\pi(g)\,dg = Sp\left(\int_G T(g) f(g) dg \right).$$

Now we come to the sketch of our method. The principal steps in it are the following (I), (II) and (III).

(I) *To obtain explicitly the characters π of irreducible unitary representations of G which may appear in the regular representation of G. Or rather, to obtain the functions $\xi^k = L\kappa^k$ on H_k' ($0 \leqq k \leqq q$) corresponding to π.*

(II) *To prove the following equality: for any $f \in C_0^\infty(G)$ and any invariant eigendistribution π on G,*

1) In § 1 of the present paper the function Δ^k is a little revised.

$$\sum_{k=0}^{q} \alpha_k \int_{H_k} \{LK_f^k(h) \cdot L\kappa^k(h) + (-1)^s K_f^k(h) L^2 \kappa^k(h)\} d_k h = 0\,, \tag{5}$$

where $s = 2^{-1}(\dim G - \text{rank}\, G)$.

We see that there exists a constant a_π such that $L^2\kappa^k = a_\pi \kappa^k$ for every k and therefore the equality (5) can be rewritten as

$$a_\pi (\pi, f) = -(-1)^s \sum_{k=0}^{q} \alpha_k \int_{H_k} LK_f^k(h) \cdot \xi^k(h) d_k h\,, \tag{6}$$

where $\xi^k = L\kappa^k$. In fact, $L^2 = L_k^2$ of $S(\mathfrak{h}_k^c)$ is invariant under the Weyl group of $(\mathfrak{g}^c, \mathfrak{h}_k^c)$ and moreover there exists a Laplace operator D such that $K_{Df}^k = L^2 K_f^k$ for every k. The constant a_π is such that $D\pi = a_\pi \pi$.

Let us remark here the following facts. The function K_f^k can be extended to a continuous function on $H_k'(I)$ which is indefinitely differentiable on the closure of every connected component of $H_k'(I)$. The function κ^k can be extended to an analytic function on $H_k'(R)$ which has a certain form on every connected component of $H_k'(R)$. Therefore integrating by parts, the left hand side of (5) can be rewritten as a sum of integrals on the boundaries H_k^I of $H_k'(I)$ and H_k^R of $H_k'(R)$ in H_k. Here the integrands on H_k^I are certain products of the limits of derived functions of K_f^k on H_k^I and the restrictions of derived functions of κ^k on H_k^I. Similarly the integrands on H_k^R are certain products of the restrictions of derived functions of K_f^k on H_k^R and the limits of those of κ^k on H_k^R. On the other hand, almost all elements of H_k^I and H_k^R are semiregular elements of non-compact type, that is, semisimple elements whose centralizers in $\mathfrak{g}$ are reductive Lie algebras of dimension rank $\mathfrak{g}+2$. Take an arbitrary semiregular element h_0 of H_k^R (H_k^I resp.), then there exists only one H_l such that some conjugate element of h_0 is contained in H_l^I (H_l^R resp.). Therefore we see that, to prove the equality (5), it will be essential to obtain some relations between the restrictions or the limits of derived functions of K_f^k (κ^k resp.) at h_0 and those of K_f^l (κ^l resp.) at the conjugate element of h_0, where h_0 is an arbitrary semiregular element of H_k^R or H_k^I and the integers k and l are as above.

Let us return to the sketch of the method. By the formula (6), we may consider that $a_\pi(\pi, f)$, as a function of the variable π, gives some integral transformation of the functions $LK_f^0, LK_f^1, \cdots, LK_f^q$. In the following, we assume that the Cartan subgroup H_0 is fundamental, i. e., the dimension of the vector part of $\mathfrak{h}_0$ is minimum. We know from the results of Harish-Chandra that for every k, the function LK_f^k can be extended to a continuous function on the whole H_k and that there exists a non-zero constant c such that

$$LK_f^0(e) = c\, f(e) \tag{7}$$

for every $f \in C_0^\infty(G)$ [**3**(c) and (e)], where e is the identity element of G. Then

(III) *To express the value* $LK_f^0(e)$ *by means of the integral transformation* $a_\pi(\pi, f)$ *of* $LK_f^0, LK_f^1, \cdots, LK_f^q$.

After solving this problem, we shall see, roughly speaking, that there exist a set Π of characters π and a complex-valued measure μ on it such that

$$LK_f^0(e) = \int_\Pi a_\pi(\pi, f) d\mu(\pi). \tag{8}$$

It follows from (7) and (8) that

$$f(e) = \int_\Pi (\pi, f) d\mu'(\pi), \tag{9}$$

where $d\mu'(\pi) = c^{-1} a_\pi d\mu(\pi)$. The measure μ' will be a positive measure on Π. Then, as a character π determines uniquely an equivalent class of irreducible unitary representations, the formula (9) is essentially the Plancherel formula for G and μ' is the Plancherel measure which is unique up to a positive constant depending only on the normalization of the Haar measure dg on G.

Let us now explain how we can pass these three steps for $G = SU(p, q)$. There is no essential difference to treat $G = U(p, q)$ instead of $SU(p, q)$. Therefore in §§1–4 of the present paper we treat $U(p, q)$ for the sake of simplicity and obtain in §3 the Plancherel formula for $U(p, q)$. The one for $SU(p, q)$ is deduced from it in §5. Put $G = G_{p,q} = U(p, q)$ $(p \leqq q)$ and let $H_0, H_1, \cdots, H_q$ be certain Cartan subgroups of G such that the dimension of the vector part of $\mathfrak{h}_k$ is equal to k $(0 \leqq k \leqq q)$.

Step I. This is the content of §1. The characters π of all irreducible unitary representations of discrete series (i.e., square-integrable ones) are essentially determined in [**3**(f)] as the tempered invariant eigendistributions on G which have a certain given form on the compact Cartan subgroup H_0. Thus the function $\kappa^0 = \kappa_\pi^0$ on H_0 corresponding to π is known. The function $\kappa^k = \kappa_\pi^k$ on another Cartan subgroups H_k (or more exactly on $H_k'(R)$) is calculated explicitly in [**4**(d), §10]. We say that this series of representations is of type 0.

We can construct as follows another series of representations which is called generally the continuous principal series. Let $G = NAK$ be an Iwasawa decomposition of G, where N is nilpotent and K is a maximal compact subgroup of G. Let K_A be the centralizer of A in K. Take a unitary character of A and an irreducible unitary representation of K_A, and consider the representation M of $R = AK_A$ which is naturally obtained from them. Inducing M from R to G, we obtain a unitary representation T^M of G. T^M is "in general" irreducible [**1**]. The character π of T^M is obtained in [**3**(g)] (cf. also [**4**(e)]). The functions κ^k on H_k' corresponding to π are such that $\kappa^0 = 0$,

$\kappa^1=0, \cdots, \kappa^{q-1}=0$ and $\kappa^q \neq 0$. Therefore we say that this series is of type q.

Let r be an integer such that $1 \leqq r \leqq q-1$. Let us construct a series of representations called of type r. Take a certain subgroup A_r of A of dimension r. Then the centralizer R of A_r in G is isomorphic to the product $G_{p-r,q-r} \times G_{r,r}$. Consider a representation M of R which is obtained naturally from a representation of $G_{p-r,q-r}$ of type 0 and a one of $G_{r,r}$ of type r. By a process described in detail in [4(e)], we can construct a so-called induced representation T^M of G. The character π of T^M can be calculated by Theorem 2 in [4(e)] and it is an invariant eigendistribution on G for which $\kappa^0=0$, $\kappa^1=0, \cdots, \kappa^{r-1}=0$, $\kappa^r \neq 0, \cdots, \kappa^q \neq 0$. It is proved that T^M is a direct sum of a finite number of irreducible unitary representations of G (Lemma 1.5). Thus we obtain the characters of irreducible unitary representations or of a certain direct sum of them (Theorem 1). We see in § 3 that these representations appear actually in the Plancherel formula for G.

Step II. This is the content of § 2. As is remarked above, we must study the relations between the restrictions or the limits of derived functions of K_f^k's (κ^k's resp.) on semiregular elements of non-compact type of the boundaries H_k^I and H_k^R of $H_k'(I)$ and $H_k'(R)$ in H_k $(0 \leqq k \leqq q)$. This has been done in [4(e)] and the results necessary for the proof of the equality (5) are summalized in Lemma 2.1 for K_f^k's and in Lemma 2.2 for κ^k's. Note that any semiregular elements of non-compact type of H_k^I and of H_{k+1}^R are conjugate to elements of $H_{k,k+1}=H_k \cap H_{k+1}$ and therefore the left hand side of (5) can be rewritten as a sum of integrals on $H_{k,k+1}$, where $k=0, 1, 2, \cdots, q-1$ (see § 2). Using Lemmas 2.1 and 2.2, we can prove the equality (5) (Theorem 2).

Step III. This is the content of § 3. The integral transformation $a_\pi(\pi, f)$ is essentially a mixed Fourier-Laplace transformation of $LK_f^0, LK_f^1, \cdots, LK_f^q$ with respect to certain co-ordinates on $H_0, H_1, \cdots, H_q$. As is remarked before, the function LK_f^k on $H_k'(I)$ can be extended to a continuous function on the whole H_k which is not in general differentiable on H_k, but indefinitely differentiable on the closure of every connected component of $H_k'(I)$ $(0 \leqq k \leqq q)$. Therefore the principal part of this step is, in a word, to obtain the inverse transformation of Fourier-Laplace transformation of continuous and piecewise differentiable functions on a product of tori and a Euclidian space. Lemmas 3.3 and 3.5, which are rather classical, solve this problem in the case of lower dimensions. We can generalize without difficulty these lemmas to the case of higher dimensions. Using them and discussing the convergence of certain integro-summations, we can pass the step III. Thus we obtain the Plancherel formula for $G=G_{p,q}=U(p, q)$ (Theorem 3).

The contents of §§ 1-3 are already explained above. Let us mention briefly the contents of other sections. In § 4 a positive constant γ in the

Plancherel formula in Theorem 3 is calculated for a certain normalization of the Haar measure dg on G. In § 5 we define a mapping of $C_0^\infty(SU(p, q))$ into $C_0^\infty(U(p, q))$ which naturally imbeds the former in the latter. Using this imbedding, the Plancherel formula for $SU(p, q)$ is deduced from that for $U(p, q)$ (Theorem 4). In Appendix some spherical functions on $U(p, q)$ are calculated. The formal degrees of the square-integrable representations corresponding to these spherical functions are calculated in § 4 and they determine the constant γ mentioned above.

The method employed here may be applied without any essential change for $Sp(p, q)$, $SO(2p, 1)$ and $SO^*(2n)$[2]. In fact, for instance, the characters of all square-integrable irreducible unitary representations of these groups are obtained explicitly as in [4(e)] from the results of [3(f)]. The Plancherel formula for $SO(2p, 1)$ has been obtained in [4(a) and (b)].

§ 1. The characters of some irreducible unitary representations.

Let p and q be integers such that $p \geqq q \geqq 1$ and put $n = p+q$. Let $U(p, q)$ be the group of all matrices of order n which satisfy

$$g^* \begin{bmatrix} 1_p & 0 \\ 0 & -1_q \end{bmatrix} g = \begin{bmatrix} 1_p & 0 \\ 0 & -1_q \end{bmatrix}, \tag{1.1}$$

where $g^* = {}^t\bar{g}$ and 1_p is the identity matrix of order p. Denote by $SU(p, q)$ the subgroup of $U(p, q)$ defined by $\det g = 1$.

We wish to obtain the Plancherel formula for $SU(p, q)$. But it is convenient to treat $U(p, q)$ first. From the Plancherel formula for $U(p, q)$, the one for $SU(p, q)$ can be obtained immediately (see § 5). Therefore we treat $U(p, q)$ in §§ 1-4. Let us begin with stating some results in [4(d), §§ 1 and 10] without proofs. Let $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the field of real and complex numbers, respectively.

1. Put $G = U(p, q)$. The following $q+1$ Cartan subgroups $H_0, H_1, \cdots, H_q$ of G form a maximal set of Cartan subgroups which are not conjugate to each other under inner automorphisms of G. For $0 \leqq k \leqq q$, let $H_k = H_k^- H_k^+$, and H_k^- and H_k^+ be the subgroups of G consisting of all matrices of the following form, respectively:

2) Here we use the notation of " S. Helgason, Differential geometry and symmetric spaces, Academic Press, 1962, p. 340."

$$
(1.2)\qquad H_k^+ : \left[\begin{array}{ccccccccc}
1_{p-k} & & & & & & & & 0 \\
& \mathrm{ch}\, t_k & & & & & & \mathrm{sh}\, t_k & \\
& & \mathrm{ch}\, t_{k-1} & & & & \mathrm{sh}\, t_{k-1} & & \\
& & & \ddots & & \cdot^{\cdot^{\cdot}} & & & \\
& & & & \mathrm{ch}\, t_1 \quad \mathrm{sh}\, t_1 & & & & \\
& & & & \mathrm{sh}\, t_1 \quad \mathrm{ch}\, t_1 & & & & \\
& & & \cdot^{\cdot^{\cdot}} & & \ddots & & & \\
& & \mathrm{sh}\, t_{k-1} & & & & \mathrm{ch}\, t_{k-1} & & \\
& \mathrm{sh}\, t_k & & & & & & \mathrm{ch}\, t_k & \\
0 & & & & & & & & 1_{q-k}
\end{array}\right]
$$

$$
(1.2')\qquad H_k^- : d(e^{i\varphi_1}, e^{i\varphi_2}, \cdots, e^{i\varphi_{p-k}}, e^{i\theta_k}, e^{i\theta_{k-1}}, \cdots, e^{i\theta_1}, e^{i\theta_1}, \cdots, e^{i\theta_{k-1}}, e^{i\theta_k}, e^{i\phi_{q-k}}, \cdots, e^{i\phi_2}, e^{i\phi_1}),
$$

where all $t_j, \varphi_j, \theta_j, \phi_j \in \boldsymbol{R}$, $\mathrm{ch}\, t = 2^{-1}(e^t + e^{-t})$, $\mathrm{sh}\, t = 2^{-1}(e^t - e^{-t})$, $i = \sqrt{-1}$, the blank of the above matrix must be filled up by 0, and $d(a_1, a_2, \cdots, a_n)$ denote the diagonal matrix with diagonal elements $a_1, a_2, \cdots, a_n$.

Let $h \in H_k$ be the product of the above two matrices. We take as the co-ordinates of h

$$
(1.3)\qquad (\Phi_1, \Phi_2, \cdots, \Phi_{p-k}, z_1, z_2, \cdots, z_k, \Psi_1, \Psi_2, \cdots, \Psi_{q-k}, z_{-1}, z_{-2}, \cdots, z_{-k}),
$$

where

$$
\Phi_j = i\varphi_j \ (1 \leqq j \leqq p-k), \qquad \Psi_l = i\phi_l \ (1 \leqq l \leqq q-k),
$$

$$
z_j = t_j + \Theta_j, \qquad z_{-j} = -\bar{z}_j = -t_j + \Theta_j \ \text{and} \ \Theta_j = i\theta_j \ (1 \leqq j \leqq k).
$$

Let $x = (x_1, x_2, \cdots, x_n)$ be the co-ordinates of $h \in H_k$ and put

$$
(1.4)\qquad \Delta^k(h) = \prod_{1 \leqq j < l \leqq n} (e^{x_j} - e^{x_l}), \quad \Delta_R^k(h) = \prod_{j=1}^{k} (1 - e^{-2t_j}), \quad \varepsilon_R^k(h) = \mathrm{sgn}\,(\Delta_R^k(h)).
$$

Let $X_j = \partial/\partial x_j$ and $X = (X_1, X_2, \cdots, X_n)$, where

$$
(1.5)\qquad \begin{aligned}
&\frac{\partial}{\partial \Phi_j} = \frac{1}{i}\frac{\partial}{\partial \varphi_j}, \qquad \frac{\partial}{\partial \Psi_l} = \frac{1}{i}\frac{\partial}{\partial \phi_l}, \\
&\frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial t_j} + \frac{1}{i}\frac{\partial}{\partial \theta_j}\Big), \qquad \frac{\partial}{\partial z_{-j}} = \frac{1}{2}\Big(-\frac{\partial}{\partial t_j} + \frac{1}{i}\frac{\partial}{\partial \theta_j}\Big).
\end{aligned}
$$

Let us consider the polynomial

$$
(1.6)\qquad L(y_1, y_2, \cdots, y_n) = \prod_{1 \leqq j < l \leqq n} (y_j - y_l)
$$

of n variables and put

$$
(1.6')\qquad L(X) = \prod_{1 \leqq j < l \leqq n} (X_j - X_l).
$$

Then this is a differential operator on H_k corresponding to the product of all positive roots of (G, H_k) with respect to a certain lexicographic order, and this will be denoted simply by L if there is no danger of misunderstanding.

Let H'_k and $H'_k(R)$ be the subsets of H_k defined by $\Delta^k(h)\neq 0$ and $\Delta^k_R(h)\neq 0$, respectively. Let W_k be the Weyl group of (G, H_k), i.e., the group of all inner automorphisms of G which make H_k stable. W_k contains (i) all permutations of the components $\Phi_1, \Phi_2, \cdots, \Phi_{p-k}$ of the co-ordinates of $h\in H_k$, (ii) all permutations of the components $\Psi_1, \Psi_2, \cdots, \Psi_{q-k}$, (iii) all permutations of the pairs $(z_1, z_{-1}), (z_2, z_{-2}), \cdots, (z_k, z_{-k})$, and (iv) for any j $(1\leqq j\leqq k)$, the permutation of z_j and z_{-j}, that is, the transformation induced by $t_j\to -t_j$. Conversely W_k is generated by the transformations in (i)-(iv). Define $\varepsilon(\omega)$ and $\varepsilon'(\omega)$ for $\omega\in W_k$ by

$$\varepsilon^k_R(\omega h)\Delta^k(\omega h)=\varepsilon(\omega)\varepsilon^k_R(h)\Delta^k(h)\ ,\quad \varepsilon^k_R(\omega h)=\varepsilon'(\omega)\varepsilon^k_R(h)\qquad (h\in H'_k)\ . \tag{1.7}$$

If a function f on H_k satisfies that for any $\omega\in W_k$,

$$f(\omega h)=\varepsilon(\omega)f(h)\qquad (f(\omega h)=\varepsilon'(\omega)f(h)\ \text{resp.})\ ,$$

then f is skew-symmetric (symmetric resp.) with respect to $(p-k)$ variables Φ_j $(1\leqq j\leqq p-k)$ and also to $(q-k)$ variables Ψ_l $(1\leqq l\leqq q-k)$, symmetric (symmetric resp.) with respect to k pairs (z_j, z_{-j}) $(1\leqq j\leqq k)$, and even (odd resp.) with respect to every t_j $(1\leqq j\leqq k)$, and vice versa.

2. An element $g\in G$ is called regular if the rank of $Ad(g)$ is the maximum of those of $Ad(g')$ $(g'\in G)$. Let G' be the subset of G of all regular elements. Then $G'\cap H_k=H'_k$. The character π of any irreducible unitary representation of G is an invariant eigendistribution (of all Laplace operators) on G and therefore is essentially a locally summable function on G which is equal to an analytic function on G' [**3**(e)]. Put

$$\kappa^k(h)=\varepsilon^k_R(h)\Delta^k(h)\pi(h)\qquad (h\in H'_k)\ . \tag{1.8}$$

Then $\kappa^k(h)$ can be extended to an analytic function on $H'_k(R)$ [**3**(e), or **4**(d), § 5]. And π is completely determined by the functions κ^k on $H'_k(R)$ for $0\leqq k\leqq q$. Let us describe the characters of all square-integrable irreducible unitary representations of G [**4**(d), § 10].

Let $c=(c_1, c_2, \cdots, c_n)$ be a row of integers such that $c_1\geqq c_2\geqq\cdots\geqq c_n$. Let $I=\{i_1, i_2, \cdots, i_p\}$ be a subset of p elements of $I_n=\{1, 2, \cdots, n\}$, and $J=\{j_1, j_2, \cdots, j_q\}$ the complement of I in I_n. Let us assume that

$$i_1<i_2<\cdots<i_p\ ,\qquad j_1<j_2<\cdots<j_q\ .$$

Let $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_k)$ and $\beta=(\beta_1, \beta_2, \cdots, \beta_k)$ be arbitrary rows of k elements of I and J, respectively. And the sets of all components of α and β are denoted respectively by

$$\bar{\alpha}=\{\alpha_1, \alpha_2, \cdots, \alpha_k\} \quad \text{and} \quad \bar{\beta}=\{\beta_1, \beta_2, \cdots, \beta_k\}.$$

Define

$$\varepsilon_I=(-1)^{pq}\,\mathrm{sgn}\begin{pmatrix}1, 2, \cdots, p, p+1, p+2, \cdots, n \\ i_1, i_2, \cdots, i_p, j_1, j_2, \cdots, j_q\end{pmatrix},$$

$$\varepsilon(\alpha, \beta, I, J)=\mathrm{sgn}\begin{pmatrix}i_1, i_2, \cdots, i_{p-k}, i_{p-k+1}, i_{p-k+2}, \cdots, i_p \\ i'_1, i'_2, \cdots, i'_{p-k}, \alpha_1, \alpha_2, \cdots, \alpha_k\end{pmatrix}$$

$$\times\,\mathrm{sgn}\begin{pmatrix}j_1, j_2, \cdots, j_{q-k}, j_{q-k+1}, j_{q-k+2}, \cdots, j_q \\ j'_1, j'_2, \cdots, j'_{q-k}, \beta_1, \beta_2, \cdots, \beta_k\end{pmatrix},$$

where $i'_1, i'_2, \cdots, i'_{p-k}$ and $j'_1, j'_2, \cdots, j'_{q-k}$ are the elements of $I-\bar{\alpha}$ and $J-\bar{\beta}$ respectively[3] lined up as

$$i'_1<i'_2<\cdots<i'_{p-k}, \qquad j'_1<j'_2<\cdots<j'_{q-k}.$$

Let us denote the exponent $\exp(a)$ by $e(a)$ for the sake of brevity. For any two different integers a, b in I_n, define a function of $z=t+i\theta\in \boldsymbol{C}$ as follows:

$$\xi(z; c; a, b)=\mathrm{sgn}(a-b)e\{-|c_a-c_b||t|+(c_a+c_b)i\theta\}. \tag{1.9}$$

Then $\xi(z; c; a, b)=-\xi(z; c; b, a)$ and

$$\xi(z; c; a, b)=\begin{cases} -e(c_a z-c_b\bar{z}) & \text{if } a<b \text{ and } t<0, \\ e(-c_a\bar{z}+c_b z) & \text{if } a>b \text{ and } t<0. \end{cases} \tag{1.9'}$$

Let $M=\{m_1, m_2, \cdots, m_r\}$ be a subset of I_n and let us assume that $m_1<m_2<\cdots<m_r$. Define for $y_1, y_2, \cdots, y_r\in \boldsymbol{C}$,

$$D(y_1, y_2, \cdots, y_r; c; M)=|y^{a_1}, y^{a_2}, \cdots, y^{a_r}|_{y=y_1,\cdots,y_r}, \tag{1.10}$$

where $a_j=c_{m_j}$ $(1\leqq j\leqq r)$ and the right hand side denotes the $r\times r$ determinant whose j-th row is $y_j^{a_1}, y_j^{a_2}, \cdots, y_j^{a_r}$ $(1\leqq j\leqq r)$.

With these notations, define for $h\in H'_k(R)$ $(0\leqq k\leqq q)$,

$$\kappa^k_{I,c}(h)=\varepsilon_I\sum_{\alpha,\beta}\varepsilon(\alpha, \beta, I, J)D(e^{\Phi_1}, e^{\Phi_2}, \cdots, e^{\Phi_{p-k}}; c; I-\bar{\alpha}) \tag{1.11}$$

$$\times D(e^{\Psi_1}, e^{\Psi_2}, \cdots, e^{\Psi_{q-k}}; c; J-\bar{\beta})\times\prod_{j=1}^{k}\xi(z_j; c; \alpha_j, \beta_j),$$

where α and β run over all rows of k elements of I and J respectively and Φ_j, Ψ_l, z_j are the components of the co-ordinates of h. And put

$$\pi_{I,c}(h)=(\varepsilon^k_R(h)\Delta^k(h))^{-1}\kappa^k_{I,c}(h) \qquad (h\in H'_k). \tag{1.12}$$

Then $\pi_{I,c}$ defines an invariant eigendistribution on G. And if $c_1>c_2>\cdots>c_n$, this distribution is the character of a square-integrable irreducible unitary representation of G. Conversely the character of any such representation is

3) $I-\bar{\alpha}$ denotes the set of elements of I not in $\bar{\alpha}$.

equal to some $\pi_{I,c}$ for which $c_1 > c_2 > \cdots > c_n$ [4(d), § 10][4]. Put

$$\kappa_c^k = \sum_I \kappa_{I,c}^k \quad (0 \leqq k \leqq q), \qquad \pi_c = \sum_I \pi_{I,c},$$

where I runs over all subset of p elements of I_n.

Let $\mathcal{S}_n$ be the symmetric group of order n. Define for $\sigma = \begin{pmatrix} 1, & 2, & \cdots, & n \\ \sigma(1), & \sigma(2), & \cdots, & \sigma(n) \end{pmatrix}$,

$$\sigma c = (c_{\sigma(1)}, c_{\sigma(2)}, \cdots, c_{\sigma(n)}).$$

And put

$$\kappa_{\sigma c}^k = \mathrm{sgn}(\sigma) \kappa_c^k, \qquad \pi_{\sigma c} = \mathrm{sgn}(\sigma) \pi_c.$$

As is easily seen, $\kappa_c^k = 0$ and $\pi_c = 0$ if $L(c) = \prod_{1 \leqq j < l \leqq n} (c_j - c_l) = 0$.

Now take any row of integers $c = (c_1, c_2, \cdots, c_n)$ and define for $h \in H_k'(R)$,

$$\eta_{I,c}^k(h) = \sum_{\sigma \in \mathcal{S}_p} \sum_{\tau \in \mathcal{S}_q} \Big[\prod_{j=1}^{p-k} e(a_{\sigma(j)} \Phi_j) \prod_{l=1}^{q-k} e(b_{\tau(l)} \Psi_l) \times \prod_{j=1}^{k} \mathrm{sgn}(t_j) e\{-|a_{\sigma(p-k+j)} - b_{\tau(q-k+j)}| |t_j| + (a_{\sigma(p-k+j)} + b_{\tau(q-k+j)}) \Theta_j\} \Big], \tag{1.13}$$

where $a_l = c_{i_l}$ $(1 \leqq l \leqq p)$ and $b_l = c_{j_l}$ $(1 \leqq l \leqq q)$. Put $\eta_c^k = \sum_I \eta_{I,c}^k$. Then

$$\eta_c^k(h) = \sum_{\tau \in \mathcal{S}_n} \delta^k(h; \tau c) \qquad (h \in H_k'(R)), \tag{1.14}$$

where

$$\delta^k(h; c) = \prod_{j=1}^{p-k} e(c_j \Phi_j) \cdot \prod_{l=1}^{q-k} e(c_{p+l} \Psi_l) \times \prod_{j=1}^{k} \mathrm{sgn}(t_j) e\{-|c_{p-k+j} - c_{n-k+j}| |t_j| + (c_{p-k+j} + c_{n-k+j}) \Theta_j\}. \tag{1.15}$$

LEMMA 1.1. *For any c and I such that $c_1 \geqq c_2 \geqq \cdots \geqq c_n$,*

$$L\eta_{I,c}^k = (-1)^{pq} L(c) \kappa_{I,c}^k \qquad (0 \leqq k \leqq q), \tag{1.16}$$

and for any c,

$$L\eta_c^k = (-1)^{pq} L(c) \kappa_c^k \qquad (0 \leqq k \leqq q). \tag{1.16'}$$

PROOF. First remark that

$$L\delta^k(h; c) = L(c) \prod_{j=1}^{p-k} e(c_j \Phi_j) \prod_{l=1}^{q-k} e(c_{p+l} \Psi_l) \times \prod_{j=1}^{k} \mathrm{sgn}(c_{n-k+j} - c_{p-k+j}) e\{-|c_{p-k+j} - c_{n-k+j}| |t_j| + (c_{p-k+j} + c_{n-k+j}) \Theta_j\},$$

4) The function $\Delta^k(h)$ in this paper is different from the one in [4(d)] by the factor $(-1)^{q(q-1)/2}$. And the definition of ε_I is a little changed accordingly.

where we put $\operatorname{sgn}(0)=1$. Therefore if $L(c)=0$, the assertion of the lemma is evident. Now let us assume that $L(c)\neq 0$. Define a function ζ on $H'_k(R)$ as follows:

$$\zeta(h)=|e^{a_1\Phi}, e^{a_2\Phi}, \cdots, e^{a_{p-k}\Phi}|_{\Phi=\Phi_1,\cdots,\Phi_{p-k}}\times|e^{b_1\Psi}, \cdots, e^{b_{q-k}\Psi}|_{\Psi=\Psi_1,\cdots,\Psi_{q-k}}$$
$$\times\prod_{j=1}^{k}\operatorname{sgn}(d_{-j}-d_j)e\{-|d_j-d_{-j}||t_j|+(d_j+d_{-j})\Theta_j\},$$

where

$$(a_1, a_2, \cdots, a_{p-k}, d_1, d_2, \cdots, d_k, b_1, b_2, \cdots, b_{q-k}, d_{-1}, \cdots, d_{-k})=(c_1, c_2, \cdots, c_n).$$

Note that

$$\zeta(\omega h)=\varepsilon(\omega)\zeta(h)\quad(\omega\in W_k), \qquad L(\tau c)=\operatorname{sgn}(\tau)L(c)\quad(\tau\in\mathcal{S}_n).$$

Then we know that

$$(L\zeta)(\omega h)=\varepsilon'(\omega)(L\zeta)(h)\qquad(\omega\in W_k),$$

and we obtain easily that $L\zeta=L(c)\eta^k_{I,c}$, where $I=\{1, 2, \cdots, p\}$. On the other hand, as is easily seen, $L^2\zeta=(L(c))^2\zeta$. Therefore $L\eta^k_{I,c}=L(c)\zeta$. The equalities of the lemma can be easily obtained from this one, taking into account that

$$L(\sigma c)\kappa^k_{\sigma c}=L(c)\kappa^k_c\qquad(0\leqq k\leqq q,\ \sigma\in\mathcal{S}_n).$$

Q. E. D.

LEMMA 1.2. *Let N_j, M_j $(1\leqq j\leqq n)$ be $2n$ integers. Then*

$$\sum_{\substack{M_j\leqq c_j\leqq N_j\\(1\leqq j\leqq n)}}\eta^k_c(h)=\sum_{\sigma\in\mathcal{S}_n}\Big\{\sum_{\substack{M_{\sigma(j)}\leqq c_j\leqq N_{\sigma(j)}\\(1\leqq j\leqq n)}}\delta^k(h;c)\Big\},\tag{1.17}$$

where the first sum runs over all c such that $M_j\leqq c_j\leqq N_j$ for $1\leqq j\leqq n$.

PROOF. This is easy to prove.

3. A unitary representation of G is called of type r if its character π is identically zero on $H_0, H_1, \cdots, H_{r-1}$, and not on H_r (π is of height r in the terminology of [4(d), § 6]). The square-integrable irreducible unitary representations are all of type 0. In this subsection, let us give the characters of a series of irreducible unitary representations of type q. This series of representations is called in general the continuous principal series.

Let $c=(c_1, c_2, \cdots, c_{p-q})$ be a row of integers such that $c_1\geqq c_2\geqq\cdots\geqq c_{p-q}$. Let $m_1, m_2, \cdots, m_q$ be integers and let $i\rho_1, i\rho_2, \cdots, i\rho_q$ $(i=\sqrt{-1}, \rho_j\in\boldsymbol{R})$ be pure imaginary numbers. Put

$$d_j=2^{-1}(m_j+i\rho_j), \quad d_{-j}=2^{-1}(m_j-i\rho_j)\qquad(1\leqq j\leqq q),$$
$$d=(d_1, d_2, \cdots, d_q, d_{-1}, d_{-2}, \cdots, d_{-q})\ \text{ and }\ \chi=(c, d).$$

An invariant eigendistribution π_χ on G is given as follows. Put as before

$$\kappa^k_\chi(h)=\varepsilon^k_R(h)\Delta^k(h)\pi_\chi(h)\qquad(h\in H'_k).$$

Then

(1.18)
$$\kappa_\chi^k(h)=0 \quad \text{for} \quad 0\leqq k\leqq q-1\,,$$
$$\kappa_\chi^q(h)=(-1)^{pq+2^{-1}q(q+1)}|e^{c_1\Phi}, e^{c_2\Phi}, \cdots, e^{c_{p-q}\Phi}|_{\Phi=\Phi_1,\cdots,\Phi_{p-q}}$$
$$\times\Big[\sum_{\sigma\in\mathcal{S}_q}\prod_{j=1}^{q} e(m_{\sigma(j)}\Theta_j)\{e(i\rho_{\sigma(j)}t_j)+e(-i\rho_{\sigma(j)}t_j)\}\Big]$$
$$=(-1)^{pq+2^{-1}q(q+1)}|e^{c_1\Phi}, e^{c_2\Phi}, \cdots, e^{c_{p-q}\Phi}|$$
$$\times\Big[\sum_{\sigma\in\mathcal{S}_q}\prod_{j=1}^{q}\{e(d_{\sigma(j)}z_j+d_{-\sigma(j)}z_{-j})+e(d_{-\sigma(j)}z_j+d_{\sigma(j)}z_{-j})\}\Big]\,.$$

Let us show that π_χ is in general the character of an irreducible unitary representation of G of type q, if $c_1>c_2>\cdots>c_{p-q}$. Let Q be the subgroup of G consisting of all elements of the form

$$\begin{bmatrix}\alpha & 0\\ 0 & 1_{2q}\end{bmatrix},$$

where $\alpha\in U(p-q)$, the unitary group of order $p-q$. Put $R=QH_q$. Then R is the centralizer of H_q^+ in G. This is a reductive Lie group and H_q is a Cartan subgroup of it. Let M be a finite-dimensional irreducible unitary representation of R whose character is given for $h\in H_q$ as

$$\Big(\prod_{j=1}^{p-q} e^{\Phi_j}\Big)^{-q}\cdot\frac{|e^{c_1\Phi}, \cdots, e^{c_{p-q}\Phi}|}{\prod_{1\leqq j<l\leqq p-q}(e^{\Phi_j}-e^{\Phi_l})}\times\Big(\prod_{j=1}^{q} e^{\Theta_j}\Big)^{-n+1}\prod_{j=1}^{q} e^{m_j\Theta_j}e^{i\rho_j t_j}\,,$$

where Φ_j, $z_l=t_l+\Theta_l$, $z_{-l}=-t_l+\Theta_l$ are the components of the co-ordinates of h. Consider the induced representation T^M of G defined in [4(e), § 1]. This is unitary. Moreover this is irreducible if $\rho_1, \rho_2, \cdots, \rho_q$ are all non-zero and different from each other [1, Théorème 7; 2]. Even if not so, T^M is a direct sum of a finite number of irreducible unitary representations [1, p. 193].

Taking into account that

$$\varepsilon_R^q(h)\Delta^q(h)=(-1)^{pq+2^{-1}q(q+1)}\Big(\prod_{j=1}^{p-q}e^{\Phi_j}\Big)^q\Big(\prod_{j=1}^{q}e^{\Psi_j}\Big)^{n-1}$$
$$\times\prod_{1\leqq j<l\leqq p-q}(e^{\Phi_j}-e^{\Phi_l})\times\Big|\prod_{\substack{1\leqq j<l\leqq n\\ p-q\leqq j \text{ or } p-q<l}}(e^{\frac{1}{2}(x_j-x_l)}-e^{-\frac{1}{2}(x_j-x_l)})\Big|\,,$$

where $(x_1, x_2, \cdots, x_n)$ is the co-ordinates of h, we see from [4(e), Theorem 2] that the character of T^M is exactly π_χ [5].

We denote sometimes κ_χ^k and π_χ by $\kappa_{c,d}^k$ and $\pi_{c,d}$, respectively. Define for $\sigma\in\mathcal{S}_{p-q}$,

$$\kappa_{\sigma c,d}^k=\operatorname{sgn}(\sigma)\kappa_{c,d}^k\,,\qquad \pi_{\sigma c,d}=\operatorname{sgn}(\sigma)\pi_{c,d}\,.$$

5) As is well known, the character of this representation T^M has been calculated in [3(g)]. But a constant factor is left uncalculated there.

If $\prod_{1\leqq j<l\leqq p-q}(c_j-c_l)=0$, then $\kappa_{c,d}^k=0$ and $\pi_{c,d}=0$.

Let $c=(c_1, c_2, \cdots, c_{p-q})$ be an arbitrary row of integers and $d=(d_1, d_2, \cdots, d_q, d_{-1}, \cdots, d_{-q})$ be as before. Define for $h\in H_q$,

$$(1.19)\qquad \delta^q(h;\chi)=\delta^q(h;c,d)$$
$$=\prod_{j=1}^{p-q} e(c_j\Phi_j)\cdot\prod_{j=1}^{q}\{e(d_jz_j+d_{-j}z_{-j})-e(d_{-j}z_j+d_jz_{-j})\}$$
$$=\prod_{j=1}^{p-q} e(c_j\Phi_j)\cdot\prod_{j=1}^{q} e(m_j\Theta_j)\{e(i\rho_jt_j)-e(-i\rho_jt_j)\}\,.$$

And put

$$(1.20)\qquad \eta_\chi^q(h)=\eta_{c,d}^q(h)=\sum_{\sigma\in\mathfrak{S}_{p-q}}\sum_{\tau\in\mathfrak{S}_q}\delta^q(h;\sigma c,\tau d)\,,$$

where $\tau d=(d_{\tau(1)}, d_{\tau(2)}, \cdots, d_{\tau(q)}, d_{-\tau(1)}, \cdots, d_{-\tau(q)})$. Then we obtain the following lemma.

LEMMA 1.3.

(a) $$L\eta_\chi^q(h)=(-1)^{pq+2^{-1}q(q+1)}L(\chi)\kappa_\chi^q(h)\qquad (h\in H_q)\,,$$

where $L(\chi)=L(c_1, c_2, \cdots, c_{p-q}, d_1, d_2, \cdots, d_q, d_{-1}, \cdots, d_{-q})$.

(b) *Let* M_j, N_j $(1\leqq j\leqq p-q)$ *be integers, then*

$$(1.21)\qquad \sum_{\substack{M_j\leqq c_j\leqq N_j\\(1\leqq j\leqq p-q)}}\eta_{c,d}^q(h)=\sum_{\sigma\in\mathfrak{S}_{p-q}}\Big\{\sum_{\substack{M_{\sigma(j)}\leqq c_j\leqq N_{\sigma(j)}\\(1\leqq j\leqq p-q)}}\sum_{\tau\in\mathfrak{S}_q}\delta^q(h;c,\tau d)\Big\}\,.$$

PROOF. This is proved analogously as Lemmas 1.1 and 1.2.

4. Now suppose that $1\leqq r\leqq q-1$. Let us consider a series of unitary representations of G of type r. Our purpose is to express explicitly the characters of these representations.

Let Q, Q' be the subgroups of G consisting respectively of all matrices of the form

$$\begin{bmatrix}\alpha & 0 & \beta\\ 0 & 1_{2r} & 0\\ \gamma & 0 & \delta\end{bmatrix},\qquad \begin{bmatrix}1_{p-r} & 0 & 0\\ 0 & \varepsilon & 0\\ 0 & 0 & 1_{q-r}\end{bmatrix},$$

where α, β, γ, δ and ε are matrices of type $(p-r)\times(p-r)$, $(p-r)\times(q-r)$, $(q-r)\times(p-r)$, $(q-r)\times(q-r)$ and $2r\times 2r$, respectively. Q (resp. Q') is isomorphic to $U(p-r, q-r)$ (resp. $U(r, r)$).

Let $c=(c_1, c_2, \cdots, c_{n-2r})$ be a row of integers such that $c_1\geqq c_2\geqq\cdots\geqq c_{n-2r}$ and I be a subset of $(p-r)$ elements of $I_{n-2r}=\{1, 2, \cdots, n-2r\}$. Let $\pi_{I,c}$ be the invariant eigendistribution on Q defined by the analogous formula as (1.11) and (1.12). Let moreover $m=(m_1, m_2, \cdots, m_r)$ be a row of integers and $\rho=(\rho_1, \rho_2, \cdots, \rho_r)$ a row of real numbers. Put as before

$$d_j=2^{-1}(m_j+i\rho_j)\,,\qquad d_{-j}=2^{-1}(m_j-i\rho_j)\,,\qquad d=(d_1, d_2, \cdots, d_r, d_{-1}, \cdots, d_{-r})$$
$$(i=\sqrt{-1})\,.$$

Let π_d be the invariant eigendistribution on Q' defined by the analogous formula as (1.18).

As is proved in [4(d), § 7], we can define an invariant eigendistribution $\pi_{I,c,d}=(-1)^{nr}\pi_{I,c}\otimes\pi_d$ on G from $\pi_{I,c}$ on Q and π_d on Q' by the following formula. Put as before

$$\kappa^k_{I,c,d}(h)=\varepsilon^k_R(h)\Delta^k(h)\pi_{I,c,d}(h)\qquad (h\in H'_k)\,.$$

Then for $h\in H'_k(R)$,

(1.22) $\kappa^k_{I,c,d}(h)=0$ for $0\leqq k<r$;

(1.22') $$\kappa^k_{I,c,d}(h)=(-1)^{(p+r)r}\sum_{\substack{\sigma\in\mathcal{S}_k\\ \sigma(1)<\cdots<\sigma(r)\\ \sigma(r+1)<\cdots<\sigma(k)}}\kappa^r_d(z_{\sigma(1)}, z_{\sigma(2)}, \cdots, z_{\sigma(r)}, z_{-\sigma(1)}, z_{-\sigma(2)}, \cdots, z_{-\sigma(r)})$$
$$\times\kappa^{k-r}_{I,c}(\Phi_1, \cdots, \Phi_{p-k}, z_{\sigma(r+1)}, \cdots, z_{\sigma(k)}, \Psi_1, \cdots, \Psi_{q-k}, z_{-\sigma(r+1)}, \cdots, z_{-\sigma(k)})^{6)},$$

for $r\leqq k\leqq q$, where σ runs over all $\sigma\in\mathcal{S}_k$ such that

$$\sigma(1)<\sigma(2)<\cdots<\sigma(r)\,,\qquad \sigma(r+1)<\sigma(r+2)<\cdots<\sigma(k)\,.$$

Put $\chi=(c, d)$ and denote $\pi_{I,c,d}$ and $\kappa^k_{I,c,d}$ also by $\pi_{I,\chi}$ and $\kappa^k_{I,\chi}$, respectively. And define

$$\pi_\chi=\pi_{c,d}=\sum_I\pi_{I,\chi}\,,\qquad \kappa^k_\chi=\kappa^k_{c,d}=\sum_I\kappa^k_{I,\chi}\,;$$
$$\pi_{\sigma c,d}=\operatorname{sgn}(\sigma)\pi_{c,d}\,,\qquad \kappa^k_{\sigma c,d}=\operatorname{sgn}(\sigma)\kappa^k_{c,d}\qquad (\sigma\in\mathcal{S}_{n-2r})\,,$$

where the sums run over all subsets I of $(p-r)$ elements of I_{n-2r}.

Define for $h\in H_k$,

(1.23) $$\delta^k(h\,;\chi)=\delta^k(h\,;c, d)$$
$$=\prod_{j=1}^{p-k}e(c_j\Phi_j)\cdot\prod_{l=1}^{q-k}e(c_{p-r+l}\Psi_l)\cdot\prod_{j=1}^{r}e(m_j\Theta_j)\{e(i\rho_jt_j)-e(-i\rho_jt_j)\}$$
$$\times\prod_{j=1}^{k-r}\operatorname{sgn}(t_{j+r})e\{-|c_{p-k+j}-c_{n-r-k+j}|\,|t_{j+r}|+(c_{p-k+j}+c_{n-r-k+j})\Theta_{j+r}\}\,.$$

Let us call $z=(z_1, z_2, \cdots, z_k, z_{-1}, z_{-2}, \cdots, z_{-k})$ the z-part of the co-ordinates of $h\in H_k$. For $\sigma\in\mathcal{S}_k$, define $\sigma z=(z_{\sigma(1)}, z_{\sigma(2)}, \cdots, z_{\sigma(k)}, z_{-\sigma(1)}, \cdots, z_{-\sigma(k)})$. And σh be the element of H_k such that the z-part of its co-ordinates is σz and the other part of its co-ordinates is equal to that of h. In this manner, $\mathcal{S}_k$ is

6) As is mentioned before, the function $\Delta^k(h)$ in this paper is different from the one in [4(d)] by the factor $(-1)^{q(q+1)/2}$. Accordingly this formula is different from the one in [4(d), § 7] by the constant factor $(-1)^{(p+r)r}$.

contained in W_k. Define

(1.24) $$\eta_\chi^k(h)=\sum_{\tau\in\mathcal{S}_{n-2r}}\sum_{\substack{\sigma\in\mathcal{S}_k\\ \sigma(r+1)<\sigma(r+2)<\cdots<\sigma(k)}}\delta^k(\sigma h\,;\tau c,d) \qquad (h\in H_k).$$

Then we obtain the following lemma analogous as Lemma 1.3.

LEMMA 1.4.

(a) $$L\eta_\chi^k=(-1)^{pq+qr+2^{-1}r(r-1)}L(\chi)\kappa_\chi^k,$$

where

$$L(\chi)=L(c_1,c_2,\cdots,c_{p-r},d_1,d_2,\cdots,d_r,c_{p-r+1},\cdots,c_{n-2r},d_{-1},\cdots,d_{-r}).$$

(b) *Let* M_j, N_j $(1\leqq j\leqq n-2r)$ *be* $n-2r$ *integers. Then*

(1.25) $$\sum_{\substack{M_j\leqq c_j\leqq N_j\\(1\leqq j\leqq n-2r)}}\eta_{c,d}^k(h)=\sum_{\tau\in\mathcal{S}_{n-2r}}\Big\{\sum_{\substack{M_{\tau(j)}\leqq c_j\leqq N_{\tau(j)}\\(1\leqq j\leqq n-2r)}}\sum_{\substack{\sigma\in\mathcal{S}_k\\ \sigma(r+1)<\cdots<\sigma(k)}}\delta^k(\sigma h\,;\tau c,d)\Big\}.$$

PROOF. This can be proved by a simple calculation.

If $c_1>c_2>\cdots>c_{n-2r}$, the distribution $\pi_{I,c,d}$ is the character of a representation of G which is a direct sum of a finite number of irreducible unitary ones. Let us prove this fact.

Put $R=QQ'$. This is the centralizer of the subgroup of H_q^+ defined by $t_1=t_2=\cdots=t_r$ and $t_{r+1}=t_{r+2}=\cdots=t_q=0$. R is a reductive Lie group and $\{H_r,H_{r+1},\cdots,H_q\}$ is a maximal family of its Cartan subgroups containing $H_r\cap Q'$ which are not conjugate to each other under inner automorphisms of R. Let M be an irreducible unitary representation of R whose character π is as follows: if $g\in R$ is conjugate to $h\in H_k'$ for some k $(r\leqq k\leqq q)$,

$$\begin{aligned}\pi(g)&=\pi(h)\\&=\Big(\prod_{j=1}^{r}e^{\Theta_j}\Big)^{-n+2r}\pi_d(z_1,z_2,\cdots,z_r,z_{-1},\cdots,z_{-r})\\&\quad\times\Big(\prod_{j=1}^{p-k}e^{\Phi_j}\cdot\prod_{l=1}^{q-k}e^{\Psi_l}\cdot\prod_{j=r+1}^{k}e^{\Theta_j}\Big)^{-r}\pi_{I,c}(\Phi_1,\Phi_2,\cdots,\Phi_{p-k},z_{r+1},\cdots,z_k,\\&\qquad\qquad\Psi_1,\cdots,\Psi_{q-k},z_{-r-1},\cdots,z_{-k}),\end{aligned}$$

and if otherwise, $\pi(g)=0$.

Let T^M be the induced representation of M defined in [4(e), §1]. Then its character can be obtained easily by [4(e), Theorem 2] and we see that $\pi_{I,c,d}$ is exactly the character of T^M. The detailed proof is ommitted here because it is only a simple calculation.

LEMMA 1.5. *The unitary representation T^M of G is a direct sum of a finite number of irreducible ones with the same infinitesimal character.*

PROOF. We know that for our group G there exist only a finite number of linearly independent invariant eigendistributions on G having a given infinitesimal character [4(d), § 8]. We know also that (1) the character of any irreducible unitary representation of G is an invariant eigendistribution on G, (2) two such representations are unitary equivalent if and only if they have the same character, (3) the characters of a finite number of non-equivalent irreducible unitary representations are linearly independent [3(a)], and (4) the character of any irreducible constituent of T^M has the same infinitesimal character as that of $\pi_{I,c,d}$. Therefore we see that the equivalent classes of irreducible constituents of T^M is only finite.

On the other hand, T^M has the following property. Let U be a maximal compact subgroup of G. Let $\mathcal{D}$ be an arbitrary equivalent class of finite-dimensional irreducible representations of U and let $d(\mathcal{D})$ be the dimension of $\mathcal{D}$. Then there exists a constant N such that the restriction of T^M on U contains irreducible constituents of class $\mathcal{D}$ at most $Nd(\mathcal{D})$ times for any $\mathcal{D}$ [4(e), § 2]. (Moreover we can take $N=1$ for any such T^M.)

The assertion of the lemma follows immediately from these facts. Q. E. D.

REMARK. It has not yet been proved that for a semisimple Lie group, the induced representations T^M in [4(e), § 1] (see also [1, § 7]) are "in general" irreducible, when M is irreducible. A partial result is found in "F. Bruhat, C. R., 240 (1955), pp. 2196-2198". For the group $G=U(p, 1)$ $(p \geqq 1)$, the characters of all quasi-simple irreducible (not necessarily unitary) representations are calculated in [4(c)] except for a few representations with singular infinitesimal characters. (The characters of all quasi-simple irreducible representations of $SO(p, 1)$ $(p \geqq 4)$ are given in [4(a)].)

Thus we obtained the following theorem.

THEOREM 1. *Suppose that* $c_1 > c_2 > \cdots > c_{n-2r}$ $(0 \leqq r \leqq q)$. *Then for any* I *and* $d=(d_1, d_2, \cdots, d_r, d_{-1}, d_{-2}, \cdots, d_{-r})$, *the distribution* $\pi_{I,c,d}$ *is the character of a unitary representation of* G *which is a direct sum of a finite number of irreducible unitary ones. In particular,* (1) *if* $r=q$, *the unitary representation corresponding to* $\pi_{I,c,d}$ *is "in general" irreducible, that is, irreducible if* $\rho_1 \rho_2 \cdots \rho_q \prod_{l<m} (\rho_l - \rho_m) \neq 0$, *where* $\rho_l = (2i)^{-1}(d_l - d_{-l})$ *for* $1 \leqq l \leqq q$, *and* (2) *if* $r=0$, *it is always irreducible and moreover square-integrable.*

§ 2. Some properties of an invariant eigendistribution on G.

Let dg be a fixed Haar measure on G. Let $d_k h$ be a Haar measure on H_k defined as

$$d_k h = d\varphi_1\, d\varphi_2 \cdots d\varphi_{p-k}\, d\phi_1\, d\phi_2 \cdots d\phi_{q-k}\, dt_1\, dt_2 \cdots dt_k\, d\theta_1\, d\theta_2 \cdots d\theta_k\,, \tag{2.1}$$

where $\varphi_j, \phi_l, t_m, \theta_m$ are the components of the co-ordinates of $h \in H_k$. Let $g \to \bar{g}$ be the natural mapping of G onto $\bar{G}^k = G/H_k$ and let $d_k\bar{g}$ be a left invariant measure on $\bar{G}^k$. We denote $d_k h$ and $d_k \bar{g}$ by dh and $d\bar{g}$ respectively if there is no danger of confusion. As is proved in [3(e), p. 493], there exist $(q+1)$ positive constants $\alpha_0, \alpha_1, \cdots, \alpha_q$ such that for any integrable function f on G,

$$\int_G f(g)\,dg = \sum_{k=0}^{q} \alpha_k \int_{H_k} \left\{ \int_{\bar{G}^k} f(ghg^{-1})\,d\bar{g} \right\} |\Delta^k(h)|^2 dh \qquad (\bar{g} = gH_k).$$

Let $C_0^\infty(G)$ denote the set of all indefinitely differentiable functions on G which vanish outside some compact sets. Define for any $f \in C_0^\infty(G)$ a function K_f^k on H_k' $(0 \leq k \leq q)$ by

$$K_f^k(h) = \varepsilon_R^k(h)\overline{\Delta^k(h)} \int_{\bar{G}^k} f(ghg^{-1})\,d\bar{g} \qquad (h \in H_k'), \tag{2.2}$$

where $\overline{\Delta^k(h)}$ is the complex conjugate of $\Delta^k(h)$ and $\bar{g} = gH_k$. Then

$$\int_G f(g)\,dg = \sum_{k=0}^{q} \alpha_k \int_{H_k} K_f^k(h)\varepsilon_R^k(h)\Delta^k(h)\,dh \qquad (f \in C_0^\infty(G)). \tag{2.3}$$

Let $H_k'(I)$ be the subset of H_k defined by

$$\prod_{j=1}^{p-k} \prod_{l=1}^{q-k} (e^{\Phi_j} e^{-\Psi_l} - 1) = 0.$$

We use in this section the following notations. Let $A(h)$ be a function on $H_k'(I)$. If h tends to an element h_0 of rank $n-1$[7] in such a manner that $\varphi_{p-k} = \theta$, $\psi_{q-k} \to \theta - 0$, then the limit of $A(h)$ is denoted by

$$A(h_0)|^{(\varphi_{p-k}, \psi_{q-k}) = (\theta, \theta-0)} = -A(h_0)|_{(\varphi_{p-k}, \psi_{q-k}) = (\theta, \theta-0)},$$

where $\varphi_{p-k}, \psi_{q-k}$ are the components of the co-ordinates of $h \in H_k'(I)$. For a function $B(h)$ on $H_k'(R)$, the analogous notation $B(h_0)|^{t_k=-0}$ is used to denote the limit of $B(h)$, as h tends to h_0 in such a manner that $t_k \to -0$. And denote by $\boldsymbol{\Phi}_j, \boldsymbol{\Psi}_l, \boldsymbol{z}_m, \boldsymbol{z}_{-m}, \boldsymbol{t}_m$ and $\boldsymbol{\Theta}_m$ the differential operators $\partial/\partial\Phi_j$, $\partial/\partial\Psi_l$, $\partial/\partial z_m$, $\partial/\partial z_{-m}$, $\partial/\partial t_m$ and $\partial/\partial\Theta_m$, respectively.

LEMMA 2.1. *Every K_f^k $(f \in C_0^\infty(G),\ 0 \leq k \leq q)$ has the following properties.*

(1) *K_f^k on H_k' can be extended to an indefinitely differentiable function on $H_k'(I)$. The restriction of the extended function on any connected component of $H_k'(I)$ may be considered as an indefinitely differentiable function on its closure*[8].

(2) $K_f^k(\omega h) = \varepsilon(\omega) K_f^k(h)$ $(\omega \in W_k,\ h \in H_k'(I))$.

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7) An element $g \in G$ is called semi-regular if the endomorphism $Ad(g)$ of the Lie algebra $\mathfrak{g}$ of G is semisimple and the centralizer $\mathfrak{z}_g$ of g in $\mathfrak{g}$ is of dimension rank $\mathfrak{g}+2$ (cf. [**3**(d), p. 554]). An element $h_0 \in H_k$ is semi-regular if and only if the rank of h_0 is $n-1$ (as an $n \times n$ matrix).

8) At the boundary of the closure, the differentiations must be considered appropriately (see [**3**(d), p. 573]).

In the following, let r and s be two non-negative integers.

(3) $(\Phi_{p-k}^{r}\Psi_{q-k}^{s}-\Psi_{q-k}^{r}\Phi_{p-k}^{s})K_{f}^{k}(h)\Big|_{(\varphi_{p-k},\psi_{q-k})=(\theta+0,\theta)}^{(\varphi_{p-k},\psi_{q-k})=(\theta-0,\theta)}=0$,

$\Phi_{p-k}^{r}\Psi_{q-k}^{s}K_{f}^{k}(h)|^{(\varphi_{p-k},\psi_{q-k})=(\theta\pm 0,\theta)}=\Phi_{p-k}^{r}\Psi_{q-k}^{s}K_{f}^{k}(h)|^{(\varphi_{p-k},\psi_{q-k})=(\theta,\theta\mp 0)}$.

(4) $z_{k}^{r}z_{-k}^{s}K_{f}^{k}(h)|^{t_{k}=0}=z_{-k}^{r}z_{k}^{s}K_{f}^{k}(h)|^{t_{k}=0}$.

(5) $i^{-1}\alpha_{k}(p-k)(q-k)\Phi_{p-k}^{r}\Psi_{q-k}^{s}K_{f}^{k}(h)\Big|_{(\varphi_{p-k},\psi_{q-k})=(\theta_{k+1}+0,\theta_{k+1})}^{(\varphi_{p-k},\psi_{q-k})=(\theta_{k+1}-0,\theta_{k+1})}$

$=\alpha_{k+1}(k+1)z_{k+1}^{r}z_{-k-1}^{s}K_{f}^{k+1}(h)|^{t_{k+1}=0}$,

where $h\in H_{k}\cap H_{k+1}$ *and is of rank* $n-1$.

PROOF. (1), (2), (3) and (4) are shown in [4(d), § 2] (see also [3(c) and (e)]).

Now let us prove (5). This is a slight extension of [4(d), Proposition 1]. Put

$$\varphi_{p-k}=\theta_{k+1}+\eta\,,\quad \psi_{q-k}=\theta_{k+1}-\eta\,,\quad Y=i\eta\,,\quad \boldsymbol{Y}=\partial/\partial Y=i^{-1}\partial/\partial\eta\,.$$

Then we know from [4(d), § 4] that for $0\leqq m<+\infty$,

$$i^{-1}\alpha_{k}(p-k)(q-k)\boldsymbol{Y}^{m}K_{f}^{k}(h)|_{\eta=+0}^{\eta=-0}=\alpha_{k+1}(k+1)\boldsymbol{t}_{k+1}^{m}K_{f}^{k+1}(h)|^{t_{k+1}=0}\,.$$

The equalities (5) follow immediately from these. Q. E. D.

Let us now summalize the elementary properties of an invariant eigendistribution π on G. Put as before

$$\kappa^{k}(h)=\varepsilon_{R}^{k}(h)\Delta^{k}(h)\pi(h)\,,\qquad \xi^{k}(h)=L\kappa^{k}(h)\qquad (h\in H_{k}')\,.$$

Then for $\pi=\pi_{\chi}$, $\xi^{k}(h)=\pm L(\chi)\eta_{\chi}^{k}$.

LEMMA 2.2. *The functions* κ^{k}, ξ^{k} *and* η_{χ}^{k} $(0\leqq k\leqq q)$ *have the following properties.*

(1) $\kappa^{k}(h)$ *on* H_{k}' *can be extended to an analytic function on* $H_{k}'(R)$. *The restriction of the extended function on any connected component of* $H_{k}'(R)$ *can be expressed as a linear combination of products of polynomial functions and exponential functions of* $x_{1}, x_{2}, \cdots, x_{n}$, *where* $(x_{1}, x_{2}, \cdots, x_{n})$ *is the co-ordinates of* h. *The functions* ξ^{k} *and* η_{χ}^{k} *have the same properties.*

(2) $\kappa^{k}(\omega h)=\varepsilon(\omega)\kappa^{k}(h)$, $\xi^{k}(\omega h)=\varepsilon'(\omega)\xi^{k}(h)$, $\eta_{\chi}^{k}(\omega h)=\varepsilon'(\omega)\eta_{\chi}^{k}(h)$ $(\omega\in W_{k},\ h\in H_{k}'(R))$.

(3) *Let r and s be arbitrary non-negative integers. Then*

$$(2.4)\qquad (\Phi_{p-k}^{r}\Psi_{q-k}^{s}+\Psi_{q-k}^{r}\Phi_{p-k}^{s})\xi^{k}(h)|^{(\varphi_{p-k},\psi_{q-k})=(\theta_{k+1},\theta_{k+1})}$$
$$=-(z_{k+1}^{r}z_{-k-1}^{s}+z_{-k-1}^{r}z_{k+1}^{s})\xi^{k+1}(h)|^{t_{k+1}=-0}\qquad (0\leqq k<q)\,,$$

where $h\in H_{k}\cap H_{k+1}$ *and is of rank* $n-1$. *The analogous relations hold between* η_{χ}^{k} *and* η_{χ}^{k+1}.

PROOF. (1) and (2) are shown in [4(d), § 2]. The equalities (3) are immediate consequences of [4(d), Theorem 1′]. In fact, it asserts that for $0<r<+\infty$,

$$(\Phi^r_{p-k}-\Psi^r_{q-k})\kappa^k(h)\,|^{(\varphi_{p-k},\psi_{q-k})=(\theta_{k+1},\theta_{k+1})}=-(z^r_{k+1}-z^r_{-k-1})\kappa^{k+1}(h)\,|^{t_{k+1}=-0}\,.$$

Since $\Phi_{p-k}=2^{-1}(\Theta_{k+1}+Y)$, $\Psi_{q-k}=2^{-1}(\Theta_{k+1}-Y)$, the above equalities are equivalent to

$$Y^{2m+1}\kappa^k(h)\,|^{\eta=0}=-t^{2m+1}_{k+1}\kappa^{k+1}(h)\,|^{t_{k+1}=-0}\qquad(0\leqq m<+\infty)\,.$$

Therefore for $0\leqq m,\ m'<+\infty$,

$$(\Phi^m_{p-k}\Psi^{m'}_{q-k}-\Psi^m_{q-k}\Phi^{m'}_{p-k})\kappa^k(h)\,|^{(\varphi_{p-k},\psi_{q-k})=(\theta_{k+1},\theta_{k+1})}$$
$$=-(z^m_{k+1}z^{m'}_{-k-1}-z^m_{-k-1}z^{m'}_{k+1})\kappa^{k+1}(h)\,|^{t_{k+1}=-0}\,.$$

Taking into account the explicit form of the differential operators L on H_k and H_{k+1}, the equalities (2.4) for $\xi^k=L\kappa^k$ and $\xi^{k+1}=L\kappa^{k+1}$ may be easily obtained from the above ones.

Now if $L(\chi)\neq 0$, then $\eta^k_\chi=\pm(L(\chi))^{-1}\xi^k$ where $\xi^k=L\kappa^k_\chi$. Therefore, in this case, the analogous equalities as (2.4) for η^k_χ and η^{k+1}_χ are immediate consequences of (2.4). We assert that the same equalities hold even if $L(\chi)=0$. This is proved by a simple calculation. Thus the assertion (3) is completely proved. Q. E. D.

REMARK. We can define the invariant eigendistribution $\pi_{I,c,d}$ even if $d_j+d_{-j}=i\rho_j$ $(1\leqq j\leqq r)$ are not pure imaginary but complex and define $\eta^k_{I,c,d}$ in the analogous fashion so as to hold $L\kappa^k_{I,c,d}=\varepsilon\eta^k_{I,c,d}$ for $0\leqq k\leqq q$ $(\varepsilon=\pm 1)$, then the analogous lemma holds also for these $\eta^k_{I,c,d}$.

We call $\chi=(c,d)$ of type r if $c=(c_1,c_2,\cdots,c_{n-2r})$ and $d=(d_1,d_2,\cdots,d_r,d_{-1},\cdots,d_{-r})$.

THEOREM 2. *For any invariant eigendistribution π on G, define $\kappa^k(h)$ as in* (1.8) *for $0\leqq k\leqq q$. Then for any $f\in C^\infty_0(G)$,*

$$(-1)^{n(n-1)/2}a_\pi\int_G f(g)\pi(g)\,dg=\sum_{k=0}^{q}\alpha_k\int_{H_k}LK^k_f(h)\cdot L\kappa^k(h)\,dh\,,$$

where a_π is a constant such that $L^2\kappa^k=a_\pi\kappa^k$ $(0\leqq k\leqq q)$. And for any $\chi=(c,d)$ of type r $(0\leqq r\leqq q)$,

$$\varepsilon_r L(\chi)\int_G f(g)\pi_\chi(g)\,dg=\sum_{k=0}^{q}\alpha_k\int_{H_k}LK^k_f(h)\cdot\eta^k_\chi(h)\,dh\,,$$

where $\varepsilon_r=(-1)^{n(n-1)/2}(-1)^{pq+pr+r(r-1)/2}$.

To prove this theorem, it needs a rather complicated calculation. We divide the calculation into four parts.

(i) Put $\xi^k=L\kappa^k$ $(0\leqq k\leqq q)$. Denote by dh the Haar measure on $H_{k,k+1}=H_k\cap H_{k+1}$ defined as

$$dh=d\varphi_1\,d\varphi_2\cdots d\varphi_{p-k-1}\,d\psi_1\,d\psi_2\cdots d\psi_{q-k}\,dt_1\,dt_2\cdots dt_k\,d\theta_1\cdots d\theta_k\,d\theta_{k+1}\,,$$

where

$$(\Phi_1, \Phi_2, \cdots, \Phi_{p-k-1}, z_1, z_2, \cdots, z_k, \Theta_{k+1}, \Psi_1, \Psi_2, \cdots, \Psi_{q-k-1}, z_{-1}, z_{-2}, \cdots, z_{-k}, \Theta_{k+1})$$

is the co-ordinates of $h \in H_{k,k+1} = H_{k+1}|^{t_{k+1}=0}$ considered as an element of H_{k+1}.

Denote by $\mathcal{D}(H_k)$ the set of all polynomials of the differential operators $\Phi_1, \Phi_2, \cdots, \Phi_{p-k}, z_1, z_2, \cdots, z_k, \Psi_1, \cdots, \Psi_{q-k}, z_{-1}, \cdots, z_{-k}$ on H_k. For a fixed k, let $X=(X_1, X_2, \cdots, X_n)$ be as in § 1, that is, $X_1 = \Phi_1$, $X_2 = \Phi_2, \cdots, X_n = z_{-k}$. For any $\sigma \in \mathcal{S}_n$, define $\sigma X = (X_{\sigma(1)}, X_{\sigma(2)}, \cdots, X_{\sigma(n)})$. Then σ induces naturally a transformation on $\mathcal{D}(H_k)$. Since $(\sigma\tau)(j) = \tau(\sigma(j))$, we see that $(\sigma\tau)X = \sigma(\tau X)$ and $(\sigma\tau)(P) = \sigma(\tau(P))$ for any $P \in \mathcal{D}(H_k)$. Every element of W_k induces naturally a permutation of the differential operators $X_1, X_2, \cdots, X_n$, and therefore we may consider that W_k is contained in the transformation group $\mathcal{S}_n$ of $\mathcal{D}(H_k)$. Let $\operatorname{sgn}(\sigma)$ be as usual. Let ρ_j^k (σ_l^k, τ_m^k, β_m^k resp.) be the transformation on $\mathcal{D}(H_k)$ induced by the permutation of Φ_j and Φ_{p-k} (Ψ_l and Ψ_{q-k}, pairs (z_m, z_{-m}) and (z_k, z_{-k}), z_m and z_{-m} resp.). Here ρ_{p-k}^k, σ_{q-k}^k and τ_k^k are the identity transformation. In the following, we omit the upper suffices k and denote them only by ρ_j, σ_l etc. Hence $\operatorname{sgn}(\rho_j) = -1$ if $1 \leq j < p-k$, and $\operatorname{sgn}(\rho_{p-k}) = 1$, when they are considered on $\mathcal{D}(H_k)$.

Let α (β resp.) be the transformation on all $\mathcal{D}(H_k)$ such that $0 \leq k < q$ ($0 \leq k \leq q$ resp.) which permutes Φ_{p-k} and Ψ_{q-k} (z_k and z_{-k} resp.) on $\mathcal{D}(H_k)$. Let γ be the transformation on all $\mathcal{D}(H_k)$ such that $0 \leq k < q$ which maps every $\mathcal{D}(H_k)$ onto $\mathcal{D}(H_{k+1})$ by replacing Φ_{p-k} and Ψ_{q-k} by z_{k+1} and z_{-k-1}, respectively. Then $\gamma\alpha = \beta\gamma$[9].

Taking into account Lemmas 2.1 (1) and 2.2 (1) and then integrating by parts, we obtain for $1 \leq r < +\infty$ and $A, B \in \mathcal{D}(H_k)$,

$$\begin{aligned}
&\int_0^{2\pi} \{\Phi_j^r A K_f^k(h) \cdot B\xi^k(h) - (-1)^r A K_f^k(h) \cdot \Phi_j^r B\xi^k(h)\}\, d\varphi_j \\
&= i^{-1} \sum_{m=1}^{r} (-1)^{m-1} \sum_{l=1}^{q-k} \Big[\Phi_j^{r-m} A K_f^k(h) \cdot \Phi_j^{m-1} B\xi^k(h) \Big]_{\varphi_j=\psi_l+0}^{\varphi_j=\psi_l-0} \\
&= i^{-1} \sum_{m=1}^{r} (-1)^{m-1} \sum_{l=1}^{q-k} \Big[\Phi_j^{r-m} A K_f^k(h) \Big]_{\varphi_j=\psi_l+0}^{\varphi_j=\psi_l-0} \times \Phi_j^{m-1} B\xi^k(h)\,|^{\varphi_j=\psi_l} .
\end{aligned}$$

It follows from Lemmas 2.1 (2) and 2.2 (2) that

$$\begin{aligned}
&\int_0^{2\pi} \cdots \int_0^{2\pi} \Phi_j^{m_1} \Psi_l^{m_2} K_f^k(h) \Big|_{\varphi_j=\psi_l+0}^{\varphi_j=\psi_l-0} \\
&\qquad \times \Phi_j^{m_3} \Psi_l^{m_4} \xi^k(h) d\varphi_1 d\varphi_2 \cdots \widehat{d\varphi_j} \cdots d\varphi_{p-k} d\psi_1 d\psi_2 \cdots d\psi_{q-k} \\
&= \operatorname{sgn}(\rho_j \sigma_l) \int_0^{2\pi} \cdots \int_0^{2\pi} \Phi_{p-k}^{m_1} \Psi_{q-k}^{m_2} K_f^k(h) \Big|_{\eta=+0}^{\eta=-0}
\end{aligned}$$

9) $\gamma\alpha$ and $\beta\gamma$ are not defined on $\mathcal{D}(H_q)$.

$$\times \Phi_{p-k}^{m_3} \Psi_{q-k}^{m_4} \xi^k(h)|^{\eta=0} d\varphi_1 d\varphi_2 \cdots d\varphi_{p-k-1} \widehat{d\varphi}_{p-k} d\psi_1 \cdots d\psi_{q-k-1} \widehat{d\psi}_{q-k} d\theta_{k+1},$$

where we put $\varphi_{p-k}=\theta_{k+1}+\eta$ and $\psi_{q-k}=\theta_{k+1}-\eta$, and $\widehat{d\varphi}_{p-k}$ means that $d\varphi_{p-k}$ is omitted. Put $E(h)=\Phi_{p-k}^{m_1}\Psi_{q-k}^{m_2}K_f^k(h)$, then

$$E(h)|^{\eta=-0}=E(h)|^{(\varphi_{p-k},\psi_{q-k})=(\theta_{k+1}-0,\theta_{k+1}+0)}=E(h)|^{(\varphi_{p-k},\psi_{q-k})=(\theta_{k+1}-0,\theta_{k+1})}.$$

Thus we obtain at last the following equality:

$$\int_{H_k}\{\Phi_j^r AK_f^k(h)\cdot B\xi^k(h)-(-1)^r AK_f^k(h)\cdot \Phi_j^r B\xi^k(h)\}dh$$

$$=i^{-1}\sum_{m=1}^{r}(-1)^{m-1}\sum_{l=1}^{q-k}\operatorname{sgn}(\rho_j\sigma_l)$$

$$\times\int_{H_{k,k+1}}\Phi_{p-k}^{r-m}\rho_j\sigma_l(A)K_f^k(h)\Big|_{\eta=+0}^{\eta=-0}\times\Phi_{p-k}^{m-1}\rho_j\sigma_l(B)\xi^k(h)|^{\eta=0}dh.$$

Analogously we obtain that

$$\int_{H_k}\{\Psi_j^r AK_f^k(h)\cdot B\xi^k(h)-(-1)^r AK_f^k(h)\cdot \Psi_j^r B\xi^k(h)\}dh$$

$$=-i^{-1}\sum_{m=1}^{r}(-1)^{m-1}\sum_{l=1}^{p-k}\operatorname{sgn}(\rho_l\sigma_j)$$

$$\times\int_{H_{k,k+1}}\Psi_{q-k}^{r-m}\rho_l\sigma_j(A)K_f^k(h)\Big|_{\eta=+0}^{\eta=-0}\times\Psi_{q-k}^{m-1}\rho_l\sigma_j(B)\xi^k(h)|^{\eta=0}dh.$$

Because

$$\Phi_{p-k}^r\Psi_{q-k}^s K_f^k(h)\Big|_{(\theta_{k+1},\theta_{k+1}+0)}^{(\theta_{k+1},\theta_{k+1}-0)}=-\Phi_{p-k}^r\Psi_{q-k}^s K_f^k(h)\Big|_{(\theta_{k+1}+0,\theta_{k+1})}^{(\theta_{k+1}-0,\theta_{k+1})}$$

$$=-\Phi_{p-k}^r\Psi_{q-k}^s K_f^k(h)\Big|_{\eta=+0}^{\eta=-0},$$

where $(\theta_{k+1},\theta_{k+1}-0)$ means that $(\varphi_{p-k},\psi_{q-k})=(\theta_{k+1},\theta_{k+1}-0)$ and so on.

Put

$$R'_k(A, B, r)$$

$$=\alpha_k\sum_{j=1}^{p-k}\operatorname{sgn}(\rho_j)\int_{H_k}\{\Phi_j^r\rho_j(A)K_f^k(h)\cdot\rho_j(B)\xi^k(h)-(-1)^r\rho_j(A)K_f^k(h)\cdot\Phi_j^r\rho_j(B)\xi^k(h)\}\,dh,$$

$$R''_k(A, B, r)$$

$$=\alpha_k\sum_{l=1}^{q-k}\operatorname{sgn}(\sigma_l)\int_{H_k}\{\Psi_l^r\sigma_l(A)K_f^k(h)\cdot\sigma_l(B)\xi^k(h)-(-1)^r\sigma_l(A)K_f^k(h)\cdot\Psi_l^r\sigma_l(B)\xi^k(h)\}\,dh,$$

and

$$R_k^-(A, B, r)=R'_k(A, B, r)-R''_k(\alpha(A), \alpha(B), r).$$

Then if $\sigma_l(A)=\pm A$, $\sigma_l(B)=\pm B$, $\sigma_l(AB)=\operatorname{sgn}(\sigma_l)AB$ $(1\leqq l\leqq q-k)$,

$$R'_k(A, B, r)$$
$$= i^{-1}\alpha_k(p-k)(q-k)\sum_{m=1}^{r}(-1)^{m-1}\int_{H_{k,k+1}} \Phi_{p-k}^{r-m} AK_f^k(h)\Big|_{\eta=+0}^{\eta=-0} \times \Phi_{p-k}^{m-1} B\xi^k(h)\Big|^{\eta=0} dh\,;$$

and if $\rho_j(A)=\pm A$, $\rho_j(B)=\pm B$, $\rho_j(AB)=\mathrm{sgn}\,(\rho_j)AB$ $(1\leqq j\leqq p-k)$,

$$R''_k(A, B, r)$$
$$= -i^{-1}\alpha_k(p-k)(q-k)\sum_{m=1}^{r}(-1)^{m-1}\int_{H_{k,k+1}} \Psi_{q-k}^{r-m} AK_f^k(h)\Big|_{\eta=+0}^{\eta=-0} \times \Psi_{q-k}^{m-1} B\xi^k(h)\Big|^{\eta=0} dh\,.$$

Therefore if A and B fulfill that

(2.5) $$\sigma_l(A)=\pm A\,,\quad \sigma_l(B)=\pm B\,,\quad \sigma_l(AB)=\mathrm{sgn}\,(\sigma_l)AB\,;\quad \rho_j\alpha(A)=\pm\alpha(A)\,,$$
$$\rho_j\alpha(B)=\pm\alpha(B)\,,\quad \rho_j\alpha(AB)=\mathrm{sgn}\,(\rho_j)\alpha(AB)\quad (1\leqq j\leqq p-k,\ 1\leqq l\leqq q-k)\,,$$

then we obtain

(2.6) $$R_k^-(A, B, r)$$
$$= i^{-1}\alpha_k(p-k)(q-k)\sum_{m=1}^{r}(-1)^{m-1}$$
$$\times\int_{H_{k,k+1}}\Big\{\Phi_{p-k}^{r-m} AK_f^k(h)\Big|_{\eta=+0}^{\eta=-0}\times\Phi_{p-k}^{m-1}B\xi^k(h)\Big|^{\eta=0}$$
$$+\Psi_{q-k}^{r-m}\alpha(A)K_f^k(h)\Big|_{\eta=+0}^{\eta=-0}\times\Psi_{q-k}^{m-1}\alpha(B)\xi^k(h)\Big|^{\eta=0}\Big\}dh$$
$$= i^{-1}\alpha_k(p-k)(q-k)\sum_{m=1}^{r}(-1)^{m-1}$$
$$\times\int_{H_{k,k+1}}\Phi_{p-k}^{r-m} AK_f^k(h)\Big|_{\eta=+0}^{\eta=-0}\times(\Phi_{p-k}^{m-1}B+\Psi_{q-k}^{m-1}\alpha(B))\,\xi^k(h)\Big|^{\eta=0}dh$$
$$= -\alpha_{k+1}(k+1)\sum_{m=1}^{r}(-1)^{m-1}$$
$$\times\int_{H_{k,k+1}} z_{k+1}^{r-m}\gamma(A)K_f^{k+1}(h)\Big|^{t_{k+1}=0}$$
$$\times(z_{k+1}^{m-1}\gamma(B)+z_{-k-1}^{m-1}\beta\gamma(B))\xi^{k+1}(h)\Big|^{t_{k+1}=-0}dh\,.$$

Because it follows from Lemmas 2.1 and 2.2 that for any $A\in\mathfrak{D}(H_k)$,

$$AK_f^k(h)\Big|_{\eta=+0}^{\eta=-0}=\alpha(A)K_f^k(h)\Big|_{\eta=+0}^{\eta=-0}\,,$$
$$(A+\alpha(A))\xi^k(h)\Big|_{\eta=+0}^{\eta=-0}=-(\gamma(A)+\beta\gamma(A))\xi^{k+1}(h)\Big|^{t_{k+1}=-0}\,,$$

and

$$i^{-1}\alpha_k(p-k)(q-k)AK_f^k(h)\Big|_{\eta=+0}^{\eta=-0}=\alpha_{k+1}(k+1)\gamma(A)K_f^{k+1}(h)\Big|^{t_{k+1}=-0}\,,$$

where $h\in H_k\cap H_{k+1}$ is of rank $n-1$.

(ii) Let $A, B\in\mathcal{D}(H_{k+1})$. As $K_f^{k+1}(h)$ and $\xi^{k+1}(h)$ are even and odd in every t_j $(1\leqq j\leqq k+1)$, respectively,

$$\begin{aligned}\int_{-\infty}^{+\infty} & AK_f^{k+1}(h)\cdot B\xi^{k+1}(h)dt_j\\ &=-\int_{-\infty}^{+\infty}\beta_j(A)K_f^{k+1}(h)\cdot\beta_j(B)\xi^{k+1}(h)dt_j\\ &=\int_{-\infty}^{0}\{AK_f^{k+1}(h)\cdot B\xi^{k+1}(h)-\beta_j(A)K_f^{k+1}(h)\cdot\beta_j(B)\xi^{k+1}(h)\}dt_j\,.\end{aligned}$$

Recall that

$$z_j=2^{-1}(\partial/\partial t_j+i^{-1}\partial/\partial\theta_j)\,,\qquad z_{-j}=2^{-1}(-\partial/\partial t_j+i^{-1}\partial/\partial\theta_j)\,,$$
$$AK_f^{k+1}(h)|^{t_j=0}=\beta_j(A)K_f^{k+1}(h)|^{t_j=0}\,.$$

Then integrating by parts, we obtain that

$$\begin{aligned}\int_{-\infty}^{+\infty}&\{z_j^r AK_f^{k+1}(h)\cdot B\xi^{k+1}(h)-(-1)^r AK_f^{k+1}(h)\cdot z_j^r B\xi^{k+1}(h)\}dt_j\\ &=2^{-1}\sum_{m=1}^{r}(-1)^{m-1}z_j^{r-m}AK_f^{k+1}(h)|^{t_j=0}\times\{z_j^{m-1}B+z_{-j}^{m-1}\beta_j(B)\}\xi^{k+1}(h)|^{t_j=-0}\,.\end{aligned}$$

Put

$$\begin{aligned}R_{k+1}^{+}&(A, B, r)\\ &=\alpha_{k+1}\sum_{j=1}^{k+1}\int_{H_{k+1}}\{z_j^r\tau_j(A)K_f^{k+1}\cdot\tau_j(B)\xi^{k+1}-z_{-j}^r\beta_j\tau_j(A)K_f^{k+1}\cdot\beta_j\tau_j(B)\xi^{k+1}\\ &\quad-(-1)^r\tau_j(A)K_f^{k+1}\cdot z_j^r\tau_j(B)\xi^{k+1}+(-1)^r\beta_j\tau_j(A)K_f^{k+1}\cdot z_{-j}^r\beta_j\tau_j(B)\xi^{k+1}\}dh\,.\end{aligned}$$

Taking into account Lemmas 2.1 (2) and 2.2 (2), we obtain that

$$\begin{aligned}R_{k+1}^{+}(A, B, r)=\alpha_{k+1}(k+1)\sum_{m=1}^{r}\int_{H_{k,k+1}}& z_{k+1}^{m-r}AK_f^{k+1}(h)|^{t_{k+1}=0}\\ &\times(z_{k+1}^{m-1}B+z_{-k-1}^{m-1}\beta(B))\xi^{k+1}(h)|^{t_{k+1}=-0}\,dh\,.\end{aligned}\tag{2.7}$$

Thus we proved the following lemma.

LEMMA 2.3. *Suppose that* $A, B\in\mathcal{D}(H_k)$ *fulfill the condition* (2.5). *Then*

$$R_k^{-}(A, B, r)+R_{k+1}^{+}(\gamma(A), \gamma(B), r)=0\qquad(0\leqq r<+\infty)\,.$$

We can immediately generalize this lemma in the following fashion.

LEMMA 2.4. *Let* A^m, B^m $(1\leqq m\leqq N)$ *be* $2N$ *elements of* $\mathcal{D}(H_k)$. *Suppose that under any* σ_l $(1\leqq l\leqq q-k)$, *the pairs* (A^m, B^m) *are transformed in such a way that there exists a permutation* w_l *of* $\{1, 2, \cdots, N\}$ *such that*

$$\sigma_l(A^m)=\pm A^{w_l(m)}\,,\quad\sigma_l(B^m)=\pm B^{w_l(m)}\,,\quad\sigma_l(A^mB^m)=\operatorname{sgn}(\sigma_l)A^{w_l(m)}B^{w_l(m)}\,.$$

Moreover suppose that the pairs $(\alpha(A^m), \alpha(B^m))$ *fulfill the analogous condition for any* ρ_j $(1\leqq j\leqq p-k)$. *Then*

$$\sum_{m=1}^{N}\{R_k^{-}(A^m, B^m, r)+R_{k+1}^{+}(\gamma(A^m), \gamma(B^m), r)\}=0\qquad(0\leqq r<+\infty)\,.$$

(iii) Put

$$I=\sum_{k=0}^{q}\alpha_k\int_{H_k}\{LK_f^k(h)\cdot\xi^k(h)-(-1)^{n(n-1)/2}K_f^k(h)\cdot L\xi^k(h)\}dh\,.$$

Let us prove that $I=0$. Recall that

$$L=\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)\sigma(X_1^{n-1}X_2^{n-2}\cdots X_{n-1})=\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)X_{\sigma(1)}^{n-1}X_{\sigma(2)}^{n-2}\cdots X_{\sigma(n-1)}\,,$$

and define for $1\leqq s\leqq n-1$,

$$I_s=\sum_{k=0}^{q}\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)\int_{H_k}\{\sigma(X_s^{n-s}X_{s+1}^{n-s-1}\cdots X_{n-1})K_f^k\cdot\sigma(X_{s-1}^{n-s+1}\cdots X_2^{n-2}X_1^{n-1})\xi^k$$
$$-(-1)^{n-s}\sigma(X_{s+1}^{n-s-1}\cdots X_{n-1})K_f^k\cdot\sigma(X_s^{n-s}X_{s-1}^{n-s+1}\cdots X_2^{n-2}X_1^{n-1})\xi^k\}\,dh\,.$$

Since $\sigma(AB)=\sigma(A)\sigma(B)$ for any $\sigma\in\mathcal{S}_n$ and $A, B\in\mathcal{D}(H_k)$, we know that $I=\sum_{s=1}^{n-1}(-1)^{(s-1)(2n-s)/2}I_s$. Hence it is sufficient to prove that $I_s=0$ for any s.

First let us consider the case of $s=1$. For every k $(0\leqq k\leqq q)$, let A_k, B_k be the following elements of $\mathcal{D}(H_k)$:

$$A_k=\sum_{\sigma(X_1)=\Phi_{p-k}}\operatorname{sgn}(\sigma)\sigma(X_2^{n-2}X_3^{n-3}\cdots X_{n-1})\,,\qquad B_k=1\,,$$

where the sum runs over all $\sigma\in\mathcal{S}_n$ such that $\sigma(X_1)=\Phi_{p-k}$. Then

$$\sigma_l(A_k)=\operatorname{sgn}(\sigma_l)A_k\,,\qquad \rho_j\alpha(A_k)=\operatorname{sgn}(\rho_j)\alpha(A_k)\,,$$
$$\sigma_l(B_k)=B_k\,,\qquad \rho_j\alpha(B_k)=\alpha(B_k)\qquad (1\leqq l\leqq q-k,\ 1\leqq j\leqq p-k)\,.$$

And considering L as an element of $\mathcal{D}(H_k)$,

$$L=\sum_{j=1}^{p-k}\operatorname{sgn}(\rho_j)\Phi_j^{n-1}\rho_j(A_k)\rho_j(B_k)-\sum_{l=1}^{q-k}\operatorname{sgn}(\sigma_l)\Psi_l^{n-1}\sigma_l\alpha(A_k)\sigma_l\alpha(B_k)$$
$$+\sum_{j=1}^{k}\{z_j^{n-1}\tau_j\gamma(A_{k-1})\tau_j\gamma(B_{k-1})-z_{-j}^{n-1}\beta_j\tau_j\gamma(A_{k-1})\beta_j\tau_j\gamma(B_{k-1})\}\,.$$

Therefore we see easily from Lemma 2.3 that

$$I_1=\sum_{k=0}^{q}\{R_k^-(A_k, B_k, n-1)+R_k^+(\gamma(A_{k-1}), \gamma(B_{k-1}), n-1)$$
$$=\sum_{k=0}^{q-1}\{R_k^-(A_k, B_k, n-1)+R_{k+1}^+(\gamma(A_k), \gamma(B_k), n-1)$$
$$=0\,,$$

where we put $R_0^+=0$ and $R_q^-=0$.

Now let us prove that $I_s=0$ for any s. Let us fix s and k $(2\leqq s\leqq n-1,\ 0\leqq k\leqq q)$. For any element $\tau\in\mathcal{S}_n$, define two elements A_k^τ, B_k^τ of $\mathcal{D}(H_k)$ as

$$A_k^\tau=\operatorname{sgn}(\tau)\tau(X_{s+1}^{n-s-1}\cdots X_{n-2}^2X_{n-1})\,,\qquad B_k^\tau=\tau(X_{s-1}^{n-s+1}\cdots X_2^{n-2}X_1^{n-1})\,,$$

then for any σ, $\tau\in\mathcal{S}_n$,

$$\sigma(A_k^\tau)=\operatorname{sgn}(\sigma)A_k^{\sigma\tau}, \qquad \sigma(B_k^\tau)=B_k^{\sigma\tau}.$$

Let $\mathcal{T}_{s,k}$ be the subset of the transformation group $\mathcal{S}_n$ of $\mathcal{D}(H_k)$ consisting of all elements τ such that $\tau(X_s)=\Phi_{p-k}$. We see that $\sigma_l\mathcal{T}_{s,k}=\mathcal{T}_{s,k}$ for $1\leqq l\leqq q-k$, $(\alpha^{-1}\rho_j\alpha)\mathcal{T}_{s,k}=\mathcal{T}_{s,k}$ and $\operatorname{sgn}(\alpha^{-1}\rho_j\alpha)=\operatorname{sgn}(\rho_j)$ for $1\leqq j\leqq p-k$. Put

$$C_k=\sum_{\tau\in\mathcal{T}_{s,k}}\Phi_{p-k}^{n-s}A_k^\tau B_k^\tau.$$

Then

$$C_k=\sum_{\tau(X_s)=\Phi_{p-k}}\operatorname{sgn}(\tau)\tau(X_1^{n-1}X_2^{n-2}\cdots X_{n-1}),$$

and L in $\mathcal{D}(H_k)$ is expressed as

$$L=\sum_{j=1}^{p-k}\operatorname{sgn}(\rho_j)\rho_j(C_k)-\sum_{l=1}^{q-k}\operatorname{sgn}(\sigma_l)\sigma_l\alpha(C_k)+\sum_{j=1}^{k}\{\tau_j\gamma(C_{k-1})-\beta_j\tau_j\gamma(C_{k-1})\}.$$

By the analogous argument as for I_1, we see from Lemma 2.4 that

$$\begin{aligned}I_s&=\sum_{k=0}^{q}\{\sum_{\tau\in\mathcal{T}_{s,k}}R_k^-(A_k^\tau, B_k^\tau, n-s)+\sum_{\tau\in\mathcal{T}_{s,k-1}}R_k^+(\gamma(A_{k-1}^\tau), \gamma(B_{k-1}^\tau), n-s)\}\\&=\sum_{k=0}^{q-1}\sum_{\tau\in\mathcal{T}_{s,k}}\{R_k^-(A_k^\tau, B_k^\tau, n-s)+R_{k+1}^+(\gamma(A_k^\tau), \gamma(B_k^\tau), n-s)\}\\&=0.\end{aligned}$$

Thus we proved that $I_s=0$.

(iv) Now let us prove Theorem 2. We know that

$$\int_G f(g)\pi(g)dg=\sum_{k=0}^{q}\alpha_k\int_{H_k}K_f^k(h)\kappa^k(h)dh \qquad (f\in C_0^\infty(G)).$$

On the other hand, there exists a constant a_π such that $L^2\kappa^k=a_\pi\kappa^k$ for every k [**4**(d), § 3]. Therefore $I=0$ means that

$$\begin{aligned}\sum_{k=0}^{q}\alpha_k\int_{H_k}LK_f^k(h)\cdot\xi^k(h)&=(-1)^{n(n-1)/2}\sum_{k=0}^{q}\alpha_k\int_{H_k}K_f^k(h)\cdot L^2\kappa^k(h)dh\\&=(-1)^{n(n-1)/2}a_\pi\int_G f(g)\pi(g)dg.\end{aligned}$$

This is nothing but the first equality of the theorem.

The second equality of the theorem is obtained immediately from the first if $L(\chi)\neq 0$. In fact, in this case,

$$L\kappa_\chi^k=(-1)^{pq+qr+r(r-1)/2}L(\chi)\eta_\chi^k, \qquad L^2\kappa_\chi^k=L(\chi)^2\kappa_\chi^k.$$

Hence $a_\pi=L(\chi)^2$. The second equality for the case where $L(\chi)=0$ can be proved by the analogous argument as above replacing ξ^k by η_χ^k. Now Theorem 2 is completely proved.

REMARK. The infinitesimal character of an arbitrary invariant eigendistribution π on G is singular if and only if $L^2\kappa^k=0$ for $0\leqq k\leqq q$. Therefore, in this case, Theorem 2 asserts that for any $f\in C_0^\infty(G)$,

$$\sum_{k=0}^{q}\alpha_k\int_{H_k}LK_f^k(h)\cdot L\kappa^k(h)dh=0.$$

But, in this case, we can prove stronger fact that $L\kappa^k=0$ for $0\leqq k\leqq q$. This is proved by a direct calculation using the explicit form of κ^k studied in [4(d), §§ 9 and 10].

§ 3. Plancherel formula for $G=U(p, q)$.

For any $f\in C_0^\infty(G)$, define

$$F_f^k(h)=\alpha_k LK_f^k(h) \qquad (h\in H_k'(I))$$

and

$$A_f^r(\chi)=\int_G f(g)\pi_\chi(g)dg\,,$$

where χ is of type r, i.e., $\chi=(c, d)$, $c=(c_1, c_2, \cdots, c_{n-2r})$, $d=(d_1, d_2, \cdots, d_r, d_{-1}, \cdots, d_{-r})$. Then it follows from Theorem 2 that

$$\varepsilon_r L(\chi)A_f^r(\chi)=\sum_{k=0}^{q}\int_{H_k}F_f^k(h)\eta_\chi^k(h)dh=\sum_{k=r}^{q}\int_{H_k}F_f^k(h)\eta_\chi^k(h)dh\,, \tag{3.1}$$

because $\eta_\chi^0=0, \eta_\chi^1=0, \cdots, \eta_\chi^{r-1}=0$.

We know the following properties of F_f^k $(0\leqq k\leqq q)$.

(1) F_f^k is zero outside some relatively compact subset of $H_k'(I)\subset H_k$ and can be extended to a continuous function on the whole H_k [3(e), Lemma 40].

(2) $F_f^k(\omega h)=\varepsilon'(\omega)F_f^k(h)$ $(h\in H_k'(I),\ \omega\in W_k)$.

(3) The restriction of F_f^k on every connected component of $H_k'(I)$ may be considered as an indefinitely differentiable function on the closure of the component.

The following lemma, due to Harish-Chandra, plays an essential role in this section.

LEMMA 3.1. *There exists a positive constant γ_0 such that*

$$LK_f^0(e)=(-1)^{pq+n(n-1)/2}\gamma_0 f(e) \qquad (f\in C_0^\infty(G))\,,$$

where e is the identity element of G.

PROOF. Since $\overline{\Delta^0(h)}=(-1)^{n(n-1)/2}(\det h)^{-n+1}\Delta^0(h)$, we obtain

$$K_f^0(h)=(-1)^{n(n-1)/2}(\det h)^{-n+1}\Delta^0(h)\int_{G/H_0}f(ghg^{-1})d\bar{g} \qquad (\bar{g}=gH_0)\,.$$

Denote the subgroup $SU(p, q)$ of G by G_0 and put $K_0=K\cap G_0$, where K is a maximal compact subgroup of G. Then G_0 is semisimple and K_0 is a maximal compact subgroup of it. As is easily seen,

$$2^{-1}(\dim(G_0/K_0)-\mathrm{rank}\,(G_0)+\mathrm{rank}\,(K_0))=pq\,.$$

Apply for G_0, [3(c), Theorem 4] and [3(e), Lemmas 17 and 18] and extend them from G_0 to G. Then the assertion of the lemma is immediately obtained.

Let $\mathfrak{S}_n$ be the set of all symmetric polynomials of n indeterminates. As is proved in [4(d), § 3], for any Laplace operator D on G, there exists a unique $P_D \in \mathfrak{S}_n$ such that[10] for any $f \in C_0^\infty(G)$, I and χ,

$$K_{Df}^k = P_D(X) K_f^k, \qquad D\pi_{I,\chi} = P_D(\chi)\pi_{I,\chi},$$

where

$$P_D(\chi) = P_D(c_1, c_2, \cdots, c_{p-r}, d_1, d_2, \cdots, d_r, c_{p-r+1}, \cdots, c_{n-2r}, d_{-1}, d_{-2}, \cdots, d_{-r}).$$

And hence for any χ of type r $(0 \leqq r \leqq q)$,

$$A_{Df}^r(\chi) = P_D(\chi) A_f^r(\chi). \tag{3.2}$$

The mapping $D \to P_D$ is one-to-one onto $\mathfrak{S}_n$.

Let us study some elementary properties of $A_f^r(\chi)$. Let $m_j = d_j + d_{-j}$, $\sqrt{-1}\rho_j = d_j - d_{-j}$ be as before. We denote sometimes $A_f^r(\chi)$ by $A_f^r(c, m, \rho)$, where $m = (m_1, m_2, \cdots, m_r)$ and $\rho = (\rho_1, \rho_2, \cdots, \rho_r)$. For $\sigma \in \mathcal{S}_r$, define $\sigma m = (m_{\sigma(1)}, m_{\sigma(2)}, \cdots, m_{\sigma(r)})$ as before.

$A_f^r(c, m, \rho)$ is symmetric in $c_1, c_2, \cdots, c_{p-r}$ and also in $c_{p-r+1}, c_{p-r+2}, \cdots, c_{n-2r}$, odd in every ρ_j $(1 \leqq j \leqq r)$, and

$$A_f^r(c, \sigma m, \sigma\rho) = A_f^r(c, m, \rho) \qquad (\sigma \in \mathcal{S}_r).$$

Define for $f \in C_0^\infty(G)$,

$$M_f = \sum_{k=0}^{q} \sup_{h \in H_k'} |F_f^k(h)| < +\infty.$$

Then there exists a constant $\alpha > 0$ such that for any χ,

$$|A_f^r(\chi)| \leqq \alpha M_f. \tag{3.3}$$

In fact, we can find a constant α fulfilling $|\eta_\chi^k(h)| \leqq \alpha$ for any k and χ.

Define

$$|\chi|^2 = \sum_{j=1}^{n-2r} c_j^2 + \sum_{l=1}^{r}(|d_l|^2 + |d_{-l}|^2), \quad |\chi|_+^2 = \sum_{j=1}^{n-2r} c_j^2 + 2^{-1}\sum_{l=1}^{r} m_l^2, \quad |\chi|_-^2 = 2^{-1}\sum_{l=1}^{r}\rho_l^2,$$

then $|\chi|^2 = |\chi|_+^2 + |\chi|_-^2$.

LEMMA 3.2. *Let $f \in C_0^\infty(G)$. For any positive integer N, there exists a positive constant $M_{N,f}$ such that for any χ,*

$$(1 + |\chi|^2)^N |A_f^r(\chi)| \leqq M_{N,f}.$$

PROOF. It follows from (3.2) and (3.3) that for any $P \in \mathfrak{S}_n$, there exists a constant $M_{P,f}$ such that for any χ of type r, $|P(\chi)A_f^r(\chi)| \leqq M_{P,f}$. Suppose that for some N,

$$\sup_\chi \{(1 + |\chi|^2)^N |A_f^r(\chi)|\} = +\infty.$$

10) In the notation of [3(c), p. 755], $\partial(\gamma'(D)) = P_D(X)$, as the differential operator on every H_k $(0 \leqq k \leqq q)$.

Then we can find a sequence $\chi^{(1)}, \chi^{(2)}, \cdots, \chi^{(m)}, \cdots$ such that

$$\lim_{j\to+\infty} |\chi^{(j)}| = +\infty, \qquad \lim_{j\to+\infty} |\chi^{(j)}|^{2N} |A_f^r(\chi^{(j)})| = +\infty.$$

Let $P \in \mathfrak{S}_n$. Then

$$|P(\chi^{(j)})^N A_f^r(\chi)| = (|P(\chi^{(j)})|/|\chi^{(j)}|^2)^N |\chi^{(j)}|^{2N} |A_f^r(\chi^{(j)})| \leqq M_{P^N,f} < +\infty.$$

Therefore $P(\chi^{(j)})/|\chi^{(j)}|^2 \to 0$ as $j \to +\infty$. Put $P(X) = P_0(X) = \sum_{j=1}^{n} X_j^2$. Since $P_0(\chi) = |\chi|_+^2 - |\chi|_-^2 = |\chi|^2 - 2|\chi|_-^2$, we see that

$$P_0(\chi^{(j)})/|\chi^{(j)}|^2 = 1 - 2|\chi^{(j)}|_-^2/|\chi^{(j)}|^2 \to 0.$$

Hence $|\chi^{(j)}|_-^2/|\chi^{(j)}|^2 \to 1/2$ and $|\chi^{(j)}|_+^2/|\chi^{(j)}|^2 \to 1/2$. Therefore we can choose a subsequence $\chi^{(j_i)}$ $(i \geqq 1)$ having the following property. Let us denote this subsequence again by $\chi^{(i)}$ and let $c_l^{(i)}$, $d_l^{(i)}$ and $d_{-l}^{(i)}$ be its components. Then every component of $\chi^{(i)}/|\chi^{(i)}|$, i.e., $c_l^{(i)}/|\chi^{(i)}|$, $d_l^{(i)}/|\chi^{(i)}|$ and $d_{-l}^{(i)}/|\chi^{(i)}|$ etc. is convergent. Denote by ξ the limit of $\chi^{(i)}/|\chi^{(i)}|$. Let $P \in \mathfrak{S}_n$ be homogeneous of degree $s \geqq 2$. Since $|\chi^{(i)}|^{sN} |A_f^r(\chi^{(i)})| \to +\infty$ as $i \to +\infty$, we see analogously as above that $P(\chi^{(i)})/|\chi^{(i)}|^s \to 0$. Therefore $P(\xi) = 0$. Hence ξ must be zero. But this contradicts $\lim_{i\to+\infty} |\chi^{(i)}|_-^2/|\chi^{(i)}|^2 = 1/2$. The lemma is now completely proved. Q. E. D.

COROLLARY. *The series*

$$\sum_{c,m} A_f^r(c, m, \rho) = \sum_{c_1=-\infty}^{+\infty} \cdots \sum_{c_{n-2r}=-\infty}^{+\infty} \sum_{m_1=-\infty}^{+\infty} \cdots \sum_{m_r=-\infty}^{+\infty} A_f^r(c, m, \rho)$$

is absolutely convergent and the convergence is uniform with respect to $\rho = (\rho_1, \rho_2, \cdots, \rho_r) \in \boldsymbol{R}^r$.

Let $\boldsymbol{Z}$ be the set of all integers and put $\boldsymbol{C}^* = \boldsymbol{C} - \{0\}$.

LEMMA 3.3. *Let F be a continuously differentiable function on* $\boldsymbol{C}^*$ *which is zero outside a compact set. Define*

$$\check{F}(a, b) = \int_{-\infty}^{+\infty}\int_0^{2\pi} F(e^z) \operatorname{sgn}(t) e\{-|a-b||t| + (a+b)i\theta\} dt\, d\theta \qquad (a, b \in \boldsymbol{Z})$$

and

$$\hat{F}(m, \rho) = \int_{-\infty}^{+\infty}\int_0^{2\pi} F(e^z) e^{i\rho t} e^{im\theta} dt\, d\theta \qquad (m \in \boldsymbol{Z},\ \rho \in \boldsymbol{R}),$$

where $z = t + i\theta$ *and* $i = \sqrt{-1}$. *Then* $\check{F}(a, b) = \check{F}(b, a)$. *If* $F(e^z)$ *is odd in* t,

$$\text{(3.4)} \qquad \sum_{a=-\infty}^{+\infty} \sum_{b=-\infty}^{+\infty} \check{F}(a, b) = \pi \int_{-\infty}^{+\infty} F(e^t) \coth(t/2) dt + \pi \int_{-\infty}^{+\infty} F(-e^t) \tanh(t/2) dt$$

$$= -i2^{-1} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{F}(m, \rho) \coth(\pi\rho) d\rho$$

$$-i2^{-1} \sum_{m=-\infty}^{+\infty} (-1)^m \int_{-\infty}^{+\infty} \hat{F}(m, \rho) \operatorname{cosech}(\pi\rho) d\rho,$$

where

$$\tanh t=\frac{e^t-e^{-t}}{e^t+e^{-t}}\,, \qquad \coth t=\frac{e^t+e^{-t}}{e^t-e^{-t}}\,, \qquad \operatorname{cosech} t=\frac{2}{e^t-e^{-t}}\,.$$

The order of the successive integro-summation in the last part can be arbitrarily changed.

PROOF. Let M, N be positive integers.

$$\sum_{a=-M}^{N}\check{F}(a,b)=\sum_{m=-M-b}^{N-b}\check{F}(b+m,b)$$
$$=2\int_{-\infty}^{0}\int_{0}^{2\pi}F(e^z)e^{2bi\theta}\Big(\sum_{m=-M-b}^{N-b}e^{|m||t|+mi\theta}\Big)dt\,d\theta\,.$$

Put $M'=M+b$, $N'=N-b$, and $\bar{z}=t-i\theta$. Then

$$\sum_{m=-M'}^{N'}e^{|m||t|+mi\theta}=\sum_{m=0}^{N'}e^{mz}+\sum_{m=1}^{M'}e^{m\bar{z}}$$
$$=\frac{1-e^{(N'+1)z}}{1-e^z}+\frac{1-e^{(M'+1)\bar{z}}}{1-e^{\bar{z}}}=\frac{1-e^{2t}}{|1-e^z|^2}-\frac{e^{(N'+1)z}}{1-e^z}-\frac{e^{(M'+1)\bar{z}}}{1-e^{\bar{z}}}\,.$$

Put $g(e^z)=(1-e^t)(1-e^{2t})|1-e^z|^{-2}$. This is bounded and continuously differentiable with respect to θ. $F(e^z)(1-e^t)^{-1}$ is also bounded and continuously differentiable with respect to θ, because $F(e^z)$ is odd in t. Therefore

$$h(e^{i\theta})=\int_{-\infty}^{0}F(e^z)(1-e^t)^{-1}g(e^z)dt=\int_{-\infty}^{0}F(e^z)\frac{1-e^{2t}}{|1-e^z|^2}dt$$

is continuously differentiable. Put

$$g_N(e^z)=F(e^z)(1-e^z)^{-1}e^{Nz}\,.$$

Then if $t<0$, $|g_N(e^z)|\leqq|F(e^z)|\,|1-e^z|^{-1}$ and $g_N(e^z)\to 0$ pointwisely as $N\to+\infty$. Since $|F(e^z)|\,|1-e^z|^{-1}$ is integrable with respect to $dt\,d\theta$, we can apply Lebesgue's theorem and obtain that

$$\int_{-\infty}^{0}\int_{0}^{2\pi}e^{2bi\theta}g_N(e^z)dt\,d\theta\to 0 \qquad (N\to+\infty)\,.$$

Analogously

$$\int_{-\infty}^{0}\int_{0}^{2\pi}e^{2bi\theta}F(e^z)(1-e^{\bar{z}})^{-1}e^{M\bar{z}}dt\,d\theta\to 0 \qquad (M\to+\infty)\,.$$

Thus we proved that

$$\sum_{a=-\infty}^{+\infty}\check{F}(a,b)=-2\int_{0}^{2\pi}e^{2bi\theta}h(e^{i\theta})d\theta\,.$$

It follows from the theory of Fourier series that

$$\sum_{b=-\infty}^{+\infty}\Big\{\sum_{a=-\infty}^{+\infty}\check{F}(a,b)\Big\}=-2\pi(h(1)+h(-1))$$

$$= -2\pi \int_{-\infty}^{0} F(e^t)\frac{1+e^t}{1-e^t}dt - 2\pi \int_{-\infty}^{0} F(-e^t)\frac{1-e^t}{1+e^t}dt$$

$$= \pi \int_{-\infty}^{+\infty} F(e^t)\coth(t/2)dt + \pi \int_{-\infty}^{+\infty} F(-e^t)\tanh(t/2)dt\,.$$

This is the first part of the equality (3.4).

To prove the second part, it is sufficient to apply the following two well known facts.

LEMMA 3.4. *Let $f(t)$ be a continuously differentiable function on $\boldsymbol{R}$ such that $f(0)=0$ and zero outside a compact set. Then*

$$\int_{-\infty}^{+\infty} f(t)\coth(t/2)dt = -i\int_{-\infty}^{+\infty} \hat{f}(\rho)\coth(\pi\rho)\,d\rho\,,$$

$$\int_{-\infty}^{+\infty} f(t)\tanh(t/2)dt = -i\int_{-\infty}^{+\infty} \hat{f}(\rho)\tanh(\pi\rho)\,d\rho\,,$$

where $\hat{f}(\rho)=\int_{-\infty}^{+\infty} f(t)e^{i\rho t}dt.$

LEMMA 3.5. *Let $F(e^{i\theta}, \alpha)$ be a continuous function on the torus $\boldsymbol{T}$ of one dimension with parameter $\alpha \in A$. Suppose that $F(e^{i\theta}, \alpha)$ is continuously differentiable on $\boldsymbol{T}$ for every α except at a finite number of fixed points of $\boldsymbol{T}$ and that $(\partial/\partial\theta)F(e^{i\theta}, \alpha)$ is uniformly bounded. Then*

$$2\pi F(1, \alpha) = \sum_{m=-\infty}^{+\infty} \int_{0}^{2\pi} F(e^{i\theta}, \alpha)e^{im\theta}d\theta \qquad (\alpha \in A)\,,$$

and the convergence is uniform with respect to $\alpha \in A$.

Now it follows from Lemma 3.5 that

$$2\pi F(e^t) = \sum_{m=-\infty}^{+\infty} \hat{F}(m, e^t)\,,$$

where $\hat{F}(m, e^t)=\int_0^{2\pi} F(e^{t+i\theta})e^{im\theta}d\theta$ and the convergence is uniform with respect to t. Therefore

$$2\pi \int_{-\infty}^{+\infty} F(e^t)e^{i\rho t}dt = \sum_{m=-\infty}^{+\infty} \hat{F}(m, \rho)\,,$$

and hence by Lemma 3.4,

$$2\pi \int_{-\infty}^{+\infty} F(e^t)\coth(t/2)\,dt = -i\int_{-\infty}^{+\infty}\Big\{\sum_{m=-\infty}^{+\infty} \hat{F}(m, \rho)\Big\}\coth(\pi\rho)\,d\rho\,.$$

Analogously we obtain that

$$2\pi\int_{-\infty}^{+\infty} F(-e^t)\tanh(t/2)\,dt = -i\int_{-\infty}^{+\infty}\Big\{\sum_{m=-\infty}^{+\infty} (-1)^m\hat{F}(m, \rho)\Big\}\,\mathrm{cosech}(\pi\rho)\,d\rho\,.$$

It remains only to prove that the order of integration and summation in the above equalities can be changed. Put

$$l(e^{i\theta})=\int_{-\infty}^{+\infty}F(e^{t+i\theta})\coth(t/2)\,dt=\int_{-\infty}^{+\infty}t^{-1}F(e^{t+i\theta})(t\coth(t/2))\,dt\,.$$

Then $l(e^{i\theta})$ is continuously differentiable with respect to θ. Changing the order of the integration, we obtain that

$$\hat{l}(m)=\int_0^{2\pi}l(e^{i\theta})e^{im\theta}d\theta=\int_{-\infty}^{+\infty}\hat{F}(m,e^t)\coth(t/2)\,dt$$
$$=-i\int_{-\infty}^{+\infty}\hat{F}(m,\rho)\coth(\pi\rho)\,d\rho\,.$$

Inserting this into the equality $2\pi l(0)=\sum_{m=-\infty}^{+\infty}\hat{l}(m)$, then we obtain

$$2\pi\int_{-\infty}^{+\infty}F(e^t)\coth(t/2)\,dt=-i\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\hat{F}(m,\rho)\coth(\pi\rho)\,d\rho\,.$$

Analogously,

$$2\pi\int_{-\infty}^{+\infty}F(-e^t)\tanh(t/2)\,dt=-i\sum_{m=-\infty}^{+\infty}(-1)^m\int_{-\infty}^{+\infty}\hat{F}(m,\rho)\,\mathrm{cosech}(\pi\rho)\,d\rho\,.$$

Now Lemma 3.3 is completely proved. Q. E. D.

Let $\boldsymbol{Z}_0$ ($\boldsymbol{Z}_1$) be the set of all even (odd) integers. For a sequence $\{a_m\}_{m\in Z}$, let us denote by $\sum_{m\in Z}a_m$, $\sum_{m\in Z_0}a_m$ and $\sum_{m\in Z_1}a_m$ the limits $\lim_{M,N\to+\infty}\sum_{m=-M}^{N}a_m$, $\lim_{M,N\to+\infty}\sum_{m=-M}^{N}a_{2m}$ and $\lim_{M,N\to+\infty}\sum_{m=-M}^{N}a_{2m+1}$, respectively.

COROLLARY. *Let F be as in Lemma* 3.3. *Then*

$$\sum_{a\in Z}\sum_{b\in Z}\check{F}(a,b)=-i2^{-1}\sum_{m\in Z_0}\int_{-\infty}^{+\infty}\hat{F}(m,\rho)\coth(\pi\rho/2)\,d\rho$$
$$-i2^{-1}\sum_{m\in Z_1}\int_{-\infty}^{+\infty}\hat{F}(m,\rho)\tanh(\pi\rho/2)\,d\rho\,. \tag{3.5}$$

PROOF. This is obvious from the equality (3.4).

Now let us return to the functions F_f^k on H_k ($0\leqq k\leqq q$, $f\in C_0^\infty(G)$). Define

$$\hat{F}_f^k(\chi\,;r)=\int_{H_k}F_f^k(h)\delta^k(h\,;\chi)dh\,, \tag{3.6}$$

where χ is of type r. And put

$$b_r^k=\frac{1}{2^r}\sum_{j_1,\cdots,j_r=0,1}\sum_{c_1\in Z}\sum_{c_2\in Z}\cdots\sum_{c_{n-2r}\in Z}\sum_{m_1\in Z_{j_1}}\cdots\sum_{m_r\in Z_{j_r}}$$
$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\hat{F}_f^k(\chi\,;r)\,e_{j_1}(\rho_1)e_{j_2}(\rho_2)\cdots e_{j_r}(\rho_r)\,d\rho_1 d\rho_2\cdots d\rho_r\,, \tag{3.7}$$

where c_j, m_l, ρ_l are the components of $\chi=(c,m,\rho)$ and

$$e_0(\rho)=-(i/2)\coth(\pi\rho/2)\,,\qquad e_1(\rho)=-(i/2)\tanh(\pi\rho/2)\,.$$

LEMMA 3.6. *The successive integro-summation in* (3.7) *is convergent for* *any $f\in C_0^\infty(G)$ and the order of the operations of integrations and summations on $F_f^k(\chi\,;r)$ can be arbitrarily changed. And for any k such that $0\leqq k\leqq q$,*

$$b_0^k=b_1^k=b_2^k=\cdots=b_k^k\,.$$

Moreover for any $f\in C_0^\infty(G)$,

$$b_0^0=(2\pi)^n n\,!\,F_f^0(e)=(-1)^{pq+n(n-1)/2}(2\pi)^n n\,!\,\alpha_0\gamma_0 f(e)\,. \tag{3.8}$$

PROOF. The first and second assertions of the lemma can be obtained by slightly extending Lemma 3.3 and its Corollary to functions of several variables. The last assertion follows from Lemma 3.1 if we know that

$$b_0^0=(2\pi)^n n\,!\,F_f^0(e)=(2\pi)^n n\,!\,\alpha_0\gamma_0 LK_f^0(e)\,.$$

But, as is remarked in the beginning of this section, the function $F_f^0(h)=\alpha_0 LK_f^0(h)$ on $H_0'(I)$ can be extended to a continuous function on the whole H_0 which is indefinitely differentiable on the closure of every connected component of $H_0'(I)$. Generalize Lemma 3.5 for the case of higher dimensions and apply it to the function $F_f^0(h)$, then we see that $b_0^0=(2\pi)^n n\,!\,F_f^0(e)$. Q. E. D.

Define

$$a^r=\frac{1}{2^r}\sum_{j_1,\cdots,j_r=0,1}\sum_{c_1\in Z}\cdots\sum_{c_{n-2r}\in Z}\sum_{m_1\in Z_{j_1}}\cdots\sum_{m_r\in Z_{j_r}}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}L(\chi)A_f^r(\chi)\prod_{l=1}^{r}e_{j_l}(\rho_l)\,d\rho_1 d\rho_2\cdots d\rho_r\,. \tag{3.9}$$

It follows from (1.23), (1.24), (1.25) and (3.1) that the order of the successive integro-summation in (3.9) can be arbitrarily changed and that for $0\leqq r\leqq q$,

$$\varepsilon_r a^r=\alpha_r^r b_r^r+\alpha_{r+1}^r b_r^{r+1}+\cdots+\alpha_{q-1}^r b_r^{q-1}+\alpha_q^r b_r^q\,, \tag{3.10}$$

where $\varepsilon_r=(-1)^{n(n-1)/2}(-1)^{pq+qr+r(r-1)/2}$ and $\alpha_k^r=(n-2r)\,!\,k\,!((k-r)\,!)^{-1}$ for $0\leqq r\leqq k$.

Put $b^k=b_0^k=b_1^k=\cdots=b_k^k$ and define

$$\boldsymbol{a}={}^t(\varepsilon_0 a^0,\varepsilon_1 a^1,\cdots,\varepsilon_q a^q)\,,\qquad \boldsymbol{b}={}^t(b^0,b^1,\cdots,b^q)\,,$$

and

$$A=\begin{bmatrix}\alpha_0^0 & \alpha_1^0 & \alpha_2^0 & \cdots & \alpha_q^0\\ 0 & \alpha_1^1 & \alpha_2^1 & \cdots & \alpha_q^1\\ 0 & 0 & \alpha_2^2 & \cdots & \alpha_q^2\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \alpha_q^q\end{bmatrix}.$$

Then

$$\boldsymbol{a}=A\boldsymbol{b}\,. \tag{3.10'}$$

Put $\beta_r=(-1)^r((n-2r)\,!\,r\,!)^{-1}$. Then $(\beta_0,\beta_1,\beta_2,\cdots,\beta_q)$ is the first row of A^{-1}. In fact, it follows from the equality:

$$\sum_{r=0}^{k}(-1)^r\frac{k!}{r!(k-r)!}=0 \qquad (k\geqq 1),$$

that

$$(\beta_0, \beta_1, \beta_2, \cdots, \beta_q)A=(1, 0, 0, \cdots, 0).$$

Therefore we obtain that $b^0=\sum_{r=0}^{q}\beta_r\varepsilon_r a^r$. Hence it follows from (3.8) that

$$(3.11) \qquad (2\pi)^n n!\,\alpha_0\gamma_0 f(e)=\sum_{r=0}^{q}\frac{(-1)^r(-1)^{qr+r(r-1)/2}}{(n-2r)!\,r!}\times a^r.$$

Let χ be of type r. Let $d_j=2^{-1}(m_j+i\rho_j)$ and $d_{-j}=2^{-1}(m_j-i\rho_j)$ be as before the components of χ. If $c_1>c_2>\cdots>c_{n-2r}$,

$$L(\chi)=L(c_1, c_2, \cdots, c_{p-r}, d_1, d_2, \cdots, d_r, c_{p-r+1}, \cdots, c_{n-2r}, d_{-1}, \cdots, d_{-r})$$
$$=(-1)^{(q-r)r}(-1)^{r(r-1)/2}\left|L(\chi)/\prod_{j=1}^{r}i\rho_j\right|\left(\prod_{j=1}^{r}i\rho_j\right).$$

Therefore for any $j_1, j_2, \cdots, j_r$ $(=0, 1)$,

$$(-1)^r(-1)^{qr+r(r-1)/2}L(\chi)e_{j_1}(\rho_1)e_{j_2}(\rho_2)\cdots e_{j_r}(\rho_r)\geqq 0.$$

Suppose also that $c_1>c_2>\cdots>c_{n-2r}$. Let $g\to T^r_{I,\chi}(g)$ be a unitary representation of G whose character is $\pi_{I,\chi}$[11]. Then it follows immediately from the definition that

$$(3.12) \qquad A^r_f(\chi)=\sum_I Sp(T^r_{I,\chi}(f)) \qquad (f\in C_0^\infty(G)),$$

where I runs over all subsets of $p-r$ elements of $I_{n-2r}=\{1, 2, \cdots, n-2r\}$ and

$$(3.13) \qquad T^r_{I,\chi}(f)=\int_G T^r_{I,\chi}(g)f(g)\,dg.$$

Insert (3.12) into (3.9) and then the latter into (3.11), then (3.11) is essentially the Plancherel formula for G.

Define for $f_1, f_2\in C_0^\infty(G)$,

$$f_1^*(g)=\overline{f_1(g^{-1})}, \qquad f_1*f_2(g)=\int_G f_1(gg_0^{-1})f_2(g_0)\,dg_0 \qquad (g\in G).$$

Then if $c_1>c_2>\cdots>c_{n-2r}$, $Sp(T^r_{I,\chi}(f*f^*))\geqq 0$. Therefore taking into account the symmetries of $A^r_f(\chi)$ and $L(\chi)$ with respect to the components of χ, we see that the series (3.9) are absolutely convergent for $0\leqq r\leqq q$ if $f=f_1*f_1^*$ for some $f_1\in C_0^\infty(G)$. And so are also, if $f=f_1*f_2$ for some $f_1, f_2\in C_0^\infty(G)$. Thus we obtained the following theorem. Put $\gamma=(2\pi)^n n!\,\alpha_0\gamma_0$.

THEOREM 3. *Let $f_1, f_2\in C_0^\infty(G)$ and put $f=f_1*f_2$. Then*

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11) An irreducible unitary representation of G is determined within unitary equivalence by its character [**3**(a), p. 250].

$$\gamma f(e)=\sum_{0\leqq r\leqq q}\frac{1}{2^r r!}\sum_{j_1,\cdots,j_r=0,1}\sum_{c_1>c_2>\cdots>c_{n-2r}}\sum_{\substack{m_s\in Z_{j_s}\\(1\leqq s\leqq r)}}$$

$$\int_{\rho\in R^r}\Big\{\sum_I Sp(T^r_{I,\chi}(f))\Big\}\,|L(\chi)e_{j_1}(\rho_1)e_{j_2}(\rho_2)\cdots e_{j_r}(\rho_r)|\,d\rho$$

$$=\sum_{0\leqq r\leqq q}\sum_{j_1,\cdots,j_r=0,1}\sum_{c_1>c_2>\cdots>c_{n-2r}}\sum_{\substack{m_s\in Z_{j_s}\\(1\leqq s\leqq r)}}$$

$$\int_{\rho_1>\rho_2>\cdots>\rho_r>0}\Big\{\sum_I Sp(T^r_{I,\chi}(f))\Big\}\,|L(\chi)e_{j_1}(\rho_1)e_{j_2}(\rho_2)\cdots e_{j_r}(\rho_r)|\,d\rho\,,$$

where I runs over all subset of $p-r$ elements of I_{n-2r}, $d\rho=d\rho_1 d\rho_2\cdots d\rho_r$ and $\chi=(c, m, \rho)$. The successive integro-summation converges absolutely.

REMARK. Let X^r_+ be the set of all $\chi=(c, m, \rho)$ of type r such that $c_1>c_2>\cdots>c_{n-2r}$ and $\rho_1>\rho_2>\cdots>\rho_r>0$. Suppose that $\chi\in X^r_+$ and $\chi'\in X^s_+$, then the unitary representations $T^r_{I,\chi}$ and $T^s_{I',\chi'}$ are equivalent if and only if $r=s$, $I=I'$ and $\chi=\chi'$. In fact, this is seen easily from the explicit form of $\pi_{I,\chi}$ and $\pi_{I',\chi'}$. In the above equality, we must take $\Big\{\sum_I S_p(T^0_{I,\chi}(f))\Big\}|L(\chi)|$ as the summand for $r=0$.

§ 4. Computation of the constant γ.

The constant γ in Theorem 3 depends on the fixed Haar measure dg on G. Let us choose a standard normalization of it. Let U be the maximal compact subgroup of G consisting of all matrices of the form

$$u=\begin{bmatrix}u_{11} & 0\\ 0 & u_{22}\end{bmatrix};\qquad u_{11}\in U(p)\,,\qquad u_{22}\in U(q)\,,$$

where $U(p)$ is the unitary group of order p. Denote the subgroup H^+_q by A. Let du ($u\in U$) be the normalized Haar measure on U such that $\int_U du=1$ and let da be the Haar measure on A defined as $da=dt_1 dt_2\cdots dt_q$, where a is the matrix in (1.2) for $k=q$.

Let P_+ be the set of all positive roots (with respect to some lexicographic order) of (G, H_q) which are not trivial on A, and, as in [4(d), § 1], denote by $\xi_\alpha(a)$ the homomorphism of A into C^* corresponding to $\alpha\in P_+$. Put

$$\begin{aligned}(4.1)\qquad \Delta(a)&=\Big|\prod_{\alpha\in P_+}(\xi_\alpha(a)-\xi_{-\alpha}(a))\Big|\\ &=\prod_{1\leqq j\leqq q}(e^{t_j}-e^{-t_j})^{2(p-q)}\cdot\prod_{1\leqq j,l\leqq q}|e^{t_j+t_l}-e^{-t_j-t_l}|\cdot\prod_{1\leqq j<l\leqq q}(e^{t_j-t_l}-e^{-t_j+t_l})^2\\ &=\prod_{1\leqq j\leqq q}(2\,\mathrm{sh}\,t_j)^{2(p-q)}\cdot\prod_{1\leqq j\leqq q}|2\,\mathrm{sh}\,2t_j|\cdot\prod_{1\leqq j<l\leqq q}(2\,\mathrm{ch}\,2t_j-2\,\mathrm{ch}\,2t_l)^2\,.\end{aligned}$$

Then it follows from [3(b), Lemma 22] that we can normalize a Haar measure dg in such a fashion that the following equality holds: for every integrable function f on G,

$$\int_G f(g)\,dg = \int_A \Delta(a)\,da \int_{U\times U} f(uav)\,du\,dv^{12)}. \tag{4.2}$$

Let us calculate the constant γ for this normalized Haar measure dg. This can be done using the formula given in [3(b), Lemma 28]. But the integral appeared in this formula is not easy to calculate explicitly. As we feel some interest in spherical functions, we proceed here in the following way. First we give the explicit form of some spherical functions of a series of square-integrable irreducible unitary representations, and second, using these spherical functions, calculate the formal degrees (see [3(b), p. 574]) of these representations, then the constant γ is obtained. In the second step, the formal degrees of the representations of this series are expressed as a polynomial function of integral variables which parametrize these representations. This fact makes us possible to calculate the formal degrees.

Let $g\to T(g)$ be an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$. Let $\mathcal{D}$ be an equivalent class of finite-dimensional irreducible representation of U. Denote by $\mathcal{H}(\mathcal{D})$ the set of all vectors in $\mathcal{H}$ which transform under $T(u)$ $(u\in U)$ according to $\mathcal{D}$. Let $E(\mathcal{D})$ be the canonical projection of $\mathcal{H}$ onto $\mathcal{H}(\mathcal{D})$. Then the function on G:

$$\phi(g) = Sp(E(\mathcal{D})T(g)) \qquad (g\in G)$$

is called the spherical function of type $\mathcal{D}$ of the representation T.

Let $\mathcal{D}_{k,l}$ be the equivalent class of the representation $u\to|u_{11}|^k|u_{22}|^l$ $(u\in U)$, where $|u_{11}|=\det u_{11}$. As is well known [3(a)], $\dim \mathcal{H}(\mathcal{D}_{k,l})\leqq 1$. Denote every element $g\in G$ in the form

$$g=\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{bmatrix}, \tag{4.3}$$

where g_{11}, g_{12}, g_{21} and g_{22} are matrices of type $p\times p$, $p\times q$, $q\times p$ and $q\times q$, respectively. Let $g_{11}^* = {}^t\bar{g}_{11}$. Then we have the following proposition.

PROPOSITION 1. *Put $n=p+q$ as before and let k and l be two integers. Then the functions*

$$\phi_{k,l}(g) = |g_{11}^*|^{-k}|g_{22}|^l = |\bar{g}_{11}|^{-k}|g_{22}|^l \qquad (k-l\geqq n) \tag{4.4}$$

and

$$\phi_{k,l}(g) = \left\{\frac{1}{p}Sp(g_{11}g_{11}^*) + \frac{p-k+l}{pq}Sp(g_{21}g_{21}^*)\right\}|g_{11}^*|^{-k}|g_{22}|^l \qquad (k-l\geqq n+2) \tag{4.5}$$

are respectively the spherical functions of type $\mathcal{D}_{k,l}$ of the square-integrable irreducible unitary representations $g\to T_{I,c}^0(g)$, where for $\phi_{k,l}$,

$$I=\{1, 2, \cdots, p\},$$

$$c=(k+p-1, k+p-2, \cdots, k+1, k, l+n-1, \cdots, l+p+1, l+p),$$

12) In the notation of [3(b), p. 592], $\Delta(a)=|\mathrm{D}(a)|^{1/2}$.

and for $\phi_{k,l}$,

$$I=\{1, 2, \cdots, p\},$$

$$c=(k+p-1, k+p-2, \cdots, k+1, k-1, l+n, l+n-2, l+n-3, \cdots, l+p+1, l+p).$$

The proof of this proposition will be given in Appendix. In this section, we use only $\phi_{k,l}$ $(k-l\geqq n)$.

It is easily proved that [3(b), Theorem 3] is also valid for our reductive Lie group G. Therefore denoting by $d_{p,q}^{k,l}$ the formal degree of $T_{I,c}^{0}$ corresponding to $\phi_{k,l}$, we see from Theorem 3 in the preceding section that

$$\gamma d_{p,q}^{k,l}=L(c)=(p-1)!(p-2)!\cdots 2!1!(q-1)!(q-2)!\cdots 2!1!\prod_{i=1}^{p}\prod_{j=1}^{q}(N-i-j+1),$$

where $N=k-l$ $(\geqq n)$. Therefore $d_{p,q}^{k,l}$ may be expressed as

$$d_{p,q}^{k,l}=c_{p,q}\prod_{i=1}^{p}\prod_{j=1}^{q}(N-i-j+1), \tag{4.6}$$

where $c_{p,q}$ is a constant independent of k and l (and of N).

On the other hand, by the definition of the formal degree, it follows from (4.2) that

$$\begin{aligned}(d_{p,q}^{k,l})^{-1}&=\int_{G}|\phi_{k,l}(g)|^{2}dg=\int_{A}|\phi_{k,l}(a)|^{2}\Delta(a)\,da\\&=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{j=1}^{q}(\text{ch } t_j)^{-2N}\cdot\prod_{m=1}^{q}(2\text{sh } t_m)^{2(p-q)}|2\text{sh } 2t_m|\\&\quad\times\prod_{1\leqq i<j\leqq q}(2\text{ch } 2t_i-2\text{ch } 2t_j)^2\,dt_1dt_2\cdots dt_q.\end{aligned}$$

Put

$$x_j=(\text{sh } t_j)^2=(\text{ch } t_j)^2-1=2^{-1}(\text{ch } 2t_j-1)\qquad(1\leqq j\leqq q).$$

Then

$$(d_{p,q}^{k,l})^{-1}=4^{pq}\int_{0}^{+\infty}\cdots\int_{0}^{+\infty}\prod_{m=1}^{q}(x_m+1)^{-N}x_m^{p-q}\cdot\prod_{1\leqq i<j\leqq q}(x_i-x_j)^2\,dx_1dx_2\cdots dx_q.$$

Since

$$\prod_{1\leqq i<j\leqq q}(x_i-x_j)=\sum_{\tau\in\mathfrak{S}_q}\text{sgn}(\tau)x_1^{\tau(q)-1}x_2^{\tau(q-1)-1}\cdots x_q^{\tau(1)-1},$$

we obtain that

$$\begin{aligned}(d_{p,q}^{k,l})^{-1}&=4^{pq}q!\int_{0}^{+\infty}\cdots\int_{0}^{+\infty}\prod_{j=1}^{q}(x_j+1)^{-N}x_j^{(p-q)+(q-j)}\cdot\prod_{1\leqq i<j\leqq q}(x_i-x_j)\,dx_1dx_2\cdots dx_q\\&=4^{pq}q!\sum_{\tau\in\mathfrak{S}_q}\text{sgn}(\tau)\prod_{j=1}^{q}B(p-j+\tau(q-j+1), N-p+j-\tau(q-j+1)),\end{aligned}\tag{4.7}$$

where $B(a, b)$ is the B-function defined as

$$B(a, N-a)=\int_{0}^{+\infty}(y+1)^{-N}y^{a-1}dy\qquad(N>a\geqq 1).$$

Let Γ be the usual Γ-function, that is, $\Gamma(s)=\int_0^{+\infty}e^{-x}x^{s-1}dx$. Then

$$B(a, b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \qquad \Gamma(0)=1, \qquad \Gamma(s)=(s-1)\Gamma(s-1).$$

Hence

$$(4.7')\quad (d_{p,q}^{k,l})^{-1}=4^{pq}q!\sum_{\tau\in\mathcal{S}_q}\operatorname{sgn}(\tau)\prod_{j=1}^{q}\Gamma(p-j+\tau(q-j+1))\frac{\Gamma(N-p+j+\tau(q-j+1))}{\Gamma(N)};$$

Now

$$\prod_{j=1}^{q}\frac{\Gamma(N-p+j-\tau(q-j+1))}{\Gamma(N)}$$
$$=\frac{\Gamma(N-p)\Gamma(N-p-1)\cdots\Gamma(N-p-q+1)}{\Gamma(N)\Gamma(N-1)\cdots\Gamma(N-q+1)}\times\frac{p_\tau(N)}{\prod_{1\leqq i<j\leqq q}(N-i)},$$

where $p_\tau(N)$ is a polynomial of N of degree $q(q-1)/2$ whose coefficient of $N^{q(q-1)/2}$ is equal to 1. As N runs over all integers $\geqq n$, we obtain by comparing (4.6) and (4.7′) that

$$c_{p,q}^{-1}\cdot\prod_{1\leqq i<j\leqq q}(N-i)=4^{pq}q!\sum_{\tau\in\mathcal{S}_q}\operatorname{sgn}(\tau)p_\tau(N)\prod_{j=1}^{q}\Gamma(p-j+\tau(q-j+1)).$$

Comparing the coefficients of $N^{q(q-1)/2}$ in the two sides of this equality, we see that

$$(4.8)\qquad c_{p,q}^{-1}=4^{pq}q!\sum_{\tau\in\mathcal{S}_q}\operatorname{sgn}(\tau)\prod_{j=1}^{q}\Gamma(p-j+\tau(q-j+1)).$$

Now let us prove that

$$(4.9)\qquad c_{p,q}^{-1}=4^{pq}q!(q-1)!(q-2)!\cdots 2!1!\cdot(p-1)!(p-2)!\cdots(p-q)!.$$

It is sufficient for this only to prove the following equality:

$$(4.10)\qquad 1=\sum_{\tau\in\mathcal{S}_q}\operatorname{sgn}(\tau)\prod_{j=1}^{q}\binom{p-j+\tau(q-j+1)-1}{p-j},$$

where $\binom{a}{b}=\frac{a!}{b!(a-b)!}$ for $a\geqq b\geqq 0$.

Denote by C the right hand side of the above equality. C is equal to the coefficient of $x_1^{p-1}x_2^{p-2}\cdots x_q^{p-q}$ in

$$\sum_{\tau\in\mathcal{S}_q}\operatorname{sgn}(\tau)(1+x_1)^{p-1+\tau(q)-1}(1+x_2)^{p-2+\tau(q-1)-1}\cdots(1+x_q)^{p-q+\tau(1)-1}$$
$$=\prod_{m=1}^{q}(1+x_m)^{p-m}\cdot\prod_{1\leqq i<j\leqq q}(x_i-x_j).$$

Therefore C is also equal to

$$\sum_{\tau\in\mathcal{S}_q}\operatorname{sgn}(\tau)\binom{p-1}{\tau(q)-1}\binom{p-2}{\tau(q-1)-1}\cdots\binom{p-q}{\tau(1)-1},$$

where we put $\binom{a}{b}=0$ if $b>a$. But this is exactly the coefficient of

$x_2x_3^2x_4^3 \cdots x_q^{q-1} = x_{\tau(q)}^{\tau(q)-1} x_{\tau(q-1)}^{\tau(q-1)-1} \cdots x_{\tau(1)}^{\tau(1)-1}$ in

$$\sum_{\tau \in \mathcal{S}_q} \operatorname{sgn}(\tau)(1+x_{\tau(q)})^{p-1}(1+x_{\tau(q-1)})^{p-2} \cdots (1+x_{\tau(1)})^{p-q}$$
$$= \prod_{m=1}^{q} (1+x_m)^{p-q} \cdot \prod_{1 \leq i < j \leq q} (x_i - x_j) .$$

Hence we see that C is equal to 1.

Thus the following result is obtained:

$$d_{p,q}^{k,l} = c_{pq} \prod_{i=1}^{q} \prod_{j=1}^{p} (N-i-j+1) , \qquad c_{p,q} = 4^{pq} q! \prod_{j=1}^{q} (j-1)!(p-j)! \tag{4.11}$$

and hence

$$d_{p,q}^{k,l} = L(c)\Big(4^{pq} \prod_{j=1}^{q} j!(p-j)!\Big)^{-1} \Big(\prod_{j=1}^{q} (j-1)! \prod_{i=1}^{p} (i-1)!\Big)^{-1}. \tag{4.11$'$}$$

PROPOSITION 2. *Let dg be the Haar measure of G normalized in such a fashion that the relation* (4.2) *holds. Then the constant γ in Theorem* 3 *is equal to*

$$\gamma = 4^{pq} \prod_{j=1}^{q} j!(p-j)! \cdot \prod_{m=1}^{q} (m-1)! \prod_{i=1}^{p} (i-1)! \tag{4.12}$$
$$= 4^{pq} q! \Big(\prod_{j=1}^{q-1} j!\Big)^2 \prod_{m=1}^{q} (p-m)! \prod_{i=1}^{p-1} i! .$$

NOTE. By the way of proving (4.11), we obtain the following equalities:

$$\sum_{\tau \in \mathcal{S}_q} \operatorname{sgn}(\tau) \prod_{j=1}^{q} B(a-j+\tau(j), b+j-\tau(j)) = (-1)^{q(q-1)/2} \prod_{j=0}^{q-1} \frac{j!\,\Gamma(a-j)\Gamma(b-j)}{\Gamma(a+b-j)} ,$$

where a and b are positive integers such that $a, b \geqq q$; and

$$\sum_{\tau \in \mathcal{S}_q} \operatorname{sgn}(\tau) \prod_{j=1}^{q} \Gamma(a-j+\tau(j)) = (-1)^{q(q-1)/2} \prod_{j=0}^{q-1} j!\,\Gamma(a-j) ,$$

where a is a positive integer such that $a \geqq q$. As is seen above, these are the consequences of the following equality:

$$1 = \sum_{\tau \in \mathcal{S}_q} \operatorname{sgn}(\tau) \binom{a}{\tau(1)-1} \binom{a+1}{\tau(2)-1} \cdots \binom{a+q-1}{\tau(q)-1} ,$$

where a is a non-negative integer.

§ 5. The Plancherel formula for $SU(p, q)$.

Let G_0 denote the subgroup $SU(p, q)$ of G and put $U_0 = G_0 \cap U$ and $A = H_q^+$. Then $G_0 = U_0 A U_0$, that is, every element $g \in G_0$ can be expressed as $g = uav$, where $u, v \in U_0$ and $a \in A$. Let du be the normalized Haar measure on U_0 such that $\int_{U_0} du = 1$ and da the Haar measure on A defined in the preceding section. There exists a unique Haar measure dg on G_0 such that the following relation holds: for any integrable function f on G_0,

$$\int_{G_0} f(g)\,dg = \int_A \Delta(a)\,da \int_{U_0\times U_0} f(uav)\,du\,dv\,. \tag{5.1}$$

Let $\boldsymbol{T}$ be the set of all complex numbers with absolute value 1. Let λg ($\lambda\in\boldsymbol{T}$, $g\in G_0$) be the usual product of λ and g. The mapping $(\lambda, g)\to\lambda g$ maps $\boldsymbol{T}\times G_0$ onto G. Put $\lambda_0=\exp(2\pi\sqrt{-1}/n)$, where $n=p+q$. Then $\lambda_0 g\in G_0$ if $g\in G_0$. Define a mapping P on $C_0^\infty(G_0)$ as

$$(Pf)(g)=f(\lambda_0 g) \qquad (f\in C_0^\infty(G_0),\ g\in G_0)$$

and let I be the identity mapping on $C_0^\infty(G_0)$. Define, as in [4(d), § 2],

$$f_j=\frac{1}{n}\prod_{\substack{0\leq k\leq n-1\\ k\neq j}}(P-\lambda_0^k I)f \qquad (0\leqq j\leqq n-1)\,.$$

Then it follows from $P^n=I$ that

$$Pf_j=\lambda_0^j f_j\,, \qquad f=f_0+f_1+f_2+\cdots+f_{n-1}\,.$$

Extend every function f_j to a function $\tilde{f}_j\in C_0^\infty(G)$ in such a way that

$$\tilde{f}_j(\lambda g)=\lambda^j f_j(g) \qquad (\lambda\in\boldsymbol{T},\ g\in G_0)$$

and put $\tilde{f}=\tilde{f}_0+\tilde{f}_1+\cdots+\tilde{f}_{n-1}$. Then, for any two functions $f, h\in C_0^\infty(G_0)$,

$$\begin{aligned}\int_{G_0} f(g)\overline{h(g)}\,dg &= \sum_{j=0}^{n-1}\int_{G_0} f_j(g)\overline{h_j(g)}\,dg \\ &= \sum_{j=0}^{n-1}\int_G \tilde{f}_j(g)\overline{\tilde{h}_j(g)}\,dg=\int_G \tilde{f}(g)\overline{\tilde{h}(g)}\,dg\,,\end{aligned} \tag{5.2}$$

where the Haar measure dg on G is the one defined in the preceding section by (4.2).

Let $T_{I,\chi}^r$ be the representation of G defined earlier, whose character is $\pi_{I,\chi}$. Let us denote by ${}^0T_{I,\chi}^r$ the representation of G_0 induced by $T_{I,\chi}^r$ of G. Let ${}^0\pi_{I,\chi}$ be the character of ${}^0T_{I,\chi}^r$. Then

$${}^0\pi_{I,\chi}(g)=\pi_{I,\chi}(g) \qquad (g\in G_0')\,, \tag{5.3}$$

where G_0' is the set of all regular elements of G_0.

Recall that if χ is of type r and $\pi_{I,\chi}$ is the character of $T_{I,\chi}^r$, then $\chi=(c, d)$ or (c, m, ρ), where

$$c_1>c_2>\cdots>c_{n-2r}\,;\quad d_j=2^{-1}(m_j+\sqrt{-1}\rho_j)\,,\quad d_{-j}=2^{-1}(m_j-\sqrt{-1}\rho_j)\quad (1\leqq j\leqq r)\,.$$

Define

$$n_\chi=\sum_{k=1}^{n-2r}c_k+\sum_{j=1}^{r}(d_j+d_{-j})=\sum_{k=1}^{n-2r}c_k+\sum_{j=1}^{r}m_j\,, \tag{5.4}$$

then

(5.5) $$\pi_{I,\chi}(\lambda g)=\lambda^{n_\chi}\pi_{I,\chi}(g) \qquad (\lambda\in \boldsymbol{T},\ g\in G).$$

For any χ and integer s, define $\chi^s=(c', d')$ as follows:

(5.6) $$c'_k=c_k+s \qquad (1\leqq k\leqq n-2r);$$
$$d'_j=d_j+s=2^{-1}(m_j+2s)+\sqrt{-1}\rho_j,\quad d'_{-j}=d_{-j}+s=2^{-1}(m_j+2s)-\sqrt{-1}\rho_j \qquad (1\leqq j\leqq r).$$

Then it follows from the definition of $\pi_{I,\chi}$ in §1 that

$$\pi_{I,\chi^s}(g)=(\det g)^s\pi_{I,\chi}(g) \qquad (g\in G)$$

and hence for any s, ${}^0\pi_{I,\chi^s}={}^0\pi_{I,\chi}$. This means that ${}^0T^r_{I,\chi^s}$ and ${}^0T^r_{I,\chi}$ are unitary equivalent. Therefore any ${}^0T^r_{I,\chi}$ is equivalent to a certain ${}^0T^r_{I,\chi'}$ for which $c'_{n-2r}=0$. Moreover when $c_{n-2r}=0$ and $c'_{n-2r}=0$ for χ and χ' respectively, ${}^0T^r_{I,\chi}$ and ${}^0T^r_{I',\chi'}$ are unitary equivalent if and only if $I=I'$ and $\chi=\chi'$.

Let $f\in C_0^\infty(G_0)$. Evidently, if $n_\chi\neq -j$,

(5.7) $$Sp(T_{I,\chi}(\tilde{f}_j))=0.$$

If $n_\chi=-j$, we obtain by a calculation analogous as in (5.2) that

(5.8) $$Sp(T^r_{I,\chi}(\tilde{f}_j))=\int_G \pi_{I,\chi}(g)\tilde{f}_j(g)\,dg$$
$$=\int_{G_0}{}^0\pi_{I,\chi}(g)f_j(g)\,dg=Sp({}^0T^r_{I,\chi}(f_j)),$$

where ${}^0T^r_{I,\chi}(f_j)=\int_{G_0}{}^0T^r_{I,\chi}(g)f_j(g)\,dg$.

Let $\mathcal{S}_{n-2r}$ be the symmetric group of order $n-2r$. For any $\chi=(c, d)$ of type r and for $\sigma\in\mathcal{S}_{n-2r}$, put $\sigma\chi=(\sigma c, d)$. Let us recall the definitions in §1: for any χ and $\sigma\in\mathcal{S}_{n-2r}$, $\pi_{\sigma\chi}=\operatorname{sgn}(\sigma)\pi_\chi$; and if $c_1\geqq c_2\geqq\cdots\geqq c_{n-2r}$ for χ, $\pi_\chi=\sum_I\pi_{I,\chi}$. Hence, if $\prod_{i\leqq j<l\leqq n-2r}(c_j-c_l)=0$, then $\pi_\chi=0$ and of course

(5.9) $$\int_G\pi_\chi(g)f(g)\,dg=0 \qquad (f\in C_0^\infty(G));$$

and if $\prod_{1\leqq j<l\leqq n-2r}(c_j-c_l)\neq 0$ and $c_{\sigma(1)}>c_{\sigma(2)}>\cdots>c_{\sigma(n-2r)}$ for χ, where $\sigma\in\mathcal{S}_{n-2r}$, then

(5.9′) $$\int_G\pi_\chi(g)f(g)\,dg=\operatorname{sgn}(\sigma)\int_G\pi_{\sigma\chi}(g)f(g)\,dg=\operatorname{sgn}(\sigma)\sum_I Sp(T^r_{I,\sigma\chi}(f)).$$

Since $L(\sigma\chi)=\operatorname{sgn}(\sigma)L(\chi)$, we see that, in the latter case,

(5.10) $$L(\chi)e_{j_1}(\rho_1)e_{j_2}(\rho_2)\cdots e_{j_r}(\rho_r)\int_G\pi_\chi(g)f(g)\,dg$$
$$=|L(\chi)e_{j_1}(\rho_1)e_{j_2}(\rho_2)\cdots e_{j_r}(\rho_r)|\sum_I Sp(T^r_{I,\sigma\chi}(f)).$$

Fix $f\in C_0^\infty(G_0)$ and apply (3.9), (3.11) and (3.12) to $\hat{f}_j\in C_0^\infty(G)$, then (3.11) is rewritten as

$$(5.11)\quad \gamma\tilde{f}_j(e)=\gamma f_j(e)$$
$$=\sum_{r=0}^{q}\frac{1}{(n-2r)!\,r!\,2^r}\sum_{j_1,\cdots,j_r=0,1}\sum_{c_1\in Z}\cdots\sum_{c_{n-2r}\in Z}\sum_{m_1\in Z_{j_1}}\cdots\sum_{m_r\in Z_{j_r}}$$
$$\int_{\rho_1\in R}\cdots\int_{\rho_r\in R}V_j(\chi\,;f_j)\,|L(\chi)e_{j_1}(\rho_1)e_{j_2}(\rho_2)\cdots e_{j_r}(\rho_r)|\,d\rho_1 d\rho_2\cdots d\rho_r\,,$$

where

$$(5.12)\qquad \gamma=(2\pi)^n n!\,\alpha_0\gamma_0=4^{pq}q!\Big(\prod_{j=1}^{q-1}j!\Big)^2\cdot\prod_{i=1}^{p-1}i!\cdot\prod_{m=1}^{q}(p-m)\,;$$

and, when $n_\chi\neq -j$, $V_j(\chi, f_j)=0$; when $n_\chi=-j$, $V_j(\chi, f_j)=0$ if $\prod_{1\leqq j<l\leqq n-2r}(c_j-c_l)=0$, and

$$V_j(\chi\,;f_j)=\sum_{I}Sp({}^0T_{I,\sigma\chi}^{\gamma}(f_j))$$

if $\prod_{1\leqq j<l\leqq n-2r}(c_j-c_l)\neq 0$ and $c_{\sigma(1)}>c_{\sigma(2)}>\cdots>c_{\sigma(n-2r)}$ for $\sigma\in\mathcal{S}_{n-2r}$. The order of the above successive integro-summation can be arbitrarily changed.

Taking into account that $Sp({}^0T_{I,\chi}^{\gamma}(f_j))=0$ if $n_\chi\not\equiv -j \pmod n$ and summing up the above equalities over $0\leqq j\leqq n-1$, we obtain that for any $f\in C_0^\infty(G_0)$,

$$(5.13)\quad \gamma f(e)=\sum_{r=0}^{q}\frac{1}{(n-2r)!\,r!\,2^r}\sum_{j_1,\cdots,j_r=0,1}\sum_{c_1\in Z}\cdots\sum_{c_{n-2r}\in Z}\sum_{m_1\in Z_{j_1}}\cdots\sum_{m_r\in Z_{j_r}}$$
$$\int_{\rho_1\in R}\cdots\int_{\rho_r\in R}V(\chi\,;f)\,|L(\chi)e_{j_1}(\rho_1)e_{j_2}(\rho_2)\cdots e_{j_r}(\rho_r)|\,d\rho_1 d\rho_2\cdots d\rho_r\,,$$

where, when $n_\chi\leqq -n$ or $n_\chi>0$, $V(\chi\,;f)=0$; when $-n+1\leqq n_\chi\leqq 0$, $V(\chi\,;f)=0$ if $\prod_{1\leqq j<l\leqq n-2r}(c_j-c_l)=0$, and

$$V(\chi\,;f)=\sum_{I}Sp({}^0T_{I,\sigma\chi}^{\gamma}(f))$$

if $\prod_{1\leqq j<l\leqq n-2r}(c_j-c_l)\neq 0$ and $c_{\sigma(1)}>c_{\sigma(2)}>\cdots>c_{\sigma(n-2r)}$ for $\sigma\in\mathcal{S}_{n-2r}$. This is essentially the Plancherel formula for $G_0=SU(p, q)$.

Let $f=h*h^*$ $(h\in C_0^\infty(G_0))$, then $Sp({}^0T_{I,\chi}^{\gamma}(f))\geqq 0$. Hence the above successive integro-summation converges absolutely. So is also when $f=f_1*f_2$, where $f_1, f_2\in C_0^\infty(G_0)$. Recall that $Sp({}^0T_{I,\chi}^{\gamma}(f))=Sp({}^0T_{I,\chi^s}^{\gamma}(f))$ for any integer s and $f\in C_0^\infty(G_0)$ and note that the set of all χ of type r such that $-n+1\leqq n_\chi\leqq 0$ corresponds one-to-one onto the set of all χ of type r such that $c_{n-2r}=0$, in such a way that $\chi\rightarrow\chi^s$, where $s=-c_{n-2r}$. Then we obtain the following theorem which gives the explicit form of the Plancherel formula for $G_0=SU(p, q)$.

THEOREM 4. *Let dg be the normalized Haar measure on G_0 for which* (5.1) *holds. Then for any $f=f_1*f_2$ $(f_1, f_2\in C_0^\infty(G_0))$,*

$$(5.14)\qquad \gamma f(e)=\sum_{r=0}^{q}\sum_{j_1,\cdots,j_r=0,1}\sum_{c_1>c_2>\cdots>c_{n-2r-1}>c_{n-2r}=0}\sum_{\substack{m_s\in Z_{j_s}\\(1\leqq s\leqq r)}}$$
$$\int_{\rho_1>\rho_2>\cdots>\rho_r>0}\sum_I Sp({}^0T^{\gamma}_{I,\chi}(f))\,|L(\chi)e_{j_1}(\rho_1)e_{j_2}(\rho_2)\cdots e_{j_r}(\rho_r)|\,d\rho\,,$$

where $d\rho=d\rho_1 d\rho_2\cdots d\rho_r$, $\chi=(c, m, \rho)$, *and* I *runs over all subset of* $p-r$ *elements of* $I_{n-2r}=\{1, 2, \cdots, n-2r\}$, *and* γ *is the constant given by* (5.12). *The successive integro-summation converges absolutely.*

Appendix. Proof of Proposition 1.

Here we use the notations in [2, § 3]. Let ω denote a complex matrix of type $q\times p$ and let $\omega^*={}^t\bar{\omega}$ be as usual. Denote by Ω the homogeneous bounded domain of dimension pq defined by $\omega\omega^*<1_q$ (i. e., $1_q-\omega\omega^*$ is positive definite). The group $G=U(p, q)$ operates on Ω on the right as

$$\omega\to(\omega g_{12}+g_{22})^{-1}(\omega g_{11}+g_{21})\qquad \left(g=\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}\in G\right),$$

where g_{ij} are as in (4.3).

M. I. Graev has constructed a series of square-integrable irreducible unitary representations of G on the Hilbert spaces of the appropriate analytic functions on Ω [2, § 3]. Let us call them the representations of Graev[13].

1. Let k and l be integers such that $k-l\geqq n$. And let $g\to T(g)$ $(g\in G)$ be a representation of Graev (see [2, p. 358]) corresponding to the parameter

$$(1)\qquad (k_1, k_2, \cdots, k_p\,;\, l_1, l_2, \cdots, l_q)=(k, k, \cdots, k\,;\, -l, -l, \cdots, -l)\,.$$

Then the Hilbert space $\mathcal{H}$ on which T is realized, is the space of all analytic functions φ on Ω which fulfill the following condition:

$$\|\varphi\|^2=\int_\Omega|\varphi(\omega)|^2 d\mu(\omega)<+\infty\,,$$

where $d\mu(\omega)$ is a measure on Ω invariant under the operation of G. T is defined as follows:

$$(2)\qquad T(g)\varphi(\omega)=\varphi[(\omega g_{12}+g_{22})^{-1}(\omega g_{11}+g_{21})]\cdot|g_{11}^*+g_{21}^*\omega|^{-k}|\omega g_{12}+g_{22}|^{l}\,,$$

where $|\omega g_{12}+g_{22}|=\det(\omega g_{12}+g_{22})$ etc.

Let φ_0 be the constant function 1 on Ω. As is easily seen, $\mathcal{H}(\mathcal{D}_{k,l})=\boldsymbol{C}\varphi_0$. Moreover we can prove that when $p\neq q$, $\mathcal{H}(\mathcal{D})=\{0\}$ if $\dim\mathcal{D}=1$ and $\mathcal{D}\neq\mathcal{D}_{k,l}$.

13) Comparing the characters of the representations of Graev and the ones $\pi_{I,c}$ of $T^0_{I,c}$, we see that any representation of Graev is equivalent to some $T^0_{I,c}$ for which the subset I of $I_n=\{1, 2, \cdots, n\}$ is $\{1, 2, \cdots, p\}$ or $\{q+1, q+2, \cdots, n\}$ and conversely any such $T^0_{I,c}$ is equivalent to a representation of Graev.

It follows from (2) that

(3) $$T(g)\varphi_0=|g_{11}^*+g_{21}^*\omega|^{-k}|\omega g_{12}+g_{22}|^l\,.$$

Let $E(\mathcal{D}_{k,l})$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}(\mathcal{D}_{k,l})$. We wish to calculate $E(\mathcal{D}_{k,l})T(g)\varphi_0$. For this purpose, we utilize the following structure of $\mathcal{H}$. Let $\mathcal{H}_s$ $(s\geqq 0)$ be the vector space of all polynomials of degree s of ω_{ij} $(1\leqq i\leqq q,\ 1\leqq j\leqq p)$, where ω_{ij} are the matrix elements of ω. Then $\mathcal{H}_s\subset\mathcal{H}$, $\mathcal{H}_s\perp\mathcal{H}_{s'}$ if $s\neq s'$, and $\mathcal{H}$ is the direct sum of these $\mathcal{H}_s$ $(s\geqq 0)$. A series $\sum_{s=0}^{+\infty}\phi_s$ $(\phi_s\in\mathcal{H}_s)$ converges in $\mathcal{H}$ if and only if it converges uniformly on every compact subset of Ω, and the limit function in the first sence is identical with the one in the second sence.

Now put $a=(g_{11}^*)^{-1}g_{21}^*=g_{12}g_{22}^{-1}$, then $aa^*<1_p$. Therefore

(4) $$\begin{cases}(g_{11}^*+g_{21}^*\omega)^{-1}=(1_p+a\omega)^{-1}(g_{11}^*)^{-1}=\left\{\sum_{s=0}^{+\infty}(-1)^s(a\omega)^s\right\}(g_{11}^*)^{-1}\,,\\ (\omega g_{12}+g_{22})^{-1}=g_{22}^{-1}(\omega a+1_q)^{-1}=g_{22}^{-1}\left\{\sum_{s=0}^{+\infty}(-1)^s(\omega a)^s\right\}\,.\end{cases}$$

These power series of matrices $a\omega$ and ωa converge absolutely and uniformly on every compact subset of Ω. The direct sum decomposition according to $\mathcal{H}=\sum_{s=0}^{+\infty}\oplus\mathcal{H}_s$ of $T(g)\varphi_0=|g_{11}^*+g_{21}^*\omega|^{-k}|\omega g_{12}+g_{22}|^l$ can be obtained by calculating formally the above power series of matrices. Therefore we see that

(5) $$E(\mathcal{D}_{k,l})T(g)\varphi_0=|g_{11}^*|^{-k}|g_{22}|^l=|g_{11}^*|^{-k}|g_{22}|^l\varphi_0\,.$$

This proves that $\phi_{k,l}(g)=|g_{11}^*|^{-k}|g_{22}|^l$ is the spherical function of type $\mathcal{D}_{k,l}$ of the representation T.

Let I and c be as in Proposition 1. The fact that T is equivalent to $T_{I,c}^0$ is proved by comparing the character of T [2, p. 375] and the one $\pi_{I,c}$ of $T_{I,c}^0$[14].

2. Now suppose that $k-l\geqq n+2$. Let us treat the case of $\phi_{k,l}$. Let T be a representation of Graev corresponding to the parameter

(6) $$(k_1,k_2,\cdots,k_p\,;\,l_1,l_2,\cdots,l_q)=(k,k,\cdots,k,k-1\,;\,-l,-l,\cdots,-l,-l-1)\,.$$

Denote by $\mathfrak{M}_p$ the set of all matrices of type $p\times p$. The Hilbert space $\mathcal{H}$ on which T is realized, is the set of all functions

$$\varphi(z^{(1)},z^{(2)},\omega)\qquad(z^{(1)}\in\mathfrak{M}_p,\ z^{(2)}\in\mathfrak{M}_q,\ \omega\in\Omega)$$

---

14) In the proof of the character formula in [2, § 5], Graev admitted tacitly the recent result of Harish-Chandra, which asserts that any invariant eigendistribution π on G has no contribution on the set of all singular elements $G-G'$, in other words, π is essentially determined when we know $\pi(f)$ for all $f\in C_0^\infty(G')$. In fact, the contribution of the characters on the subset $G-G'$ is not discussed there.

fulfilling the following three conditions. (i) φ is analytic in ω. (ii) Let x_j (y_l resp.) be the (p, j) ((q, l) resp.) cofactor of $z^{(1)}$ ($z^{(2)}$ resp.). Then φ contains the variables $z^{(1)}$ and $z^{(2)}$ in such a manner that it is homogeneous linear with respect to $x=(x_1, x_2, \cdots, x_p)$ and to $y=(y_1, y_2, \cdots, y_q)$, respectively. (Even when $q=1$, we introduce the variable y in order to discuss this case together with the others.) (iii) The integral (3.12) in [**2**, p. 357] is convergent. The inner product in $\mathcal{H}$ is defined by this integral (3.12).

Let us denote $\varphi(z^{(1)}, z^{(2)}, \omega)$ by $\phi(x, y, \omega)$. Let $\mathcal{H}_s$ be the set of all $\phi(x, y, \omega)$ which is homogeneous linear with respect to x and to y respectively and is a homogeneous polynomial of degree s of ω_{ij}'s. Then $\mathcal{H}_s$ $(s\geqq 0)$ are the mutually orthogonal subspaces of $\mathcal{H}$ and $\mathcal{H}$ is the direct sum of them. The same statement as in Subsection 1 holds for the convergence of a series $\sum_{s=0}^{+\infty}\phi_s$ $(\phi_s\in\mathcal{H}_s)$. The operator $T(g)$ is defined as

$$T(g)\phi(x, y, \omega)=\phi[x(\bar{g}_{11}+{}^t\omega\bar{g}_{21}), y(\omega g_{12}+g_{22}), (\omega g_{12}+g_{22})^{-1}(\omega g_{11}+g_{21})] \times|g_{11}^*+g_{21}^*\omega|^{-k}|\omega g_{12}+g_{22}|^{l}. \tag{7}$$
[15]

Put $\phi_0(x, y, \omega)=y\omega{}^tx$. Then $\mathcal{H}(\mathcal{D}_{k,l})=\boldsymbol{C}\phi_0$. Moreover we can see that when $p\neq q$, $\mathcal{H}(\mathcal{D})=\{0\}$ if $\dim\mathcal{D}=1$ and $\mathcal{D}=\mathcal{D}_{k,l}$.

Let us calculate $E(\mathcal{D}_{k,l})T(g)\phi_0$. First of all,

$$T(g)\phi_0=y(\omega g_{11}+g_{21})(g_{11}^*+g_{21}^*\omega)^t x\cdot|g_{11}^*+g_{21}^*\omega|^{-k}|\omega g_{12}+g_{22}|^{l}. \tag{8}$$

It follows from (4) that

$$|g_{11}^*+g_{21}^*\omega|^{-1}\equiv|g_{11}^*|^{-1}\{1-Sp(a\omega)\},$$
$$|\omega g_{12}+g_{22}|^{-1}\equiv|g_{22}|^{-1}\{1-Sp(\omega a)\}=|g_{22}|^{-1}\{1-Sp(a\omega)\}$$

modulo power series of ω_{ij} of degree higher than 1. Hence

$$|g_{11}^*+g_{21}^*\omega|^{-k}|\omega g_{12}+g_{22}|^{l}\equiv|g_{11}^*|^{-k}|g_{22}|^{l}\{1+(-k+l)Sp(a\omega)\}.$$

Therefore, in the sence of modulo $\sum_{s\neq 1}\oplus\mathcal{H}_s$,

$$\begin{aligned}T(g)\phi_0&\equiv y(\omega g_{11}+g_{21})(g_{11}^*+g_{21}^*\omega)^t x\cdot\{1+(-k+l)Sp(a\omega)\}|g_{11}^*|^{-k}|g_{22}|^{l}\\&\equiv|g_{11}^*|^{-k}|g_{22}|^{l}\{y(\omega g_{11}g_{11}^*+g_{21}g_{21}^*\omega)^t x+(-k+l)yg_{21}g_{11}^{*}{}^t x\cdot Sp(a\omega)\}\\&=\phi_1 \quad \text{(say)}.\end{aligned}$$

Since every $\mathcal{H}_s$ is invariant under $T(u)$ $(u\in U)$, we see that

$$E(\mathcal{D}_{k,l})T(g)\phi_0=E(\mathcal{D}_{k,l})\phi_1,$$

and by definition,

15) The formula (3.23) in [**2**, p. 358] of $T(g)\varphi$ is transformed into this one under $\varphi\to\phi$.

$$E(\mathcal{D}_{k,l})\phi_1=\int_U T(u)\phi_1\cdot|u_{11}|^{-k}|u_{22}|^{-l}du\,.$$

Let du_{11} ($u_{11}\in U(p)$) and du_{22} ($u_{22}\in U(q)$) be the normalized Haar measures on $U(p)$ and on $U(q)$ such that $\int_{U(p)}du_{11}=\int_{U(q)}du_{22}=1$, respectively. Then $du=du_{11}du_{22}$. Therefore

$$\begin{aligned}&\{|g^*_{11}|^{-k}|g_{22}|^{l}\}^{-1}E(\mathcal{D}_{k,l})T(g)\phi_0\\&=\int_{U(p)}y\omega u_{11}g_{11}g^*_{11}u_{11}^{-1}{}^t x\,du_{11}+\int_{U(q)}yu_{22}g_{21}g^*_{21}u_{22}^{-1}\omega\,{}^t x\,du_{22}\\&\quad+(-k+l)\int_{U(p)\times U(q)}yu_{22}g_{21}g^*_{11}u_{11}^{-1}{}^t x\,Sp(au_{22}^{-1}\omega u_{11})\,du_{11}\,du_{22}\,.\end{aligned}$$

By an easy calculation, we see that this is equal to

$$\begin{aligned}&\Big\{\frac{1}{p}Sp(g_{11}g^*_{11})+\frac{1}{q}Sp(g_{21}g^*_{21})+\frac{-k+l}{pq}Sp(g_{21}g^*_{11}a)\Big\}\,y\omega\,{}^t x\\&=\Big\{\frac{1}{p}Sp(g_{11}g^*_{11})+\frac{p-k+l}{pq}Sp(g_{21}g^*_{21})\Big\}\phi_0\\&=\{|g^*_{11}|^{-k}|g_{22}|^{l}\}^{-1}\phi_{k,l}(g)\phi_0\,.\end{aligned}$$

This proves that $\phi_{k,l}$ is the spherical function of type $\mathcal{D}_{k,l}$ of the representation T.

The unitary equivalence of T and $T^0_{I,c}$ given in Proposition 1 is proved analogously as in Subsection 1. Now the proof of Proposition 1 is complete.

REMARK. We can prove the following fact. The representation of Graev corresponding to the parameter $(k_1,k_2,\cdots,k_p\,;\,l_1,l_2,\cdots,l_q)$ contains $\mathcal{D}_{k,l}$ (i.e., $\mathcal{H}(\mathcal{D}_{k,l})\neq\{0\}$) if and only if there exist q non-negative integers $m_1,m_2,\cdots,m_q$ such that $0\leqq m_1\leqq m_2\leqq\cdots\leqq m_q$ and

$$\begin{aligned}&(k_1,k_2,\cdots,k_p\,;\,l_1,l_2,\cdots,l_q)\\&\quad=(k,k,\cdots,k,k-m_1,k-m_2,\cdots,k-m_q\,;\,-l-m_1,-l-m_2,\cdots,-l-m_q)\,.\end{aligned}$$

In this case, put $s=m_1+m_2+\cdots+m_q$, then $\mathcal{H}(\mathcal{D}_{k,l})\subset\mathcal{H}_s$, where $\mathcal{H}_s$ is defined analogously as above (see [2, § 4]).

In this Appendix we treated the simplest cases where $(m_1,m_2,\cdots,m_q)=(0,0,\cdots,0)$ or $(0,0,\cdots,0,1)$.

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